Munich 2010 Lecture on Topological String Theory

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Plan
1. Definition
2. Computation
3. Applications

1. Definition

Start with $\mathcal{N}=(2,2)$ SCFT on 2d world sheet Σ.

(e.g. sigma-model $\Sigma \to M$
M : Calabi-Yau n-fold)

$\mathcal{N}=(2,2)$ SCA : $\underline{T}, \underline{G}^+, \underline{G}^-, \underline{J}$ (left)
$\bar{T}, \bar{G}^+, \bar{G}^-, \bar{J}$ (right)

$$J(z)J(w) \sim \frac{\hat{c}}{(z-w)^2}$$

Virasoro $c = 3\hat{c}$

For the sigma-model, $\hat{c} = \dim_{\mathbb{C}} M$

Topological twisting

$$T \rightarrow \tilde{T} = T + \frac{1}{2}\partial J$$

(we can also choose "$-$")

This changes conformal dimensions:

$$G^+ : \left(\tfrac{3}{2}, 0\right) \rightarrow (1, 0)$$
$$G^- : \left(\tfrac{3}{2}, 0\right) \rightarrow (2, 0)$$

Since G^+ is of $(1, 0)$, it is natural to define:

$$Q_{BRST} = \oint G^+$$

- $Q_{BRST}^2 = 0$
- $\tilde{T} = \{ Q_{BRST}, G^- \}$

This is just as if G^- is the anti-ghost b of the bosonic string.

What does the twist mean for the worldsheet theory?

The topological twist

$\Updownarrow$

Coupling of J to an $U(1)$ gauge field $\bar{A}$ as

$$\int_\Sigma \bar{A} J$$

and setting $\bar{A} = -\frac{1}{2}\bar{\omega}$ ω : spin connection

$$(\text{Similarly } A = +\frac{1}{2}\omega)$$

If the EM tensor for $\bar{A}=0$ is T,

the one for $\bar{A} = -\frac{1}{2}\bar{\omega}$ is $\tilde{T} = T + \frac{1}{2}\partial J$

Q : Show this.

Hint : Show $\delta\omega_{\bar{z}} \sim g_{z\bar{z}}\,\partial_z\,\delta g^{zz}$

∘ Sigma model $\Sigma \to CY_n$

What is CY ?

- complex : can define x^i $(i=1,\ldots,n)$
- Kähler : $g_{ij}=0$, $g_{\bar{i}\bar{j}}=0$
  $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$
- Ricci-flat : $R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det g = 0$

$\Updownarrow$

$\exists 1$ holomorphic n-form $\Omega_{i_1 \cdots i_n}$

s.t. $\det g = \Omega \bar{\Omega}$.

Sigma-model

$X^i : \Sigma \to CY_n$

ψ^i : fermion

$$J = g_{i\bar{j}} \psi^i \psi^{\bar{j}}$$

$$G^+ = g_{i\bar{j}} \psi^i \partial X^{\bar{j}} \ , \ G^- = g_{i\bar{j}} \psi^{\bar{j}} \partial X^i$$

$\bar{J}, \bar{G}^+, \bar{G}^-$: $\partial \to \bar{\partial}$, $\psi \to \bar{\psi}$

If we twist $\tilde{T} = T + \frac{1}{2} \partial J$

$\tilde{\bar{T}} = \bar{T} + \frac{1}{2} \bar{\partial} \bar{J}$

$G^+ : (1,0)$, $\bar{G}^+ : (0,1)$

$$Q_{BRST} = \oint G^+ dz + \oint \bar{G}^+ d\bar{z}$$

$\psi^i : (\frac{1}{2}, 0) \to (1, 0)$

$\psi^{\bar{i}} : (\frac{1}{2}, 0) \to (0, 0)$: scalar.

$\delta X^i = [\varepsilon Q_{BRST}, X^i] = \varepsilon \psi^i$

$\delta X^{\bar{i}} = \varepsilon \bar{\psi}^{\bar{i}}$

$\delta \psi^{\bar{i}} = \varepsilon \partial X^{\bar{i}}$, $\delta \bar{\psi}^i = \varepsilon \bar{\partial} X^i$

BRST inv. configuration : $X : \Sigma \longrightarrow CY_n$ holomorphic

This is the A-model.

There is another way to twist the model.

(There are 2 more, but complex conjugate ...)

$$\tilde{T} = T - \frac{1}{2}\partial J, \quad \bar{\tilde{T}} = \bar{T} + \frac{1}{2}\bar{\partial}\bar{J}.$$

$$G^- : (0,1), \quad \bar{G}^+ : (1,0)$$
$$Q_{BRST} = \oint G^- + \oint \bar{G}^+.$$

$$\psi^{\bar{i}} : (0,0), \quad \bar{\psi}^{\bar{i}} : (0,0)$$

$\Rightarrow$ BRST transformation :

$$\delta X^i = 0, \quad \delta X^{\bar{i}} = \varepsilon(\psi^{\bar{i}} + \bar{\psi}^{\bar{i}})$$
$$\delta\psi^i = \varepsilon\,\partial X^i, \quad \delta\bar{\psi}^i = \varepsilon\,\bar{\partial} X^i$$

BRST invariant configuration :

$$X : \Sigma \to CY_n \qquad \begin{array}{l}\partial X = 0\\ \bar{\partial} X = 0\end{array}$$

X : constant map.

This is the B-model.

Mirror pair of CY_n : (M, W)

A-model on M = B-model on W.

- BRST cohomology.

Since $\widetilde{T} = \{ Q_{BRST}, * \}$,

we look for operators of dim $(0,0)$.

A-model :

$$\omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}(X) \psi^{i_1} \cdots \psi^{i_p} \bar{\psi}^{\bar{j}_1} \cdots \bar{\psi}^{\bar{j}_q}$$

Q_{BRST} acts as $d = \partial + \bar{\partial}$

$$\Rightarrow \bigoplus_{p,q=0}^{n} H^{p,q}(M)$$

(p, q) = $J, \bar{J}$ charges

B-model.

Since $\delta \chi^{\bar{i}} = \varepsilon (\psi^{\bar{i}} + \bar{\psi}^{\bar{i}})$

define $\eta^{\bar{i}} = \psi^{\bar{i}} + \bar{\psi}^{\bar{i}}$

$\theta_i = g_{i\bar{j}} (\psi^{\bar{j}} - \bar{\psi}^{\bar{j}})$

$V_{\bar{i}_1 \cdots \bar{i}_p}{}^{j_1 \cdots j_q} \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p} \theta_{j_1} \cdots \theta_{j_q}$

Q_{BRST} acts as $\bar{\partial}$ w/ coefficients θ's.

$\Rightarrow \bigoplus_{p,q} H_{\bar{\partial}}^{(0,p)} (M, \wedge^q T^{(1,0)} M)$

Thus, in particular

(M, W) : mirror pair

$H^{(p,q)}(M) \sim H_{\bar{\partial}}^{(0,p)} (W, \wedge^q T^{(1,0)} W)$

- marginal deformations
  $\Rightarrow$ moduli space.

A-model

$k_a \in H^{(1,1)}(M)$

$\left(a = 1, \cdots, h^{1,1} = \dim H^{1,1}(M) \right)$

Define $\phi_a = k_{a i \bar{j}} \psi^i \psi^{\bar{j}}$

$\left([Q_{BRST}, \phi_a] = 0, \ \phi_a \neq \{Q_{BRST}, *\} \right)$

$$G_{-1}^{-} \bar{G}_{-1}^{-} \phi_a = k_{a i \bar{j}} \partial X^i \bar{\partial} X^{\bar{j}} + \cdots$$

- (1,1) form on Σ
- $\delta_{BRST}(\ \ ) =$ total derivative on Σ

$$\sum_a t^a \int_\Sigma G_{-1}^{-} \bar{G}_{-1}^{-} \phi_a$$

t^a : complexified Kähler moduli

To keep the action real,
we add BRST trivial piece:

$$\sum_a \bar{t}^a \int \underbrace{G_0^+ \bar{G}^+ \bar{\Phi}_a}_{= \{Q^L_{BRST}, [Q^R_{BRST}, \bar{\Phi}_a]\}}$$

$$\Downarrow$$

$$(t^a + \bar{t}^a) \int k_{a i \bar{j}} (\partial X^i \bar{\partial} X^{\bar{j}} + \partial X^{\bar{j}} \bar{\partial} X^i)$$

$$+ (t^a - \bar{t}^a) \int k_{a i \bar{j}} (\,{}''\, - \,{}''\,)$$

Kähler class   B-field   worldsheet instanton term.

B-model

$$H^{(0,1)}_{\bar{\partial}} (M, T^{(1,0)} M) \rightarrow \text{complex moduli}$$

(analogue of the Beltrami differentials on Σ)

A BRST invarian amplitude may depend on t but not on $\bar{t}$.

