String Theory 101

Lectures at the International School of Strings and Fundamental Physics

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1 Introduction: Why String Theory?

The so-called Standard Model of Particle Physics is the most successful scientific theory of Nature in the sense that no other theory has such a high level of accuracy over such a complete range of physical phenomena using such a modest number of assumptions and parameters. It is unreasonably good and was never intended to be so successful. Since its formulation around 1970 there has not been a single experimental result that has produced even the slightest disagreement. Nothing, despite an enormous amount of effort. But there are skeletons in the closet. Let me mention just three.

The first is the following: Where does the Standard Model come from? For example it has quite a few parameters which are only fixed by experimental observation. What fixes these? It postulates a certain spectrum of fundamental particle states but why these? In particular these particle states form three families, each of which is a copy of the others, differing only in their masses. Furthermore only the lightest family seems to have much to do with life in the universe as we know it, so why the repetition? It is somewhat analogous to Mendelev's periodic table of the elements. There is clearly a discernible structure but this wasn't understood until the discovery of quantum mechanics. We are looking for the underlying principle that gives the somewhat bizarre and apparently ad hoc structure of the Standard Model. Moreover the Standard Model also doesn't contain Dark Matter that constitutes most of the 'stuff' in the observable universe.

The second problem is that, for all its strengths, the Standard Model does not include gravity. For that we must use General Relativity which is a classical theory and as such is incompatible with the rules of quantum mechanics. Observationally this is not a problem since the effect of gravity, at the energy scales which we probe, is smaller by a factor of 10^{-40} than the effects of the subnuclear forces which the Standard Model describes. You can experimentally test this assertion by lifting up a piece of paper with your little finger. You will see that the electromagnetic forces at work in your little finger can easily overcome the gravitational force of the entire earth which acts to pull the paper to the floor.

However this is clearly a problem theoretically. We can't claim to understand the universe physically until we can provide one theory which consistently describes gravity and the subnuclear forces. If we do try to include gravity into QFT then we encounter problems. A serious one is that the result is non-renormalizable, apparently producing an infinite series of divergences which must be subtracted by inventing an infinite series of new interactions, thereby removing any predictive power. Thus we cannot use the methods of QFT as a fundamental principle for gravity.

The third problem I want to mention is more technical. Quantum field theories generically only make mathematical sense if they are viewed as a low energy theory. Due to the effects of renormalization the Standard Model cannot be valid up to all energy scales, even if gravity was not a problem. Mathematically we know that there must be something else which will manifest itself at some higher energy scale. All we can say is that such new physics must arise before we reach the quantum gravity scale, which is some 10^{17} orders of magnitude above the energy scales that we have tested to date. To the physicists who developed the Standard Model the surprise is that we have not already seen such new physics many years ago. And we are all hoping to see it soon at the LHC.

With these comments in mind this course will introduce string theory, which, for good or bad, has become the dominant, and arguably only, framework for a complete theory of all known physical phenomena. As such it is in some sense a course to introduce the modern view of particle physics at its most fundamental level. Whether or not String Theory is ultimately relevant to our physical universe is unknown, and indeed may never be known. However it has provided many deep and powerful ideas. Certainly it has had a profound effect upon pure mathematics. But an important feature of String Theory is that it naturally includes gravitational and subnuclear-type forces consistently in a manner consistent with quantum mechanics and relativity (as far as anyone knows). Thus it seems fair to say that there is a mathematical framework which is capable of describing all of the physics that we know to be true. This is no small achievement.

However it is also fair to say that no one actually knows what String Theory really is. In any event this course can only attempt to be a modest introduction that is aimed at students with no previous knowledge of String Theory. There will be much that we will not have time to discuss: most notably the Veneziano amplitude, anomaly cancellation and compactification. The reader will undoubtably benefit from the other courses in the School, in particular the notes of Ralph Blumenhagen on D-branes.

These notes are a variation on a course "String Theory and Branes (7ccmms34)" that I have given at King's College London. You can find the notes here:

http://www.mth.kcl.ac.uk/courses/cmms34.html

We will first discuss the Bosonic string in some detail. Although this theory is unphysical in several ways (it has a tachyon and no Fermions) it is simpler to study than the superstring but has all the main ideas built-in. We then add worldsheet Fermions and supersymmetry to obtain the superstring theories that are used in current research but our discussion will be relatively brief.

2 Classical and Quantum Dynamics of Point Particles

2.1 Classical Action

We want to describe a single particle moving in spacetime. For now we simply consider flat D-dimensional Minkowski space

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + ... + (dx^{D-1})^2 \tag{2.1}$$

A particle has no spatial extent but it does trace out a curve - its worldline - in spacetime. Furthermore in the absence of external forces this will be a straight line (geodesic if you know GR). In other words the equation of motion should be that the length of the worldline is extremized. Thus we take

$$\begin{aligned} S_{pp} &= -m\int ds \\ &= -m\int \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}d\tau \end{aligned} \tag{2.2}$$

where τ parameterizes the points along the worldline and $X^\mu(\tau)$ gives the location of the particle in spacetime, *i.e.* the embedding coordinates of the worldline into spacetime.

Let us note some features of this action. Firstly it is manifestly invariant under spacetime Lorentz transformations $X^\mu \to \Lambda^\mu{}_\nu X^\nu$ where $\Lambda^T\eta\Lambda = \eta$. Secondly it is reparameterization invariant under $\tau \to \tau'(\tau)$ for any invertible change of worldline coordinate

$$d\tau = \frac{d\tau}{d\tau'}d\tau' \ , \qquad \dot{X}^\mu = \frac{dX^\mu}{d\tau} = \frac{d\tau'}{d\tau}\frac{dX^\mu}{d\tau'} \tag{2.3}$$

thus

$$\begin{aligned} S_{pp} &= -m\int \sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}d\tau \\ &= -m\int \sqrt{-\eta_{\mu\nu}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dX^\mu}{d\tau'}\frac{dX^\nu}{d\tau'}}\frac{d\tau}{d\tau'}d\tau' \\ &= -m\int \sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\tau'}\frac{dX^\nu}{d\tau'}}d\tau' \end{aligned} \tag{2.4}$$

Thirdly we can see why the m appears in front and with a minus sign by looking at the non-relativistic limit. In this case we choose a gauge for the worldline reparameterization invariance such that $\tau = X^0$ *i.e.* worldline 'time' is the same as spacetime 'time'. This is known as static gauge. It is a gauge choice since, as we have seen, we are free to take any parameterization we like. The nonrelativistic limit corresponds to assuming that $\dot{X}^i << 1$. In this case we can expand

$$S_{pp} = -m\int \sqrt{1-\delta_{ij}\dot{X}^i\dot{X}^j}d\tau = \int -m + \frac{1}{2}m\delta_{ij}\dot{X}^i\dot{X}^j d\tau + \ldots \tag{2.5}$$

where the ellipses denotes terms with higher powers of the velocities $\dot{X}^i$. The second term is just the familiar kinetic energy $\frac{1}{2}mv^2$. The first term is simply a constant and doesn't affect the equations of motion. However it can be interpreted as a constant potential energy equal to the rest mass of the particle. Thus we see that the m and minus signs are correct.

Moving on let us consider the equations of motion and conservation laws that follow from this action. The equations of motion follow from the usual Euler-Lagrange equations applied to S_{pp}:

$$\frac{d}{d\tau}\left(\frac{\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}}\right) = 0 \tag{2.6}$$

These equations can be understood as conservations laws since the Lagrangian is invariant under constant shifts $X^\mu \to X^\mu + b^\mu$. The associated charge is

$$p^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}} \tag{2.7}$$

so that indeed the equation of motion is just $\dot{p}^\mu = 0$. Note that I have called this a charge and not a current. In this case it doesn't matter because the Lagrangian theory we are talking about, the worldline theory of the point particle, is in zero spatial dimensions. So I could just as well called p^μ a conserved current with the conserved charge being obtained by integrating the temporal component of p^μ over space. Here there is no space p^μ only has temporal components.

Warning: We are thinking in terms of the worldline theory where the index μ labels the different scalar fields X^μ, it does not label the coordinates of the worldline. In particular p^0 is not the temporal component of p^μ from the worldline point of view. This confusion between worldvolume coordinates and spacetime coordinates arises throughout string theory

If we go to static gauge again, where $\tau = X^0$ and write $v^i = \dot{X}^i$ then we have the equations of motion

$$\frac{d}{d\tau}\frac{v^i}{\sqrt{1-v^2}} = 0 \tag{2.8}$$

and conserved charge

$$p^i = m\frac{v^i}{\sqrt{1-v^2}} \tag{2.9}$$

which is simply the spatial momentum. These are the standard relativistic expressions.

We can solve the equation of motion in terms of the constant of motion p^i by writing

$$\frac{v^i}{\sqrt{1-v^2}} = p^i/m \iff p^2/m^2 = \frac{v^2}{1-v^2} \iff v^2 = \frac{p^2}{p^2+m^2} \tag{2.10}$$

hence

$$X^i(\tau) = X^i(0) + \frac{p^i\tau}{\sqrt{p^2+m^2}} \tag{2.11}$$

and we see that v^i is constant with $v^2 < 1$.

2.2 Electromagnetic field

Next we can consider a particle interacting with an external electromagnetic field. An electromagnetic field is described by a vector potential A_μ and its field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The natural action of a point particle of mass m and charge q in the presence of such an electromagnetic field is

$$S_{pp} = -m \int \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} d\tau + q \int A_\mu(X)\dot{X}^\mu d\tau \tag{2.12}$$

For those who know differential geometry the vector potential is a connection one-form on spacetime and $A_\mu \dot{X}^\mu d\tau$ is simply the pull-back of A_μ to the worldline of the particle.

The equation of motion is now

$$-m\frac{d}{d\tau}\left(\frac{\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda \dot{X}^\rho}}\right) - q\frac{d}{d\tau}A_\mu + q\partial_\mu A_\nu \dot{X}^\nu = 0 \tag{2.13}$$

which we rewrite as

$$m\frac{d}{d\tau}\left(\frac{\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda \dot{X}^\rho}}\right) = qF_{\mu\nu}\dot{X}^\nu \tag{2.14}$$

To be more concrete we could choose static gauge again and we find

$$m\frac{d}{d\tau}\left(\frac{v^i}{\sqrt{1-v^2}}\right) = qF_{i0} + qF_{ij}v^j \tag{2.15}$$

Here we see the Lorentz force magnetic law arising as it should from the second term on the right hand side. The first term on the right hand side shows that an electric field provides a driving force.

At this point we should pause to mention a subtlety. In addition to (2.15) there is also the equation of motion for $X^0 = \tau$. However the reparameterization gauge symmetry implies that this equation is automatically satisfied. In particular the X^0 equation of motion is

$$-m\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) = qF_{0i}v^i \tag{2.16}$$

Problem: Show that if (2.15) is satisfied then so is (2.16)

Problem: Show that, in static gauge $X^0 = \tau$, the Hamiltonian for a charged particle is

$$H = \sqrt{m^2 + (p^i - qA^i)(p^i - qA^i)} - qA_0 \tag{2.17}$$

2.3 Quantization

Next we'd like to quantize the point particle. This is made difficult by the highly non-linear form of the action. To this end we will consider a new action which is classically equivalent to the old one. In particular consider

$$S_{HT} = -\frac{1}{2}\int d\tau e\left(-\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2\right) \tag{2.18}$$

Here we have introduced a new field $e(\tau)$ which is non-dynamical, *i.e.* has no kinetic term. This action is now just quadratic in the fields X^μ. The point of it is that it reproduces the same equations of motion as before. To see this consider the e equation of motion:

$$\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2 = 0 \tag{2.19}$$

we can solve this to find $e = m^{-1}\sqrt{-\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu}}$. We now compute the X^μ equation of motion

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\frac{1}{e}\dot{X}^\mu\right) \\ &= m\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^\lambda\dot{X}^\rho\eta_{\lambda\rho}}}\right) \end{aligned} \tag{2.20}$$

This is exactly what we want. Thus we conclude that S_{HT} is classically equivalent to S_{pp}.

One way to think about this action is that we have introduced a worldline metric $\gamma_{\tau\tau} = -e^2$ and its inverse $\gamma^{\tau\tau} = -1/e^2$ so that infinitesimal distances along the worldline have length

$$ds^2 = \gamma_{\tau\tau}d\tau^2 \tag{2.21}$$

Note that previously we never said that $d\tau$ represented the physical length of a piece of worldline, just that τ labeled points along the worldline.

There is another advantage to this form of the action; we can smoothly set $m^2 = 0$ and describe massless particles, which was impossible with the original form of the action.

Now the action is quadratic in the fields X^μ we calculate the Hamiltonian and quantize more easily. The first step here is to obtain the momentum conjugate to each of the X^μ

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{X}^\mu} \\ &= \frac{1}{e}\eta_{\mu\nu}\dot{X}^\nu \end{aligned} \tag{2.22}$$

There is no conjugate momentum to e, it acts as a constraint and we will deal with it later. The Hamiltonian is

$$\begin{aligned} H &= p_\mu \dot{X}^\mu - L \\ &= \frac{e}{2}\left(\eta_{\mu\nu}p^\mu p^\nu + m^2\right) \end{aligned} \tag{2.23}$$

To quantize this system we consider wavefunctions $\Psi(X,\tau)$ and promote X^μ and p_μ to the operators

$$\hat{X}^\mu \Psi = X^\mu \Psi \qquad \hat{p}_\mu \Psi = -i\frac{\partial \Psi}{\partial X^\mu} \tag{2.24}$$

We then arrive at the Schrödinger equation

$$i\frac{\partial \Psi}{\partial \tau} = \frac{e}{2}\left(-\eta^{\mu\nu}\frac{\partial^2 \Psi}{\partial X^\mu \partial X^\nu} - m^2\Psi\right) \tag{2.25}$$

Lastly we must deal with e which we saw has no conjugate momentum. Classically its equation of motion imposes the constraint

$$p^\mu p_\mu + m^2 = 0 \tag{2.26}$$

which is the mass-shell condition for the particle. Quantum mechanically we realize this by restricting our physical wavefunctions to those that satisfy the corresponding constraint

$$-\eta^{\mu\nu}\frac{\partial^2 \Psi}{\partial X^\mu \partial X^\nu} + m^2\Psi = 0 \tag{2.27}$$

However this is just the condition that $\hat{H}\Psi = 0$ so that the wavefunctions are τ independent. If you trace back the origin of this time-independence it arises as a consequence of the reparameterization invariance of the worldline theory. It simply states that wavefunctions must also be reparameterization invariant, *i.e.* they can't depend on τ. This is deep issue in quantum gravity. In effect it says that there is no such thing as time in the quantum theory.

This equation should be familiar if you have learnt quantum field theory. In particular if we consider a free scalar field Ψ in D-dimensional spacetime the action is

$$S = -\int d^D x \left(\frac{1}{2}\partial_\mu \Psi^* \partial^\mu \Psi + \frac{1}{2}m^2 \Psi^* \Psi\right) \tag{2.28}$$

and the corresponding equation of motion is

$$\partial^2 \Psi - m^2 \Psi = 0 \tag{2.29}$$

which is the same as our Schrodinger equation (when restricted to the physical Hilbert space).

