

Topological quantum field theories

Lecture notes (Claude typed version of handwritten notes)

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Lecture 1: from σ -models to the axioms

The aim of this first part is to arrive at the list of structures in Section 7 *from* the path integral description of topological sigma models, so that the axioms introduced on the slides afterwards (Sections 8–10) look inevitable rather than invented. Everything before Section 4 is heuristic; from Section 4 on, every integral written down is an honest finite sum.

1 Quantum field theory and the path integral

Fix $n \in \mathbb{N}$. An n -dimensional quantum field theory assigns to a closed n -manifold M , equipped with some geometric structure (for example a Riemannian metric $g_{\mu\nu}$) a number, the *partition function*

$$(M, g_{\mu\nu}) \mapsto Z(M) \text{ “} = \text{” } \int_{\Phi \in \mathcal{F}(M, g_{\mu\nu})} e^{-S[\Phi]}, \quad (1)$$

where $\mathcal{F}(M, g_{\mu\nu})$ is the space of field configurations and S the action functional. The quotation marks are essential: for most theories no measure on $\mathcal{F}$ exists. We use (1) only as a source of *formal* properties, and then check those properties against a class of examples in which the integral really is defined.

More generally the theory depends on background fields, and one writes

$$Z(M, -): \mathcal{B}(M) \longrightarrow \mathbb{C}, \quad (2)$$

where $\mathcal{B}(M)$ is the collection of background fields on M (the metric usually among them).

Definition 1.1 (physics definition). A quantum field theory is *topological* if $Z(M, -)$ is locally constant on $\mathcal{B}(M)$, i.e. in the case of metrics

$$\frac{\delta}{\delta g^{\mu\nu}} Z(M, -) = 0,$$

and likewise for the other quantities of the theory (correlation functions, etc.).

Plan for today. Derive axioms for such theories from formal properties of the path integral, guided throughout by one class of specific examples: *topological σ -models*.

Remark 1.2. Sometimes a theory is not itself topological, but its low-energy behaviour is described by a TQFT — one then speaks of *IR-TQFTs*. These TQFTs are often topological σ -models with target T_{eff} . However, for a non-topological σ -model with target T , in general $T_{\text{eff}} \neq T$.

2 σ -models

Fix a *target space*: a Riemannian manifold (X, g^T) . The fields on M are

$$\mathcal{F}(M) = \text{Map}(M, X),$$

and the action is

$$S[\Phi] = c \int_M \left(g_{IJ}^T(\Phi(x)) \partial^\mu \Phi^I \partial_\mu \Phi^J + V(\Phi) \right) \text{dvol}_g. \quad (3)$$

Remark 2.1. Neither term in (3) can survive if we want Definition 1.1 to hold: the kinetic term needs the metric g on M (and the metric g^T on X), and the potential term needs dvol_g . What *can* survive is a term whose value does not change when Φ is deformed — i.e. one depending only on the homotopy class of Φ . That is the term we keep, in Definition 4.9.

3 Topological sigma models and general targets

Allow a more general target X — a topological space, taken up to homotopy, rather than a manifold with a metric.

Topological σ -models are important, among other reasons, because they form the symmetry TQFTs for (finite, higher-form) symmetries. For a finite 0-form symmetry G of a d -dimensional theory this is the $(d+1)$ -dimensional Dijkgraaf–Witten theory below, with target BG and ω the anomaly; for a finite abelian p -form symmetry A the target is $B^{p+1}A$.

Example: $X = BG$ for G a finite group. BG is connected, and maps into it classify bundles:

$$\Phi: M \longrightarrow BG \rightsquigarrow \text{a principal } G\text{-bundle on } M, \quad \text{Map}(M, BG) \simeq \text{Bun}_G(M),$$

the groupoid of principal G -bundles and gauge transformations. This gives *Dijkgraaf–Witten theory*, i.e. finite gauge theory.

Because the action is constant on connected components of $\text{Map}(M, X)$, the path integral should be a weighted sum: one sums over these components, weighted by their sizes,

$$\int_{\Phi \in \text{Map}(M, X)} e^{-S[\Phi]} \text{ “} = \text{”} \sum_{\Phi \in \pi_0 \text{Map}(M, X)} |C(\Phi)| e^{-S[\Phi]}, \quad (4)$$

where $|C(\Phi)|$ is the “volume” of the connected component of Φ .

For Dijkgraaf–Witten theory this becomes a sum over the gauge-equivalence classes of principal bundles, and the volume divides by the residual gauge transformations,

$$\int_{P \in \text{Bun}_G(M)} e^{-S[P]} \text{ “} = \text{”} \sum_{P \in \pi_0 \text{Bun}_G(M)} \frac{1}{|\text{Aut}(P)|} e^{-S[P]}. \quad (5)$$

Section 4 turns (4) into a definition: the volume of a component is its homotopy cardinality (Definition 4.2), and for $X = BG$ this gives back (5).

Example 3.1 ($M = S^1$, $X = BG$). Evaluating on the circle,

$$\text{Map}(S^1, BG) \simeq G//G \quad (G \text{ acting on itself by conjugation}),$$

so that, for $g \in G$ representing a component,

$$\pi_0 = \text{Conj}(G), \quad \pi_1(\text{Map}(S^1, BG), g) = C_G(g), \quad \pi_k = 0 \quad (k > 1).$$

A point of the mapping space is a bundle on S^1 , classified by its holonomy g ; the automorphisms of that point are the gauge transformations commuting with g , i.e. the centraliser $C_G(g)$. In this case the partition function with trivial action is 1.

4 π -finite spaces and homotopy cardinality

Definition 4.1. A space S is π -finite if

- (i) $|\pi_i(S, x)| < \infty$ for all $i \in \mathbb{N}$ and all $x \in S$ (including $|\pi_0(S)| < \infty$), and
- (ii) there is $k \in \mathbb{N}$ with $\pi_i(S, x) = 0$ for all $i > k$ and all $x \in S$.

Definition 4.2 (homotopy cardinality). For a space S ,

$$|S| = \sum_{x \in \pi_0(S)} \frac{\prod_{i \geq 1} |\pi_{2i}(S, x)|}{\prod_{i \geq 0} |\pi_{2i+1}(S, x)|} = \sum_{x \in \pi_0(S)} \prod_{i \geq 1} |\pi_i(S, x)|^{(-1)^i}.$$

Lemma 4.3. S π -finite $\implies |S| < \infty$.

Example 4.4. $|BG| = 1/|G|$ for G finite. As a check on Example 3.1,

$$|G//G| = \sum_{[g] \in \text{Conj}(G)} \frac{1}{|C_G(g)|} = \frac{1}{|G|} \sum_{[g] \in \text{Conj}(G)} |[g]| = 1$$

by orbit–stabiliser.

Remark 4.5. The sum in Definition 4.2 sometimes converges even when S is not π -finite: for $S = \coprod_{n \in \mathbb{N}} B\mathfrak{S}_n$ ($\mathfrak{S}_n$ the symmetric group) one gets $|S| = \sum_{n \geq 0} \frac{1}{n!} = e$.

Exercise 4.6. Let X be π -finite and M compact. Then $\text{Map}(M, X)$ is π -finite.

Definition 4.7 (integration). For a π -finite space S and a locally constant function $f: S \rightarrow \mathbb{C}$ set

$$\int_S f := \sum_{x \in \pi_0(S)} |C_x| f(x),$$

where $C_x \subseteq S$ is the connected component of x . In particular $\int_S 1 = |S|$.

Remark 4.8. For S the classifying space of an essentially finite groupoid $\mathcal{G}$ this is exactly the groupoid integral $\int_{g \in \mathcal{G}} f(g) = \sum_{g \in \pi_0 \mathcal{G}} f(g)/|\text{Aut}(g)|$.

Definition 4.9 (partition function). Let X be a π -finite target, $\omega \in H^n(X; \mathbb{R}/\mathbb{Z})$, and M a closed oriented n -manifold. Then

$$Z_X^\omega(M) = \int_{\Phi \in \text{Map}(M, X)} \exp\left(2\pi i \int_M \Phi^* \omega\right) \in \mathbb{C}.$$

This is well defined: $\text{Map}(M, X)$ is π -finite by Exercise 4.6, and the integrand is locally constant because $\int_M \Phi^* \omega$ depends only on the homotopy class of Φ . This is the term promised in Section 2 — the one unchanged under deformations of Φ .

Notation for the rest of the lecture. In Sections 5–6 we keep writing $e^{-S[\Phi]}$ for the weight, as on the board; for the theories above it is the phase of Definition 4.9.

