

II Defect Calculus in Finite Homotopy Theories

gauge theory for finite higher groups; $X \rightarrow \Omega X$
 $BG \leftarrow G$

Recall (Finite homotopy) Correspondence category, $\text{Cor}_{\leq 1}$

Obs: $X, \sum |\pi_i X| < \infty$; $\text{Hom}(X, Y) = \sum_{Y \leftarrow t \rightarrow X} \text{finitary}$

Composition = homotopy fiber product

(Exercise: this stays in the finite homotopy category)

Quantizes to: vector spaces & linear maps

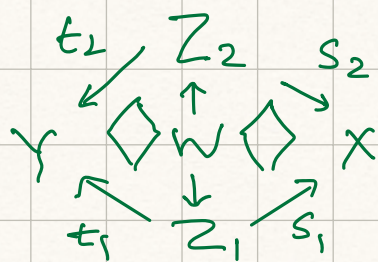
$$X \mapsto H^0(X; \mathbb{C}),$$

$$Z \rightarrow t_* s^* \text{ with weights } = \prod_i |\pi_i (t^{-1})|^{(-1)^i}$$

(Exercise: Weights are needed for composition to work out)

Obs Can continue to raise the category level arbitrarily:

2-morphisms



with commuting squares.
similar composition law
for 2-morphisms.

and so forth to a n -category Cor_n or even Cor_{∞} .

Of course if representing in an n -category we truncate at Cor_n .

The Mapping space functor $M \mapsto \text{Map}(M; X)$ gives an
(non-oriented) TQFT into $\text{Cor}_{\leq n}$ for any fixed X .

The Quantization functor

[Freed, Hopkins, Lunie, T] define inductively quantizations

$$Q_m : \text{Cor}_{\leq m+1} \longrightarrow (m\text{-Algebras})$$

$\uparrow$ NOT "Em algebras"

but algebra objects in

(Algebra objects, bimodule objects ... in

(Algebra objects, ...)

$\vdots$

in complex vector spaces.

Example A 2-algebra structure on an algebra A is

a multiplication $A - A \otimes A$ bimodule ${}_A M_{A \otimes A}$

a unit $A - \mathbb{C}$ bimodule ${}_A U_{\mathbb{C}}$

with extra associativity/unital data and conditions

e.g. ${}_A M_{A \otimes A} \otimes_A {}_A M_{A \otimes A} \cong {}_A M_{A \otimes A} \otimes_A {}_A M_{A \otimes A} \quad (+ \text{coherence})$

$\underbrace{\quad}_1 \quad \underbrace{\quad}_2 \quad \underbrace{\quad}_3 \quad \underbrace{\quad}_3 \quad \underbrace{\quad}_1 \quad \underbrace{\quad}_2$
 $"x_1(x_2 x_3) = (x_1 x_2) x_3"$

Proposition A 2-algebra structure on A

$\Leftrightarrow$ (co-continuous) tensor structure on $A\text{-Mod}$.

Pf Define $E \otimes F := {}_A M_{A \otimes A} \otimes_{A \otimes A} (E \otimes_{\mathbb{C}} F)$

Examples * Fusion Categories. ($A \cong \text{finite } \oplus \mathbb{C}$.)

* Hopf algebras. ${}_H M_{H \otimes H} = H \otimes H, {}_H U_{\mathbb{C}} = \mathbb{C}$

Comment: [FHLT] did not specify the target category tightly enough to prove functoriality. See Johnson-Freyd & Scheimbauer. In possible model, ongoing Scheimbauer-Walck..

[FHLT] gave an iterated colimit procedure guaranteed to give the correct functors, if they exist.

[See CMSA Lecture, July 2025 for discussion]

Remark Functoriality is needed to get a TQFT valued in $(n-1)\text{-Alg}$ from the one in $\text{Cor}_{\leq n}$.

Remark This generalizes to include Dijkgraaf-Witten twists $\tau \in H^n(X; \mathbb{C}^\times)$ and oriented TQFTs.

Remark These are not the universal TQFTs.

The universal TQFT target must include $\mathbb{T}_{\mathbb{C}^\times}$ as its spectrum of units (Pontrjagin dual of sphere)

see eg [Freed, Scheimbauer, T] Appendix A for an example of a 4D gauge theory on $\mathbb{Z}_2$ which **cannot** be realized in "fusion 2-categories" (3-algebras).