Thus we see that there is a natural identification of a free scalar field with a quantum point particle. In particular the quantum states of the point particle are in a one-to-one correspondence with the classical solutions of the free spacetime effective action.

However one important difference should be stressed. The quantum point particle gave a Schrodinger equation which could be identified with the classical equation of motion for the scalar field. In quantum field theory one performs a second quantization whereby particles are allowed to be created and destroyed. This is beyond the quantization of the point particle that we considered since by default we studied the effective action on the worldline of a single particle: it would have made no sense to create or destroy particles. Thus the second quantized spacetime action provides a more complete physical theory.

Here we also can see that the quantum description of a point particle in one dimension leads to a classical spacetime effective action in D-dimensions. This is an important concept in String theory where the quantum dynamics of the two-dimensional worldvolume theory, with interactions included, leads to interesting and non-trivial spacetime effective actions.

Problem: Find the Schödinger equation, constraint and effective action for a quantized particle in the background of a classical electromagnetic field using the action

$$S_{pp} = -\int \frac{1}{2}e\left(-\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2\right) - A_\mu\dot{X}^\mu \tag{2.30}$$

3 Classical and Quantum Dynamics of Strings

3.1 Classical Action

Having studied point particles from their worldline perspective we now turn to our main subject: strings. Our starting point will be the action the worldvolume of a string, which is two-dimensional. The natural starting point is to consider the action

$$S_{string} = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det(\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu})} \tag{3.1}$$

which is simply the area of the two-dimensional worldvolume that the string sweeps out. Here $\sigma^\alpha, \alpha = 0,1$ labels the spatial and temporal coordinates of the string: τ,σ. Here $\sqrt{\alpha'}$ is a length scale that determines the size of the string.

Again we don't want to work directly with such a highly non-linear action. We saw above that we could change this by coupling to an auxiliary worldvolume metric $\gamma_{\alpha\beta}$.

Problem: Show that by solving the equation of motion for the metric $\gamma_{\alpha\beta}$ on a d-dimensional worldsheet the action

$$S_{HT} = -\frac{1}{2}\int d^d\sigma\sqrt{-\gamma}\left(\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} + m^2(d-2)\right) \tag{3.2}$$

one finds the action

$$S_{NG} = m^{2-d}\int d^d\sigma\sqrt{-\det\left(\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\right)} \tag{3.3}$$

for the remaining fields X^μ, *i.e.* calculate and solve the $\gamma_{\alpha\beta}$ equation of motion and then substitute the solution back into S_{HT} to obtain S_{NG}. Note that the action S_{HT} is often referred to as the Howe-Tucker form for the action whereas S_{NG} is the Nambu-Goto form. (Hint: You will need to use the fact that $\delta\sqrt{-\gamma}/\delta\gamma^{\alpha\beta} = -\frac{1}{2}\gamma_{\alpha\beta}\sqrt{-\gamma}$). If you have not yet learnt much about metrics just consider the case of $d = 1$ where $\gamma_{\alpha\beta}$ just has a single component $\gamma_{\tau\tau}$.

So we might instead start with

$$S_{string} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \tag{3.4}$$

where we have taken $d = 2$ in (3.2).

Problem: What transformation law must $\gamma_{\alpha\beta}$ have to ensure that S_{string} is reparameterization invariant? (Hint: Use the fact that

$$\frac{\partial\sigma'^\gamma}{\partial\sigma^\alpha}\frac{\partial\sigma^\beta}{\partial\sigma'^\gamma} = \delta_\alpha^\beta \tag{3.5}$$

why?)

However this case is very special. If we evaluate the $\gamma_{\alpha\beta}$ equation of motion we find

$$T_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\partial_\gamma X^\mu\partial_\delta X^\nu\eta_{\mu\nu} = 0 \tag{3.6}$$

Once again we see that $\gamma_{\alpha\beta} = b\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}$ for some b. However in this case nothing fixes b, it is arbitrary. This occurs because there is an addition symmetry of the action that is unique to two-dimensions: it is conformally invariant. That means that under a worldvolume conformal transformation

$$\gamma_{\alpha\beta} \to e^{2\varphi}\gamma_{\alpha\beta} \tag{3.7}$$

(here φ is any function of the worldvolume coordinates) the action is invariant.

There are other features that are unique to two-dimensions. The first is that, up to a reparameterization, we can always choose the metric $\gamma_{\alpha\beta} = e^{2\rho}\eta_{\alpha\beta}$. To see this we note that under a reparameterization we have

$$\gamma'_{\alpha\beta} = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta}\gamma_{\gamma\delta} \tag{3.8}$$

Thus we simply choose our new coordinates to fix $\gamma'_{01} = 0$ and $\gamma'_{00} = -\gamma'_{11}$. This requires that

$$\frac{\partial\sigma^\gamma}{\partial\sigma'^0}\frac{\partial\sigma^\delta}{\partial\sigma'^1}\gamma_{\gamma\delta} = 0 \tag{3.9}$$

and

$$\frac{\partial\sigma^\gamma}{\partial\sigma'^1}\frac{\partial\sigma^\delta}{\partial\sigma'^1}\gamma_{\gamma\delta} + \frac{\partial\sigma^\gamma}{\partial\sigma'^0}\frac{\partial\sigma^\delta}{\partial\sigma'^0}\gamma_{\gamma\delta} = 0 \tag{3.10}$$

These are two (complicated) differential equation for two functions $\sigma^0(\sigma'^0, \sigma'^1)$ and $\sigma^1(\sigma'^0, \sigma'^1)$. Hence there will be a solution (at least locally).

The second feature is that in two-dimensions the Einstein equation

$$R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}R = 0 \tag{3.11}$$

vanishes identically. The reason for this is that in two-dimensions there is only one independent component for the Riemann tensor: $R_{0101} = -R_{0110} = -R_{1001} = R_{1010}$. Therefore $R_{00} = R_{0101}\gamma^{11}$, $R_{11} = R_{0101}\gamma^{00}$ and $R_{01} = -R_{0101}\gamma^{01}$. Thus we see that

$$\begin{aligned} R &= 2R_{0101}(\gamma^{00}\gamma^{11} - \gamma^{01}\gamma^{01}) \\ &= 2R_{0101}\det(\gamma^{-1}) \\ &= \frac{2}{\det(\gamma)}R_{0101} \end{aligned} \tag{3.12}$$

Now we note that

$$\begin{pmatrix} \gamma^{00} & \gamma^{01} \\ \gamma^{01} & \gamma^{11} \end{pmatrix} = \frac{1}{\det(\gamma)}\begin{pmatrix} \gamma_{11} & -\gamma_{01} \\ -\gamma_{01} & \gamma_{00} \end{pmatrix} \tag{3.13}$$

and the result follows.

Thus Einstein's equation

$$R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}R = T_{\alpha\beta} \tag{3.14}$$

will imply that $T_{\alpha\beta} = 0$. Hence even if we include two-dimensional gravity the $\gamma_{\alpha\beta}$ equation of motion imposes the constraint that the energy-momentum tensor vanishes

$$T_{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial\gamma^{\alpha\beta}} = 0 \tag{3.15}$$

Thus we can use worldsheet diffeomorphisms to set $\gamma_{\alpha\beta} = e^{2\rho}\eta_{\alpha\beta}$ and then use worldsheet conformal invariance to set $\rho = 0$, *i.e.* $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. This means that the worldvolume metric $\gamma_{\alpha\beta}$ actually decouples from the fields X^μ. This conformal invariance of two-dimensional gravity coupled to the embedding coordinates (viewed as scalar fields) will be our fundamental principle. It allows us to simply set $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. Thus to consider strings propagating in flat spacetime we use the action (known as the Polyakov action)

$$S_{string} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \tag{3.16}$$

subject to the constraint (3.15) which becomes

$$\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu\partial_\delta X^\nu\eta_{\mu\nu} = 0 \tag{3.17}$$

3.2 Spacetime Symmetries and Conserved Charges

We should also pause to outline how the spacetime symmetries lead to conserved currents and hence conserved charges in the worldsheet theory.

First we summarize Noether's theorem. Suppose that a Lagrangian $\mathcal{L}(\Phi_A, \partial_\alpha \Phi_A)$, where we denoted the fields by Φ_A, has a symmetry: $\mathcal{L}(\Phi_A) = \mathcal{L}(\Phi_A + \delta\Phi_A)$. This implies that

$$\frac{\partial \mathcal{L}}{\partial \Phi_A}\delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\delta\partial_\alpha\Phi_A = 0 \tag{3.18}$$

This allows us to construct a current:

$$J^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\delta\Phi_A \tag{3.19}$$

which is conserved

$$\begin{aligned}
\partial_\alpha J^\alpha &= \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\right)\delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\partial_\alpha\delta\Phi_A \\
&= \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\right)\delta\Phi_A - \frac{\partial \mathcal{L}}{\partial \Phi_A}\delta\Phi_A \\
&= 0
\end{aligned} \tag{3.20}$$

by the equation of motion. This means that the integral over space of J^0 is a constant defines a charge

$$Q = \int_{space} \sigma J^0 \tag{3.21}$$

which is conserved

$$\begin{aligned}
\frac{dQ}{dt} &= \int_{space} \partial_0 J^0 \\
&= -\int_{space} \partial_i J^i \\
&= 0
\end{aligned}$$

Let us now consider the action

$$S_{string} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{3.22}$$

This has the spacetime Poincare symmetries: translations $\delta X^\mu = a^\mu$ and Lorentz transformations $\delta X^\mu = \Lambda^\mu{}_\nu X^\nu$. In the first case the conserved current is

$$P^\alpha_{a^\mu} = -\frac{1}{2\pi\alpha'}\partial^\alpha X_\mu a^\mu \tag{3.23}$$

The associated conserved charge is just the total momentum along the direction a^μ and in particular there are D independent choices

$$p_\mu = \frac{1}{2\pi\alpha'}\int d\sigma \dot{X}_\mu \tag{3.24}$$

We can also consider the spacetime Lorentz transformations which lead to the conserved currents

$$J^\alpha_\Lambda = -\frac{1}{2\pi\alpha'}\partial^\alpha X_\mu \Lambda^\mu{}_\nu X^\nu \tag{3.25}$$

The independent conserved charges are therefore given by (here $Q_\Lambda = \int d\sigma J^0_\Lambda = M^\mu{}_\nu \Lambda^\nu{}_\mu$)

$$M^\mu{}_\nu = \frac{1}{4\pi\alpha'}\int d\sigma \dot{X}^\mu X_\nu - X^\mu \dot{X}_\nu \tag{3.26}$$

The Poisson brackets of these worldsheet charges will, at least at the classical level, satisfy the algebra Poincare algebra. In the quantum theory they are lifted to operators that commute with the Hamiltonian.

3.3 Quantization

Next we wish to quantize this action. Unlike the point particle this action is a field theory in (1+1)-dimensions. As such we must use the quantization techniques of quantum field theory rather than simply constructing a Schrodinger equation. There are several ways to do this. The most modern way is the path integral formulation and Fadeev-Popov ghosts. However this requires some techniques that are possibily unfamiliar. So here we will use the method of canonical quantization.

Canonical quantization is essentially the Heisenberg picture of quantum mechanics where the fields X^μ and their conjugate momenta P_μ are promoted to operators which satisfy the equal time commutation relations

$$\begin{aligned}
[\hat{X}^\mu(\tau,\sigma), \hat{P}_\nu(\tau,\sigma')] &= i\delta(\sigma-\sigma')\delta^\mu_\nu \\
[\hat{X}^\mu(\tau,\sigma), \hat{X}^\nu(\tau,\sigma')] &= 0 \\
[\hat{P}_\mu(\tau,\sigma), \hat{P}_\nu(\tau,\sigma')] &= 0
\end{aligned} \tag{3.27}$$

as well as the Heisenberg equation

$$\frac{d\hat{X}^\mu}{d\tau} = i[\hat{H}, \hat{X}^\mu] \qquad \frac{d\hat{P}_\mu}{d\tau} = i[\hat{H}, \hat{P}_\mu] \tag{3.28}$$

In the case at hand we have

$$\hat{L} = \frac{1}{4\pi\alpha'}\int d\sigma \eta_{\mu\nu}\dot{\hat{X}}^\mu \dot{\hat{X}}^\nu - \eta_{\mu\nu}\hat{X}'^\mu \hat{X}'^\nu \tag{3.29}$$

hence

$$\hat{P}_\mu = \frac{1}{2\pi\alpha'}\eta_{\mu\nu}\dot{\hat{X}}^\nu \tag{3.30}$$

and

$$
\begin{aligned}
\hat{H} &= \int d\sigma \hat{P}_\mu \dot{\hat{X}}^\mu - \hat{L} \\
&= \int d\sigma 2\pi\alpha' \eta^{\mu\nu} \hat{P}_\mu \hat{P}_\nu - \int d\sigma \frac{1}{4\pi\alpha'}(2\pi\alpha')^2 \eta^{\mu\nu} \hat{P}_\mu \hat{P}_\nu + \frac{1}{4\pi\alpha'} \eta_{\mu\nu} \hat{X}'^\mu \hat{X}'^\nu \\
&= \int d\sigma \pi\alpha' \eta^{\mu\nu} \hat{P}_\mu \hat{P}_\nu + \frac{1}{4\pi\alpha'} \eta_{\mu\nu} \hat{X}'^\mu \hat{X}'^\nu
\end{aligned}
\tag{3.31}
$$

We can now calculate

$$
\begin{aligned}
\dot{\hat{X}}^\mu(\sigma) &= i[\hat{H}, \hat{X}^\mu(\sigma)] \\
&= \pi\alpha' i \int d\sigma' \eta^{\lambda\nu} [\hat{P}_\lambda(\sigma')\hat{P}_\nu(\sigma'), \hat{X}^\mu(\sigma)] \\
&= 2\pi\alpha' i \int d\sigma' \eta^{\lambda\nu} \hat{P}_\lambda(\sigma')[\hat{P}_\nu(\sigma'), \hat{X}^\mu(\sigma)] \\
&= 2\pi\alpha' \int d\sigma' \eta^{\lambda\nu} \hat{P}_\lambda(\sigma') \delta^\mu_\nu \delta(\sigma - \sigma') \\
&= 2\pi\alpha' \eta^{\mu\nu} \hat{P}_\nu(\sigma)
\end{aligned}
\tag{3.32}
$$

which we already knew. But also we can now calculate

$$
\begin{aligned}
\dot{\hat{P}}_\mu(\sigma) &= i[\hat{H}, \hat{P}_\mu(\sigma)] \\
&= \frac{i}{4\pi\alpha'} \int d\sigma' \eta_{\lambda\nu} [\hat{X}'^\lambda(\sigma')\hat{X}'^\nu(\sigma'), \hat{P}_\mu(\sigma)] \\
&= \frac{i}{2\pi\alpha'} \int d\sigma' \eta_{\lambda\nu} \hat{X}'^\lambda(\sigma')[\hat{X}'^\nu(\sigma'), \hat{P}_\mu(\sigma)] \\
&= \frac{i}{2\pi\alpha'} \int d\sigma' \eta_{\lambda\nu} \hat{X}'^\lambda(\sigma') \frac{\partial}{\partial\sigma'}[\hat{X}^\nu(\sigma'), \hat{P}_\mu(\sigma)] \\
&= -\frac{i}{2\pi\alpha'} \int d\sigma' \eta_{\lambda\nu} \hat{X}''^\lambda(\sigma')[\hat{X}^\nu(\sigma'), \hat{P}_\mu(\sigma)] \\
&= \frac{1}{2\pi\alpha'} \int d\sigma' \eta_{\lambda\nu} \hat{X}''^\lambda(\sigma') \delta^\nu_\mu \delta(\sigma - \sigma') \\
&= \frac{1}{2\pi\alpha'} \eta_{\mu\nu} \hat{X}''^\nu(\sigma)
\end{aligned}
\tag{3.33}
$$

or equivalently

$$
-\ddot{\hat{X}}^\mu + \hat{X}''^\mu = 0 \tag{3.34}
$$

Of course this is just the classical equation of motion reinterpreted in the quantum theory as an operator equation. In two-dimensions the solution to this is simply that

$$
\hat{X}^\mu = \hat{X}^\mu_L(\tau + \sigma) + \hat{X}^\mu_R(\tau - \sigma) \tag{3.35}
$$

i.e. we can split $\hat{X}^\mu$ into a left and right moving part.