Example 4.10 (Dijkgraaf–Witten theory on the torus). Let $n = 2$, $X = BG$ and $\omega = 0$. A G -bundle on $T^2 = S^1 \times S^1$ is determined by its holonomies around the two generators of $\pi_1(T^2) = \mathbb{Z}^2$, which commute:

$$\text{Map}(T^2, BG) \simeq \text{Hom}(\mathbb{Z}^2, G) // G = \{(g, h) \in G \times G \mid gh = hg\} // G,$$

with G acting by simultaneous conjugation. By Remark 4.8 and orbit–stabiliser,

$$Z_{BG}(T^2) = \frac{|\{(g, h) \mid gh = hg\}|}{|G|} = \frac{1}{|G|} \sum_{g \in G} |C_G(g)| = \sum_{[g] \in \text{Conj}(G)} \frac{|[g]| |C_G(g)|}{|G|} = |\text{Conj}(G)|.$$

So although the weights $1/|\text{Aut}|$ are fractions, the answer is a natural number: the number of conjugacy classes of G . We will see why once we have state spaces: cutting T^2 open along a circle exhibits $Z(T^2)$ as the trace of the identity of $Z(S^1)$, i.e. as $\dim Z(S^1) = \dim \text{Cl}(G)$ (Section 5).

5 Manifolds with boundary: state spaces

Ideally we want Z for manifolds with boundary as well. So let M be a compact n -manifold with $\partial M = \Sigma$. Restriction of fields gives

$$\text{res}: \text{Map}(M, X) \longrightarrow \text{Map}(\Sigma, X).$$

The boundary values are *not* integrated over, so the path integral no longer produces a number but a function of them:

$$f \longmapsto \int_{\substack{\Phi \in \text{Map}(M, X) \\ \Phi|_\Sigma \cong f}} e^{-S[\Phi]}, \quad \text{a function on } \mathcal{F}(\Sigma) = \text{Map}(\Sigma, X). \quad (6)$$

Or more correctly: if one goes through the computation carefully, one finds a section of a certain line bundle, encoding how these integrals transform when the field on the boundary is changed by a homotopy. We will mostly ignore this point from now on. For vanishing action, where we integrate the constant function 1, (6) is an honest locally constant function. Said differently, in general the function in (6) is actually not locally constant, since it changes when we follow a non-contractible loop in $\text{Map}(\Sigma, X)$; rather, it is a section of a certain flat line bundle $\mathcal{L}$ over $\text{Map}(\Sigma, X)$.

Definition 5.1 (state space). For a closed $(n-1)$ -manifold Σ and $\omega = 0$ set

$$\begin{aligned} Z(\Sigma) &:= \text{Fun}(\text{Map}(\Sigma, X)) = \{\text{locally constant functions } \text{Map}(\Sigma, X) \rightarrow \mathbb{C}\} \\ &= \text{Fun}(\pi_0 \text{Map}(\Sigma, X)). \end{aligned}$$

By (6), $Z(M) \in Z(\Sigma)$.

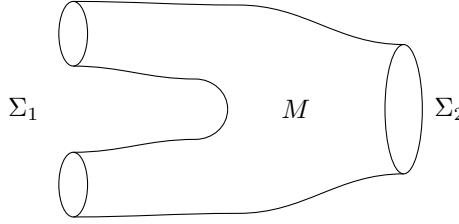
Example 5.2. For $\Sigma = S^1$ and $X = BG$, Example 3.1 gives

$$Z(S^1) = \text{Fun}(\text{Map}(S^1, BG)) = \text{Fun}(G//G) = \text{Cl}(G),$$

the class functions on G .

6 Bordisms, disjoint union and gluing

Let $M: \Sigma_1 \rightarrow \Sigma_2$ be a bordism, with restriction maps $r_i: \text{Map}(M, X) \rightarrow \text{Map}(\Sigma_i, X)$.



Definition 6.1.

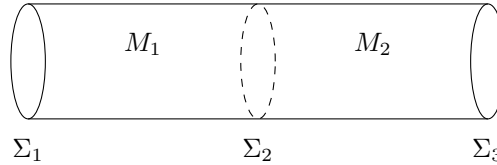
$$Z(M): Z(\Sigma_1) \longrightarrow Z(\Sigma_2), \quad \psi \longmapsto \left(f_2 \mapsto \int_{\substack{\Phi \in \text{Map}(M, X) \\ \Phi|_{\Sigma_2} \cong f_2}} \psi(\Phi|_{\Sigma_1}) e^{-S[\Phi]} \right).$$

Proposition 6.2 (disjoint union). $Z(\Sigma_1 \sqcup \Sigma_2) \cong Z(\Sigma_1) \otimes Z(\Sigma_2)$.

Indeed $\text{Map}(\Sigma_1 \sqcup \Sigma_2, X) \cong \text{Map}(\Sigma_1, X) \times \text{Map}(\Sigma_2, X)$, and $\text{Fun}(A \times B) \cong \text{Fun}(A) \otimes \text{Fun}(B)$. The last isomorphism uses finiteness of π_0 ; for infinite-dimensional function spaces such as $L^2(\mathcal{F}_\Sigma)$ one needs a completed tensor product instead.

Proposition 6.3 (gluing). For $M_1: \Sigma_1 \rightarrow \Sigma_2$ and $M_2: \Sigma_2 \rightarrow \Sigma_3$,

$$Z(M_2) \circ Z(M_1) = Z(M_2 \circ_{\Sigma_2} M_1).$$



Computation. For $\psi \in Z(\Sigma_1)$ and $f_3 \in \text{Map}(\Sigma_3, X)$,

$$\begin{aligned} (Z(M_2) \circ Z(M_1))(\psi)(f_3) &= \int_{\substack{\Phi_2 \in \text{Map}(M_2, X) \\ \Phi_2|_{\Sigma_3} \cong f_3}} Z(M_1)(\psi)(\Phi_2|_{\Sigma_2}) e^{-S[\Phi_2]} \\ &= \int_{\substack{\Phi_2 \in \text{Map}(M_2, X) \\ \Phi_2|_{\Sigma_3} \cong f_3}} \int_{\substack{\Phi_1 \in \text{Map}(M_1, X) \\ \Phi_1|_{\Sigma_2} \cong \Phi_2|_{\Sigma_2}}} \psi(\Phi_1|_{\Sigma_1}) e^{-S[\Phi_1] - S[\Phi_2]} \\ &= \int_{\substack{\Phi \in \text{Map}(M_2 \circ_{\Sigma_2} M_1, X) \\ \Phi|_{\Sigma_3} \cong f_3}} \psi(\Phi|_{\Sigma_1}) e^{-S[\Phi]} = Z(M_2 \circ_{\Sigma_2} M_1)(\psi)(f_3). \end{aligned}$$

The two inputs to the last step are: *locality of the action*, $S[\Phi_1] + S[\Phi_2] = S[\Phi]$ for the glued field Φ , and the fact that a pair (Φ_1, Φ_2) agreeing on Σ_2 is the same thing as a single field on the glued manifold. Both deserve a word: the second is a homotopy fibre product rather than a strict one (Definition 14.1), and exchanging the two integrals is Fubini for homotopy fibres.

7 What we have extracted

- a vector space $Z(\Sigma)$ for every closed $(n-1)$ -manifold Σ ;
- a linear map $Z(M): Z(\Sigma_1) \rightarrow Z(\Sigma_2)$ for every bordism $M: \Sigma_1 \rightarrow \Sigma_2$;
- $Z(\Sigma_1 \sqcup \Sigma_2) \cong Z(\Sigma_1) \otimes Z(\Sigma_2)$ and $Z(\emptyset) \cong \mathbb{C}$;
- $Z(M_2) \circ Z(M_1) = Z(M_2 \circ_\Sigma M_1)$;
- $Z(\Sigma \times [0, 1]) = \text{id}_{Z(\Sigma)}$: both restriction maps of the cylinder are homotopy equivalences, so all homotopy fibres are contractible;
- consequently, for closed M viewed as a bordism $\emptyset \rightarrow \emptyset$, $Z(M)$ is again a number — the partition function of Definition 4.9.