Quantization recipe: from finite homotopy X

$$Q_{-1}(X) := \sum_{p \in \pi_0 X} \pi_1(X, p)^{(-1)^i} \quad (\text{Kontsevich formula})$$

$$Q_0(X) := H^0(X; \mathbb{C}) \quad (\text{vector space})$$

If Q_{m-1} is defined

then $Q_m(X) = \bigoplus_{p \in \pi_0 X} Q_{m-1}(\Omega_p X)$

this is a group!
So that becomes an algebra object in $(m-1)$ -algebras!

Examples ^{2D} $Q_1(X) = \bigoplus_{p \in \pi_0 X} H^0(\Omega_p X; \mathbb{C}) = \bigoplus_{p \in \pi_0 X} \mathbb{C} \langle \pi_1(X, p) \rangle$

group ring

This gives gauge theory for the groupoid $\coprod_p \pi_1(X, p)$

Ex ^{3D} $Q_2(X) = \bigoplus_{p_0 \in \pi_0 X} \bigoplus_{p_1 \in \pi_1(X, p_0)} \mathbb{C} \langle \pi_2(X, p_0) \rangle$

group algebra

Second multiplication from group law on ΩX

On each $\pi_2(X, p_0)$, that agrees with the π_2 group law, so second multiplication on $\mathbb{C} \langle \pi_2 \rangle$ agrees with the first, exploiting commutativity of π_2

Result: $\bigoplus_{p_0 \in \pi_0 X} \mathbb{C} \langle \pi_1(X, p_0) \rangle \otimes \mathbb{C} \langle \pi_2(X, p_0) \rangle$

$\mathbb{C}[\pi_2^v]$ w/ pointwise mult.

better understood as

$$\bigoplus_{\pi_0 X} (\text{Group ring of } \pi_1 \text{ with coeff. in } \text{Vec}[\pi_2^v])$$

⊗ cat with pointwise multiplication

$$\cong \bigoplus_{p_0 \in \pi_0 X} \bigoplus_{\alpha \in \pi_2^v / \pi_1} \text{Vec}^{K(X)} \langle Z_{\pi_1}(X) \rangle$$

← action of π_1 on π_2 Dijkgraaf-Witten twist possible

Remark $\pi_0 \neq *$ or $\pi_2 \neq \langle 0 \rangle \Rightarrow$ theory is not simple

Have one point op for each elt of $\coprod_{\pi_0} \pi_2^\vee$,

if action of π_1 on π_2 is trivial.

(Else, get the orbits of the action on $\pi_2^\vee$)

Instance of Electromagnetic Duality.

Remark k -invariant of X could conceal a nontrivial
associator for $\text{Vec} \langle \pi_1 \rangle$ - a DW twist.

Example ^{3D} $\pi_0 = *$, $\pi_1 = \pi_2 = \mathbb{Z}/2$, $k^3 \in H^3(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$, not 0.

$$\mathcal{Q}_2(X) = \bigoplus_{\epsilon = \pm 1} \text{Vec}^\epsilon \langle \mathbb{Z}/2 \rangle = \text{Vec}^+ \langle \mathbb{Z}/2 \rangle \oplus \text{Vec}^- \langle \mathbb{Z}/2 \rangle$$

Standard Dijkgraaf-Witten
3d gauge theory twisted theory

General: a $\chi: \pi_2 \rightarrow \mathbb{C}^\times$ lives $\chi \circ k^3 \in H^3(B\pi_1; \mathbb{C}^\times)$.

Observation

In 3D, we always get (Dijkgraaf-Witten twisted) gauge theories for finite groups - not more exotic fusion categories.

This is also true in 4D - but there, the expectation is that we did not miss anything.

Example^{4D} BFCs from X with $\pi_2 = A, \pi_3 = B$.

Such a space is completely determined by its k -invariant

$$k^4 \in H^4(B^2 A; B)$$

space w/ $\pi_2 = A$
no other htpy

Theorem (Eilenberg, MacLane) This group classifies the homogeneous quadratic maps $A \rightarrow B$.

$$q(a+a') - q(a) - q(a') \text{ bilinear}$$

Proposition Such maps are in bijection with iso classes of braided tensor structures on the category

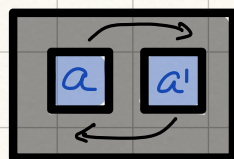
$$\text{Objects} = A, \text{Hom}(a, a') = 0 \text{ if } a \neq a', \text{End}(a) = B$$

which lift the obvious tensor structure on iso classes of objects.