$\Rightarrow$ holomorphic in the moduli space

$\begin{cases} \text{Kähler moduli for A-model} \\ \text{Complex structure moduli for B-model.} \end{cases}$

- A-model :

$\bar{t} \to \infty$: $k a_{i\bar{j}} \partial X^{\bar{j}} \bar{\partial} X^{i}$

$\bar{\partial} X = 0$: instanton approx is exact

- B-model :

no Kähler moduli dependence

Large volume limit of M

$\to$ only constant maps contribute.

- CY_3 is special.

$$\int \bar{A} J + A \bar{J}$$

$$\Rightarrow \qquad \bar{\partial} J \sim \hat{c} F_{z\bar{z}}$$

chiral anomaly

Since we set $A = -\frac{i}{2}\omega$,

we have $\bar{\partial} J \sim \hat{c} R \sqrt{g}$.

- Another way to see this:

Set $J = i\sqrt{\hat{c}}\, \partial\phi$

$$J(z) J(w) \sim \frac{\hat{c}}{(z-w)^2}$$

$$\Rightarrow \phi(z)\phi(w) \sim \log \frac{1}{z-w}$$

$\Rightarrow$ action for $\phi = \int \frac{1}{2} \partial\phi \bar{\partial}\phi$

Add

$$\sqrt{\hat{c}} \int \frac{1}{2} \bar{\omega}\, \partial \Phi + \frac{1}{2} \omega\, \bar{\partial} \Phi$$

$$\to \frac{\sqrt{\hat{c}}}{2} \int R\, \Phi \sqrt{g}$$

Since $\frac{1}{2} \int R \sqrt{g} = 1 - g$ ← genus of Σ.

the total amount of J-charge violation is $\hat{c}(1-g)$.

(Yet anothe way to see this:

ψ^i : $(0,0)$, $\Rightarrow$ # zero modes $= n$

$\psi^{\bar{i}}$: $(1,0)$ $\Rightarrow$ # zero modes $= ng$

$\Rightarrow n(1-g)$ $\quad\quad n = \dim_{\mathbb{C}} M$.

The index does not change by perturbation).

In the following, we use the A-model notation.

$$Q_{BRST} = \oint G^+ + \oint \bar{G}^+ .$$

If $\hat{c} = 3$, we can define

$$\langle \prod_{i=1}^{3g-3} G^-(z_i)\, \bar{G}^-(\bar{w}_i) \rangle$$

The Beltrami diff's $\eta^z{}_{\bar{z}}$

$\Leftrightarrow \; T\mathcal{M}_g \leftarrow$ moduli space of Σ.

Since $\int_\Sigma G^- \eta$ is well defined.

$\langle (G^-)^{3g-3} (\bar{G}^-)^{3g-3} \rangle$ is

the top form on $\mathcal{M}_g$.

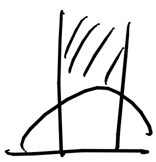
$$\Rightarrow F_g = \int_{\mathcal{M}_g} \langle (G^-)^{3g-3} (\bar{G}^-)^{3g-3} \rangle$$

- For $g=1$, we need to fix translational invariance

$$\frac{\partial F_{g=1}}{\partial t^a} = \int_{\mathcal{M}_{g=1,\,n=1}} \langle G^- \bar{G}^- \phi_a(0) \rangle$$

This can be integrated to give

$$F_g = \int \langle J \bar{J} \rangle$$



$$\langle J \bar{J} \rangle = \mathrm{tr}\, (-1)^{J_0 + \bar{J}_0} J_0 \bar{J}_0 \, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}$$

- For $g=0$, we need to fix 3 pt's.

$$C_{abc} = \langle \phi_a(0) \phi_b(1) \phi_c(\infty) \rangle$$

It turns out, $\exists$ prepotential $F_0(t)$

$$C_{abc} = \frac{\partial^3 F_0}{\partial t^a \partial t^b \partial t^c}$$

- Special Geometry of the moduli space of CY_3.

The moduli space $\mathcal{M}$ is locally a product $\mathcal{M}_K \times \mathcal{M}_c$

Kähler   complex

A-model   B-model

- $\mathcal{M}_C$

Since the hol 3-form Ω is unique up to scale, it defines a line bundle $\mathcal{L}$ over $\mathcal{M}_C$. (= subbundle of the Hodge bundle)

with a metric

$$\|\Omega\|^2 = i \int_M \Omega \wedge \bar{\Omega}$$

Define $K = -\log \|\Omega\|^2$

then, $\mathcal{M}_c$ is a Kähler mfd

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K .$$

- flat coordinates on $\mathcal{M}_c$.

Choose a basis $\{\alpha_I, \beta^I\}_{I=0,1,\cdots h^{1,2}}$

of $H_3(M;\mathbb{Z})$

$(\dim H_3 = 2 + 2h^{1,2})$

Define the period integrals

$$X^I = \int_{\alpha_I} \Omega , \quad F_I = \int_{\beta^I} \Omega .$$

$F_I = F_I(X)$; homogeneous, weight 1.

$\exists\ F(X)$,

$$F_I = \frac{\partial F}{\partial X^I}$$

$$t^i = \frac{X^i}{X^0} \qquad i = 1, \cdots, h^{1,2}$$

flat coordinates

If W is the mirror of M,

t^i's give the Kähler moduli of W.

$F(X)$ is not globally defined on $\mathcal{M}_c$

but $C_{ijk} = \dfrac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}$ is

One can show

$$K = -\log\left(4F - 4\bar{F} + \bar{t}^a \partial_a F - t^a \bar{\partial}_a \bar{F}\right)$$

and then

$$(*) \quad R_{a\bar{b}c\bar{d}} = G_{a\bar{d}} G_{\bar{b}c} + G_{a\bar{b}} G_{c\bar{d}} - e^{2K} C_{ace} \bar{C}_{\bar{b}\bar{d}\bar{f}} G^{e\bar{f}}$$

$$\left(\Leftrightarrow [\nabla - C, \bar{\nabla} - \bar{C}] = 0\right)$$

The special geometry relations has been derived from

- complex geometry
- topological string theory
- spacetime $\mathcal{N}=2$ SUSY for $\mathbb{R}^4 \times CY_3$

A-model,

Same story, except

$$C_{abc} = \int_M k_a \wedge k_b \wedge k_c + \sum_n n_a n_b n_c N_{0,n} \times \frac{e^{2\pi i n t}}{1 - e^{2\pi i n t}}$$

$$\left(\begin{array}{l} \tilde{N}_{0,dm}: \text{ Gromov-Witten invariants} \\ = \sum_{k|d} \frac{1}{k^3} N_{0,dm/k} \end{array} \right)$$

∘ Higher genus

$$F_g = \int_{\mathcal{M}_g} \langle (G^-)^{3g-3} (\bar{G}^-)^{3g-3} \rangle$$

$$\frac{\partial}{\partial \bar{t}^a} \to \text{insertion of } \int G^+ \bar{G}^+ \phi_a$$

Since $\{ \oint G^+, G^- \} = \tilde{T}$,

this gives rise to total der on $\mathcal{M}_g$.

$$\frac{\partial}{\partial \bar{t}^a} F_g = 0 \quad \text{if we can ignore } \partial \mathcal{M}_g.$$

In fact

$$\frac{\partial}{\partial \bar{t}^a} F_g = \frac{1}{2} \bar{C}_{\bar{a}\bar{b}\bar{c}} e^{2K} G^{b\bar{b}} G^{c\bar{c}} \times \left\{ D_b D_c F_{g-1} + \sum_{r=1}^{g-1} D_b F_r D_c F_{g-r} \right\}$$

This can be used to compute F_g exactly.

A-model

Since $\bar{t}$ dependence is controled, we can choose $\bar{t}$ as we like.

$\bar{t} \to \infty$; WS instanton computation becomes exact.

Chern class of M

$$\frac{\partial F_1}{\partial t^a} = -\frac{1}{24}\int k_a \wedge c_2$$

$$- \sum_n n_a N_{1,n} \sum_m \frac{e^{2\pi i m n t}}{1-e^{2\pi i m n t}}$$

$$- \frac{1}{12} \sum_n n_a N_{0,1} \frac{e^{2\pi i n t}}{1-e^{2\pi i n t}}$$

bubbling effect

$$F_g = \frac{1}{2}\chi(M)\int_{M_g} c_{g-1}^3$$

Chern class for the Hodge bundle

$$+ \sum_n N_{g,n}\, e^{2\pi i n t} + \cdots$$

$$\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{(-1)^{g-1}}{(2\pi)^{2g-2}} 2\zeta(2g-2)\chi_g$$

$$\chi_g = \frac{(-1)^{g-1}}{2g(2g-2)} B_g$$: Euler characteristics

B-model

$$F_{g=1} = \frac{1}{2}\sum_{p,q} (-1)^{p+q} pq \log\det\Delta^{p,q}$$

$$\Delta^{p,q} = \bar\partial^\dagger\bar\partial \quad \text{on } \Omega^{p,q}(M)$$

Ray-Singer torsion

- indep of Kähler moduli
- obey the Quillen anomaly

(as expected from top. string)

D branes, preserve $\frac{1}{2}$ of BRST.