Switch to slides: the bordism category, symmetric monoidal categories, and then the mathematical axiomatisation

$$Z: \text{Bord}_n \longrightarrow \text{Vect} \quad \text{symmetric monoidal.}$$

Lecture 1, part II (slides): categories and Atiyah's axioms

8 Categories and bordisms

Definition 8.1. A *category* $\mathcal{C}$ consists of

- a collection of *objects* $X, Y, \dots$;
- for every pair of objects a set of *morphisms* $\mathcal{C}(X, Y)$;
- composition maps $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ and identities $\text{id}_X \in \mathcal{C}(X, X)$,

such that composition is associative and unital. A morphism $f \in \mathcal{C}(X, Y)$ is an *isomorphism* if $gf = \text{id}_X$ and $fg = \text{id}_Y$ for some $g \in \mathcal{C}(Y, X)$.

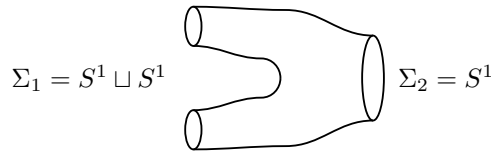
Example 8.2.

- **Set:** sets and maps. **Vect:** complex vector spaces and linear maps.
- A group G : one object, morphisms the elements of G , composition the product. Every morphism is invertible — a *groupoid*.

Definition 8.3. Let Σ_1, Σ_2 be closed oriented $(n-1)$ -manifolds. A *bordism* $M: \Sigma_1 \rightarrow \Sigma_2$ is a compact oriented n -manifold M together with an orientation-preserving diffeomorphism

$$\partial M \cong \overline{\Sigma_1} \sqcup \Sigma_2,$$

where $\overline{\Sigma}$ is Σ with the reversed orientation.



The orientation reversal on the incoming end is what makes gluing orientation-compatible.

Definition 8.4. Bord_n is the category with

- **objects:** closed oriented $(n-1)$ -manifolds Σ ;
- **morphisms** $\Sigma_1 \rightarrow \Sigma_2$: diffeomorphism classes of bordisms $M: \Sigma_1 \rightarrow \Sigma_2$ (the diffeomorphisms being compatible with the identifications of the boundary).

Composition is by gluing: for $M_1: \Sigma_1 \rightarrow \Sigma_2$ and $M_2: \Sigma_2 \rightarrow \Sigma_3$, set

$$M_2 \circ M_1 := M_1 \cup_{\Sigma_2} M_2,$$

as in the picture after Proposition 6.3. The identity on Σ is the cylinder $\Sigma \times [0, 1]$. To glue smoothly one chooses collars of the boundaries; the glued manifold is well defined up to diffeomorphism, which is one reason why the morphisms are diffeomorphism classes.

Example 8.5 ($n = 1$). Objects: finite disjoint unions of points, each $+$ or $-$. A compact oriented 1-manifold is a union of intervals and circles, so the intervals give

$$\begin{array}{ccc}
 \begin{array}{c} + \bullet \text{---} \bullet + \\ \text{id}_+ \end{array} & \begin{array}{c} + \bullet \text{---} \bullet - \\ \text{ev} \end{array} & \begin{array}{c} - \bullet \text{---} \bullet + \\ \text{coev} \end{array}
 \end{array}$$

Put $V = Z(+)$, $W = Z(-)$ for a TQFT Z as in Section 10, and read each boundary top to bottom: $\text{ev}: V \otimes W \rightarrow \mathbb{C}$ and $\text{coev}: \mathbb{C} \rightarrow W \otimes V$. In Bord_1 :

$$\begin{array}{c} + \bullet \text{---} \bullet + \\ \text{id}_+ \end{array} = \begin{array}{c} + \bullet \text{---} \bullet + \end{array}$$

Exercise 8.6. Deduce from this identity that $\dim_{\mathbb{C}} V < \infty$, and compute $Z(S^1)$. (This is taken up in Section 11.)

Definition 8.7. A *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ assigns to every object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$ and to every morphism $f: X \rightarrow Y$ a morphism $F(f): F(X) \rightarrow F(Y)$, such that

$$F(g \circ f) = F(g) \circ F(f), \quad F(\text{id}_X) = \text{id}_{F(X)}.$$

Definition 8.8. A *natural transformation* $\eta: F \Rightarrow G$ between functors $\mathcal{C} \rightarrow \mathcal{D}$ is a family $\eta_X: F(X) \rightarrow G(X)$ such that

$$\begin{array}{ccc}
 F(X) & \xrightarrow{\eta_X} & G(X) \\
 F(f) \downarrow & & \downarrow G(f) \\
 F(Y) & \xrightarrow{\eta_Y} & G(Y)
 \end{array}$$

commutes for every $f: X \rightarrow Y$; a *natural isomorphism* if every η_X is an isomorphism.

9 Symmetric monoidal categories

Bord_n carries more structure: disjoint union of manifolds and of bordisms,

$$\sqcup: \text{Bord}_n \times \text{Bord}_n \longrightarrow \text{Bord}_n, \quad \text{unit } \emptyset.$$

But $(\Sigma_1 \sqcup \Sigma_2) \sqcup \Sigma_3$ and $\Sigma_1 \sqcup (\Sigma_2 \sqcup \Sigma_3)$ are *not* the same manifold — they are canonically diffeomorphic. Likewise $\emptyset \sqcup \Sigma \cong \Sigma$.

The tensor product of vector spaces behaves the same way: $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are different vector spaces, identified by the canonical isomorphism $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$; and $\mathbb{C} \otimes V \cong V$ rather than $= V$. So $\sqcup$ and $\otimes_{\mathbb{C}}$ are associative and unital only up to coherent isomorphism. That is precisely the notion of a monoidal category.

Definition 9.1. A *monoidal category* is a category $\mathcal{C}$ with a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object $\mathbf{1} \in \mathcal{C}$, and natural isomorphisms

$$\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \quad \lambda_X: \mathbf{1} \otimes X \xrightarrow{\sim} X, \quad \rho_X: X \otimes \mathbf{1} \xrightarrow{\sim} X,$$

subject to the *pentagon* and *triangle* identities:

$$\begin{array}{ccc}
 & (W \otimes X) \otimes (Y \otimes Z) & \\
 \alpha \nearrow & & \searrow \alpha \\
 ((W \otimes X) \otimes Y) \otimes Z & & W \otimes (X \otimes (Y \otimes Z)) \\
 \alpha \otimes \text{id} \downarrow & & \uparrow \text{id} \otimes \alpha \\
 (W \otimes (X \otimes Y)) \otimes Z & \xrightarrow{\alpha} & W \otimes ((X \otimes Y) \otimes Z)
 \end{array}$$

$$\begin{array}{ccc}
(X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\alpha} & X \otimes (\mathbf{1} \otimes Y) \\
& \searrow \rho \otimes \text{id} \quad \swarrow \text{id} \otimes \lambda & \\
& X \otimes Y &
\end{array}$$

Example 9.2. Vect with $\otimes_{\mathbb{C}}$ and $\mathbf{1} = \mathbb{C}$; Bord $_n$ with $\sqcup$ and $\mathbf{1} = \emptyset$; Set with $\times$ and $\mathbf{1} = \{*\}$.

Theorem 9.3 (Mac Lane's coherence theorem). *Pentagon and triangle force every formal diagram built from α , λ , ρ to commute; equivalently, every monoidal category is equivalent to a strict one.*

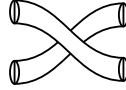
We do not prove this. So we suppress the associators from here on and write $X \otimes Y \otimes Z$.

Definition 9.4. A monoidal category is *symmetric* if it carries a natural isomorphism $\tau_{X,Y}: X \otimes Y \xrightarrow{\sim} Y \otimes X$ with $\tau_{Y,X} \circ \tau_{X,Y} = \text{id}_{X \otimes Y}$, subject to the *hexagon* identities (associators suppressed):

$$\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{\tau_{X,Y} \otimes \text{id}} & Y \otimes X \otimes Z \\
& \searrow \tau_{X,Y \otimes Z} & \downarrow \text{id} \otimes \tau_{X,Z} \\
& & Y \otimes Z \otimes X
\end{array}
\quad
\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{\text{id} \otimes \tau_{Y,Z}} & X \otimes Z \otimes Y \\
& \searrow \tau_{X \otimes Y, Z} & \downarrow \tau_{X,Z} \otimes \text{id} \\
& & Z \otimes X \otimes Y
\end{array}$$

Remark 9.5. For a *braiding* both hexagons are axioms; in the symmetric case the second follows from the first together with $\tau^2 = \text{id}$, which is why many texts state only one.