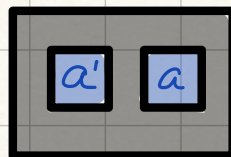
Eg $B = \mathbb{C}^\times$: this is the same as braided $\otimes$ structures on $\text{Vec}[A]$ lifting the convolution.

Construction Define $B = \Omega^2 X$ and take π_0 on objects & morphism flat vector bundles, with convolution.

Braided structure: standard proof of commutativity of π_2 :



$\rightsquigarrow$



a : holonomy

Conversely, from a braiding $\beta: A \times A \rightarrow B$ on objects, get the quadratic map $a \mapsto \beta_{a,a}: A \rightarrow B$.

Can check these constructions are mutually inverse.

Theorem ^{4D} $\mathcal{Q}_3(X) \cong \mathbb{C}[A \rtimes B]$

with the first two multiplications pointwise, and the third multiplication

= a second multiplication on $\text{Vec}[A^\vee \rtimes B^\vee]$

agrees with the 1st (pointwise tensoring)

but the commutation data between the 2nd and 1st tensoring given by the following braiding:

For each $b^\vee \in B^\vee$, use the braiding on $\text{Vec}[A^\vee]$

given by $b^\vee \circ k^4 \in H^4(B^2 A; \mathbb{C}^\times)$.

Remark When the braiding is a nondegenerate $A \rtimes A \rightarrow \mathbb{C}^\times$ that component of the algebra $\mathcal{Q}_3$ is Morita-invertible.

The TQFT can then be trivialized on framed manifolds by enlarging the ambient category of TQFTs.

This same construction also converts a general component of $\mathcal{Q}_3$ into the

4D gauge theory based on the dual of the kernel of the braiding form.

This relies on a 4D Electromagnetic duality which trades π_2 for an addition of $\pi_2^\vee$ to π_1 .

2 Internal (Bulk) Defects

Recall: fully local p -fects in an $(n+1)$ dim TQFT $\mathcal{T}$ valued in $\mathcal{C}$

$\longleftrightarrow$ boundary conditions for $\mathcal{T}/S^p$

$\longleftrightarrow$ Morphisms $\mathbb{1} \rightarrow \mathcal{T}(S^p)$ in $\Omega^p \mathcal{C}$.

Our target is $\text{Alg}(\text{Alg}(\dots \text{Vect}))$ and $\mathcal{T}_X = \mathcal{Q}_n(X)$
for a fin. ltpy space X .

$$\text{Now } \mathcal{T}(S^p) = \mathcal{Q}_{n-p}(\text{Map}(S^p; X)) = \mathcal{Q}_{n-p}(X^{S^p})$$

We have a fiber bundle $\Omega^p X \hookrightarrow X^{S^p} \xrightarrow{\text{eval.}} X$
constant maps

$$\Rightarrow \text{splittings } \pi_k X^{S^p} \cong \pi_k X \oplus \pi_k \Omega^p X = \pi_k X \oplus \pi_{k+p} X$$

Careful at $k=1$, $\pi_1 \Omega^p X \cong \pi_1 X \times \pi_{p+1} X$

and $k=0$, $\pi_0 X^{S^p} \cong$ orbits of $\pi_1 X$ action on $\pi_p X$

Moreover, other components of X^{S^p} may not have sections
so splitting may not apply

In nice cases (eg X a group) we do have a splitting
for all components

$$\pi_k X^{S^p} = \begin{matrix} * & 0 & 1 & 2 & \dots & n \\ \pi_0 X & \pi_1 X & \pi_2 X & \dots & \pi_n X \\ \oplus & \oplus & \oplus & & \oplus \\ \pi_p X & \pi_{p+1} X & \pi_{p+2} X & & \pi_{n+1} X \end{matrix}$$

Rest do not matter
for quantization

So fully local p -fects are modules in $(n-p-1)$ algebras
over $\mathcal{Q}_{n-p}$ of this space $\underline{\mathbb{Z}}$

3 Boundary defects

Thankfully the classification is simpler

Boundary conditions: interfaces from the trivial theory
Can be obtained by quantizing spaces $Y \rightarrow X$.