A branes (branes in the A-model)

$$G^+ = \pm \bar{G}^+ , \quad J = \bar{J}$$

B branes

$$G^+ = \pm \bar{G}^- , \quad J = -\bar{J} .$$

Assuming the gauge field on the branes are turned off.

A brane on γ $\Leftrightarrow$ $k|_\gamma = 0$

$$k = i g_{i\bar{j}} \, dx^i \wedge dx^{\bar{j}}$$

B brane on γ $\Leftrightarrow$ $\gamma \subset M$ holomorphic

$\boxed{\text{Q: Show this.}}$

If, in addition, we want spacetime SUSY, we need $\Omega = e^{i\sqrt{\hat{c}}\,\phi}$ to be glued nicely left / right.

(SUSY generator $\sim e^{i\sqrt{\hat{c}}/2\,\phi}$)

A + SUSY $\Rightarrow$ special Lagrangian

$$\Omega|_\gamma = \text{volume form}|_\gamma$$

(note $\Omega \wedge \bar{\Omega} = \det g$)

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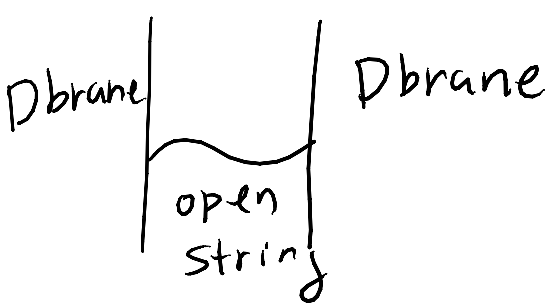
more general definition.

D brane : object in a category

open string : morphism



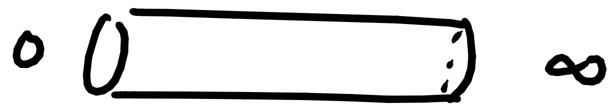
Topological String Theory

2. Computation

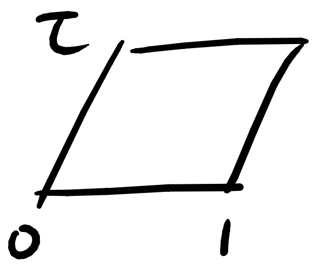
- examples of CY

<u>1d</u>

- $\mathbb{C}$
- $\mathbb{C}^\times = \mathbb{C} \setminus \{0\} \sim$ cylinder



- $T^2 = \mathbb{C} / \{\mathbb{Z} + \tau \mathbb{Z}\}$



τ: complex structure

$t = i\theta + r$: Kähler moduli

(θ: B field, r = area T^2)

T-duality $\tau \longleftrightarrow t$

(mirror symmetry)

2d

- ALE (asymptotically locally Euclidean)

$$\mathbb{C}^2/G \ , \quad G \subset \underbrace{SU(2)}$$

"resolve the singularity" needed to preserve a holomorphic 2 form

e.g. $G = \mathbb{Z}_n$.

$(n-1)$ resolved $\mathbb{C}P^1$'s

$$C_1, \cdots, C_{n-1}$$

$$C_i \cap C_j = -2,$$

$$C_i \cap C_j = \begin{cases} 1 & |i-j|=1 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ Cartan matrix of A_{n-1} .

- $K3$: compact CY_2

e.g. start with $\mathbb{C}P^3$

$$(z_1, \cdots, z_4) \sim (\lambda z_1, \cdots, \lambda z_4)$$

$$\lambda \in \mathbb{C}^\times$$

Consider $\underbrace{P}_{\text{homogeneous of degree } d}(z_1, \cdots, z_4) = 0$.

In general,

a hypersurface X in $\mathbb{C}P^{k-1}$

defined by $\underbrace{P}_{\deg d}(z_1, \cdots, z_k) = 0$

has $c_1(X) \sim (d-k)\, c_1(\mathbb{C}P^{k-1})$

So $d=k$.

In the above, $d=4$ for $K3$.

$d=3$

- local $\mathbb{C}P^2$

$\mathbb{C}P^2$: not CY_3

Consider a line bundle over $\mathbb{C}P^2$.
The 1^{st} chern class of the fiber should cancel that of the base.

$\Downarrow$

Start with $(x, z_1, z_2, z_3) \in \mathbb{C}^4 \setminus \{0\}$

Identify (x, z_1, z_2, z_3)

$$\sim (\lambda^{-3} x, \lambda z_1, \lambda z_2, \lambda z_3)$$

It is the total space of

$$\mathcal{O}(-3) \to \mathbb{C}P^2.$$

This describe the geometry of CY_3 near a 4-cycle $\sim \mathbb{C}P^4$.

- local $\mathbb{C}P^1$

(x_1, x_2, z_1, z_2)

$\sim (\lambda^{-1} x_1, \lambda^{-1} x_2, \lambda z_1, \lambda z_2)$

This is the total space of

$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P1.$

also known as the resolved conifold.

<u>Conifold</u>

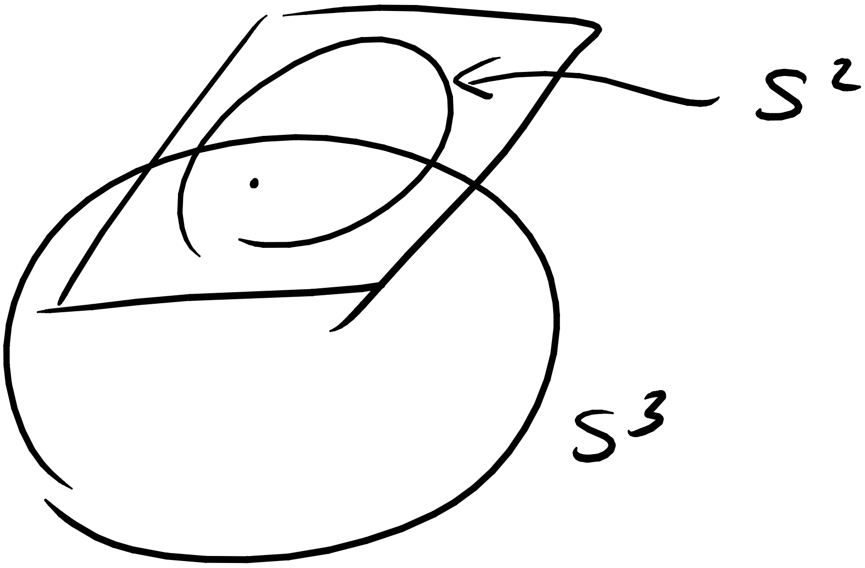
$(x, y, w, z) \in \mathbb{C}^4$

$xy - wz = \mu$, μ : complex moduli

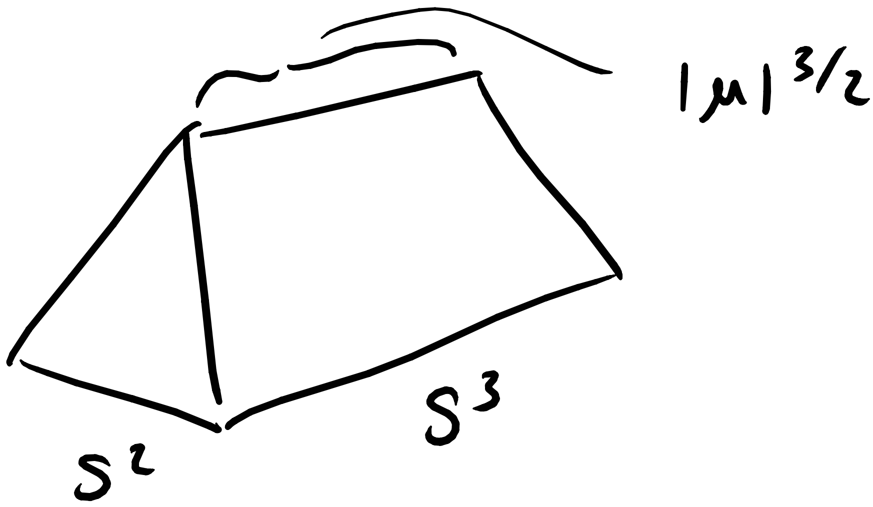
gives the deformed conifold.

$\sim T^* S^3$

$(\text{vol } S^3 \sim |\mu|^{3/2})$



The infinity of T^*S^3 looks like $S^3 \times S^2$.