Example 9.6. Bord $_n$ is symmetric monoidal: $\otimes = \sqcup$ and $\mathbf{1} = \emptyset$, with $\tau_{\Sigma_1, \Sigma_2}$ the bordism interchanging the two cylinders. Here $\tau^2 = \text{id}$ holds already on diffeomorphism classes.



Example 9.7. Vect is symmetric monoidal:

$$\otimes = \otimes_{\mathbb{C}}, \quad \mathbf{1} = \mathbb{C}, \quad \tau_{V,W}(v \otimes w) = w \otimes v.$$

Definition 9.8. A *symmetric monoidal functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor together with natural isomorphisms

$$\mu_{X,Y}: F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y), \quad \varepsilon: \mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} F(\mathbf{1}_{\mathcal{C}}),$$

compatible with α , λ , ρ and τ :

$$\begin{array}{ccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\alpha} & F(X) \otimes (F(Y) \otimes F(Z)) \\
\mu \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \mu \\
F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\
\mu \downarrow & & \downarrow \mu \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha)} & F(X \otimes (Y \otimes Z))
\end{array}$$

$$\begin{array}{ccc}
\mathbf{1}_{\mathcal{D}} \otimes F(X) & \xrightarrow{\varepsilon \otimes \text{id}} & F(\mathbf{1}_{\mathcal{C}}) \otimes F(X) \\
\lambda_{F(X)} \downarrow & & \downarrow \mu \\
F(X) & \xleftarrow{F(\lambda_X)} & F(\mathbf{1}_{\mathcal{C}} \otimes X)
\end{array}
\quad
\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\mu_{X,Y}} & F(X \otimes Y) \\
\tau_{F(X), F(Y)} \downarrow & & \downarrow F(\tau_{X,Y}) \\
F(Y) \otimes F(X) & \xrightarrow{\mu_{Y,X}} & F(Y \otimes X)
\end{array}$$

(associativity, unit and symmetry). The right unit condition is the mirror image, with ρ in place of λ .

Definition 9.9. A *monoidal natural transformation* $\eta: F \Rightarrow G$ between symmetric monoidal functors is a natural transformation compatible with the structure maps: $\eta_{X \otimes Y} \circ \mu_{X,Y}^F = \mu_{X,Y}^G \circ (\eta_X \otimes \eta_Y)$ and $\eta_{\mathbf{1}} \circ \varepsilon^F = \varepsilon^G$. These are the morphisms of TQFTs in Lecture 2.

10 Atiyah's axioms

Definition 10.1. An n -dimensional topological quantum field theory is a symmetric monoidal functor

$$Z: \text{Bord}_n \longrightarrow \text{Vect}.$$

Unpacking:

Z on objects	the state spaces $Z(\Sigma)$
Z on morphisms	the operators $Z(M)$
functoriality	gluing, $Z(M_2 \circ M_1) = Z(M_2) \circ Z(M_1)$
$Z(\text{id}_\Sigma) = \text{id}$	the cylinder acts trivially
monoidality μ, ε	$Z(\Sigma_1 \sqcup \Sigma_2) \cong Z(\Sigma_1) \otimes Z(\Sigma_2)$, $Z(\emptyset) \cong \mathbb{C}$
symmetry	the swap bordism acts by $v \otimes w \mapsto w \otimes v$
morphisms are classes	diffeomorphism invariance

Compare with the list in Section 7: every item there is now an axiom.

First consequences.

- A closed oriented n -manifold M is a bordism $\emptyset \rightarrow \emptyset$, so

$$Z(M): \mathbb{C} \cong Z(\emptyset) \longrightarrow Z(\emptyset) \cong \mathbb{C}$$

is multiplication by a number: the partition function is recovered.

- Cutting M along a codimension-one Σ computes $Z(M)$ from the pieces — locality, now a theorem rather than a hope.
- $\text{Diff}(\Sigma)$ acts on $Z(\Sigma)$, and isotopic diffeomorphisms act equally: every TQFT produces mapping class group representations. (A diffeomorphism f acts through its mapping cylinder, a bordism $\Sigma \rightarrow \Sigma$.)

Remark 10.2 (what is *not* required). No Hilbert space structure, no positivity, no unitarity, no Hamiltonian, and no metric anywhere. Those are extra conditions — and each of them is a real restriction.

Theorem 10.3 (Quinn '95). *Let X be π -finite and $\omega \in H^n(X; \mathbb{R}/\mathbb{Z})$. Then there is an n -dimensional TQFT $Z_X^\omega: \text{Bord}_n \rightarrow \text{Vect}$ with partition functions as in Definition 4.9. For $\omega = 0$ it is given by*

$$\Sigma \longmapsto \text{Fun}(\text{Map}(\Sigma, X)), \quad Z_X(M)(\psi)(f_2) = \int_{\substack{\Phi \in \text{Map}(M, X) \\ \Phi|_{\Sigma_2} \cong f_2}} \psi(\Phi|_{\Sigma_1})$$

— the formulas of Sections 5–6. For $\omega \neq 0$ the state spaces are the sections of the flat line bundle $\mathcal{L}$ of Section 5, and the integrand acquires the phase $e^{2\pi i \int_M \Phi^* \omega}$, defined using a cocycle representative of ω .

Quinn's *finite total homotopy TQFT* (1995). For $X = BG$ with G finite it is Dijkgraaf–Witten (1990) and Freed–Quinn (1993); for the modern and extended treatment see Freed–Hopkins–Lurie–Teleman (2010). For the fully extended version: ongoing work by Scheimbauer and Walde.

Lecture 2: Basic properties and defects

11 Dualizable objects and 1-dimensional TQFTs

Bord_1 is generated, under $\sqcup$ and composition, by the two points $+$ and $-$ and the bordisms

$$\begin{array}{ccc}
 \begin{array}{c} + \\ \cdot \\ \text{---} \\ \cdot \\ - \end{array} & \begin{array}{c} \cdot \\ \text{---} \\ \cdot \\ + \end{array} & \begin{array}{c} + \quad - \\ \cdot \quad \cdot \\ \diagdown \quad \diagup \\ \cdot \quad \cdot \\ - \quad + \end{array} \\
 \text{ev}: + \sqcup - \rightarrow \emptyset & \text{coev}: \emptyset \rightarrow - \sqcup + & \tau: + \sqcup - \rightarrow - \sqcup +
 \end{array}$$

(time runs from left to right, and each boundary is read from top to bottom), subject to the relations

$$\begin{array}{ccc}
 \begin{array}{c} + \\ \cdot \\ \text{---} \\ \cdot \\ + \end{array} & = & + \cdot + \\
 \begin{array}{c} \cdot \\ \text{---} \\ \cdot \\ - \end{array} & = & - \cdot -
 \end{array}$$

In addition the circle is the composite

$$\bigcirc = \bigcirc$$

i.e. $\text{ev} \circ \tau_{-,+} \circ \text{coev}: \emptyset \rightarrow \emptyset$, with $\tau_{-,+}: - \sqcup + \rightarrow + \sqcup -$ the symmetry. This is *not* an additional relation: it follows from the relations above together with the fact that Bord_1 is symmetric monoidal.

Theorem 11.1. *1-dimensional TQFTs $Z: \text{Bord}_1 \rightarrow \text{Vect}$ are classified by finite-dimensional vector spaces: $Z \mapsto Z(+)$ is an equivalence from the groupoid of 1-dimensional TQFTs to the groupoid of finite-dimensional vector spaces and linear isomorphisms.*

Proof sketch. A TQFT Z is determined by $V = Z(+)$ and $V' = Z(-)$ together with the maps

$$Z(\text{ev}): V \otimes V' \rightarrow \mathbb{C}, \quad Z(\text{coev}): \mathbb{C} \rightarrow V' \otimes V,$$

and the relations say

$$\begin{aligned}
 \left(V \xrightarrow{\text{id} \otimes Z(\text{coev})} V \otimes V' \otimes V \xrightarrow{Z(\text{ev}) \otimes \text{id}} V \right) &= \text{id}_V, \\
 \left(V' \xrightarrow{Z(\text{coev}) \otimes \text{id}} V' \otimes V \otimes V' \xrightarrow{\text{id} \otimes Z(\text{ev})} V' \right) &= \text{id}_{V'}.
 \end{aligned}$$

So V' is a dual of V in the sense of Definition 11.2, and Exercise 11.3 finishes the argument. This uses the presentation of Bord_1 above, which we do not prove. $\square$

Definition 11.2. An object V of a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is *dualizable* if there are an object $V^\vee$ and morphisms $\text{ev}: V \otimes V^\vee \rightarrow \mathbf{1}$ and $\text{coev}: \mathbf{1} \rightarrow V^\vee \otimes V$ such that

$$(\text{ev} \otimes \text{id}_V) \circ (\text{id}_V \otimes \text{coev}) = \text{id}_V, \quad (\text{id}_{V^\vee} \otimes \text{ev}) \circ (\text{coev} \otimes \text{id}_{V^\vee}) = \text{id}_{V^\vee}.$$

Exercise 11.3. The dualizable objects of Vect are exactly the finite-dimensional vector spaces, with $V^\vee := \text{Hom}(V, \mathbb{C})$ and

$$\text{ev}: V \otimes V^\vee \rightarrow \mathbb{C}, \quad v \otimes f \mapsto f(v), \quad \text{coev}: \mathbb{C} \rightarrow V^\vee \otimes V, \quad 1 \mapsto \sum_i e^i \otimes e_i$$

for a basis (e_i) of V with dual basis (e^i) .