The loop space ΩX acts on $F \hookrightarrow Y$ (the htpy fiber)
and this leads to an action of

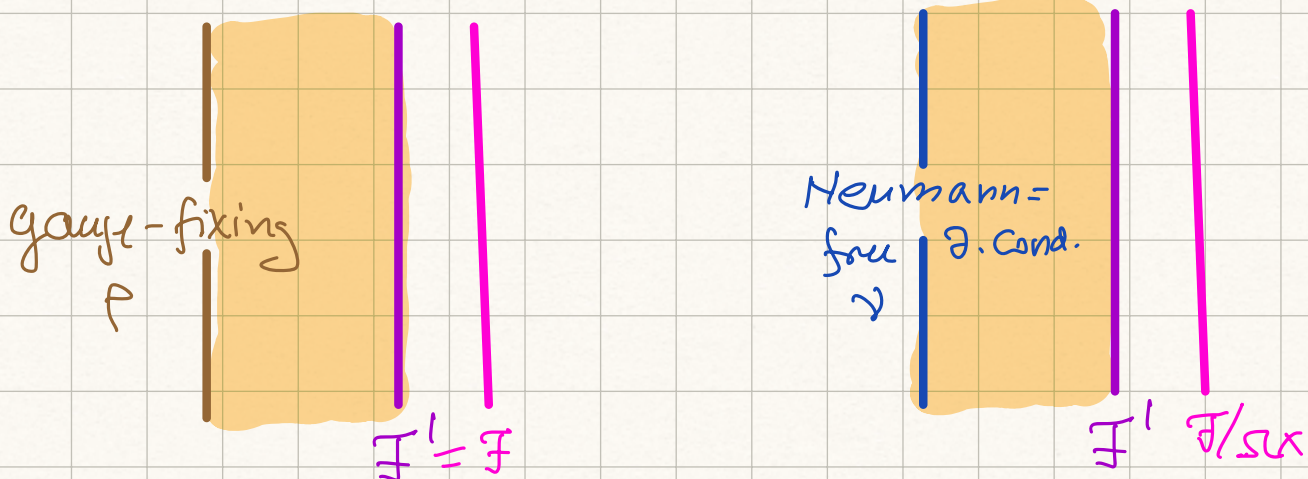
$$\mathbb{Q}_n(X) \text{ on } \mathbb{Q}_{n-1}(F)$$

($\mathbb{Q}_n(X)$ is the group ring of ΩX in n -algebras;
General boundary conditions are n -dim TFTs
with ΩX symmetry)

$$X \text{ connected, } * \in X$$

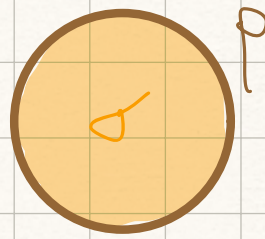
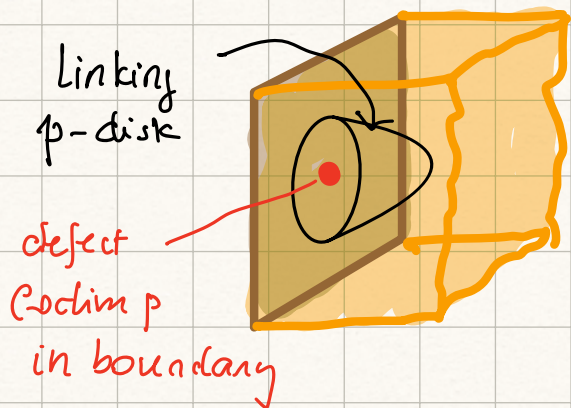
If T_X is the bulk σ of the quiche, take $Y = * \hookrightarrow X$
so $F = \Omega X$, ρ becomes the regular module for σ

$\Rightarrow$ The standard gauge theory picture (left) for group ΩX
which also allows gauging the theory (right)



Repeating the construction of bulk defects:

fully local defects in the boundary are determined by the link



Linking disk LD
"screening" the defect

(p, σ) gives an object in $\Omega^p \mathcal{C}$. The datum for local defect is a morphism $\mathbb{1} \longrightarrow (p, \sigma) \left(\text{disk} \right)$ in $\Omega^p \mathcal{C}$.