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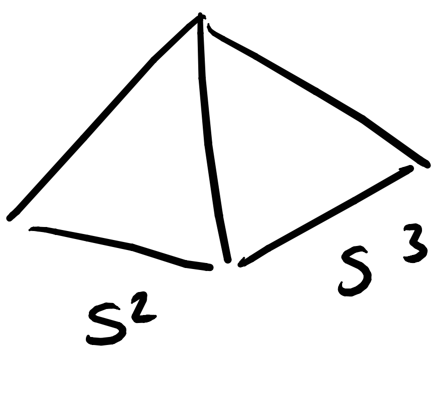
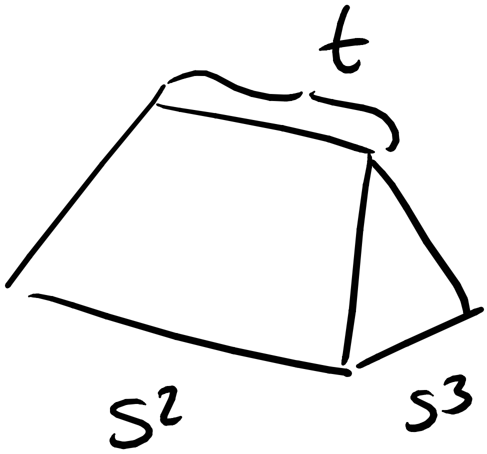


$\mu \to 0$

conifold singularity



resolved conifold.

Toric CY_3

Start with $\mathbb{C}^{N+3} \setminus \{0\}$

$\ni (z_1, \cdots, z_{N+3})$

- Divide by $U(1)^N$

  z_i : charge $(Q_i^1, \cdots, Q_i^N)$

  gauge symmetry

- $\sum_i Q_i^a |z_i|^2 = t^a$; Kähler moduli

  Gauss law constraints

CY3 condition :

$$\sum_i Q_i^a = 0 \qquad a = 1, \cdots, N.$$

e.g. local $\mathbb{C}P^2$

- (z_0, z_1, z_2, z_3)
  $\sim (e^{-3i\theta} z_0, e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3)$

- $-3|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = r.$

Kähler form

$$k = \sum_{i=0,1,2,3} dz_i \wedge d\bar{z}_i$$

$$= \sum_i dp_i \wedge d\phi_i$$

$$z_i = \sqrt{p_i}\, e^{i\phi_i}$$

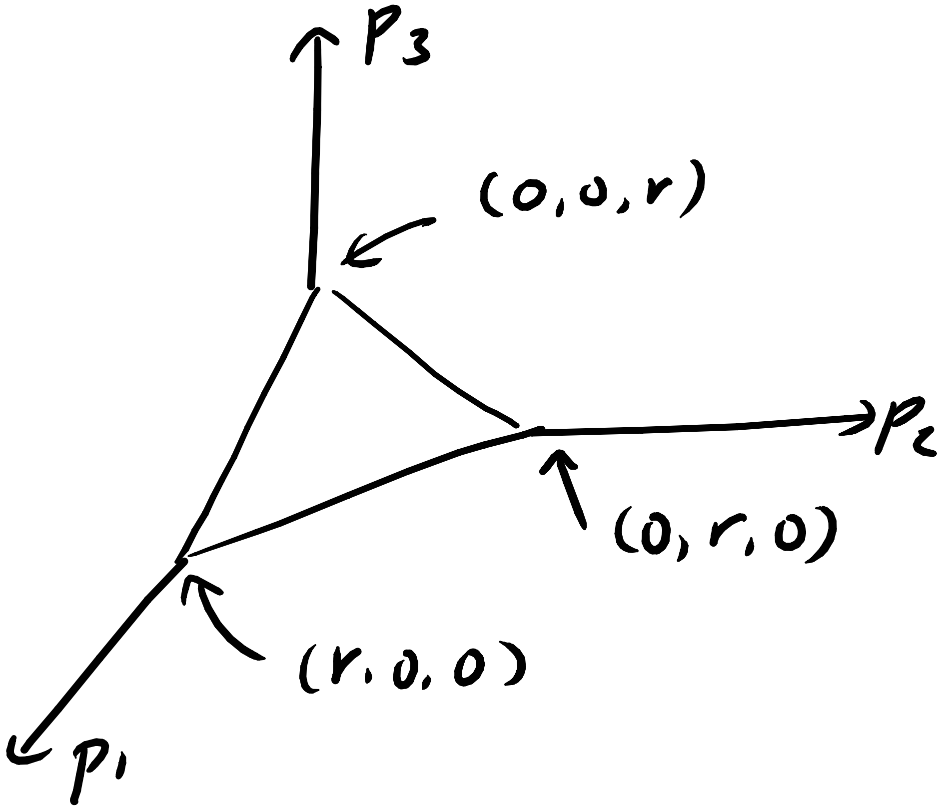
$$\begin{cases} -3p_0 + p_1 + p_2 + p_3 = r \\ (\phi_0, \phi_1, \phi_2, \phi_3) \end{cases}$$

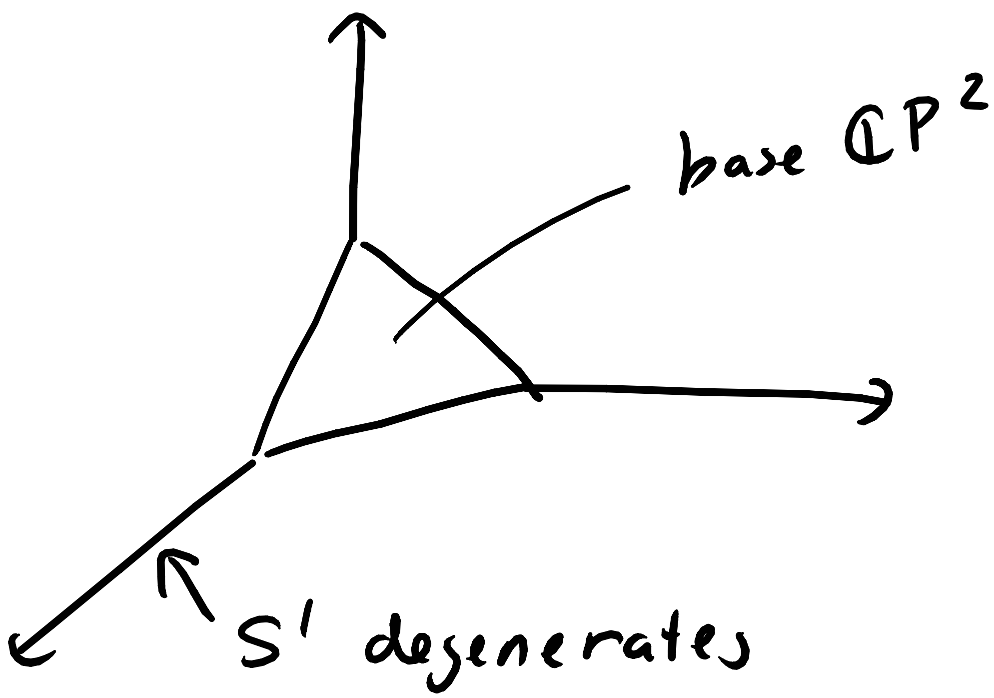
$$\sim (\phi_0 - 3\theta, \phi_1 + \theta, \phi_2 + \theta, \phi_3 + \theta)$$

We can eliminate (p_0, ϕ_0) and consider T^3 fiber over (p_1, p_2, p_3).

$$p_1 + p_2 + p_3 > r$$

$$p_1, p_2, p_3 > 0$$





- local CP^1

$$-p_0 - p_1 + p_2 + p_3 = 0$$

$$(\phi_0, \phi_1, \phi_2, \phi_3)$$

$$\sim (\phi_0 - \theta, \phi_1 - \theta, \phi_2 + \theta, \phi_3 + \theta)$$

Q: What is the toric diagram?

∘ local $\mathbb{CP}^1 \times \mathbb{CP}^1$

$$(Q) = \begin{pmatrix} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

What is the tonic diagram?

Mirror Symmetry

$(X, \tilde{X})$: mirror pair of CY_3's

$\Updownarrow$

A model on X

= B model on $\tilde{X}$.