Morphisms of TQFTs are symmetric monoidal natural transformations (Definition 9.9).

Exercise 11.4. Every symmetric monoidal natural transformation between 1-dimensional TQFTs is invertible.

12 Dimensional reduction

Let Y be a closed oriented ℓ -dimensional manifold. Then

$$(-) \times Y: \text{Bord}_k \longrightarrow \text{Bord}_{k+\ell}, \quad \Sigma \mapsto \Sigma \times Y, \quad M \mapsto M \times Y$$

is a symmetric monoidal functor.

Definition 12.1. Let $Z: \text{Bord}_{k+\ell} \rightarrow \text{Vect}$ be a TQFT. Its *dimensional reduction along Y* is

$$\text{red}_Y Z: \text{Bord}_k \xrightarrow{(-) \times Y} \text{Bord}_{k+\ell} \xrightarrow{Z} \text{Vect}, \quad \Sigma \mapsto Z(\Sigma \times Y).$$

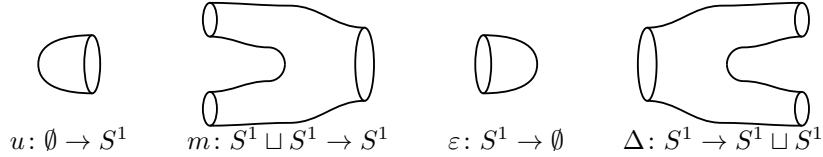
Corollary 12.2.

- (i) *The state spaces of a TQFT are finite-dimensional.*
- (ii) *The category of TQFTs is a groupoid, i.e. every morphism is invertible.*

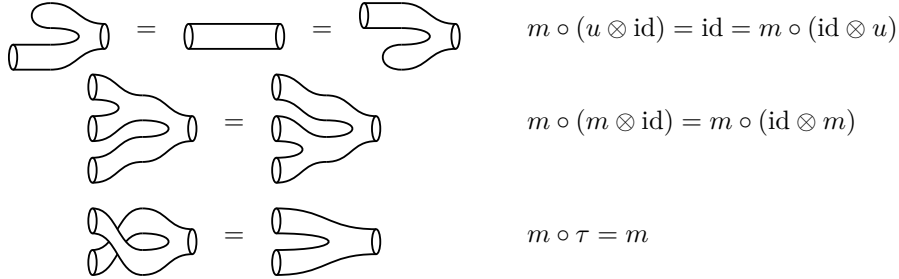
Proof. Let Σ be a closed $(n-1)$ -manifold. Then $\text{red}_\Sigma Z$ is a 1-dimensional TQFT with $(\text{red}_\Sigma Z)(+) = Z(\Sigma)$, so (i) follows from Theorem 11.1. For (ii), the component η_Σ of a morphism $\eta: Z \Rightarrow Z'$ is the component at $+$ of $\text{red}_\Sigma \eta: \text{red}_\Sigma Z \Rightarrow \text{red}_\Sigma Z'$, which is invertible by Exercise 11.4. $\square$

13 The 2-dimensional classification

Generators:



together with the cylinder id_{S^1} and the symmetry τ . Relations: unit, associativity and commutativity,



their reflections,

$$(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta, \quad (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \quad \tau \circ \Delta = \Delta,$$

and the Frobenius relation

$$\Delta \circ m = (m \otimes \text{id}) \circ (\text{id} \otimes \Delta) = (\text{id} \otimes m) \circ (\Delta \otimes \text{id}).$$

Theorem 13.1. $Z \mapsto Z(S^1)$ is an equivalence between the groupoid of 2-dimensional TQFTs and the groupoid of commutative Frobenius algebras. (Dijkgraaf; Abrams; see Kock's book.)

Here $A = Z(S^1)$ has multiplication $Z(m)$, unit $Z(u)$, comultiplication $Z(\Delta)$ and counit $Z(\varepsilon)$. The pairing $Z(\varepsilon) \circ Z(m): A \otimes A \rightarrow \mathbb{C}$ is non-degenerate: by the Frobenius and counit relations the cylinder is the zig-zag composite of the two elbows $\varepsilon \circ m: S^1 \sqcup S^1 \rightarrow \emptyset$ and $\Delta \circ u: \emptyset \rightarrow S^1 \sqcup S^1$, so S^1 is self-dual in the sense of Definition 11.2.

14 Computations for finite homotopy theories

Let X be π -finite and connected, and take $\omega = 0$.

Definition 14.1. For maps $f: A \rightarrow C$ and $g: B \rightarrow C$, the *homotopy pullback* $A \times_C^h B$ is the space of triples (a, b, γ) with $a \in A$, $b \in B$ and γ a path from $f(a)$ to $g(b)$ in C ; for groupoids, γ is an isomorphism $f(a) \rightarrow g(b)$. The homotopy fibre of f over $c \in C$ is $A \times_C^h \{c\}$.

The homotopy fibre of r_2 over $f_2 \in \text{Map}(\Sigma_2, X)$ is what the integral in Definition 6.1 really runs over.

Example 14.2. $* \times_X^h * \simeq \Omega X$, $X \times_{X \times X}^h X \simeq \mathcal{L}X$ (along the diagonal), $* \times_{BG}^h * \simeq G$.

Remark 14.3 (spaces versus groupoids). We pass freely between spaces and groupoids. For 1-types this is a theorem: sending a space Y to its fundamental groupoid $\Pi_1(Y)$, and a groupoid $\mathcal{G}$ to its classifying space $B\mathcal{G}$, gives an equivalence between spaces with $\pi_i = 0$ for $i \geq 2$ and groupoids, each up to equivalence. For example BG corresponds to the groupoid with one object and automorphism group G , and $\text{Map}(S^1, BG)$ to $G//G$. Grothendieck's *homotopy hypothesis* is the slogan that in the same way all spaces are “ ∞ -groupoids”, the higher homotopy groups being higher automorphisms. For us this means two things: homotopy pullbacks of 1-types are computed by the groupoid version of Definition 14.1, and anything valued in a 1-category such as Vect (as in Section 21) only sees the fundamental groupoid.

The circle.

$$Z(S^1) = \text{Fun}(\text{Map}(S^1, X)) \cong \text{Fun}(\pi_0 \mathcal{L}X) \cong \text{Fun}(\pi_1(X)/\pi_1(X)) \cong \text{Cl}(\pi_1(X)),$$

with $\pi_1(X)$ acting on itself by conjugation. To see this, use the fibre sequence

$$\Omega X \longrightarrow \mathcal{L}X = \text{Map}(S^1, X) \longrightarrow X,$$

the second map evaluating at the base point. On π_0 it gives $\pi_1(X) = \pi_0(\Omega X) \twoheadrightarrow \pi_0(\mathcal{L}X) \rightarrow \pi_0(X) = *$, and two elements of $\pi_1(X)$ have the same image iff they are conjugate: free homotopy classes of loops are conjugacy classes. For $X = BG$ this is $G \rightarrow G//G \rightarrow BG$ (Example 3.1).