Alternative: Reducing $\mathcal{T}$ on LD gives $(n+1-p)$ theory $\mathcal{T}/_{D^p}$
Defect datum is a boundary condition for this

The Transparent Defect is $\mathcal{T} \left(\text{cone} \right): \mathcal{T}/_{D^p}(\emptyset) \rightarrow \mathcal{T}/_{D^p}(\text{pt})$

Example $n=2$, $\sigma = 3D$ G gauge theory, $p = \text{Dirichlet}$
($X = BG$ $Y = *$)

$p=1$: boundary line defect

$$(p, \sigma) \left(\text{line} \right) = \mathcal{Q}_1(G) = \mathcal{C}[G]$$

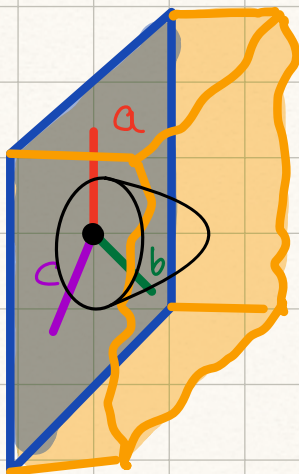
Defect labels $\in \text{Vect}[G]$ (as linear category)

Effect on $\mathcal{F}$ of a line supported at $g \in G$
 $\rightarrow g$ -transformation across the line

Example, ctd: With Neumann (free) condition for ρ :
get Wilson lines in the gauged theory $\mathcal{T}/G$

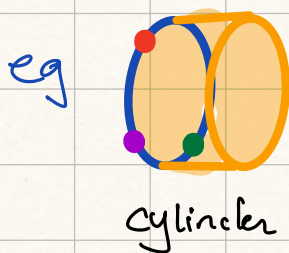
$$(\nu, \sigma) (\bullet \text{---} \bullet) = Q_c(BG) = \mathbb{C}\langle G \rangle$$

defect labels $\in \text{Rep}(G)$



Labels: $a \ b \ c \in \text{Rep}(G)$

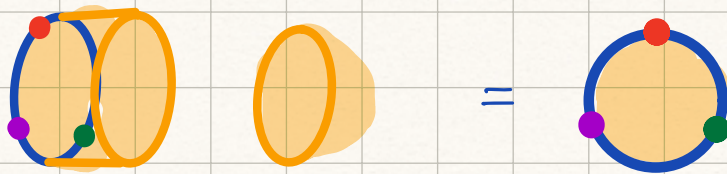
$$\bullet \in \mathcal{T} \left(\text{circle with dots } a, b, c \right) = (a \otimes b \otimes c)^G$$



$$= a \otimes b \otimes c \in \text{Vec}_1^G[G]$$

sitting over 1

Capping the cylinder = integration over $BG = G$ invariants



Standard Dirichlet boundary Condition $\ast \rightarrow X$:

Defects supported in the boundary of the $(d+1)$ -dim $\mathcal{Q}_d(X)$

Reduced theories: $\mathcal{Q}_d(X)|_{D^p} = \mathcal{Q}_{d-p}(\Omega^p X)$

$\pi_p = \pi_p(X)$, brackets indicate degree in which they live

Codimension of defects (in crust)	Quantum defect labels are modules over...	Contributing homotopy groups, placed in dim $[k]$ all/ new defects	Condensed WAST DAT
d (points)	$\mathcal{Q}_0(\Omega^d X)$	$\pi_d[0]$	None
$d-1$ (lines)	$\mathcal{Q}_1(\Omega^{d-1} X)$	$\pi_{d-1}[0]$, $\pi_d[1]$	$\pi_d[1]$
$d-2$ (surfaces)	$\mathcal{Q}_2(\Omega^{d-2} X)$	$\pi_{d-2}[0]$, $\pi_{d-1}[1]$, $\pi_d[2]$	$\pi_{d-1}[1]$, $\pi_d[2]$
k	$\mathcal{Q}_{d-k}(\Omega^k X)$	$\pi_k[0]$, $\pi_{k+1}[1]$, $\pi_{k+2}[2]$, ...	$\pi_{k+1}[1]$, $\pi_{k+2}[2]$, ...
2	$\mathcal{Q}_{d-2}(\Omega^2 X)$	$\pi_2[0]$, $\pi_3[1]$..., $\pi_d[d-2]$	$\pi_3[1]$..., $\pi_d[d-2]$
1 (interfaces)	$\mathcal{Q}_{d-1}(\Omega X)$	$\pi_1[0]$, $\pi_2[1]$, ..., $\pi_d[d-1]$	$\pi_2[1]$, $\pi_3[2]$..., $\pi_d[d-1]$
$\circ$ new ∂ condition!	$\mathcal{Q}_d(X)$	$\pi_0[0]$, $\pi_1[1]$, ..., $\pi_d[d]$	

Point operator in $\mathcal{Q}_p(\Omega^{d-p} X)$: $\mathbb{C}[\pi_{d-p} \times \pi_d^y]$ ($0 < p < d$)