X : toric CY_3

(Q_i^a, t^a)

$\Updownarrow$

$\tilde{X}$: $uv + F(e^x, e^y) = 0$

$$F = \sum_{i=1}^{N+3} c_i(t)\, e^{n_i x + m_i y}$$

$$\begin{cases} \sum_i Q_i^a n_i = 0 \\ \sum_i Q_i^a m_i = 0 \\ \sum_i Q_i^a \log c_i = -t^a \end{cases}$$

To be precise t^a's are FI parameters and receive WS instanton corrections

For the conifold,

$$F = e^{x+y} + e^{x} + e^{y} + e^{-t}$$

$uv + F = 0$: mirror of conifold

When $t \to 0$

$$F = (e^{x}+1)(e^{y}+1)$$

ref : Mirror Book

D branes

Remember :

A-branes : Lagrangian 3 cycles

B-branes : holomorphic cycles

$$F_{g,n} = \int_{\mathcal{M}_{g,n}} \langle (G^-)^{6g-6+n} \rangle$$

g = genus

n = # boundaries

String field theory

$$S = \frac{1}{2} \langle \Phi | Q_{BRST} | \Phi \rangle + \frac{1}{3} \langle \Phi | \Phi * \Phi \rangle$$

Witten
9207094

B-model ;

SFT $\to$ field theory

A-model :

SFT $\to$ field theory

if there are no compact 2cycles
($\to$ no Kähler moduli)

e.g. A-model on T^*S^3

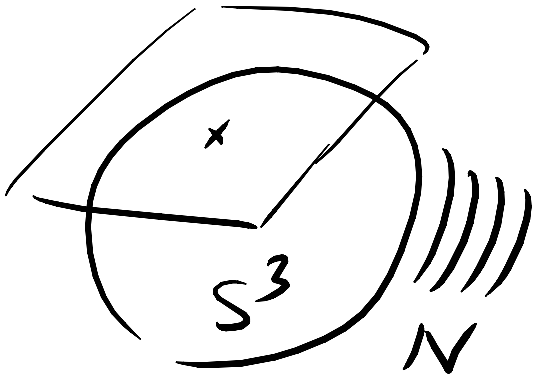
The base S^3 is a Lagrangian

$(\because k = dp \wedge dx, \quad x \in S^3$
$\qquad\qquad\qquad\qquad p \in T^*_x S^3)$

N A branes

$\Downarrow$

SFT = CS gauge theory, U(N)



$$\exp\Big(-\sum_{g,n} F_{g,n}\, \lambda^{2g-2+n} N^n\Big)$$

$$\lambda = \frac{1}{k+N} \qquad k: \text{level}$$

We should include the gauge group volume vol $U(N)$.

$$Z_{CS}(S^3) = \frac{e^{\frac{\pi}{8} i N(N-1)}}{(k+N)^{N/2}} \sqrt{\frac{k+N}{N}}$$

$$\times \prod_{s=1}^{N-1} \left(2 \sin\left(\frac{s\pi}{k+N}\right)\right)^{N-s}$$

We can expand this in powers of

$$\lambda = \frac{1}{k+N}$$

$$Z(S^3) = \exp\left(-\sum_g \lambda^{2g-2} F_g(t)\right)$$

$$F_0(t) = \frac{i}{12} t^3 + \cdots + \sum_{n=1}^{\infty} n^{-3} e^{-nt}$$

$$F_1(t) = \frac{1}{24} t + \frac{1}{12} \log(1 - e^{-t})$$

$$F_g(t) = \int_{\mathcal{M}_g} c_{g-1}^3 - \frac{\chi_g}{(2g-3)!} \sum_{n=1}^{\infty} n^{2g-3} e^{-nt}$$

$$t = N\lambda.$$

These are top string partition functions on a resolved conifold.

Gopakumar - Vafa 9811131

What is going on?

Large N duality:

k-types of D branes

$F_{g;n_1\cdots n_k}$

$\Downarrow$

$$F_g(t) \equiv \sum_{n_1\cdots n_k} F_{g;n_1\cdots n_k} t_1^{n_1}\cdots t_k^{n_k}$$

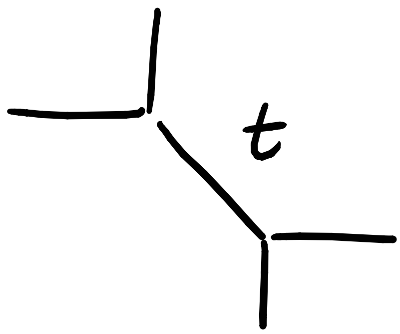
$\exists\, M(t)$: a family of CY_3's

s.t.

$F_g(t)$ is a closed string partition function on $M(t)$.

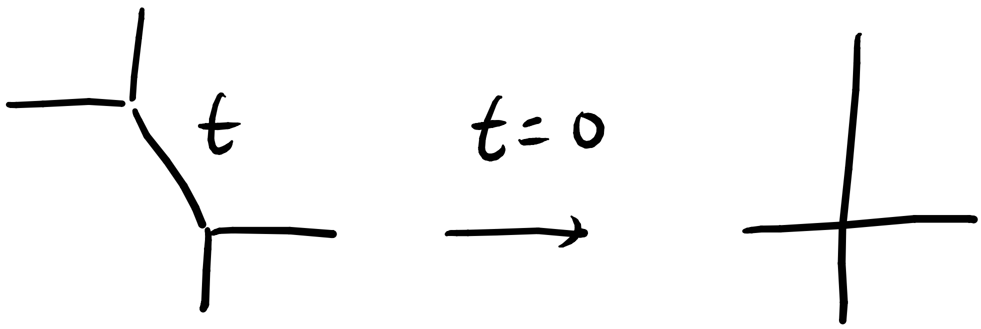
Geometric Transition

resolved conifold

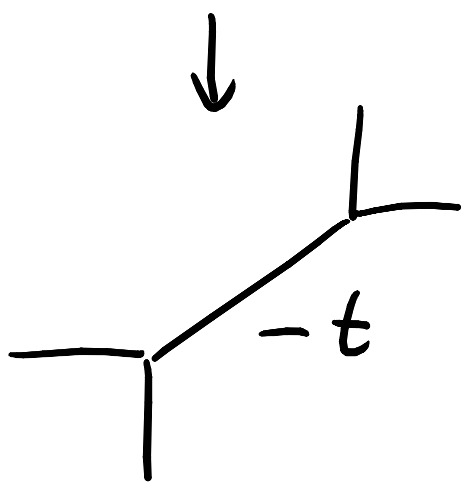


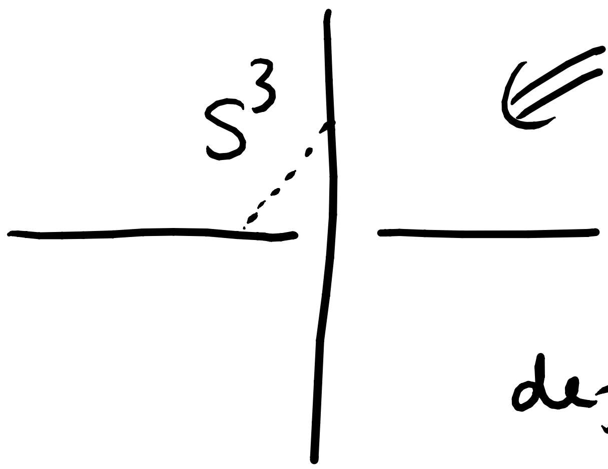
slightly different from the previous diagram

$T^2 \times \mathbb{R}$ fibration over $\mathbb{R}^3$



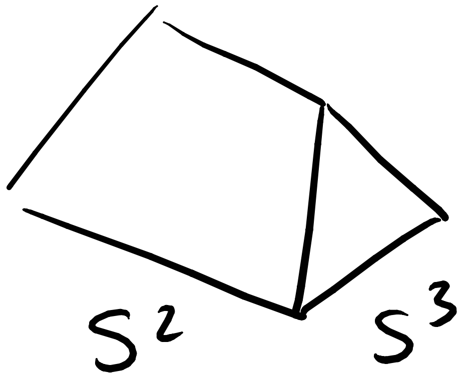
flop



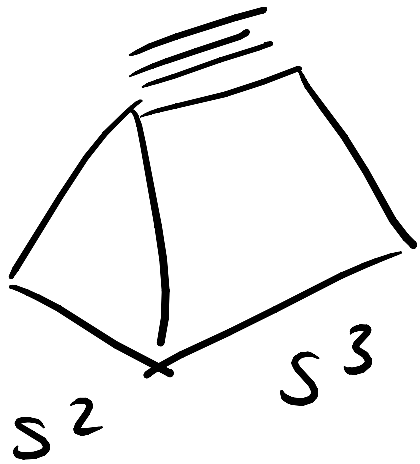


$\Leftarrow$ no Kähler moduli

deformed conifold



$\Longleftrightarrow$



closed string on resolved conifold

Chern-Simons on S^3

$$t = N g_s$$

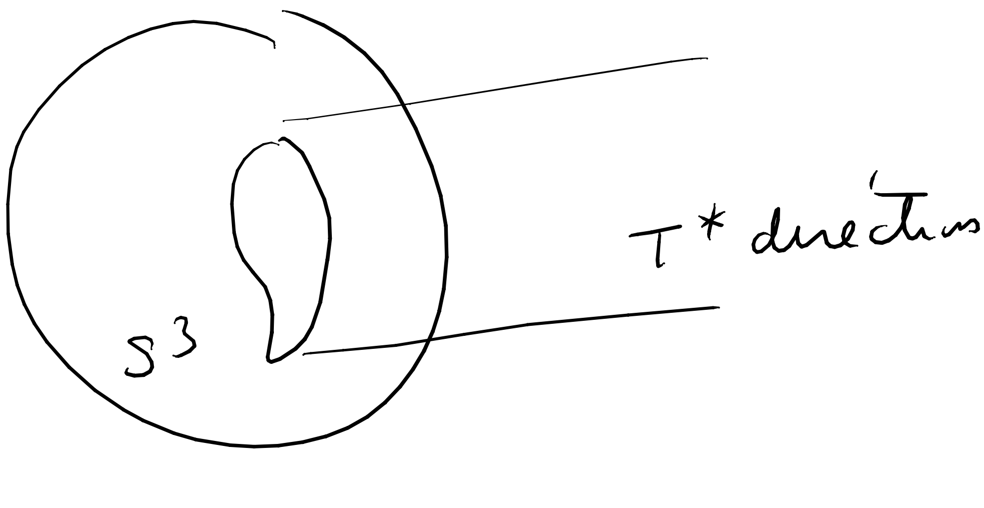
Not only the partition function $Z(S^3)$, we can also compute knot invariants