To proceed we expand the string in a Fourier series

$$\hat{X}^\mu = x^\mu + \alpha' p^\mu \tau + \sqrt{\frac{\alpha'}{2}} i \sum_{n\neq 0} \left(\frac{a_n^\mu}{n} e^{-in(\tau+\sigma)} + \frac{\tilde{a}_n^\mu}{n} e^{-in(\tau-\sigma)} \right) \tag{3.36}$$

The various factors of n and α' will turn out to be useful later on. We have also included linear terms since $\hat{X}^\mu$ need not be periodic (more on this later). Or if you prefer

$$\begin{aligned} \hat{X}_L^\mu &= x_L^\mu + \frac{1}{2}\alpha' p^\mu (\tau+\sigma) + \sqrt{\frac{\alpha'}{2}} i \sum_{n\neq 0} \frac{a_n^\mu}{n} e^{-in(\tau+\sigma)} \\ \hat{X}_R^\mu &= x_R^\mu + \frac{1}{2}\alpha' p^\mu (\tau-\sigma) + \sqrt{\frac{\alpha'}{2}} i \sum_{n\neq 0} \frac{\tilde{a}_n^\mu}{n} e^{-in(\tau-\sigma)} \end{aligned} \tag{3.37}$$

Note that we have dropped the hat on the operators a^μ and $\tilde{a}^\mu$ since they will appear frequently. But don't forget that they are operators! Note also that we haven't yet said what n is, *e.g.* whether or not it is an integer, we will be more specific later. The a_n^μ and $\tilde{a}_n^\mu$ have the interpretation as left and right moving oscillators. Just as in quantum mechanics and quantum field theory these will be related to particle creation and annihilation operators.

Since X^μ is an observable we require that it is Hermitian in the quantum theory. This in turn implies that

$$(a_n^\mu)^\dagger = a_{-n}^\mu \ , \qquad (\tilde{a}_n^\mu)^\dagger = \tilde{a}_{-n}^\mu \tag{3.38}$$

and $(x^\mu)^\dagger = x^\mu, (p^\mu)^\dagger = p^\mu$. In this basis

$$\begin{aligned} \hat{P}^\mu &= \frac{1}{2\pi\alpha'} \dot{\hat{X}}^\mu \\ &= \frac{1}{2\pi\alpha'} \left(\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{-n\neq 0} a_n^\mu e^{-in(\tau+\sigma)} + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \tilde{a}_n^\mu e^{-in(\tau-\sigma)} \right) \end{aligned} \tag{3.39}$$

We can work out the commutator. First we take $x^\mu = p^\mu = 0$

$$\begin{aligned} [\hat{X}^\mu(\tau,\sigma), \hat{P}_\nu(\tau,\sigma')] &= \frac{i}{4\pi} \sum_n \sum_m \frac{1}{n} e^{-i(n+m)\tau} e^{-i(n\sigma+m\sigma')} [a_n^\mu, a_m^\nu] \\ &+ \frac{i}{4\pi} \sum_n \sum_m \frac{1}{n} e^{-i(n+m)\tau} e^{i(n\sigma+m\sigma')} [\tilde{a}_n^\mu, \tilde{a}_m^\nu] \\ &+ \frac{i}{4\pi} \sum_n \sum_m \frac{1}{n} e^{-i(n+m)\tau} e^{i(n\sigma-m\sigma')} [\tilde{a}_n^\mu, a_m^\nu] \\ &+ \frac{i}{4\pi} \sum_n \sum_m \frac{1}{n} e^{-i(n+m)\tau} e^{-i(n\sigma-m\sigma')} [a_n^\mu, \tilde{a}_m^\nu] \end{aligned} \tag{3.40}$$

In order for the τ-dependent terms to cancel we see that we need the commutators to vanish if $n \neq -m$. The sum now reduces to

$$
\begin{aligned}
[\hat{X}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] &= \frac{i}{4\pi}\sum_n \frac{1}{n} e^{-in(\sigma-\sigma')}[a^\mu_n, a^\nu_{-n}] \\
&+ \frac{i}{4\pi}\sum_n \frac{1}{n} e^{in(\sigma-\sigma')}[\tilde{a}^\mu_n, \tilde{a}^\nu_{-n}] \\
&+ \frac{i}{4\pi}\sum_n \frac{1}{n} e^{in(\sigma+\sigma')}[\tilde{a}^\mu_n, a^\nu_{-n}] \\
&+ \frac{i}{4\pi}\sum_n \frac{1}{n} e^{-in(\sigma+\sigma')}[a^\mu_n, \tilde{a}^\nu_{-n}]
\end{aligned}
\tag{3.41}
$$

Next translational invariance implies that the $\sigma + \sigma'$ terms vanish and hence

$$
[a^\mu_n, \tilde{a}^\nu_m] = 0 \tag{3.42}
$$

A slight rearrangement of indices shows that we are left with

$$
[\hat{X}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] = \frac{i}{4\pi}\sum_n \frac{1}{n} e^{-in(\sigma-\sigma')}([a^\mu_n, a^\nu_{-n}] + [\tilde{a}^\mu_n, \tilde{a}^\nu_{-n}]) \tag{3.43}
$$

In a Fourier basis

$$
\delta(\sigma-\sigma') = \frac{1}{2\pi}\sum_n e^{-in(\sigma-\sigma')} \tag{3.44}
$$

Note that there is a contribution from $n = 0$ here that doesn't come from the oscillators, we'll deal with it in a moment. Therefore we see that we must take

$$
[a^\mu_n, a^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m} \ , \qquad [\tilde{a}^\mu_n, \tilde{a}^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m} \tag{3.45}
$$

Next it remains to consider the zero-modes (including the $n = 0$ contribution in (3.44)).

Problem: Show that if $x^\mu, p^\mu \neq 0$ then we also have

$$
[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{3.46}
$$

with the other commutators vanishing.

We also have to consider the constraint $\hat{T}_{\alpha\beta} = 0$. Its components are

$$
\begin{aligned}
\hat{T}_{00} &= \frac{1}{2}\dot{\hat{X}}^\mu \dot{\hat{X}}^\nu \eta_{\mu\nu} + \frac{1}{2}\hat{X}'^\mu \hat{X}'^\nu \eta_{\mu\nu} \\
\hat{T}_{11} &= \frac{1}{2}\hat{X}'^\mu \hat{X}'^\nu \eta_{\mu\nu} + \frac{1}{2}\dot{\hat{X}}^\mu \dot{\hat{X}}^\nu \eta_{\mu\nu} \\
\hat{T}_{01} &= \dot{\hat{X}}^\mu \hat{X}'^\nu \eta_{\mu\nu}
\end{aligned}
\tag{3.47}
$$

It is helpful to change coordinates to

$$\begin{array}{l}\sigma^+ = \tau + \sigma \\ \sigma^- = \tau - \sigma\end{array} \iff \begin{array}{l}\tau = \frac{\sigma^+ + \sigma^-}{2} \\ \sigma = \frac{\sigma^+ - \sigma^-}{2}\end{array} \tag{3.48}$$

Problem: Show that in these coordinates

$$\begin{aligned}\hat{T}_{++} &= \partial_+ \hat{X}^\mu \partial_+ \hat{X}^\nu \eta_{\mu\nu} \\ \hat{T}_{--} &= \partial_- \hat{X}^\mu \partial_- \hat{X}^\nu \eta_{\mu\nu} \\ \hat{T}_{+-} &= T_{-+} = 0\end{aligned} \tag{3.49}$$

Let us now calculate T_{++} in terms of oscillators. We have

$$\partial_+ \hat{X}^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} a_n^\mu e^{-in(\tau+\sigma)} \tag{3.50}$$

where we have introduced

$$a_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu + \sqrt{\frac{1}{2\alpha'}} w^\mu \tag{3.51}$$

thus

$$\begin{aligned}\hat{T}_{++} &= \frac{\alpha'}{2} \sum_{nm} a_n^\mu a_m^\nu e^{-i(n+m)(\tau+\sigma)} \eta_{\mu\nu} \\ &= \alpha' \sum_n L_n e^{-in(\tau+\sigma)}\end{aligned} \tag{3.52}$$

with

$$L_n = \frac{1}{2} \sum_m a_{n-m}^\mu a_m^\nu \eta_{\mu\nu} \tag{3.53}$$

where again we've dropped a hat on L_n, even though it is an operator. Similarly we find

$$T_{--} = \alpha' \sum_n \tilde{L}_n e^{-in(\tau-\sigma)} \tag{3.54}$$

with

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{a}_{n-m}^\mu \tilde{a}_m^\nu \eta_{\mu\nu} \tag{3.55}$$

and

$$\tilde{a}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu - \sqrt{\frac{2}{\alpha'}} w^\mu \tag{3.56}$$

We can rewrite the commutators (3.45) using (3.38) as

$$[a_n^\mu, a_n^{\nu\dagger}] = n\eta^{\mu\nu} \qquad [\tilde{a}_n^\mu, \tilde{a}_n^{\nu\dagger}] = n\eta^{\mu\nu} \tag{3.57}$$

with $n > 0$. Thus we can think of a^μ_n and $\tilde{a}^\mu_n$ annihilation operators and $a^{\mu\dagger}_n$ and $\tilde{a^\mu_n}^\dagger$ as creation operators. Following the standard practice of QFT we consider the ground state $|0>$ to be annihilated by a_n and $\tilde{a}_n$:

$$a_n|0>=0\ ,\qquad \tilde{a}_n|0>=0\ ,\qquad n>0 \tag{3.58}$$

The zero modes also act on the ground state. Since x^μ and p^μ don't commute we can only chose $|0>$ to be an eigenstate of one, we take

$$\hat{p}^\mu|0>=p^\mu|0> \tag{3.59}$$

when we want to be precise we label the ground state $|0;p>$. You will have to excuse the clumsy notion where I have reintroduce a hat on an operator to distinguish it from its eigenvalue acting on a state. We can now construct a Fock space of multi-particle states by acting on the ground state with the creation operators a^μ_{-n} and $\tilde{a}^\mu_{-n}$. For example

$$a^\mu_{-1}\tilde{a}^\nu_{-1}|0>\ ,\qquad a^\mu_{-2}\tilde{a}^\lambda_{-1}\tilde{a}^\rho_{-1}|0>\ ,\qquad etc. \tag{3.60}$$

These elements should be familiar from the study of the harmonic oscillator. In a string theory each classical vibrational mode is mapped in the quantum theory to an individual harmonic oscillator with the same frequency.

Note that we really should considering normal ordered operators, where the annihilation operators always appear to the right of the creation operators. For L_n and $\tilde{L}_n$ with $n \neq 0$ there is no ambiguity as a^μ_m and a^ν_{n-m} will commute. However for L_0 and $\tilde{L}_0$ one finds

$$L_0 = \frac{1}{2}a^\mu_0 a^\nu_0\eta_{\mu\nu} + \sum_{m>0} a^\mu_{-m}a^\nu_m\eta_{\mu\nu} - \frac{1}{2}\sum_{m>0}[a^\mu_{-m}, a^\nu_m]\eta_{\mu\nu} \tag{3.61}$$

The last term is an infinite divergent sum

$$\frac{D}{2}\sum_{m>0} m \tag{3.62}$$

This can be thought of as sum over the zero-point energies of the infinite number of harmonic oscillators. We must renormalize. Clearly $\tilde{L}_0$ has the same problem and this introduces the same sum. Since this is just a number the end result is that we define the normal ordered L_0 and $\tilde{L}_0$ to be

$$\begin{aligned} :L_0: &= \frac{1}{2}a^\mu_0 a^\nu_0\eta_{\mu\nu} + \alpha'\sum_{m>0} a^\mu_{-m}a^\nu_m\eta_{\mu\nu} \\ :\tilde{L}_0: &= \frac{1}{2}\tilde{a}^\mu_0 \tilde{a}^\nu_0\eta_{\mu\nu} + \alpha'\sum_{m>0} \tilde{a}^\mu_{-m}\tilde{a}^\nu_m\eta_{\mu\nu} \end{aligned} \tag{3.63}$$

In string theory $:L_n:$ and $:\tilde{L}_n:$ play a central role.

How do we deal with constraints in the quantum theory? We should proceed by reducing to the so-called physical Hilbert space of states which are those states that are animated by $:\hat{T}_{\alpha\beta}:$. However this turns out to be too strong a condition and would remove all states. Instead we impose that the positive frequency components of $:\hat{T}_{\alpha\beta}:$ annihilates any physical state

$$
:L_n:|phys>=:\tilde{L}_n:|phys>=0\,,n>0 \qquad (:L_0:-a)|phys>=(:\tilde{L}_0:-a)|phys>=0 \tag{3.64}
$$

Here we have introduced a parameter a since $:L_0:$ differs from L_0 by an infinite constant that we must regularize to the finite value a. For historical reasons the parameter a is called the intercept (and α' the slope). However it is not a parameter but rather is fixed by consistency conditions. Indeed it can be calculated by a variety of methods (such as ζ-function regularization or by using the modern BRST approach to quantization). We will see that the correct value is $a=1$.

This is then sufficient to show that the expectation value of $:\hat{T}_{\alpha\beta}:$ vanishes

$$
<phys|:L_n:|phys>=<phys|:\tilde{L}_n:|phys>=0 \qquad \forall n\neq 0 \tag{3.65}
$$

since the state on the right is annihilated by the postiche frequency parts where as by taking the Hermitian conjugates one sees that the state on the left is annihilated by the negative frequency part.