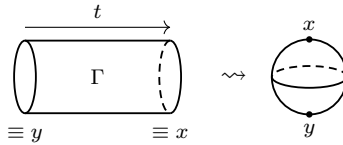
The disc. Read as $D^2: \emptyset \rightarrow S^1$. Since $\text{Map}(D^2, X) \simeq X$ and restriction to the boundary is the inclusion of constant loops, the span is

$$* \longleftarrow X \xrightarrow{\text{const}} \mathcal{L}X,$$

so $Z(D^2): \mathbb{C} \rightarrow Z(S^1)$ sends 1 to the function f with $f([\gamma]) = |\Omega^2 X| \delta_{[\gamma], [\text{const}]}$. Here $|\Omega^2 X|$ is the homotopy cardinality of the homotopy fibre of $\text{const}: X \rightarrow \mathcal{L}X$ over the constant loop c_x at $x \in X$ (this fibre is what the integral in Definition 6.1 runs over), and this fibre is the double loop space:

Lemma 14.4. $\text{hofib}_{c_x}(X \xrightarrow{\text{const}} \mathcal{L}X) \cong \Omega_x^2 X := \{\sigma: S^2 \rightarrow X \mid \sigma(\text{north pole}) = x\}$.

Proof. By Definition 14.1, a point of the homotopy fibre is a pair (y, Γ) of a point $y \in X$ and a path Γ in $\mathcal{L}X$ from the constant loop c_y to c_x . A path of loops is a map $\Gamma: [0, 1] \times S^1 \rightarrow X$, $(t, s) \mapsto \Gamma_t(s)$, with $\Gamma_0 \equiv y$ and $\Gamma_1 \equiv x$; in particular $y = \Gamma_0(s)$ is determined by Γ . A map out of the cylinder that is constant on each of its two boundary circles is the same as a map out of the cylinder with both boundary circles collapsed to points, and this quotient is the 2-sphere:



So the homotopy fibre is the space of maps $S^2 \rightarrow X$ sending the north pole to x , with the south pole free. This is $\Omega_x^2 X$. $\square$

Abstractly, this is the statement that the homotopy fibre of a section of a fibration is the loop space of the fibre: here const is a section of $\text{ev}_0: \mathcal{L}X \rightarrow X$, whose fibre over x is $\Omega_x X$, so the homotopy fibre is $\Omega(\Omega_x X)$. For the cardinality, $\pi_k(\Omega^2 X) = \pi_{k+2}(X)$, so $|\Omega^2 X| = |\pi_2 X| |\pi_3 X|^{-1} |\pi_4 X| \cdots$, and for 1-types such as BG the space $\Omega^2 X$ is contractible and the factor is 1. Read as $S^1 \rightarrow \emptyset$, the disc gives $f \mapsto \int_{\text{Map}(D^2, X)} f(\Phi|_{S^1}) = |X| f([\text{const}])$.

The pair of pants. $P \simeq S^1 \vee S^1$, so $\text{Map}(P, X) \simeq \mathcal{L}X \times_X \mathcal{L}X$ (two loops at a common base point), and the span is

$$\mathcal{L}X \times \mathcal{L}X \xleftarrow{(\gamma_1, \gamma_2)} \mathcal{L}X \times_X \mathcal{L}X \xrightarrow{\gamma_1 * \gamma_2} \mathcal{L}X.$$

Hence $Z(P)$ is a convolution product,

$$(f * g)(\gamma) = \int_{\{(\gamma_1, \gamma_2) : \gamma_1 * \gamma_2 \simeq \gamma\}} f(\gamma_1) g(\gamma_2),$$

the integral taken over the homotopy fibre (Definition 4.7).

Dijkgraaf–Witten theories. For $X = BG$,

$$Z^{DW}(S^1) = \{f : G \rightarrow \mathbb{C} \mid f(hgh^{-1}) = f(g)\} = \text{Cl}(G).$$

The pair of pants gives the span

$$G // G \times G // G \longleftarrow (G \times G) // G \longrightarrow G // G, \quad (g_1, g_2) \longmapsto g_1 g_2,$$

hence the convolution product

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2),$$

and the disc read as $S^1 \rightarrow \emptyset$ gives

$$f \longmapsto \frac{1}{|G|} f(e).$$

Together with the unit δ_e from the disc read as $\emptyset \rightarrow S^1$, this makes $\text{Cl}(G)$ — the centre of the group algebra $\mathbb{C}[G]$ — a commutative Frobenius algebra, as Theorem 13.1 says it must be.

15 Topological defects

Including topological defects:

$$\text{Bord}_n \rightsquigarrow \text{Bord}_n^D \quad \begin{array}{|c|c|} \hline & D \\ \hline Z_1 & Z_2 \\ \hline \end{array}$$

— bordisms carry codimension-1 submanifolds, the regions are labelled by theories and the walls by defects between them.

Semi-classical defects for finite homotopy theories

A defect between (X_1, ω_1) and (X_2, ω_2) is a span

$$\begin{array}{ccc} & (Y, \theta) & \\ s \swarrow & & \searrow t \\ (X_1, \omega_1) & & (X_2, \omega_2) \end{array} \quad \text{with} \quad d\theta = s^* \omega_1 - t^* \omega_2.$$

Here the ω_i have to be cocycles rather than cohomology classes, and θ is an $(n-1)$ -cochain on the π -finite space Y .

For $M = M_1 \cup_\Sigma M_2$ with the defect along Σ , the fields are the homotopy pullback

$$\begin{array}{ccc} \begin{array}{|c|c|} \hline \Sigma \\ \hline M_1 & M_2 \\ \hline \end{array} & \begin{array}{ccc} F & \xrightarrow{\quad} & \text{Map}(\Sigma, Y) \\ \downarrow & & \downarrow (s,t) \circ - \\ \text{Map}(M_1, X_1) \times \text{Map}(M_2, X_2) & \xrightarrow{\quad} & \text{Map}(\Sigma, X_1 \times X_2) \end{array} \end{array}$$

and

$$Z(M_1 \cup_\Sigma M_2) = \int_{(f_1, f_2, f_\Sigma) \in F} \exp \left(2\pi i \left[\int_{M_1} f_1^* \omega_1 + \int_{M_2} f_2^* \omega_2 + \int_\Sigma f_\Sigma^* \theta \right] \right).$$

Examples in Dijkgraaf–Witten theory

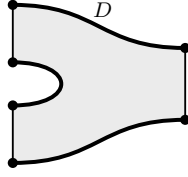
The *Dirichlet* boundary condition D is the span $* \leftarrow * \rightarrow BG$, a defect between the trivial theory and Dijkgraaf–Witten theory. For M with boundary Σ carrying D ,

$$\begin{array}{ccc} \text{Diagram of } M \text{ with boundary } \Sigma & & \begin{array}{ccc} F & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow \\ \text{Map}(M, BG) & \longrightarrow & \text{Map}(\Sigma, BG) \end{array} \end{array}$$

so $F = \{G\text{-bundles on } M \text{ with a trivialisation on } \Sigma\}$. For $n = 2$: the interval with D at both ends, $\xrightarrow{D} \xrightarrow{D}$, has $F \simeq G$, so

$$Z^{DW}(\xrightarrow{D} \xrightarrow{D}) = \mathcal{O}(G),$$

and the bordism



(merging two such intervals into one; the thick part of the boundary carries D) gives the convolution product on $\mathcal{O}(G)$.

Composition of defects

Parallel defects fuse:

$$\begin{array}{|c|c|c|} \hline D_1 & D_2 & \\ \hline Z_1 & Z_2 & Z_3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline D_1 \otimes_{Z_2} D_2 & \\ \hline Z_1 & Z_3 \\ \hline \end{array}$$

For semi-classical defects the composite is the homotopy pullback of spans,

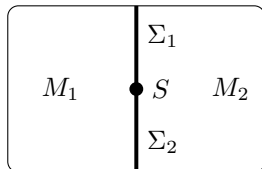
$$\begin{array}{ccccc} & & Y_1 \times_{X_2}^h Y_2 & & \\ & \swarrow & & \searrow & \\ & Y_1 & & Y_2 & \\ \swarrow & & & & \searrow \\ X_1 & & X_2 & & X_3 \end{array}$$

with the pulled-back cochains added.