Suppose

$$C = \{ q^i(s) \mid q(s) \in S^3, \quad 0 \le s \le 2\pi, \; q(0) = q(2\pi) \}$$

$$\Downarrow$$

$$\gamma_C = \{ (q(s), p) \in T^*S^3 \mid p_i \frac{dq^i}{ds} = 0 \}$$



C_γ is a Lagrangian

$$\sim S^1 \times \mathbb{R}^2$$

We can turn on a holonomy $U \subset U(N)$ on $S^1 \subset C_\gamma$.

$$Z_C(S^3; U)$$

$$= \langle \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} U^{-n}$$

$$\times \operatorname{tr} W(C)^n\Big)\rangle_{CS}$$

where $W(C) = P \exp \oint_C A$

in fundamental rep.

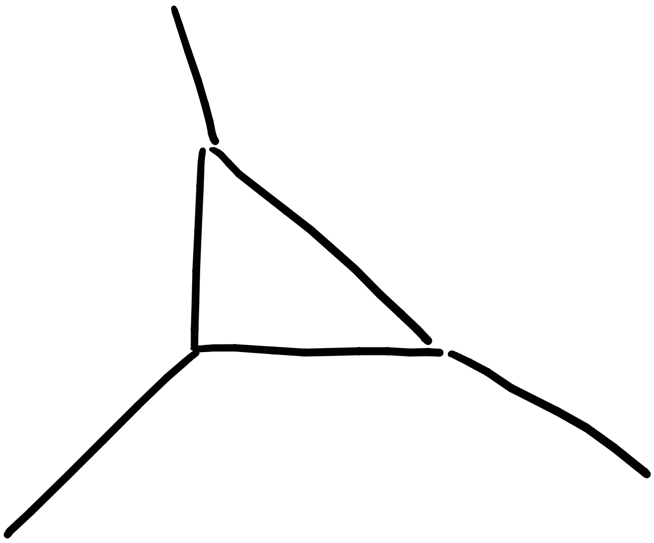
This is a generating function

of $\langle \operatorname{tr}_R W(C) \rangle_{CS}$

for any rep R of $U(N)$

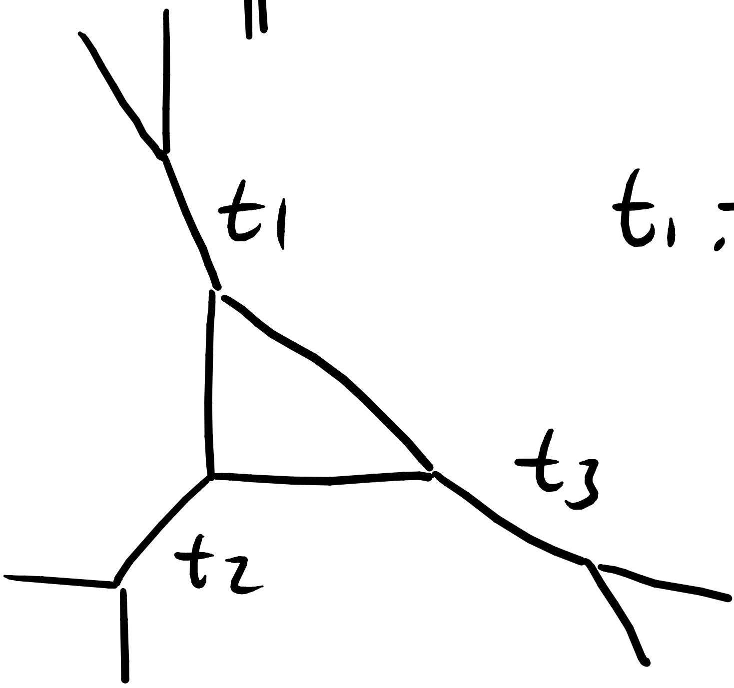
Ooguri - Vafa 9912123

- local $\mathbb{C}P^2$



X

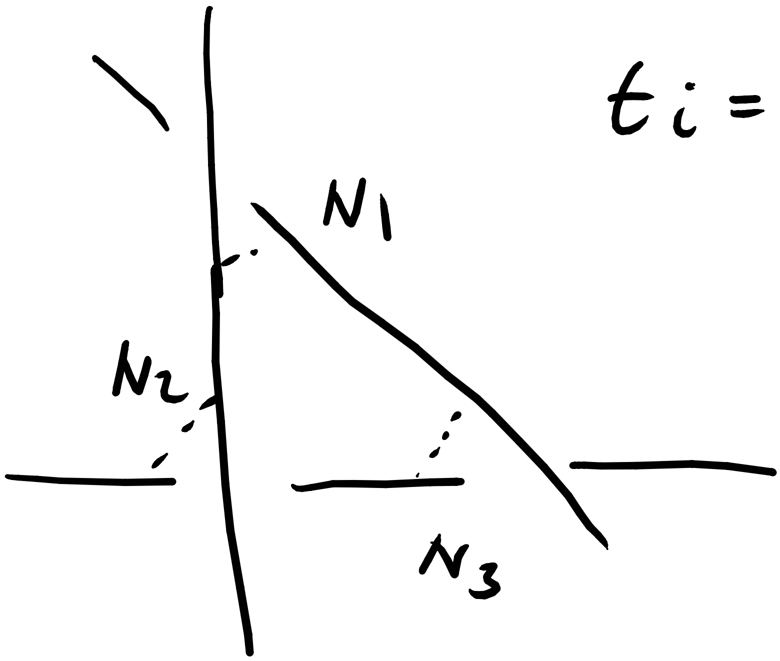
$\Uparrow$



$t_1, t_2, t_3 \to \infty$

If we can combute this for all t_1 t_2 t_3, we can complete $Z_{top}(X)$.

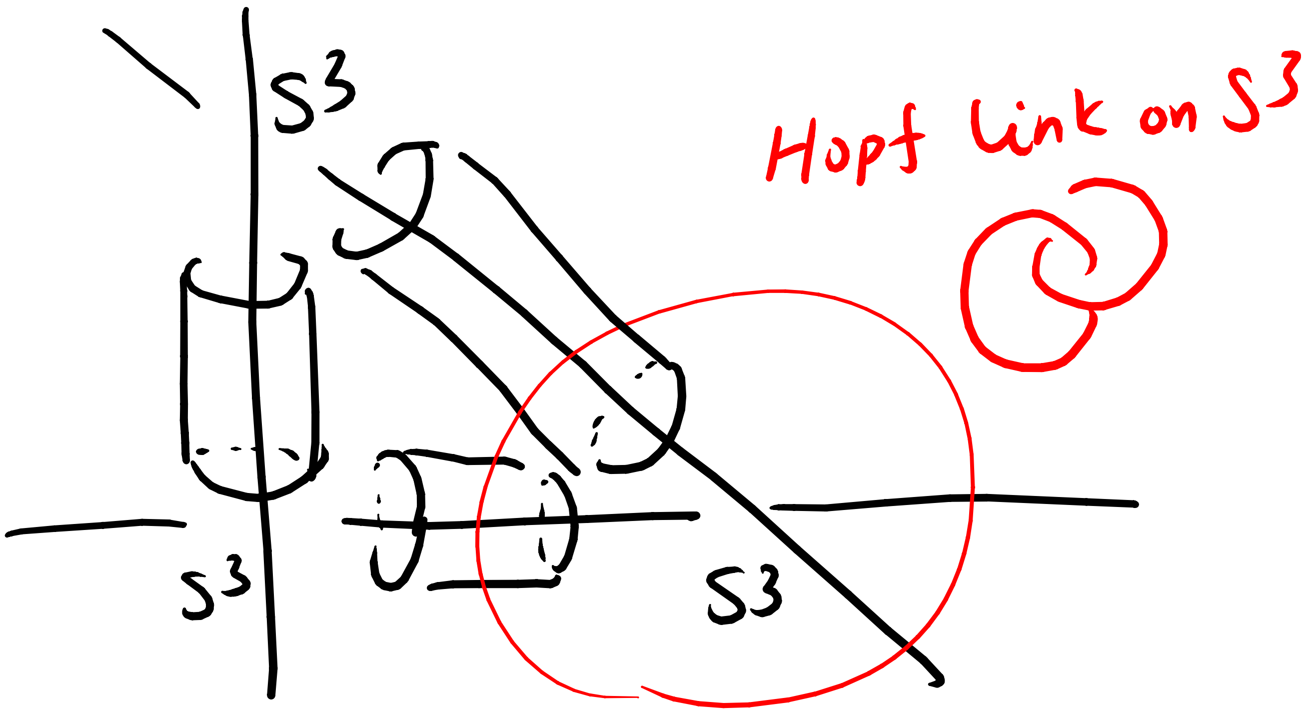
Geometric transition in t_i's



$$t_i = N_i g_s$$

No Kähler moduli

$\Rightarrow$ purely open string theory



By Ooguri + Vafa (hepth/9912123)