It is helpful to calculate the commutator $[:L_m:,:L_n:]$. There will be a similar expression for $[:\tilde{L}_m:,:\tilde{L}_m:]$ and clearly one has $[:L_m:,:\tilde{L}_n:]=0$. To do this we first consider the case without worrying about normal orderings

$$
\begin{aligned}
[L_m,L_n] &= \frac{1}{4}\sum_{pq}[a^\mu_{m-p}a^\nu_p,a^\lambda_{n-q}a^\rho_q]\eta_{\mu\nu}\eta_{\lambda\rho} \\
&= \frac{1}{4}\sum_{pq}\eta_{\mu\nu}\eta_{\lambda\rho}\Big([a^\mu_{m-p}a^\nu_p,a^\lambda_{n-q}]a^\rho_q+a^\lambda_{n-q}[a^\mu_{m-p}a^\nu_p,a^\rho_q]\Big) \\
&= \frac{1}{4}\sum_{pq}\eta_{\mu\nu}\eta_{\lambda\rho}\Big(a^\mu_{m-p}[a^\nu_p,a^\lambda_{n-q}]a^\rho_q+[a^\mu_{m-p},a^\lambda_{n-q}]a^\nu_p a^\rho_q \\
&\quad +a^\lambda_{n-q}a^\mu_{m-p}[a^\nu_p,a^\rho_q]+a^\lambda_{n-q}[a^\mu_{m-p},a^\rho_q]a^\nu_p\Big) \\
&= \frac{1}{4}\sum_{p}\eta_{\mu\rho}\Big(pa^\mu_{m-p}a^\rho_{n+p}+(m-p)a^\mu_p a^\rho_{n+m-p} \\
&\quad +pa^\rho_{n+p}a^\mu_{m-p}+(m-p)a^\rho_{n+m-p}a^\mu_p\Big) \\
&= \frac{1}{2}\sum_{p}\eta_{\mu\rho}\Big((p-n)a^\mu_{m+n-p}a^\rho_p\ +(m-p)a^\mu_p a^\rho_{n+m-p}\ \eta_{\mu\rho}\Big)
\end{aligned} \tag{3.66}
$$

Here we have used the identities

$$
[A,BC]=[A,B]C+B[A,C]\ , \qquad [AB,C]=A[B,C]+[A,C]B \tag{3.67}
$$

and shifted the p=variable in the sum. Thus we find

$$[L_m, L_n] = (m-n)L_{m+n} \tag{3.68}$$

This is called the classical Virasoro algebra and is of crucial importance in string theory and conformal field theory in general. Recall that it is the algebra of constraints that arose from the condition $\hat{T}_{\alpha\beta} = 0$ which is the statement of conformal invariance.

In the quantum theory we must consider the issues associated with normal ordering. We saw that this only affected $: L_0 :$. It follows that the only effect this can have on the Virasoro algebra is in terms with an $: L_0 :$. Since the effect on $: L_0 :$ is a shift by an infinite constant it won't appear in the commutator on the left hand side. Thus any new terms can only appear with $: L_0 :$ on the right hand side. Thus the general form is

$$[: L_m :, : L_n :] = (m-n) : L_{m+n} : + C(n)\delta_{m\ -n} \tag{3.69}$$

The easiest way to determine the $C(n)$ is to note the following (one can also perform a direct calculation but it is notoriously complicated and messy). First one imposes the Jacobi identity

$$[: L_k :, [: L_m :, : L_n :]] + [: L_m :, [: L_n :, : L_k :]] + [: L_m :, [: L_n :, : L_k :]] = 0 \tag{3.70}$$

If we impose that $k + m + n = 0$ with $k, m, n \neq 0$ (so that no pair of them adds up to zero) then this reduces to

$$(m-n)C(k) + (n-k)C(m) + (k-m)C(n) = 0 \tag{3.71}$$

If we pick $k = 1$ and $m = -n - 1$ one finds

$$-(2n+1)C(1) + (n-1)C(-n-1) + (n+2)C(n) = 0 \tag{3.72}$$

Now we note that $C(-n) = -C(n)$ by definition. Hence we learn that $C(0) = 0$ and

$$C(n+1) = \frac{(n+2)C(n) - (2n+1)C(1)}{n-1} \tag{3.73}$$

This is just a difference equation and given $C(2)$ it will determine $C(n)$ for $n > 1$ (note that it can't determine $C(2)$ given $C(1)$). We can look for a solution to this by considering polynomials. Since it must be odd in n the simplest guess is

$$c(n) = c_1 n^3 + c_2 n \tag{3.74}$$

In this case the right hand side becomes

$$\begin{aligned} \frac{(n+1)(c_1 n^3 + c_2 n) - (2n+1)(c_1 + c_2)}{n-1} &= \frac{c_1 n^4 + 2c_1 n^3 + c_2 n^2 - 2c_1 n - (c_1 + c_2)}{n-1} \\ &= \frac{(n-1)(c_1 n^3 + 3c_1 n^2 + (3c_1 + c_2)n + c_1 + c_2)}{n-1} \end{aligned} \tag{3.75}$$

Expanding out the left hand side gives

$$c_1(n+1)^3 + c_2(n+1) = c_1 n^3 + 3c_1 n^2 + (3c_1 + c_2)n + c_1 + c_2 \tag{3.76}$$

and hence they agree.

Note that if we shift L_0 by a constant l then $C(n)$ is shifted by $2nl$ (note that in so doing we'd have to shift a as well). This means that we can change the value of c_2. Therefore we will fix it to be $c_1 = -c_2$. Finally we must calculate c_1. To do this we consider the ground state with no momentum $|0;0,0>$ This state is annihilated by $:L_n:$ for all $n \geq 0$. Thus we have

$$\begin{aligned} <0,0;0| :L_2::L_{-2}: |0;0,0> &= <0,0;0|[:L_2:,:L_{-2}:]|0;0,0> \\ &= 4<0,0;0|:L_0:|0;0,0> + 6c_1 <0,0;0|0;0,0> \\ &= 6c_1 \end{aligned} \tag{3.77}$$

where we assume that the ground state has unit norm.

Problem: Show that

$$<0,0;0| :L_2::L_{-2}: |0;0,0> = \frac{D}{2} \tag{3.78}$$

So we deduce that

$$[:L_m:,:L_n:] = (m-n):L_{m+n}: + \frac{D}{12}(m^3 - m)\delta_{m\ -n} \tag{3.79}$$

Of course there is a similar expression for $[:\tilde{L}_m:,:\tilde{L}_m:]$. This is called the central extension of the Virasoro algebra and D is the central charge which has arisen as a quantum effect. From now on we will always take operators to be normal ordered and we will drop the :: symbol, unless otherwise stated.

Let us return to our Fock space of states. It is built up out of the ground state which we take to have unit norm $<0|0>=1$. One sees that the one-particle state $a^\mu_{-1}|0>$ has norm

$$<0|a^\mu_1 a^\mu_{-1}|0> = <0|[a^\mu_1, a^\mu_{-1}]|0> = \eta^{\mu\mu} \tag{3.80}$$

where we do not sum over μ. Thus the state $a^0_{-1}|0>$ has negative norm!

Problem: Show that the state $(a^0_{-1} + a^1_{-1})|0>$ has zero norm.

Thus the natural innerproduct on the Fock space is not positive definite because the time-like oscillators come with the wrong sign. This also occurs in other quantum theories such as QED and doesn't necessarily represent any kind of sickness.

There are stranger states still. A physical state $|\chi>$ that satisfies $<\chi|phys>=0$ for all physical states is called null (or spurious if it only satisfies the $n=0$ physical state condition). It then follows that a null state has zero norm (as it must be orthogonal to itself).

There are many such states. To construct an example just consider

$$|\chi >= L_{-1}|0;p> \quad \text{with} \quad p^2 = 0 \tag{3.81}$$

Note that the zero-momentum ground state satisfies $L_n|0;0>=0$ and for all $n \geq 0$ and this remains true if for $|0;p>$ if $p^2 = 0$. First we verify that $|\chi>$ is physical. We have For $m \geq 0$

$$\begin{aligned} L_m|\chi> &= L_m L_{-1}|0;p> \\ &= [L_m, L_{-1}]|0;p> \\ &= (m+1)L_{m-1}|0;p> + \frac{D}{12}(m^3 - m)\delta_{m1}|0;p> \end{aligned} \tag{3.82}$$

The last term will vanish automatically whereas the first term can only be non-zero for $m = 0$ (since $L_n|0;p>=0$ for all $n \geq 0$). Here we find $L_0|\chi>=|\chi>$ which is the physical state condition for $a = 1$ which will turn out to be the case. Next we see that $<\chi|phys>=<0|L_1|phys>=0$. Note that we could have used any state instead of $|0;p>$ that was annihilated by L_n for all $n \geq 0$ to construct a null state.

Thus if we calculate some amplitude between two physical states $<phys'|phys>$ we can shift $|phys> \rightarrow |phys> + |\chi>$ where $|\chi>$ is a null state. The new state $|phys> + |\chi>$ is still physical but the amplitude will remain the same - for any other choice of physical state $|phys'>$. Thus we have a stringy gauge symmetry whereby two physical states are equivalent if their difference is a null state. This will turn out to be the origin of Yang-Mills and other gauge symmetries within string theory. And furthermore one can prove a no-ghost theorem which asserts that there are no physical states with negative norm (at least for $a = 1$ and $D = 26$).

3.4 Open Strings

Strings come in two varieties: open and closed. To date we have tried to develop as many formulae and results as possible which apply to both. However now we must make a decision and proceed along slightly different but analogous roots. Open strings have two end points which traditionally arise at $\sigma = 0$ and $\sigma = \pi$. We must be careful to ensure that the correct boundary conditions are imposed. In particular we must choose boundary conditions so that the boundary value problem is well defined. This requires that

$$\eta_{\mu\nu}\delta X^\mu \partial_\sigma X^\nu = 0 \tag{3.83}$$

at $\sigma = 0, \pi$.

Problem: Show this!

There are essentially two boundary conditions that one can impose. The first is Dirichlet: we hold X^μ fixed at the end points so that $\delta X^\mu = 0$. The second is Neumann:

we set $\partial_\sigma X^\mu = 0$ at the end points. The first condition implies that somehow the end points of the string are fixed in spacetime, like a flag to a flag pole. At first glance this seems unphysical and we will ignore it for now, although such boundary conditions turn out to be profoundly important. So we will start by considering second boundary condition, which states that no momentum leaks off the ends of the string.

The condition that $\partial_\sigma \hat{X}^\mu(\tau, 0) = 0$ implies that

$$a_n^\mu = \tilde{a}_n^\mu \tag{3.84}$$

i.e. the left and right oscillators are not independent. If we look at the boundary condition at $\sigma = \pi$ then we determine that

$$\sum_{n\neq 0} a_n^\mu e^{in\tau}\sin(n\pi) = 0 \tag{3.85}$$

Thus n is indeed an integer. The mode expansion is therefore

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'}i\sum_{n\neq 0}\frac{a_n^\mu}{n}e^{in\tau}\cos(n\sigma) \tag{3.86}$$

(Note the slightly redefined value of p^μ as compared to before.)

For the open string the physical states are constrained to satisfy

$$L_n|phys>= 0\ , n > 0 \qquad \text{and} \qquad (L_0 - 1)|phys>= 0 \tag{3.87}$$

in particular there is only one copy of the constraints required since the $\tilde{L}_n$ constraints will automatically be satisfied. The second condition is the most illuminating as it gives the spacetime mass shell condition. To see this we note that translational invariance $X^\mu \to X^\mu + x^\mu$ gives rise to the conserved current $\hat{P}^\mu = \frac{1}{2\pi\alpha'}\dot{X}^\mu$. This is a worldsheet current and hence the conserved charge (from the worldsheet point of view) is

$$\begin{aligned} p^\mu &= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \dot{X}^\mu \\ &= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma 2p^\mu + \sqrt{2\alpha'}\sum_{n\neq 0} a_n^\mu e^{in\tau}\cos(n\sigma) \\ &= p^\mu \end{aligned} \tag{3.88}$$

where again we have abused notation and confused the operator $\hat{p}^\mu$ that appears in the mode expansion of X^μ with its eigenvalue p^μ which we have now identified with the conserved charge. In any case we do this because we have shown that p^μ is indeed the spacetime momentum of the string. Note that this also explains why we put in the extra factor of 2 in front of $p^\mu\tau$ in the mode expansion.

Next we let

$$N = \sum_{n>0}\eta_{\mu\nu}a_{-n}^\mu a_n^\nu \tag{3.89}$$

Which is the analogue of the number operator that appears in the Harmonic oscillator. Again this is an operator even though we are being lazy and dropping the hat. It is easy to see that for $m > 0$

$$\begin{aligned}
[N, a^\lambda_{-m}] &= \sum_{n>0} \eta_{\mu\nu} a^\mu_{-n} [a^\nu_n, a^\lambda_{-m}] \\
&= m a^\lambda_{-m}
\end{aligned} \tag{3.90}$$

Thus if $|n>$ is a state with $N|n>= n|n>$ then

$$\begin{aligned}
N a^\lambda_{-m}|n> &= ([N, a^\lambda_{-m}] + a^\lambda_{-m} N)|n> \\
&= (m a^\lambda_{-m} + a^\lambda_{-m} n)|n> \\
&= (m+n) a^\lambda_{-m}|n>
\end{aligned} \tag{3.91}$$

Therefore $a^\lambda_{-m}|n>$ is a state with N-eigenvalue $n+m$. You can think of N as counting the number of oscillator modes in a given state.

With this definition we can write the physical state condition $(L_0 - 1)|phys>= 0$ as

$$(p_\mu p^\mu + \frac{1}{\alpha'}(N-1))|phys>= 0 \tag{3.92}$$

Thus we can identify the spacetime mass-squared of a physical state to be the eigenvalue of

$$M^2 = \frac{1}{\alpha'}(N-1) \tag{3.93}$$

We call the eigenvalue of N the level of the state. In other words the higher oscillator modes give more and more massive states in spacetime. In practice one takes $a^{-1/2}$ to be a very high mass scale so that only the massless modes are physically relevant. Note that the number of states at level n grows exponentially in n as the number of possible oscillations will be of order of the number of partitions of n into smaller integers. This exponentially growing tower of massive modes a unique feature of strings as opposed to point particles.

Of course we must not forget the other physical state condition $L_n|phys>= 0$ for $n > 0$. This constraint will take the form of a gauge fixing condition . Let us consider the lowest lying states.

At level zero we have the vacuum $|0; p>$. We see that the mass-shell condition is

$$p^2 - \alpha'^{-1} = 0 \tag{3.94}$$

The other constraint, $L_n|0; p>= 0$ with $n > 0$, is automatically satisfied. This has a negative mass-squared! Such a mode is called a Tachyon. Tachyons arise in field theory if rather than expanding a scalar field about a minimum of the potential one expands about a maximum. Thus they are interpreted as instabilities. The problem is that no one knows in general whether or not the instability associated to this open string tachyon is ever stabilized. We will simply ignore the tachyon. Our reason for doing this is that it naturally disappears once one includes worldsheet Fermions and considers the superstring theories. However the rest of the physics of Bosonic strings remains useful in the superstring. Hence we continue to study it.