Example 15.1. Fusing two Dirichlet boundaries gives $* \times_{BG}^h * \simeq G$:

$$\begin{array}{ccccc} & & G & & \\ & \swarrow & & \searrow & \\ & * & & * & \\ \swarrow & & & & \searrow \\ * & & BG & & * \end{array}$$

16 Codimension-2 defects



$$\begin{array}{ccccc} & & (Y_1, \theta_1) & & \\ & \swarrow & \uparrow r_1 & \searrow & \\ X_1 & & (W, \lambda) & & X_2 \\ & \swarrow & \downarrow r_2 & \searrow & \\ & & (Y_2, \theta_2) & & \end{array} \quad d\lambda = r_2^* \theta_2 - r_1^* \theta_1.$$

The fields F are the homotopy limit of the diagram of restriction maps

$$\begin{array}{ccccc}
 & & \text{Map}(M_1, X_1) \times \text{Map}(M_2, X_2) & & \\
 & \swarrow & & \searrow & \\
 \text{Map}(\Sigma_1, X_1 \times X_2) & & & & \text{Map}(\Sigma_2, X_1 \times X_2) \\
 \uparrow & & & & \uparrow \\
 \text{Map}(\Sigma_1, Y_1) & & & & \text{Map}(\Sigma_2, Y_2) \\
 \searrow & & & \swarrow & \\
 \text{Map}(S, Y_1) & & & & \text{Map}(S, Y_2) \\
 \swarrow & & \text{Map}(S, W) & & \searrow \\
 & \xleftarrow{r_1 \circ -} & & \xrightarrow{r_2 \circ -} &
 \end{array}$$

and

$$Z\left(M_1 \bigcup_{\Sigma_1, \Sigma_2}^S M_2\right) := \int_F e^{2\pi i S[-]}.$$

Definition 16.1. A pair of boundary conditions D, N is *transverse* if

$$\begin{array}{|c|} \hline D \quad N \\ \hline X \\ \hline \end{array} = \mathbf{1}.$$

Here $\mathbf{1}$ is the trivial $(n-1)$ -dimensional theory: the slab with D and N on its two sides, viewed as an $(n-1)$ -dimensional theory, is trivial.

Example 16.2. For $X = BG$, $D = (* \leftarrow * \rightarrow BG)$ and the Neumann boundary condition $N = (BG \leftarrow BG \rightarrow *)$ are transverse:

$$\begin{array}{ccccc}
 & & * & & \\
 & \swarrow & & \searrow & \\
 & * & & BG & \\
 \swarrow & & & & \searrow \\
 * & & BG & & *
 \end{array} \quad * \times_{BG}^h BG \simeq *.$$

By contrast, D is not transverse to itself: by Example 15.1 the sandwich gives G .

Lecture 3: Extended operator–state correspondence for TQFTs (extended TQFTs)

17 Point operators and states

In a TQFT, cut a small ball D out of a closed n -manifold M around an inserted point operator $\mathcal{O}$:

$$\begin{array}{c} M \\ \text{---} \mathcal{O} \text{---} \\ \text{---} D \text{---} \end{array} \rightsquigarrow |\mathcal{O}\rangle \in Z(S^{n-1}),$$

the state obtained by evaluating Z on the ball with the operator inside. The correlation function is recovered by gluing the rest of M back on:

$$Z(M) = Z(M \setminus D)[|\mathcal{O}\rangle] \in \mathbb{C}. \quad (7)$$

This gives a map

$$\{\text{point operators}\} \longrightarrow Z(S^{n-1}).$$

- **Injective:** if two operators generate the same state, they agree in all correlation functions, hence are physically the same.
- **Surjective:** given a state $\Psi \in Z(S^{n-1})$, equation (7) defines an operator.

So on physical grounds there is a one-to-one correspondence

$$\Psi \in Z(S^{n-1}) \longleftrightarrow \text{point operators}.$$

Q: Does this generalise to extended operators?

18 Extended operators and tubular neighbourhoods

Let $L \subset M$ be a line operator with tubular neighbourhood T_L :

$$\begin{array}{c} M \\ \text{---} L \text{---} \\ \text{---} T_L \text{---} \end{array} \rightsquigarrow |L\rangle \in Z(\partial T_L), \quad Z(M, L) = Z(M \setminus T_L)[|L\rangle].$$

This is *not* a bijection:

$$Z\left(\left(\begin{array}{c} \text{---} \\ \text{---} L \text{---} \end{array}\right)\right) = \dim Z(N, p) \in \mathbb{N},$$

where the manifold is $N \times S^1$ for a closed $(n-1)$ -manifold N and the defect runs along $L = \{p\} \times S^1$ (drawn for $n=2$, $N=S^1$). Here $Z(N, p)$ is the state space of N with the point p where L crosses it, and the value on $N \times S^1$ is the trace of its identity map, i.e. its dimension; so one cannot multiply L by a scalar. There is more to L than just the correlation functions; e.g. in 2D

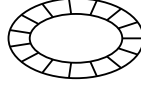
$$Z\left(\begin{array}{c} \bullet \\ \text{---} L \text{---} \bullet \end{array}\right) \in \text{Vect},$$

the vector space $\text{Hom}(L, L)$ of point operators on L : a small circle around a point of L meets L in two points, and its state space is, by the operator–state correspondence, the space of point operators sitting on L .

We would like to identify $|L\rangle \in Z(\partial T_L)$ with a state in $Z(S^{n-2} \times S^1)$. The ways of doing so are equivalent to the choice of an identification

$$T_L \cong D^{n-1} \times S^1$$

preserving the orientation and the position of L at $\{0\} \times S^1$. In 3D these are parametrized by *framings* — turning L into a ribbon:



The identification takes the meridian A to the contractible cycle and B to the boundary of the ribbon. They all differ by Dehn twists along A , i.e. by the winding of the ribbon.

Observation. In 3D, $|L\rangle$ is invariant under these choices:

$$|L\rangle \in Z(T^2)^{\mathbb{Z}}, \quad T^2 = S^1 \times S^1,$$

with $\mathbb{Z}$ acting by the Dehn twists. Hence the choice does not matter. Alternatively, *framed* line defects lead to (arbitrary) states in $Z(T^2)$.

19 The tubular neighbourhood category

Let us generalise this. Let D be a k -dimensional manifold. For an embedding $D \hookrightarrow M$ there always exists a tubular neighbourhood, i.e. an open $V \subseteq M$ with $V \cong N(D)$, the normal bundle. The disc bundle $D(N(D))$ is a bordism

$$\emptyset \longrightarrow S(N(D)).$$

Definition 19.1. $\text{TubN}(D)$ is the (higher) category with

- **objects:** the possible ways φ of viewing $D(N(D))$ as a bordism $\emptyset \rightarrow S(N(D))$;
- **morphisms:** diffeomorphisms of $D(N(D))$ preserving D .

This is a subcategory of the slice category under $\emptyset$ in Bord_n^D (a diffeomorphism is sent to its mapping cylinder).

Now any TQFT defines a functor $Z_D: \text{TubN}(D) \rightarrow \text{Vect}$,

$$\begin{array}{ccc} \varphi & \longmapsto & Z(S(N(D))_{\varphi}) \\ f \downarrow & & \downarrow Z(f) \\ \varphi' & \longmapsto & Z(S(N(D))_{\varphi'}) \end{array}$$

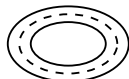
and the disc bundle gives compatible states

$$\varphi \longmapsto |D\rangle_{\varphi} := Z(D(N(D))_{\varphi}) \in Z(S(N(D))_{\varphi}), \quad Z(f)(|D\rangle_{\varphi}) = |D\rangle_{\varphi'}.$$

Hence an extended operator supported on D gives an element of $\lim_{\text{TubN}(D)} Z_D$. (These are what correspond to global defects.)

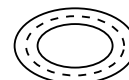
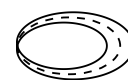
Example 19.2.

- In 2D, for a point: $\text{TubN}(*) \simeq B\text{Diff}^+(S^1)$. Since $\text{Diff}^+(S^1)$ is connected, $\lim_{\text{TubN}(*)} Z_* = Z(S^1)$ and we are good.
- In general, for a point in dimension n : $\text{TubN}(*) \simeq B\text{Diff}^+(D^n)$, so $\lim_{\text{TubN}(*)} Z_*$ is the subspace of $Z(S^{n-1})$ fixed by $\pi_0 \text{Diff}^+(D^n)$. In 4D it is not known whether $\text{Diff}^+(D^4)$ is connected, so even for point operators we might need to take fixed points. (The corresponding statement is true for PL and topological manifolds.)

- In 2D, for a circle: $\text{TubN}(S^1) \ni$  is discrete, so line operators are elements of

$$Z(S(N(S^1))) = Z(S^1 \sqcup S^1) \cong Z(S^1) \otimes Z(S^1).$$

- Let us drop the orientation for a moment. The unoriented version is

$$\text{TubN}(S^1) = \text{} \circlearrowleft \mathbb{Z}_2 \sqcup \text{} \circlearrowright \mathbb{Z}_2,$$

so line defects are

$$\mathrm{Sym}^2 Z(S^1) \oplus Z(S^1)^{\mathbb{Z}_2}.$$

For the annulus, $\mathbb{Z}_2$ swaps the two boundary circles, and the invariants in $Z(S^1) \otimes Z(S^1)$ are $\mathrm{Sym}^2 Z(S^1)$. For the Möbius band the sphere bundle is a single circle, the connected double cover of the core, and $\mathbb{Z}_2$ acts on it by the deck transformation.