Aganagic, Marino + Vafa (0206164)

$$Z_{top}(X) = \sum_{R_1 R_2 R_3} e^{-t|R_1|} S_{R_1 R_2} \, e^{-t|R_2|} S_{R_2 R_3} \, e^{-t|R_3|} S_{R_3 R_1}$$

R_i : irrep's of $U(N)$ $(N \to \infty)$

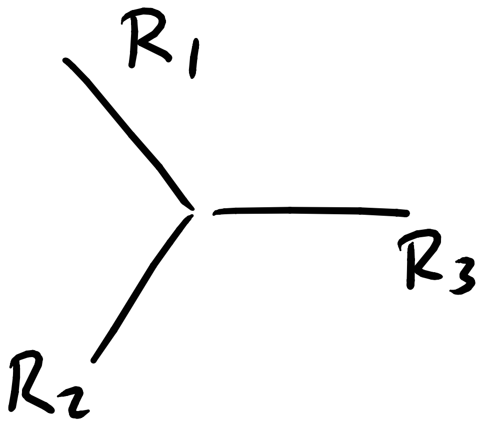
$|R|$: # of boxes in the Young diagram

$S_{R,R'} = \Big\langle$ 

$\Big\rangle$

This formula is for all genus.

Topological Vertex



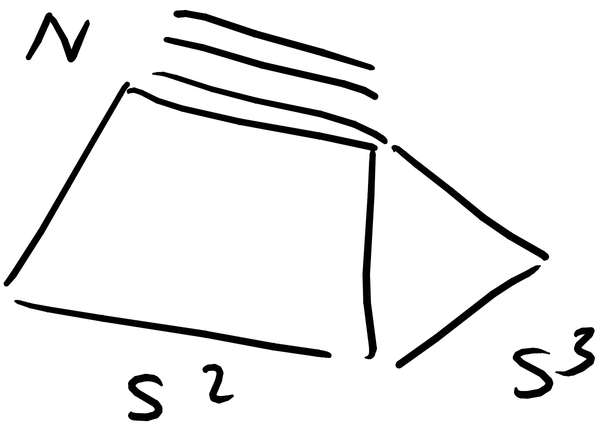
$$C_{R_1 R_2 R_3}(g_s)$$

$$e^{-t|R| + m\, C_2(R)}$$

orientation

Dijkgraaf - Vafa
0206255

Matrix model



resolved conifold
in B model

S^2 : holomorphic 2 cycle
$\leftarrow$ B brane.

In this case

the SFT = Gaussian matrix model.

$$Z_G = \frac{1}{\mathrm{vol}\,U(N)} \int dM \, e^{-\frac{1}{2\lambda} \mathrm{tr} M^2}$$

$$= \frac{(2\pi)^{N^2 + \frac{N}{2}}}{G_s(N+1)} \lambda^{N^2/2}$$

$$F_G = \log Z_G$$

$$= \frac{N^2}{2}\left(\log N\lambda - \frac{3}{2}\right)$$

$$- \frac{1}{12}\log N + \zeta'(-1)$$

$$+ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g}$$

$$= \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)$$

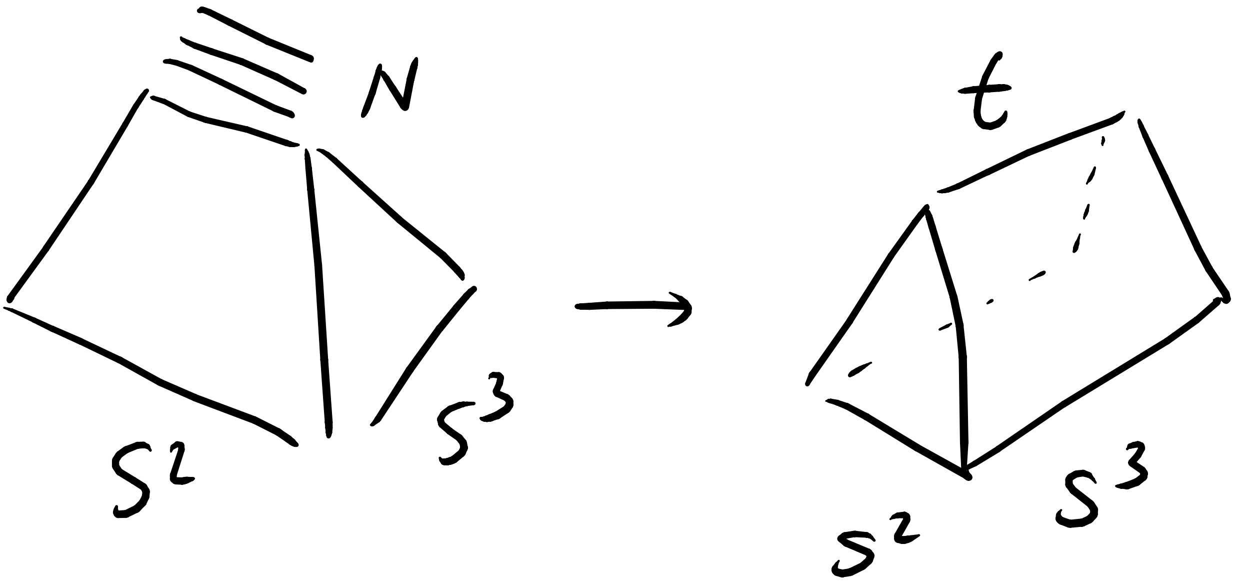
$$t = N\lambda$$

$$F_0 = \frac{1}{2} t^2 \left(\log t - \frac{3}{2}\right)$$

$$F_1 = -\frac{1}{12}\log t + \cdots$$

$$F_g = \frac{B_{2g}}{2g(2g-2)} \frac{1}{t^{2g-2}}$$

This is the closed B-model amplitude for the deformed conifold



$$uv + zw = t$$

$$(t = N\lambda)$$

3. Applications

Type II A/B on $\mathbb{R}^4 \times CY_3$

with or without branes

$\Rightarrow$ $\mathcal{N}=1$ or 2 effective action

$\Rightarrow$ BPS counting

- Type IIA on $\mathbb{R}^4 \times CY_3$

  $\Rightarrow$ $\mathcal{N}=2$ SUSY in $\mathbb{R}^4$

$\mathcal{N}=2$ SUGRA ($\ni$ gravi-photon)

Vector multiplet $h^{1,1}$

Hyper multiplet $h^{2,1} + \underbrace{1}_{\text{dilaton-axion}}$

For II B, $h^{1,1} \to h^{2,1}$
$h^{2,1} \to h^{1,1}$

Important fact:

The dilaton is in hyper.

Since there are no non-derivative coupling between vector and hyper, some vector multiplet terms can be computed exactly.

$N=2$ chiral superfield for Vectors.

$$T^i(\theta) = \underbrace{t^i}_{\text{Kähler moduli for IIA}} + \cdots$$

$$W_{\alpha\beta}(\theta) = \underbrace{F_{\alpha\beta}}_{\text{self-dual part of gravi-photon field strength.}} + R_{\alpha\beta\gamma\delta}\,\theta_1^\gamma \theta_2^\delta + \cdots$$

Consider

$$\int d^2\theta_1 d^2\theta_2 \,(W_{\alpha\beta} W^{\alpha\beta})^{2g} F_g(T)$$

$$= R_+^2 \, F_+^{2g-2} \, F_g(t) + \cdots$$

F_g : genus-g topological string partition function.

<u>Why?</u>

If we write $J = i\sqrt{\hat{c}}\,\partial\phi$

$$\Rightarrow \frac{i}{2}\sqrt{\hat{c}} \int_\Sigma R\sqrt{g}\,\phi\, d^2z$$

We can choose $R\sqrt{g} = -\sum_{i=1}^{2g-2} \delta^{(2)}(z-z_i)$

$\Downarrow$

Insertion of $\prod_{i=1}^{2g-2} \underbrace{e^{-\frac{i}{2}\sqrt{\hat{c}}\,\phi(z_i)}}_{\text{Ramond vertex}}$

BCOV 9309140

In particular, $F_0(t)$ is the low energy prepotential of the $N=2$ vector multiplet.