Next consider level 1. Here we have

$$|A_\mu>= A_\mu(p)a^\mu_{-1}|0;p> \tag{3.95}$$

Since these modes have $N = 1$ it follows from the mass shell condition that they are massless (for $a = 1!$), *i.e.* the L_0 constraint implies that $p^2A_\mu = 0$. Note that this depends crucially on the fact that $a = 1$. If $a > 1$ then $|A_\mu>$ would be tachyonic whereas if $a < 1$ $|A_\mu>$ would be massive. In either case there is no known constituent theory of a massive (or tachyonic) vector field.

But we must also check that $L_n|A>= 0$ for $n > 0$. Thus

$$\begin{aligned} L_n|A_\mu> &= \frac{1}{2}A_\mu\sum_m \eta_{\nu\lambda}a^\nu_{n-m}a^\lambda_m a^\mu_{-1}|0;p> \\ &= \frac{1}{2}A_\mu\eta_{\nu\lambda}\sum_{m\le 1} a^\nu_{n-m}a^\lambda_m a^\mu_{-1}|0;p> \\ &= \frac{1}{2}A_\mu\eta_{\nu\lambda}\sum_{n-1\le m\le 1} a^\nu_{n-m}a^\lambda_m a^\mu_{-1}|0;p> \end{aligned} \tag{3.96}$$

In the second line we've noted that if $m > 1$ we can safely commute a^λ_m past a^μ_{-1} where it annihilates the vacuum. In the third line we've observed that if $n-m > 1$ then we can safely commute a^ν_{n-m} through the other two oscillators to annihilate the vacuum (recall that for $n > 0$ a^ν_{n-m} always commutes through a^λ_m). Thus for $n > 1$ we automatically have $L_n|A_\mu>= 0$. For $n = 1$ we find just two terms

$$\begin{aligned} L_1|A> &= \frac{1}{2}A_\mu\eta_{\nu\lambda}(a^\nu_1 a^\lambda_0 a^\mu_{-1} + a^\nu_0 a^\lambda_1 a^\mu_{-1})|0;p> \\ &= A_\mu a^\mu_0|0;p> \\ &= \sqrt{2\alpha'}p^\mu A_\mu|0;p> \end{aligned} \tag{3.97}$$

Thus we see that $|A_\mu>$ is represent a massless vector mode with $p^\mu A_\mu = 0$. In position space this is just $\partial^\mu A_\mu = 0$ and this looks like the Lorentz gauge condition for an electromagnetic potential.

Indeed recall that before we found the null state, with $p^2 = 0$,

$$\begin{aligned} |\Lambda> &= i\Lambda(p)L_{-1}|0;p> \\ &= i\eta_{\mu\nu}\Lambda a^\mu_0 a^\nu_{-1}|0;p> \\ &= i\sqrt{2\alpha'}p_\mu\Lambda a^\mu_{-1}|0;p> \end{aligned} \tag{3.98}$$

provided that $p^2 = 0$. Thus we must identify $A_\mu \equiv A_\mu + i\sqrt{2\alpha'}p_\mu\Lambda$ which in position space is the electromagnetic gauge symmetry $A_\mu \equiv A_\mu + \sqrt{2\alpha'}\partial_\mu\Lambda$. Again this occurs precisely when $a = 1$, otherwise $L_{-1}|0;p>$ is not a null state and their would not be a gauge symmetry.

There is one more thing that can be done. Since and open string has two preferred points, its end points, we can attach discrete labels to the end points so that the ground state, of the open string carries two indices

$$|0;p,ab> \tag{3.99}$$

where $a = 1,..,N$ refers the $\sigma = 0$ end and $b = 1,...,N$ refers to the $\sigma = \pi$ end. It then follows that all the Fock space elements built out of $|0;p,ab>$ will carry these indices. These are called Chan-Paton indices. The level one states now have the form

$$|A^{ab}_\mu >= A^{ab}_\mu a^\mu_{-1}|0;p,ab> \tag{3.100}$$

The null states take the form

$$|\Lambda^{ab} >= i\Lambda^{ab}L_{-1}|0;p,ab> \tag{3.101}$$

and the gauge symmetry is

$$A^{ab}_\mu \equiv A^{ab}_\mu + \sqrt{2\alpha'}\partial_\mu\Lambda^{ab} \tag{3.102}$$

These are the gauge symmetries of a non-Abelian Yang-Mills field with gauge group $U(N)$ (at lowest order in the fields). Thus we see that we can obtain non-Abelian gauge field dynamics from open strings.

3.5 Closed Strings

Let us now consider a closed string, so that $\sigma \sim \sigma + 2\pi$. The resulting "boundary condition" is more simple: we simply demand that $\hat{X}^\mu(\tau,\sigma+2\pi) = \hat{X}^\mu(\tau,\sigma)$. This is achieved by again taking n to be an integer. However we now have two independent sets of left and right moving oscillators. Thus the mode expansion is given by

$$X^\mu = x^\mu + \alpha' p^\mu\tau + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\left(\frac{a^\mu_n}{n}e^{-in(\tau+\sigma)} + \frac{\tilde{a}^\mu_n}{n}e^{-in(\tau-\sigma)}\right) \tag{3.103}$$

note the absence of the factor of 2 in front of $p^\mu\tau$. The total momentum of such a string is calculated as before to give

$$\begin{aligned} p^\mu &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\dot{X}^\mu \\ &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma p^\mu + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0} a^\mu_n e^{-in(\tau+\sigma)} + \tilde{a}^\mu_n e^{-in(\tau-\sigma)} \\ &= p^\mu \end{aligned} \tag{3.104}$$

so again p^μ is the spacetime momentum of the string.

We now have double the constraints:

$$
\begin{aligned}
(L_0 - 1)|phys> &= (\tilde{L}_0 - 1)|phys>= 0 \\
L_n|phys &= \tilde{L}_n|phys>= 0
\end{aligned}
\tag{3.105}
$$

with $n > 0$. If we introduce the right-moving number operator $\tilde{N}$

$$
\tilde{N} = \sum_{n>0} \eta_{\mu\nu} \tilde{a}^\mu_{-n} \tilde{a}^\nu_n \tag{3.106}
$$

then the first conditions can be rewritten as

$$
(p_\mu p^\mu + \frac{4}{\alpha'}(N-1))|phys>= 0 \qquad (N - \tilde{N})|phys>= 0 \tag{3.107}
$$

where we have recalled that, if $w^\mu = 0$, $a^\mu_0 = \tilde{a}^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu$ and $L_0 = \frac{1}{2}\eta_{\mu\nu} a^\mu_0 a^\nu_0 + N$, $L_0 = \frac{1}{2}\eta_{\mu\nu}\tilde{a}^\mu_0 \tilde{a}^\nu_0 + \tilde{N}$. The second condition is called level matching. It simply says that any physical state must be made up out of an equal number of left and right moving oscillators. Again the remaining constraints will give gauge fixing conditions.

Let us consider the lowest modes of the closed string. At level 0 (which means level 0 on both the left and right moving sectors by level matching) we simply have the ground state $|0; p>$. This is automatically annihilated by both L_n and $\tilde{L}_n$ with $n > 0$. For $n = 0$ we find

$$
p^2 - \frac{4}{\alpha} = 0 \tag{3.108}
$$

Thus we again find a tachyonic ground state. No one knows what to do with this instability. It turns out to be much more serious than the open string tachyon that we saw, which can sometimes be dealt with. Most people today would say that the Bosonic string is inconsistent although this hasn't been demonstrated. However for us the cure is the same as for the open string: in the superstring this mode is projected out. So we continuing by simply ignoring it, as our discussion of the other modes still holds in the superstring.

ext we have level 1. Here the states are of the form

$$
|G_{\mu\nu}>= G_{\mu\nu} a^\mu_{-1} \tilde{a}^\nu_{-1} |0; p> \tag{3.109}
$$

Just as for the open string these will be massless, *i.e.* $p^2 = 0$ (again only if $a = 1$). Next we consider the constraints $L_m|G_{\mu\nu}>= \tilde{L}_m|G_{\mu\nu}>= 0$ with $m > 0$.

Problem: Show that these constraints imply that $p^\mu G_{\mu\nu} = p^\nu G_{\mu\nu} = 0$

The matrix $G_{\mu\nu}$ is a spacetime tensor. Under the Lorentz group $SO(1, D-1)$ it will decompose into a symmetric traceless, anti-symmetric and trace part. What this means is that under spacetime Lorentz transformations the tensors $g_{\mu\nu}, b_{\mu\nu}$ and ϕ will transform into themselves. Here

$$\begin{aligned} g_{\mu\nu} &= G_{(\mu\nu)} - \frac{1}{D}\eta^{\lambda\rho}G_{\lambda\rho}\eta_{\mu\nu} \\ b_{\mu\nu} &= G_{[\mu\nu]} \\ \phi &= \eta^{\lambda\rho}G_{\lambda\rho} \end{aligned} \tag{3.110}$$

i.e. $G_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu} + \frac{1}{D}\eta_{\mu\nu}\phi$.

Problem: Show this.

Thus from the spacetime point of view there are three independent modes labeled by $g_{\mu\nu}$, $b_{\mu\nu}$ and ϕ. Just as for the open string there is a gauge symmetry

$$|G_{\mu\nu}> \to |G_{\mu\nu}> + i\xi_\mu L_{-1}\tilde{a}^\mu_{-1}|0;p> + i\zeta_\mu \tilde{L}_{-1}a^\mu_{-1}|0;p> \tag{3.111}$$

where we have used the fact that $\xi_\mu L_{-1}\tilde{a}^\mu_{-1}|0;p>$ and $\zeta_\mu\tilde{L}_{-1}a^\mu_{-1}|0;p>$ are null states, provided that $p^2 = 0$. The proof of this is essentially the same as it was for the open string. We need only ensure that the level matching condition is satisfied, which is clear, and that $\tilde{L}_n L_{-1}\tilde{a}_{-1}|0;p> = L_n\tilde{L}_{-1}a_{-1}|0;p> = 0$ for $n > 0$. Thus we need only check that

$$L_n\tilde{L}_{-1}a^\mu_{-1}|0;p> = \frac{1}{2}\tilde{L}_{-1}\sum_m \eta_{\lambda\rho}a^\lambda_{n+m}a^\rho_{-m}a^\mu_{-1}|0;p> = 0 \tag{3.112}$$

Just as before the $n > 1$ terms will vanish automatically. So we need only check

$$\begin{aligned} L_1\tilde{L}_{-1}a^\mu_{-1}|0;p> &= \frac{1}{2}\tilde{L}_{-1}\sum_m \eta_{\lambda\rho}a^\lambda_{1+m}a^\rho_{-m}a^\mu_{-1}|0;p> \\ &= \tilde{L}_{-1}\eta_{\lambda\rho}a^\lambda_0 a^\rho_1 a^\mu_{-1}|0;p> \\ &= \tilde{L}_{-1}\eta_{\lambda\rho}a^\lambda_0[a^\rho_1, a^\mu_{-1}]|0;p> \\ &= \tilde{L}_{-1}a^\mu_0|0;p> \\ &= \frac{\sqrt{\alpha'}}{2}\tilde{L}_{-1}p^\mu|0;p> \end{aligned} \tag{3.113}$$

Similarly for $\tilde{L}_n L_{-1}\tilde{a}^\mu_{-1}|0;p>$. Thus we also find that $p^\mu\xi_\mu = p^\mu\zeta_\mu = 0$. This of course is required to preserve the condition $p^\mu G_{\mu\nu} = p^\nu G_{\mu\nu} = 0$.

In terms of $G_{\mu\nu}$ this implies that

$$G_{\mu\nu} \to G_{\mu\nu} + i\sqrt{\frac{\alpha'}{2}}p_\mu\xi_\nu + i\sqrt{\frac{\alpha'}{2}}p_\nu\zeta_\mu \tag{3.114}$$

or, switching to coordinate space representations and the individual tensor modes, we find

$$
\begin{aligned}
g_{\mu\nu} &\rightarrow g_{\mu\nu} + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi_\nu + \zeta_\nu) + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\nu(\xi_\mu + \zeta_\mu) \\
b_{\mu\nu} &\rightarrow B_{\mu\nu} + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi_\nu - \zeta_\nu) - \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\nu(\xi_\mu - \zeta_\mu) \\
\phi &\rightarrow \phi + 2\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi^\mu + \zeta^\mu)
\end{aligned}
\tag{3.115}
$$

If we let $v_\mu = \frac{1}{2}\sqrt{\frac{\alpha'}{2}}(\xi_\mu + \zeta_\mu)$ and $\Lambda_\mu = \frac{1}{2}\sqrt{\frac{\alpha'}{2}}(\xi_\mu - \zeta_\mu)$ and use $\partial^\mu \xi_\mu = p^\mu \zeta_\mu = 0$ then we find

$$
\begin{aligned}
g_{\mu\nu} &\rightarrow g_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu \\
b_{\mu\nu} &\rightarrow b_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \\
\phi &\rightarrow \phi
\end{aligned}
\tag{3.116}
$$

The first term line gives the infinitesimal form of a diffeomorphism, $x^\mu \rightarrow x^\mu - v^\mu$ and thus we can identify $g_{\mu\nu}$ to be a metric tensor. The second line gives a generalization of and electromagnetic gauge transformation. The analogue of the gauge invariant field strength is

$$
H_{\lambda\mu\nu} = \partial_\lambda b_{\mu\nu} + \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} \tag{3.117}
$$

Thus the massless field content at level 1 consists of a graviton mode $g_{\mu\nu}$, an anti-symmetric tensor field $b_{\mu\nu}$ and a scalar ϕ, subject to the gauge transformations (3.115). Finally the massless condition $p^2 G_{\mu\nu} = 0$ leads to

$$
\begin{aligned}
\partial^2 g_{\mu\nu} &= 0 \\
\partial^2 b_{\mu\nu} &= 0 \\
\partial^2 \phi &= 0
\end{aligned}
\tag{3.118}
$$

The conditions $p^\mu G_{\mu\nu} = p^\nu G_{\mu\nu} = 0$ now reduce to the linearized equations

$$
\begin{aligned}
\partial^\mu g_{\mu\nu} + \partial_\nu \phi &= 0 \\
\partial^\mu b_{\mu\nu} &= 0
\end{aligned}
\tag{3.119}
$$

These equations can be viewed as gauge fixing conditions (in effect $\phi = \frac{1}{D} g_{\lambda\sigma}\eta^{\lambda\sigma}$). The fields $g_{\mu\nu}$, $b_{\mu\nu}$ and ϕ are known as the graviton (metric), Kalb-Ramond (b-field) and dilaton respectively.