Those in Red vanish on the transparent defect:

= regular module $\mathcal{Q}_{p-1}(\Omega^{d-p-1} X)$ for $\mathcal{Q}_p((\Omega^{d-p} X)_\circ)$

From fibration sequence $\Omega^{d-p+1} X \hookrightarrow X \sim X^{D^{d-p+1}} \twoheadrightarrow \Omega^{d-p} X$

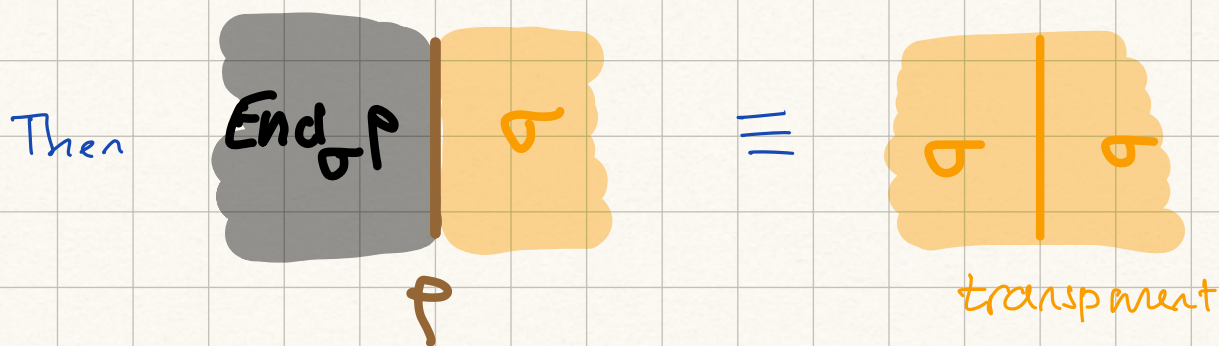
4 General Dirichlet Boundary condition

Moral definition ρ is a Dirichlet boundary cond for σ if the algebra $\text{End}_\sigma(\rho)$ generates a TQFT $\equiv \sigma$.

This needs care if σ is not generated by an algebra object
But in FHT case, $\sigma = \mathcal{Q}_n(X)$ is an algebra

so we ask $\text{End}_{\mathcal{Q}_n(X)}(\rho) \stackrel{\rho}{=} \mathcal{Q}_n(X)$

should be a Morita equivalence



In particular: Can reconstruct σ from $(\text{End}_\sigma(\rho), \rho)$.

Theorem (FHT case) ρ is Dirichlet for $\mathcal{Q}_d(X)$ iff
no point ops in the bulk vanish on the boundary.

For spaces $Y \rightarrow X$: $\pi_0 Y \rightarrow \pi_0 X$, $\pi_n X \rightarrow \pi_n(X, Y)$

Bulk pt ops for X

$$\bigoplus_{\pi_0 X} \mathbb{C}[(\pi_n X)^v]$$

Boundary pt ops:

$$\bigoplus_{\pi_0 Y} \mathbb{C}[\pi_n(X, Y)^v]$$

Bulk to
Boundary
Map

We want Dirichlet ends of defects

Open problem (topology)

Describe the image of $\text{Cor}_{\leq n}$ in $n\text{-Alg}$ under $\mathcal{Q}$.

Quantization is not faithful!

Additional symmetry: **Electromagnetic Duality**.

In dimension n : If X has a singl π_X , $\pi_p X = A$
and $X^\vee$ has a singl $\pi_{X^\vee}$, $\pi_{n-p} X^\vee = A^\vee$

Then $\mathcal{Q}_n X \equiv \mathcal{Q}_n X^\vee$ as theories on oriented manifolds

(In topology: $X = K(A, p)$, $X^\vee = K(A^\vee; n-p-1)$)

Open problem in Category theory

Describe all semi-simplish TQFTs.

("Universal target category")

(Ongoing program by Johnson-Freyd, Reutter)

Eg In 2D, gauge theories are equivalent to σ -models

In 3D, gauge theories for Pontryagin dual groups are $\equiv$

In 4D, abelian gauge theories are equivalent to
gerbe theories (BFCs) for the dual groups.

Not true that $\mathcal{Q}_n$ gives all of them!

Additional ones in dim. $3 \bmod 4$ ("Chern-Simons" type)

Additional ones in dim $1 \bmod 4$ (gauging under EM)