- Geometric Engineering.

  Start with $\mathbb{R}^4 \times CY_3$

  Take $l_s \to 0$.

  At the same time, focus on a local geometry of CY_3

  $\Rightarrow$ rigid $N=2$ theory (gravity decoupled)

  $\Rightarrow$ F_0 : Seiberg-Witten solution

Kachru-Vafa 9505105

∘ $K_3 \times \mathbb{R}^6$

local geometry ALE

2 cycles $\sim \mathbb{CP}^1$ intersecting
with the Cartan matrix of $G \sim A, D, E$.

e.g. $G = A_n = SU(n+1)$

n Kähler moduli $\to$ n $U(1)$'s

n $\mathbb{CP}^1$ $\to$ $D2$ on them
charged w.r.t. the $U(1)$'s

$\Rightarrow$ $SU(n)$ gauge theory
(6 supercharges).

further compactify on T^2

$ALE \times T^2 \times \mathbb{R}^4$

$\Rightarrow$ $\mathcal{N}=4$ SYM with

$$\frac{1}{g_{YM}^2} = vol(T^2)$$

T-duality on T^2 = S-duality
of $\mathcal{N}=4$ SYM

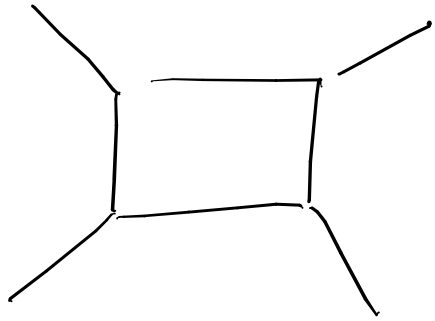
∘ ALE fibration.

We can turn this into $\mathcal{N}=2$ by fibering ALE over a 2d base.

e.g. A_1 geometry ($\mathbb{C}^2/\mathbb{Z}_2$) over $\mathbb{CP}^1$ $\Rightarrow$ $\mathcal{N}=2$ SU(2) pure SYM on $\mathbb{R}^4$

This is toric!

$$\mathbb{C}^5, \quad Q = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$



$\mathcal{O}(-2,-2) \to \mathbb{CP}^1 \times \mathbb{CP}^1$

$\Updownarrow$ mirror

$$uv + F(x,y) = 0$$

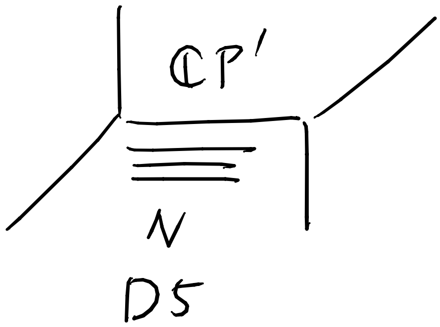
$F(x,y) = 0$; Seiberg-Witten curve

- Further reduction to $\mathcal{N}=1$.

  - with branes
  - with fluxes

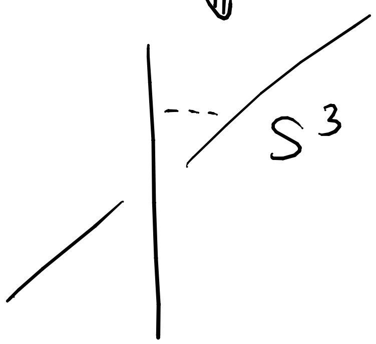
  $\circlearrowleft$ large N duality

e.g. IIB on resolved conifold



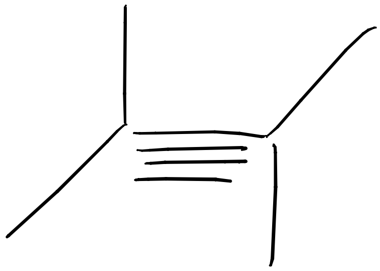
$\Rightarrow$ $\mathcal{N}=1$ $U(N)$ SYM on $\mathbb{R}^4$

$\Updownarrow$



$\Rightarrow$ N unit of D5 fluxes through S^3.

D brane side



open topological string

$F_{g,n}$

$$F_g(S) = \sum_{n=0}^{\infty} F_{g,n}\, S^n$$

glueball superfield $\quad S = \mathrm{tr}\, \psi_\alpha \psi^\alpha + \cdots$

gluino.

$\mathcal{N}=1$ F-term

$$\int d^2\theta \left(N \frac{\partial F_0}{\partial S} + \underbrace{\tau S}_{\text{YM kinetic term}} \right)$$

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}$$

BCOV '93

flux side

$II B$ on $\mathbb{R}^4 \times CY_3 \Rightarrow \mathcal{N}=2$

$$\int d^2\theta \, d^2\bar{\theta} \; F_0(X^I)$$

$$X^I = x^I + \cdots + (\theta-\bar{\theta})^2 N^I$$

$$x^I = \int_{A_I} \Omega$$

N^I = flux through A_I

$$= \int d^2\theta \; N^I \frac{\partial F_0}{\partial X^I}(X)$$

If we have τ_I flux on B^I cycle

$$N^I \frac{\partial F_0}{\partial X^I} + \tau_I N^I$$

With branes

$$W = N^I \frac{\partial F_0}{\partial S^I} + \tau_I S^I$$

with fluxes

$$W = N^I \frac{\partial F_0}{\partial X^I} + \tau_I X^I$$

It is natural to identify

$$S^I = X^I .$$

<u>example</u>

D5 on the resolved conifold

$\Downarrow$ geometric transition

N flux through deformed conifold

$$X = \int_A \Omega$$

$$F_X = \int_B \Omega = X \log X$$

Since $F_X = X \log X = \frac{\partial F_0}{\partial X}$

$$F_0 = \frac{1}{2} X^2 \log X$$

glueball superpotential

$$W(S) = N \frac{\partial F_0}{\partial S} - 2\pi i \tau S$$

$$= N S \log S - 2\pi i \tau S$$

Veneziano - Yankielowicz glueball superpotential.

- Use of the matrix model.

For the conifold

$$uv + y^2 + x^2 = 0$$

we can use the Gaussian matrix model to evaluate

$$F_{g,n}.$$

For a more general

$$uv + y^2 + W'(x)^2 = 0,$$

we can use

$$\int dM \, e^{-\frac{1}{g_s} W(M)}$$

Dijkgraaf - Vafa

BPS counting.

- M Theory on $\mathbb{R}^{1,4} \times CY_3$

5d BPS states $\Leftarrow M_2$ on $\beta \in H_2(CY_3)$

$SU(2)_L \times SU(2)_R$ little group

$\Rightarrow$ # BPS states $= N_\beta^{(m_L, m_R)}$

$m = 2j$

$$N_\beta^{m_L} \equiv \sum_{m_R} (-1)^{m_R} N_\beta^{(m_L, m_R)}$$

"elliptic genus"

$$\left(\text{c.f. 2d CFT} \quad \mathrm{tr}\left[(-1)^{J_R} Q^{J_L} q^H\right] \right)$$

Gopakumar - Vafa
9809187

Then, it is known

$$Z_{top} = \exp\left(\sum_g \lambda^{2g-2} F_g(t) \right)$$

A-model on CY_3

t : flat coords of the Kähler moduli space

$$= \prod_{\substack{\beta \in H_2 \\ m}} \prod_{k=1}^{\infty} \left(1 - q^{k+m} Q^{\beta}\right)^{k N_\beta^m}$$

$$q = e^{-\lambda}, \quad Q = e^{-t}$$

$\prod_{k=1}^{\infty}$ comes from $\mathbb{R}^4$

orbital angular momenta

This follows from the identity:

$$-\log \prod_{k=1}^{\infty} (1-q^k Q^\beta)^k$$

$$= \sum_n \frac{1}{n} \frac{Q^{n\beta}}{(2\sinh \frac{n\lambda}{2})^2}$$

Schwinger one-loop determinant

$\log\det(\Delta + m^2)$

For $\lambda \to 0$

$$\text{RHS} \sim \frac{1}{\lambda^2} \underbrace{\sum_n \frac{1}{n^3} Q^{n\beta}}_{}$$

multicovering formula

for $S^2 \to S^2$ in F_0

- type IIB on $\mathbb{R}^4 \times CY_3$

α_I, β^I : $H_3(CY_3, \mathbb{Z})$

q_I D3 on α_I
p^I D3 on β^I

$\Omega(q, p)$ = Witten index for bound states

$$\sum_q \Omega(q, p) e^{-q_I \phi^I}$$

$$= |Z_{top}(X^I = p^I + i\phi^I)|^2$$

$$Z_{top}(X) = \exp\left(\sum_g F_g(X)\right)$$

OSU formula 0405146