4 Light-cone gauge

So far we have quantized a string in flat D-dimensional spacetime. Apart from D we have the parameters a and α'. In fact α' is not a parameter, it is a dimensional quantity - it has the dimensions of length-squared - and simply sets the scale. What is important are unitless quantities such as $p^2\alpha'$. For example small momentum means $p^2\alpha' << 1$.

We are left with D and a but actually these are fixed: quantum consistency demands that $D = 26$ and $a = 1$. We have seen that things would go horribly wrong if $a \neq 1$.

The easiest way to see this is to introduce light-cone gauge. Recall that the action we started with had diffeomorphism symmetry. We used this symmetry to fix $\gamma_{\alpha\beta} = e^{2\rho}\eta_{\alpha\beta}$. However there is still a residual symmetry. In particular in terms of the coordinates $\sigma^\pm$ then under a transformation

$$\sigma'^+ = \sigma'^+(\sigma^+) \qquad \sigma'^- = \sigma'^-(\sigma^-) \tag{4.120}$$

we see that $\gamma'_{\alpha\beta} = e^{2\rho'}\eta_{\alpha\beta}$ with

$$\rho' = \rho + \frac{1}{2}\ln\left(\frac{\partial\sigma^+}{\partial\sigma'^+}\frac{\partial\sigma^-}{\partial\sigma'^-}\right) \tag{4.121}$$

i.e. this preserves the conformal gauge. In terms of the worldsheet coordinates σ, τ we see that

$$\tau' = \frac{1}{2}(\sigma'^+ + \sigma'^-) \tag{4.122}$$

and since $\sigma'^\pm$ are arbitrary functions of $\sigma^\pm$ we see that any τ that solves the two-dimensional wave equation can be obtained by such a diffeomorphism. Therefore, without loss of generality, we can choose the worldsheet 'time' coordinate τ to be any of the spacetime coordinates (since these solve the two-dimensional wave-equation). Of course there are many choices but the usual one is to define

$$\hat{X}^+ = \frac{1}{2}(X^0 + X^{D-1}) \qquad \hat{X}^- = \frac{1}{2}(X^0 - X^{D-1}) \tag{4.123}$$

and then take

$$\hat{X}^+ = x^+ + \alpha' p^+\tau \tag{4.124}$$

This is called light cone gauge.

Next we evaluate the conformal symmetry constraints (3.15). We observe that in these coordinates the spacetime $\eta_{\mu\nu}$ is

$$\eta_{-+} = \eta_{+-} = -2 \qquad \eta_{ij} = \delta_{ij} \tag{4.125}$$

Thus we find that

$$\begin{aligned} T_{00} = T_{11} &= -2\alpha' p^+\dot{X}^- + \frac{1}{2}\dot{X}^i\dot{X}^j\delta_{ij} + \frac{1}{2}X'^iX'^j\delta_{ij} = 0 \\ T_{01} = T_{10} &= -2\alpha' p^+\hat{X}'^- + \dot{X}^iX'^j\delta_{ij} = 0 \end{aligned} \tag{4.126}$$

where $i, j = 1, 2, 3, ..., D-2$. This allows one to explicitly solve for X^- in term of the mode expansions for X^i.

Problem: Show that with our conventions

$$X^- = x^- + \alpha' p^- \tau + i \left(\sum_{n \neq 0} \frac{a_n^-}{n} e^{-in\sigma^+} + \frac{\tilde{a}_n^-}{n} e^{-in\sigma^-} \right) \tag{4.127}$$

where

$$a_n^- = \frac{1}{2p^+} \sum_m a_{n-m}^i a_m^j \delta_{ij} \tag{4.128}$$

and the massshell constraint is

$$-4\alpha' p^+ p^- + \alpha' p^i p^j \delta_{ij} + 2(N + \tilde{N}) = 0 \tag{4.129}$$

with

$$N + \tilde{N} = \frac{1}{2} \delta_{ij} \sum_{n \neq 0} a_n^i a_{-n}^j + \tilde{a}_n^i \tilde{a}_{-n}^j \tag{4.130}$$

To continue we note that in the quantum theory there is a normal ordering ambiguity in the definition of $N + \tilde{N}$ and we must include our constant a again into the definition. Hence we must take (temporarily putting in the :: symbols for normal ordering)

$$: N + \tilde{N} := \delta_{ij} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^j + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^j \tag{4.131}$$

However since we have dropped an infinite constant, the intercept a will now show up in the mass shell constraint as

$$-4\alpha' p^+ p^- + \alpha' p^i p^j \delta_{ij} + 2(N + \tilde{N} - 2a) = 0 \tag{4.132}$$

Note that $-4p^+ p^- + p^i p^j \delta_{ij} = \eta_{\mu\nu} p^\mu p^\nu$ so this really just tells us that the mass of a state is

$$M^2 = \frac{2}{\alpha'} (N + \tilde{N} - 2a) \tag{4.133}$$

Where we have dropped the :: to indicate normal ordering.

We still have a level matching condition for closed strings

$$N = \tilde{N} \tag{4.134}$$

This arises because we only have one spacetime momentum p^μ (not separate ones for left and right moving modes).

Note that this breaks the $SO(1, D-1)$ symmetry of our flat target space since we choose X^0 and X^{D-1} whereas any pair will do (so long as one is timelike). Thus we will not see a manifest $SO(1, D-1)$ symmetry but just an $SO(D-2)$ symmetry from rotations of the $\tilde{X}^i$. However it is important to realize that the $SO(1, D-1)$ symmetry is not really broken, we have merely performed a kind of gauge fixing (recall there was this underlying gauge symmetry of the string spectrum). It is just no longer manifest.

4.1 $D = 26,\ a = 1$

On the other hand the benefit of this procedure is that the physical Hilbert space is manifestly postive definite because we remove the oscillators $a^0_n, \tilde{a}^0_n, a^{D-1}_n, \tilde{a}^{D-1}_n$. This is often a helpful way to determine the physical spectrum of the theory.

For example we can reconsider the low lying states that we constructed above. The ground states are unchanged as they do not involve any oscillators. For the open string we find the $D-2$ states at level one

$$|A_i >= a^i_{-1}|0;p> \tag{4.135}$$

These are the transverse components of a massless gauge field. For the closed string we find, at level one,

$$|G_{ij} >= G_{ij} a^i_{-1}\tilde{a}^j_{-1}|0;p> \tag{4.136}$$

These correspond to the physical components, in a certain gauge, of the metric, Kalb-Ramond field and dilaton Note however that there is no remnant at all of gauge symmetry which is a crucial feature of dynamics

Now formally a is given by

$$\begin{aligned} a &= -\frac{1}{2}\sum_{m=1}^{\infty}[a^i_m, a^j_{-m}]\delta_{ij} \\ &= -\frac{D-2}{2}\sum_{m=1}^{\infty} m \end{aligned} \tag{4.137}$$

This is divergent however it can be regularized in the following manner. We note that

$$a = -\frac{D-2}{2}\zeta(-1) \tag{4.138}$$

where $\zeta(s)$ is the Riemann ζ-function

$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s} \tag{4.139}$$

This is analytic for complex s with $\mathrm{Re}(s) > 1$. Thus it can be extended to a holomorphic function of the complex plane, with poles at a discrete number of points. Analytically continuing to $s = -1$ one finds $\zeta(-1) = -1/12$ and hence

$$a = \frac{D-2}{24} \tag{4.140}$$

We have seen that in order to have a sensible theory we must take $a = 1$ (otherwise there are no massless states or nice gauge invariances). Hence we must take $D = 26$.

This is not a very satisfactory derivation of the dimension of spacetime. A more convincing argument is the following. Light cone gauge is just a gauge. Therefore although the manifest spacetime Lorentz symmetry is no longer present there is still an $SO(1, D-1)$ Lorentz symmetry, even though only an $SO(D-2)$ subgroup is manifest in light cone gauge. In light cone gauge the spacetime Lorentz generators $M^{\mu}{}_{\nu}$ split into

$$M^{i\prime}{}_{j} \qquad M^{+}{}_{j}, \qquad M^{-}{}_{j}, \qquad M^{+}{}_{-} \tag{4.141}$$

The quantization procedure preserves $SO(D-2)$ so the commutators $[M^{i}{}_{j}, M^{k}{}_{l}]$ are as they should be. However problems can arise with $[M^{i}{}_{j}, M^{+}{}_{k}]$ *etc.*. It is too lengthy a calculation to do here, but one can show that the full $SO(1, D-1)$ Lorentz symmetry, generated by the charges (3.26), is preserved in the quantum theory, *i.e.* once normal ordering is taken into account, if and only if $a = 1$ and $D = 26$. You are urged to read the section 2.3 of Green Schwarz and Witten or section 12.5 of Zwiebach where this is shown more detail.

4.2 Partition Function

A useful concept is the notion of a partition function which 'counts' the physical states. Since light cone gauge only contains physical states this is most easily computed here.

Let us start with an open string and define

$$Z = \sum q^{L_0 - 1} \tag{4.142}$$

where the sum is over states (at zero momentum) and $q = e^{-2\pi t}$ is 'place-holder'. First note that for a string in flat spacetime L_0 is a sum of 24 independent free Bosons. Thus

$$Z = (Z_1)^{24} \tag{4.143}$$

where

$$\begin{aligned} Z_1 &= \sum q^{\sum_l a_{-l} a_l - \frac{1}{24}} \\ &= \sum q^{-\frac{1}{24}} \prod_l q^{a_{-l} a_l} \end{aligned} \tag{4.144}$$

For a single Boson we have the oscillators $a_{-1}, a_{-2}, \ldots.$ Each oscillator a_{-l} can be used k times in which case $a_{-l}a_l$ contributes k to the exponent. We need to sum over all k and using $\sum_{k=0}^{\infty} q^{kl} = (1 - q^l)^{-1}$ we find

$$Z_1 = q^{-\frac{1}{24}} \prod_{l=1}^{\infty} (1 - q^l)^{-1} \tag{4.145}$$

and hence

$$Z = q^{-1} \prod_{l=1}^{\infty} (1 - q^l)^{-24} = \eta(t) \tag{4.146}$$

where $\eta(t)$ is known as the Dedekind eta-function. It can be extended to the upper half complex plane $\tau = \theta + it$, $t > 0$ and is known to posses the following property:

$$\eta(-\tau^{-1}) = \eta(\tau) \tag{4.147}$$

In particular it is invariant under $t \to 1/t$ when $\theta = 0$. This property is known as modular invariance. This is a crucial feature of strings (and requires that we have 24 physical oscillators - another important feature of $D = 26$).

We can provide a physical interpretation of Z by noting that

$$\int_0^\infty dte^{-2\pi t(L_0-1)} = \frac{1}{2\pi}\frac{1}{L_0 - 1} \tag{4.148}$$

and $1/(L_0 - 1)$ is the propagator. Thus Z has the interpretation of a vacuum one-loop diagram:

$$Z = \text{Tr} < 0| \left(\frac{1}{2\pi}\frac{1}{L_0 - 1}\right) |0 > \tag{4.149}$$

(hence the restriction to zero momentum). The variable t arises in the Schwinger proper time formalism. The worldsheet of an open string is a cylinder of radius R and length L. By conformal invariance the only parameter that matters is $t = R/L$. In the large t limit the open string is relatively short compared to the size of the loop. In this case the important states that propagate around the loop are the light modes, corresponding to the IR limit of open strings. Indeed here we see, taking $q \to 0$

$$Z \sim q^{-1} + 24q^0 + \mathcal{O}(q) \tag{4.150}$$

here we see the tachyon dominants, followed by the 24 massless modes and then the massive spectrum gives ever vanishing corrections.

What about the $t \to 0$ limit? In this case the the cylinder has a very short radius compared to its length. This corresponds to the UV behaviour of the open string and the massive states dominate. Here we can use modular invariance to evaluate

$$\lim_{t\to 0}\eta(e^{-2\pi t}) = \lim_{t\to 0}\eta(e^{-2\pi/t}) = \lim_{\tilde{t}\to\infty}\eta(e^{-2\pi\tilde{t}}) \sim \tilde{q}^{-1} + 24\tilde{q}^0 + \mathcal{O}(\tilde{q}) \tag{4.151}$$

where $\tilde{q} = e^{-2\pi\tilde{t}}$. An alternative interpretation of such a diagram is that it can be viewed as a closed string of radius R propagating at tree-level along a distance L, *i.e.* no loops. Again Z dominated by the closed string tachyon and then the 24 (left-right symmetric) massless closed string modes.

This is one of the most important features of string theory. The UV description of open strings has a dual interpretation in terms of an IR propagation of closed strings and *vice-versa*.

Problem: Show that for a periodic Fermion, where $L_0 = \sum_l d_{-l}d_l + \frac{1}{24}$ and $\{d_n, d_m\} = n\delta_{n,-m}$, one has

$$Z_1 = q^{\frac{1}{24}}\prod_{l=1}^{\infty}(1 + q^l) \tag{4.152}$$

and for an anti-periodic Fermion, where $L_0 = \sum_r b_{-r}b_r - \frac{1}{48}$, $\{b_r, b_s\} = r\delta_{r,-s}$ and $r, s \in \mathbf{Z} + \frac{1}{2}$, one has

$$Z_1 = q^{-\frac{1}{48}} \prod_{l=1}^{\infty}(1 + q^{l-\frac{1}{2}}) \tag{4.153}$$

5 Curved Spacetime and an Effective Action

5.1 Strings in Curved Spacetime

We have considered quantized strings propagating in flat spacetime. This lead to a spectrum of states that included the graviton as well as other modes. More generally a string should be allowed to propagate in a curved background with non-trivial values for the metric and other fields. Our ansatz will be to consider the most general two-dimensional action for the embedding coordinates X^μ coupled to two-dimensional gravity subject to the constraint of conformal invariance. This later condition is required so that the two-dimensional worldvolume metric decouples from the other fields. We will consider only closed strings in this section. The reason for this is that these days one views open strings as description soliton like objects, called Dp-branes, that naturally sit inside the closed string theory.

Before proceeding we note that

$$S_{EH} = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\gamma}R = \chi \tag{5.1}$$

is a topological invariant called the Euler number, *i.e.* the integrand is locally a total derivative. Thus we could add the term S_{EH} to the action and not change the equations of motion.

With this in mind the most general action we can write down for a closed string is

$$S_{closed} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\alpha'\sqrt{-\gamma}\phi(X)R + \sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu b_{\mu\nu}(X) \tag{5.2}$$

where ϕ is a scalar, $g_{\mu\nu}$ symmetric and $b_{\mu\nu}$ antisymmetric. These are precisely the correct degrees of freedom to be identified with the massless modes of the string. One can think of this worldsheet theory as two-dimensional quantum gravity coupled to some matter in the form of scalar fields. More generally one can think of and conformal field theory (with central charge equal to 26) as defining the action for a string.

Furthermore this action has the diffeomorphism symmetry $X^\mu \to X'^\mu(X)$

$$\partial_\alpha X'^\mu = \frac{\partial X'^\mu}{\partial X^\nu}\partial_\alpha X^\nu \qquad g'_{\mu\nu} = \frac{\partial X^\lambda}{\partial X'^\mu}\frac{\partial X^\rho}{\partial X'^\nu}g_{\lambda\rho} \qquad b'_{\mu\nu} = \frac{\partial X^\lambda}{\partial X'^\mu}\frac{\partial X^\rho}{\partial X'^\nu}b_{\lambda\rho} \qquad \phi' = \phi \tag{5.3}$$

automatically built in. It also incorporates the b-field gauge symmetry

$$b'_{\mu\nu} = b_{\mu\nu} + \partial_\mu\lambda_\nu - \partial_\mu\lambda_\nu \tag{5.4}$$

however to see this we note that

$$
\begin{aligned}
\delta S_{closed} &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\partial_\mu\lambda_\nu \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma\partial_\alpha(\epsilon^{\alpha\beta}\partial_\beta X^\nu\lambda_\nu) \\
&= 0
\end{aligned}
\tag{5.5}
$$

where we used the fact that $\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta X^\nu = 0$ in the second to last line and the fact that the worldsheet is a closed manifold in the last line, *i.e.* the periodic boundary conditions.

Notice something important. If the dilaton ϕ is constant then the first term in the action is a topological invariant, the Euler number. In the path integral formulation the partition function for the full theory is defined by summing over all worldsheet topologies

$$
Z = \sum_{g=0}^{\infty}\int D\gamma DX e^{-S}
\tag{5.6}
$$

Here the path integral is over the worldsheet fields $\gamma_{\alpha\beta}$ and X^μ. Now each genus g worldsheet will appear suppressed by the factor $e^{-\phi\chi_g} = e^{-2\phi(g-1)}$. Thus $g_s = e^\phi$ can be thought of as the string coupling constant which counts which genus surface is contributing to a calculation. In particular for $g_s \to 0$ one can just consider the leading order term where the worldsheet is a sphere.

However if one wants to consider the splitting and joining of strings then one must take $g_s > 0$ and include higher genus surfaces. In particular the first non-trivial string interactions arise when the worldsheet is a torus. To see the analogy with quantum field theory note that a torus can be thought of as the worldvolume of a closed string that has gone around in a loop. Thus it is analogous to 1-loop processes in quantum field theory. Similarly higher genus surfaces incorporate higher loop processes. One of the great features of string theory is that each of these contributions is finite. So this defines a finite perturbative expansion of a quantum theory which includes gravity!

As stated above our general principle is the conformal invariance of the worldsheet theory, which ensures that the worldsheet metric $\gamma_{\alpha\beta}$ decouples. The action we just wrote down is conformal as a classical action. However this will not generically be the case in the quantum theory. Divergences in the quantum theory require regularization and renormalization and these effects will break conformal invariance by introducing an explicit scale: the renormalization group scale. It turns out that conformal invariance is more or less equivalent to finiteness of the quantum field theory. This restriction leads to equations of motions for the spacetime fields $\phi, g_{\mu\nu}$ and $b_{\mu\nu}$ (which from the worldvolume point of view are just fancy coupling constants). It is beyond the scope of these lectures to show this but the constraints of conformal invariance at the one loop level give equations of motion

$$
\begin{aligned}
0 &= R_{\mu\nu} + \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} - 2D_\mu D_\nu\phi + \mathcal{O}(\alpha') \\
0 &= D^\lambda H_{\lambda\mu\nu} - 2D^\lambda\phi H_{\lambda\mu\nu} + \mathcal{O}(\alpha') \\
0 &= 4D^2\phi + 4(D\phi)^2 - R - \frac{1}{12}H^2 + \mathcal{O}(\alpha')
\end{aligned}
\tag{5.7}
$$

where $H_{\mu\nu\lambda} = 3\partial_{[\mu}b_{\nu\lambda]}$. In general there will be corrections to these equations coming from all orders in perturbation theory, *i.e.* higher powers of α'. However such terms will be higher order spacetime derivatives and can be safely ignored at energy scales below the string scale.

5.2 A Spacetime Effective Action

A string propagating in spacetime has an infinite tower of massive excitations. However all but the lightest (massless) modes will be too heavy to observe in any experiment that we do. Thus in many cases one really just wants to consider the dynamics of the massless modes. This introduces the concept of an effective action. This is a very general concept (ubiquitous in quantum field theory) whereby we introduce an action for the light modes that we are interested in (below some scale M). The action is constructed so that it has all the correct symmetries of the full theory and its equations of motion reproduce the correct scattering amplitudes of the light modes that the full theory predicts. In general effective actions need not be renormalizable and they are not expected to be valid at energy scales above the scale M where the massive modes we've ignored can be excited and can no longer be ignored. Often one says that the massive modes have been integrated out. Meaning that one has performed the path integral over modes with momenta larger than M and is just left with a path integral over the low momentum modes.

In our case we have considered a string propagating in a curved spacetime that can be thought of as a background coming from a non-trivial configuration of its massless modes. In particular in our discussion we implicitly assumed that the massive modes were set to zero. The result was that quantum conformal invariance predicted the equations of motion (5.7). These are the on-shell conditions for a string to propagate in spacetime as derived in the full quantum theory. Note that they pick up an infinite series of α' corrections and also an infinite series of g_s corrections (where we allow the splitting an joining of strings). In other words, at lowest order in α' and g_s these are the equations of motion for the spacetime fields. Furthermore these equations of motion can be derived from the spacetime action

$$S_{effective} = -\frac{1}{2\alpha'^{12}} \int d^{26}x\sqrt{-g}e^{-2\phi}\left(R - 4(\partial\phi)^2 + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) + \ldots \tag{5.8}$$

Problem: Show that the equations of motion of (5.8) are indeed (5.7). You may need to recall that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $g^{\mu\nu}\delta R_{\mu\nu} = D_\mu D_\nu \delta g^{\mu\nu} - g_{\mu\nu}D^2\delta g^{\mu\nu}$.

This is therefore the effective action for the massless modes of a closed string. It plays the same role that the free scalar equation played for the point particle (although $S_{effective}$ does not include the infinite tower of string states which isn't there for the point particle). The ellipsis denotes contributions from higher loops which will contain higher numbers of derivatives and which are suppressed by higher powers of α'. Note that string theory also predicts corrections to the effective action from string loops, that is from higher genus Riemann surfaces. These terms will come with factors of $e^{-2g\phi}$ where $g = 0, -1, -2, ...$ and can be ignored if the string coupling $g_s = e^{\phi}$ is small.

6 Superstrings

In the final section let us try to extend the pervious sections to the superstring. Conceptually not much changes but there are several additional bells and whistles that need to be considered.

6.1 Type II strings

The starting point for the superstring is include Fermions ψ^μ on the worldsheet so as to construct a supersymmetric action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\eta^{\alpha\beta} + i\bar{\psi}^\mu\gamma^\alpha\partial_\alpha\psi^\nu\eta_{\mu\nu} \tag{6.9}$$

where $\bar{\psi} = \psi^T\gamma_0$ and γ^α are real 2×2 matrices that satisfy $\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}$. A convenient choice is $\gamma^0 = i\sigma^2$ and $\gamma^1 = \sigma^1$. This action is also conformally invariant and in addition has the supersymmetry

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu \ , \qquad \delta\psi^\mu = \gamma^\alpha\partial_\alpha X^\mu\epsilon \tag{6.10}$$

for any constant ϵ.

Problem: Show this.

The mode expansion for the X^μ remains as before with the a_n^μ and $\tilde{a}_n^\mu$ oscillators. When we expand the Fermionic fields we can allow for two types of boundary conditions (let us just consider boundary conditions consistent with a closed string where $\sigma \sim \sigma+2\pi$ and $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$):

$$\begin{aligned} \text{R}: \quad \psi^\mu(\tau,\sigma+2\pi) &= \psi^\mu(\tau,\sigma) \\ \text{NS}: \quad \psi^\mu(\tau,\sigma+2\pi) &= -\psi^\mu(\tau,\sigma) \end{aligned} \tag{6.11}$$

these are known as the Ramond and Neveu-Schwarz sectors respectively. Thus we find

$$\begin{aligned} \text{R}: \quad \psi^\mu(\tau,\sigma+2\pi) &= \sum_{n\in\mathbf{Z}} d_n e^{-in\sigma^+} + \tilde{d}_n e^{-in\sigma^-} \\ \text{NS}: \quad \psi^\mu(\tau,\sigma+2\pi) &= \sum_{r\in\mathbf{Z}+\frac{1}{2}} b_r e^{-ir\sigma^+} + \tilde{b}_r e^{-ir\sigma^-} \end{aligned} \tag{6.12}$$

One finds that these satisfy the anti-commutation relations

$$\begin{aligned}
\{d^\mu_m, d^\nu_n\} &= \eta^{\mu\nu}\delta_{m,-n} \qquad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s} \\
\{\tilde{d}^\mu_m, \tilde{d}^\nu_n\} &= \eta^{\mu\nu}\delta_{m,-n} \qquad \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}
\end{aligned} \tag{6.13}$$

with all other anti-commutators vanishing.

One important consequence of supersymmetry is that the algebra of constraints generated by L_n is enhanced to a super-Virasoro algebra with odd generators G_r and F_n (depending on whether or not one is in the NS or R sector respectively). The super-Virasoro algebra turns our to be (see the references)

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m,-n} \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}
\end{aligned} \tag{6.14}$$

in the NS sector and

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m,-n} \\
[L_m, F_n] &= \left(\frac{m}{2} - n\right) F_{m+n} \\
\{F_n, F_m\} &= 2L_{m+n} + \frac{D}{2}m^2\delta_{m,-n}
\end{aligned} \tag{6.15}$$

in the R sector. Here all operators are normal ordered. Just as before this only affects L_0 and F_0 however there is no associated intercept a for F_0 since it is Fermionic (and in addition this is not allowed by the $\{F_0, F_0\}$ anti-commutator). Note that the Fermionic generators are in effect the 'square-root' of L_n, as we expect in a supersymmetric theory. We won't go into more details here but we must impose the physical constraints for the positive modded generators. Just as L_0 gives a spacetime Klein-Gordon equation, F_0 gives a spacetime Dirac equation.

Let us compute the intercept a. As before we go to light-cone gauge where we fix two of the coordinates X^μ and their superpartners ψ^μ. We then compute the vacuum energy of the remaining $D-2$ Bosonic and Fermionic oscillators. The result depends on the boundary conditions we use. Noting that the sign of the Fermionic contribution is opposite to that of a Boson one finds

$$\begin{aligned}
a_R &= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{2}\sum_{n=1}^{\infty} n \\
&= -\frac{D-2}{2}\left(-\frac{1}{12} + \frac{1}{12}\right) \\
&= 0
\end{aligned} \tag{6.16}$$

The vanishing of a_R is a direct consequence of the fact that there is a Bose-Fermi degeneracy in the R-sector. In particular each periodic Fermion contributes $-\frac{1}{24}$ to a. In the NS sector we find

$$\begin{aligned}
a_{NS} &= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{2}\sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right) \\
&= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{4}\sum_{n=odd}^{\infty} n \\
&= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{4}\left(\sum_{n=1}^{\infty} n - \sum_{n=even}^{\infty} n\right) \\
&= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{4}\left(\sum_{n=1}^{\infty} n - \sum_{m=1}^{\infty} 2m\right) \\
&= -\frac{D-2}{2}\sum_{n=1}^{\infty} n - \frac{D-2}{4}\sum_{n=1}^{\infty} n \\
&= (D-2)\left(\frac{1}{24}+\frac{1}{48}\right) \\
&= \frac{D-2}{16}
\end{aligned} \tag{6.17}$$

Note that this shows that each anti-periodic Fermion contributes $\frac{1}{48}$ to a. Having determined the incepts we can now go out of Light cone gauge and consider the covariant theory.

Let us now look at the lightest states. There is a different ground state for each sector which we denote by $|R;p>$ and $|NS;p>$ where p^μ labels the spacetime momentum. As before we assume that these states are annihilated by any oscillator with positive frequency.

We see that $|R;p>$ is massless and hence all the higher level states created from it by the action of a creation operator will be massive with a mass of order the string scale. However the Ramond ground state $|R;p>$ is degenerate. In particular we see that there are Fermion zero-modes d_0^μ which satisfy $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$, $\mu,\nu = 0,...,D-1$ in light cone gauge. This is a Clifford algebra and it is known that there is a unique representation and it is $2^{[\frac{D}{2}]}$-dimensional. Thus the Ramond ground state is in fact a spinor with $2^{[\frac{D}{2}]}$ independent components.

Let us look at the Neveu-Schwarz ground state $|NS,p>$. It is clear that since $a_{NS} > 0$ this state is a tachyon. We can then consider the higher level states (for simplicity we just consider open strings)

$$\begin{aligned}
a_{-1}^\mu|NS,p> \qquad M^2 &= 1-\frac{D-2}{16} \\
b_{-\frac{1}{2}}^\mu|NS,p> \qquad M^2 &= \frac{1}{2}-\frac{D-2}{16}
\end{aligned}$$

Thus the next lightest state is $b^\mu_{-\frac{1}{2}}|NS,p>$ and its mass-squared is $M^2 = -\frac{D-10}{16}$. Thus if $D < 10$ then these states are also tachyonic. However as before the magic (that is gauge symmetries from null states) happens when these states are massless, *i.e.* $D = 10$. In this case the states $a^\mu_{-1}|NS,p>$ are massive. Thus we take $D = 10$ and $a_{NS} = 1/2$. Indeed as before this is forced upon us if we want the $SO(1, D-1)$ Lorentz symmetry of spacetime to be preserved in the quantum theory.

Nevertheless we are still left with some bad features. For one the Neveu-Schwarz ground state is still a tachyon. There is also another puzzling feature: $|NS,p>$ is a spacetime scalar and hence it must be a Boson. We can then construct the spacetime vector $b^\mu_{-\frac{1}{2}}|NS,p>$. From the spacetime point of view this state should be a Boson since it transforms under Lorentz transformations as a vector. However it is created from $|NS,p>$ by a Fermionic operator and thus will obey Fermi-statistics. This is contradictory.

The solution to both these problems is to project out the odd states and in particular $|NS,p>$. This is known as the GSO projection. More specifically we declare that $|NS,p>$ is a Fermionic state. Mathematically we introduce the operator $(-1)^F$ which acts as $(-1)^F|NS,p> = -|NS,p>$ and $\{\psi^\mu,(-1)^F\} = 0$, $[X^\mu,(-1)^F] = 0$. We then project out all Fermionic states, *i.e.* states in the eigenspace $(-1)^F = -1$. Thus $|NS,p>$ and $a^\mu_{-1}|NS,p>$ are removed from the spectrum but the massless states $b^\mu_{-\frac{1}{2}}|NS,p>$ remain.

Let us now consider the Ramond sector states. We already saw that the ground state here is massless but degenerate. Indeed it is a spinor of $SO(1,9)$, that is to say it can be represented by a vector in the 32-dimensional vector space that furnishes a representation of the Clifford algebra relation $\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu}$, $\mu,\nu = 0,...,9$. We need to discuss how $(-1)^F$ acts here. There is a natural candidate where we take $(-1)^F = \pm\Gamma_{11} = \pm\Gamma_0\Gamma_1...\Gamma_9$, the chirality operator in the 10-dimensional Clifford algebra. Thus after the GSO projection $|R,p>$ is a chiral spinor with 16 independent components. More generally in the Ramond sector we project out states with $(-1)^F = -1$. The GSO projection is also required to ensure modular invariance.

In the Ramond sector of the open superstring either choice of sign is equivalent to the other, it is just a convention. Thus for the open superstring the lightest states are massless and consist of a spacetime vector (and hence a Boson) $b^\mu_{-\frac{1}{2}}|NS,p>$ along with a spacetime Fermion $|R,p>$ which can be identified with a chiral spinor. Note that there is a Bose-Fermi degeneracy: onshell, and gauged fixed we find 8 Bosonic and 8 Fermionic states (Why? - you can see this in lightcone gauge).

Let us consider closed strings. Here the states are essentially obtained by taking a tensor product of left and right moving modes and hence there are four possibilities:

$$\begin{array}{rcl} |NS>_L & \otimes & |NS>_R \\ |R>_L & \otimes & |R>_R \\ |NS>_L & \otimes & |R>_R \\ |R>_L & \otimes & |NS>_L \end{array}$$

$$(6.18)$$

In this case the relative sign taken in the GSO projection is important. There are two choices: either we chose the same chirality projector for the left and right moving modes or the opposite. This leads to two distinction theories known as the type IIB and type IIA superstring respectively. The states one find are of the form

$$\begin{array}{ccc} |NS>_L & \otimes & |NS>_R \\ |R+>_L & \otimes & |R->_R \\ |NS>_L & \otimes & |R->_R \\ |R+>_L & \otimes & |NS>_L \end{array} \tag{6.19}$$

for type IIA and

$$\begin{array}{ccc} |NS>_L & \otimes & |NS>_R \\ |R+>_L & \otimes & |R+>_R \\ |NS>_L & \otimes & |R+>_R \\ |R+>_L & \otimes & |NS>_L \end{array} \tag{6.20}$$

for type IIB. Here the $\pm$ sign corresponds to the different choice of GSO projector for the left and right moving modes.

The spacetime Bosons come from either the NS-NS or R-R sectors whereas the spacetime Fermions from the NS-R or R-NS sectors. One sees that in the type IIA theory there are Fermionic states with both spacetime chiralities but in the type IIB theory only one chirality appears.

Let us look more closely at the massless Bosonic states. The NS-NS sector is essentially the same as the spectrum of the Bosonic string only now they are created from the vacuum by $b^\mu_{-\frac{1}{2}}$ and $\tilde{b}^\mu_{-\frac{1}{2}}$ rather than a^μ_{-1} and $\tilde{a}^\mu_{-1}$. In particular we still find a graviton, Kalb-Ramond field and a dilaton. This sector is universal to all closed string theories.

However we also have R-R fields. These arise as a tensor product of a left and right spinor ground state. As such they form a 'bi-spinor':

$$F_{\alpha\beta} = |R\pm>_{L\alpha} \quad \otimes \quad |R\pm>_{R\beta} \tag{6.21}$$

Any bi-spinor can be expanded in terms of the associated Γ-matrices:

$$F_{\alpha\beta} = \sum_{p=0}^{10} F_{\mu_1\dots\mu_p} (\Gamma^{\mu_1\dots\mu_p}\Gamma^0)_{\alpha\beta} \tag{6.22}$$

Here we have used the fact that $\{1, \Gamma^\mu, \Gamma^{\mu_1\mu_2}, ..., \Gamma^{\mu_1\dots\mu_{10}}\}$ form a basis of 32×32 matrices and used $C^{-1} = \Gamma^0$ to lower the spinor index. Next we note that

$$\Gamma^{11}\Gamma^{\mu_1\dots\mu_p} = \frac{1}{(10-p)!}\varepsilon^{\mu_1\dots\mu_p\nu_1\dots\nu_{10-p}}\Gamma_{\nu_1\dots\nu_{10-p}} \tag{6.23}$$

Using the GSO projection on the left movers implies that $(\Gamma^{11})_{\gamma}{}^{\alpha}F_{\alpha\beta} = F_{\gamma\beta}$ and hence we see that

$$F^{\mu_1\ldots\mu_p} = \frac{1}{(10-p)!}\varepsilon^{\mu_1\ldots\mu_p\nu_1\ldots\nu_{10-p}}F_{\nu_1\ldots\nu_{10-p}} \tag{6.24}$$

This implies that only the fields with $p \leq 5$ are independent of each other. In addition $F_{\mu_1\ldots\mu_5}$ is self-dual. Finally the GSO projection on the right movers tells us that $F_{\alpha\gamma}(\Gamma^{11})^{\gamma}{}_{\beta} = \pm F_{\alpha\beta}$ where the sign is $-$ for type IIA and $+$ for type IIB. This implies that $p = even$ for type IIA and $p = odd$ for type IIB. The physical state conditions, in particular the vanishing of F_0 and $\tilde{F}_0$, imply that $\partial_{[\mu_{p+1}}F_{\mu_1\ldots\mu_p]} = 0$ and $\partial^{\mu_1}F_{\mu_1\ldots\mu_p} = 0$.

We motivated superstrings by considering a worldsheet action that was supersymmetric. However it turns out that, after the GSO projection, these theories also have spacetime supersymmetry with 32 supersymmetry generators, the maximum possible. In particular the massless Fermionic states arising from the NS-R and R-NS sectors give two gravitini and a dilitino.

6.2 Type I and Heterotic String

There are three other possibilities. For example one can introduce open strings. Since open strings can combine into a closed string this theory must also contain closed strings but the presence of open strings leads to $SO(32)$ gauge fields in spacetime. This is known as the type I string. It is further complicated by the fact that the worldsheets of the strings are not oriented. The resulting theory has half as much spacetime supersymmetry as the type II theories. Indeed these days the type I string is generally viewed as an 'orientifold' of the type IIB string in the presence of so-called D9-branes. It is also thought to be dual to the Heterotic $SO(32)$ string.

A more bizarre construction is to exploit the fact the left and right moving modes sectors of the string worldsheet do not talk to each other (in a closed string). Thus one could take the left moving modes of a superstring living in 10 dimensions and tensor them with the right moving modes of a Bosonic string, which live in 26 dimensions. Remarkably this can be made to work and leads to two types of string theories known as the Heterotic strings. These theories contain $E_8 \times E_8$ or $SO(32)$ spacetime gauge fields.

Thus the right moving sector contains 16 extra Bosons. A fact about two-dimensions is that a right moving Boson is the same as a pair of right moving Fermions (since the Lorentz group in two dimension splits into two commuting, Abelian, parts that act on left and right movers respectively). This is known as Bosonization (or sometimes Fermionization, depending on your point of view). Since a right moving Fermion is more natural than a right moving Boson we will work with 10 scalars X^μ and left-moving Fermions ψ^μ_-, $\mu = 0, 1, ..., 9$ along with 32 right moving Fermions λ^A_+, $A = 1, ..., 32$. In this case left and right moving means:

$$\gamma_{01}\psi^\mu_- = -\psi^\mu_- \qquad \gamma_{01}\lambda^A_+ = \lambda^A_+ \tag{6.25}$$

The worldsheet action of a Heterotic string is now given by

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial^\beta X^\nu \eta_{\mu\nu} + i\bar{\psi}^\mu_- \gamma^\alpha \partial_\alpha \psi^\nu_- \eta_{\mu\nu} + i\bar{\lambda}^A_+ \gamma^\alpha \partial_\alpha \lambda^B_+ \delta_{AB} \tag{6.26}$$

This has $(1,0)$ supersymmetry:

Problem: Show that this action is invariant under

$$\begin{aligned} \delta X^\mu &= i\bar{\epsilon}_+ \psi^\mu_- \\ \delta\psi^\mu_+ &= \gamma^\alpha \partial_\alpha X^\mu \epsilon_+ \\ \delta\lambda^A_- &= 0 \end{aligned} \tag{6.27}$$

provided that $\gamma_{01}\epsilon_+ = \epsilon_+$.

Problem: Show that the action can be written as

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial^\beta X^\nu \eta_{\mu\nu} + i(\psi^\mu_-)^T(\partial_\tau - \partial_\sigma)\psi^\nu_- \eta_{\mu\nu} + i(\lambda^A_+)^T(\partial_\tau + \partial_\sigma)\lambda^B_+ \delta_{AB} \tag{6.28}$$

So that ψ^μ_- and λ^A_+ are indeed left and right-moving respectively.

Quantization proceeds much as before, but with all the bells and whistles turned on. The scalars are expanded in terms left and right moving oscillators a^μ_n and $\tilde{a}^\mu_n$. The ψ^μ_- have NS and R sectors with left moving oscillators b^μ_r and d^μ_n. And λ^A_+ has an expansion in terms of right moving oscillators $\tilde{b}^A_r$ and $\tilde{d}^A_n$ for NS and R sectors respectively. In the left moving sector we have $a_{NS} = 1/2$ and $a_R = 0$, just as for the type II superstrings. In the right moving sector we have (going to light cone gauge removes two X^μ fields but none of the λ^A_+ fields)

$$\begin{aligned} \tilde{a}_{NS} &= 8 \cdot \frac{1}{24} + 32 \cdot \frac{1}{48} = 1 \\ \tilde{a}_R &= 8 \cdot \frac{1}{24} - 32 \cdot \frac{1}{24} = -1 \end{aligned} \tag{6.29}$$

In particular we see that the right moving Ramond vacuum is massive.

Again the GSO projection is need to give modular invariance and to get rid of the tachyons. Let us look at the massless modes. For the left moving sector again we must take states of the form $b^\mu_{-\frac{1}{2}}|NS>_L$ and $|R>_L$, where again $|R>_L$ is a degenerate spinor ground state with 8 physical states. However in the right moving sector we need only consider the NS states of the form $\tilde{a}^\mu_{-1}|NS>_R$ and $b^A_{-\frac{1}{2}}b^B_{-\frac{1}{2}}|NS>_R$.

Looking at the massless spacetime Bosons we find the metric, dilaton and Kalb-Ramond field from $b^\mu_{-\frac{1}{2}}|NS>_L \otimes \tilde{a}^\mu_{-1}|NS>_R$. However we also obtain a vector state $b^\mu_{-\frac{1}{2}}|NS>_L \otimes b^A_{-\frac{1}{2}}b^B_{-\frac{1}{2}}|NS>_R$. This vector state has index structure $A_\mu{}^{AB}$ and can indeed be identified with a 10-dimensional gauge field. The Fermionic states then give gravitini, dilitino and gauginos. The resulting theory has 16 spacetime supersymmetries: half of the maximum of 32 that the type II theories enjoy.

Finally modular invariance and anomaly cancelation (the spacetime spectrum is chiral and for a general gauge group has anomalies) fixes the possible gauge groups to be either $E_8 \times E_8$ or $SO(32)$.

6.3 The Spacetime Effective Action

The superstrings have a spacetime supersymmetry and include gravity. Therefore their low energy effective actions are those of a supergravity. Such theories are so tightly constrained by their symmetries that, at least to lowest order in derivatives, their action is unique and known. In particular the Bosonic section of these theories is given by

$$\begin{aligned} S_{IIA} &= \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R+4(\partial\phi)^2-\frac{1}{12}H_3^2)-\frac{1}{4}F_2^2-\frac{1}{48}F_4^2\right)+\dots \\ S_{IIB} &= \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R+4(\partial\phi)^2-\frac{1}{12}H_3^2)-\frac{1}{2}F_1^2-\frac{1}{12}F_3^2-\frac{1}{240}F_5^2\right)+\dots \end{aligned}$$

where the ellipsis denotes additional terms (known as Chern-Simons terms) and the subscript $n = 1, 2, 3, 4, 5$ indicates the number of anti-symmetric indices of the field strength $F_n = F_{\mu_1\dots\mu_n}$. Note that in the S_{IIB} case there is field strength $F_\mu = \partial_\mu a$ which can be thought of as arising from an additional scalar. In addition the equation of motion that arises from S_{IIB} must be supplemented by the constraint that the five-index field strength $F_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ is self-dual:

$$F_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \frac{1}{5!}\sqrt{-g}\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\nu_1\nu_2\nu_3\nu_4\nu_5}F^{\nu_1\nu_2\nu_3\nu_4\nu_5} \tag{6.30}$$

We can also construct (in limited detail) the effective action for the Heterotic and type I superstrings. These are fixed by supersymmetry and gauge symmetry to be of the form

$$S_I = \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}e^{-2\phi}\left(R+4(\partial\phi)^2-\frac{1}{12}H_3^2-\frac{1}{4}\mathrm{tr}(F)^2+\dots\right) \tag{6.31}$$

where again the ellipsis denotes Fermionic and Green-Schwarz terms that are crucial for anomaly cancelation.

When compactified on a circle the Bosonic string admits a new duality known as T-duality. In the superstring case one finds that type IIA string theory on a circle of radius R is equivalent to type IIB string theory on a circle of radius α'/R. However one finds more remarkable dualities. It turns out that the type IIB supergravity has a symmetry $\phi \leftrightarrow -\phi$.[1] From the point of view of the string theory this is suggests a

[1] This is simplifying things if the R-R-scalar a is not zero but a more general statement is true in that case.

duality between strongly coupled strings with g_s large and weakly coupled stings with g_s small. This self-duality of the type IIB string is known as S-duality.

What happens in the strong coupling limit, $g_s \to \infty$ of the type IIA superstring? Well is it conjectured that $\sqrt{\alpha'}e^{2\phi/3}$ can be interpreted as the radius of an extra, eleventh, dimension. There is a unique supergravity theory in eleven dimensions and indeed the type IIA string effective action comes from dimensional reduction of this theory on a circle. However there is now a great deal of evidence that the whole of type IIA string theory arises as an expansion of an eleven-dimensional theory about zero-radius (in on of its dimensions). This theory is known as M-theory and is rather poorly understood. However it's existence does seem be justified. The lowest order term is in a derivative expansion is fixed by supersymmetry to be

$$S_M = \frac{1}{\kappa^9}\int d^{11}x\sqrt{-g}(R - \frac{1}{48}G_4^2) + \dots \tag{6.32}$$

where again the ellipsis denotes Chern-Simons and Fermionic terms. One also finds the Heterotic $E_8 \times E_8$ string by compactification of M-theory on a line interval.

Furthermore it promises to be very powerful as it controls not only the strong coupling limit of the type IIA string but, as a consequence of duality, the strong coupling limit of all the five known string theories. Thus one no longer thinks of there being five separate string theories but instead one unique theory, M-theory, which contains five different perturbative descriptions depending on what one considers to be a small parameter.

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