- Now in 3D, for a circle: $\mathrm{TubN}(S^1) = \text{(two concentric circles)} \circlearrowleft \mathbb{Z}$, so lines are $Z(S^1 \times S^1)^{\mathbb{Z}}$.

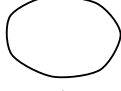
Framing data := a trivialization of the tubular neighbourhood. Then

$$\mathrm{TubN}^{\mathrm{fr}}(D) \simeq *,$$

so framed defects give states with no invariance condition. If the framing data is just a framing, i.e. a trivialization of the normal bundle, then whether this holds is open already for a point in 4D! (By the previous example it would require $\mathrm{Diff}^+(D^4) \simeq SO(4)$.)

20 Extended TQFT

To make the operator–state correspondence work, extend Z to higher codimension:

M 	Σ 	S 
$\downarrow$	$\downarrow$	$\downarrow$
$Z(M) \in \mathbb{C}$	$Z(\Sigma) \in \mathrm{Vect}$	$Z(S) \in \mathrm{LinCat}, \mathrm{Alg}$

Bordisms between bordisms, i.e. n -manifolds with corners, give natural transformations. For example, for $n = 2$ and S, S' each consisting of two points:

$$\begin{array}{ccc}
 & Z(\text{) } \text{ ()} & \\
 & \curvearrowright & \\
 Z(S) & \Downarrow Z(\text{) } \text{ ()} & Z(S') \\
 & \curvearrowleft & \\
 & Z(\text{ — } \text{ — }) &
 \end{array}$$

21 Finite homotopy theories as extended TQFTs

In finite topological σ -models:

$$Z(S) := \mathrm{Fun}(\mathrm{Map}(S, X), \mathrm{Vect}) = \text{“vector bundles on } \mathrm{Map}(S, X)\text{”}$$

= local systems on $\mathrm{Map}(S, X)$ [flat vector bundles] — here $\mathrm{Map}(S, X)$ is regarded as a groupoid; since Vect is a 1-category only its fundamental groupoid matters (Remark 14.3). (Or in general a “twisted” version depending on ω .)

Example 21.1. $X = BG$, $S = S^1$, $\omega = 0$. Since $\mathrm{Map}(S^1, BG) \simeq G//G$ (conjugation action), an object

$$F \in \mathrm{Fun}(G//G, \mathrm{Vect}) : \quad F(g) \xrightarrow{F(h)} F(hgh^{-1})$$

is a G -equivariant vector bundle on G . Equivalently: for each conjugacy class $[g]$ a representation of its stabilizer (the centralizer of g). Algebra description: this is the category of modules over the Drinfeld double $D(G) = \mathbb{C}(G) \rtimes \mathbb{C}[G]$ — the “Alg” entry of the table in Section 20.

Value on manifolds. A bordism $\Sigma: S_1 \rightarrow S_2$ gives a span

$$\begin{array}{ccc} & \text{Map}(\Sigma, X) & \\ r_1 \swarrow & & \searrow r_2 \\ \text{Map}(S_1, X) & & \text{Map}(S_2, X) \end{array}$$

and

$$Z(\Sigma): F \mapsto Z(\Sigma)[F], \quad Z(\Sigma)[F](f) = \bigoplus_{f_2 \in \pi_0(r_2^{-1}[f])} F(r_1 f_2)^{\text{Aut}(f_2)}.$$

In words: pull back along r_1 , then “integrate” over the homotopy fibre $r_2^{-1}[f]$ of r_2 over $f \in \text{Map}(S_2, X)$ — a direct sum over its components, taking invariants under their automorphisms (over $\mathbb{C}$, invariants and coinvariants of a finite group agree). In dimension n it is a natural transformation whose components are integrals weighted by the action functional, as in Lecture 1.

Example 21.2. The pair of pants : $S^1 \sqcup S^1 \rightarrow S^1$ for $X = BG$:

$$\begin{array}{ccc} G//G \times G//G & \longleftarrow & (G \times G)//G \longrightarrow G//G \\ (g_1, g_2) & \longleftarrow & (g_1, g_2) \longrightarrow g_1 g_2 \end{array}$$

so that

$$V_\bullet \boxtimes W_\bullet \mapsto \left(\bigoplus_{h_1 h_2 = g} V_{h_1} \otimes W_{h_2} \right)_{g \in G}.$$

The homotopy fibres of $(G \times G)//G \rightarrow G//G$ are the discrete sets $\{(h_1, h_2) \mid h_1 h_2 = g\}$, so no invariants appear. This is the convolution product which makes $Z(S^1)$ a monoidal category — in fact a braided one, see Section 22.

Exercise 21.3. Show that for a closed $(n-1)$ -dimensional manifold Σ the functor

$$\text{Vect} \xrightarrow{Z(\Sigma)} \text{Vect}$$

is tensoring with the state space defined in the first lecture. (Here Σ is a bordism $\emptyset \rightarrow \emptyset$ and $Z(\emptyset) = \text{Fun}(*, \text{Vect}) = \text{Vect}$.)

Fully extended: even more codimensions $\implies$ classification and the cobordism hypothesis. (The cobordism hypothesis, conjectured by Baez–Dolan, proof sketched by Lurie: evaluation at the point is an equivalence between framed fully extended TQFTs with values in $\mathcal{C}$ and fully dualizable objects of $\mathcal{C}$.)

22 The extended operator–state correspondence

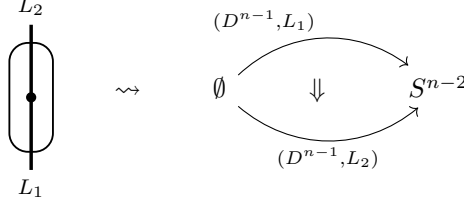
The same formalism applies to defects (many open problems and details are missing! cf. [William Stewart’s thesis](#) (UT Austin, 2024) and others).

For line defects:

$$\begin{array}{c} L \\ \bullet \\ \bigcirc \end{array} \quad \emptyset \xrightarrow{D^{n-1}} S^{n-2}, \quad Z(D^{n-1}, L) \in Z(S^{n-2}).$$

Here D^{n-1} is a small disc transverse to L , viewed as a bordism $\emptyset \rightarrow S^{n-2}$; so $Z(D^{n-1}, L)$ is a functor $\text{Vect} \rightarrow Z(S^{n-2})$, i.e. an object of $Z(S^{n-2})$. Hence there is an operator–state map

$$\begin{aligned} \{\text{line defects}\} &\longrightarrow \text{objects of } Z(S^{n-2}), \\ \{\text{point defects between lines}\} &\longrightarrow \text{morphisms of } Z(S^{n-2}) : \end{aligned}$$



The ball around the point is a 2-morphism $(D^{n-1}, L_1) \Rightarrow (D^{n-1}, L_2)$, so Z of it is a morphism $Z(D^{n-1}, L_1) \rightarrow Z(D^{n-1}, L_2)$ in $Z(S^{n-2})$.

This is again believed to be an equivalence.

$\rightsquigarrow$ **Extended operator–state correspondence:**

$$\left\{ k\text{-category of framed } k\text{-dimensional defects} \right\} \Longleftrightarrow \text{value } Z(S^{n-1-k}).$$

For $k = 0$ this is Section 17, and for line defects ($k = 1$) it is the statement above about $Z(S^{n-2})$. The same subtleties related to TubN apply.

As before, framed defects are in 1-to-1 correspondence with $Z(S^{n-1-k})$, i.e. for line defects with elements (objects) of the category $Z(S^{n-2})$.

Example 22.1. 3D framed line defects $\Longleftrightarrow Z(S^1)$. What if we want to remove the framing?

$$\text{TubN}(\bullet) = \bullet \circ S^1 \mapsto Z(S^1) \begin{array}{c} \xrightarrow{\text{id}} \\ \Downarrow \theta \\ \xrightarrow{\text{id}} \end{array} Z(S^1) \quad \theta : \text{twist}$$

$\text{TubN}(\bullet)$ has one object, the normal disc of the line, acted on by the rotations $S^1 = SO(2)$. The generating loop of S^1 is sent to a natural automorphism θ of $\text{id}_{Z(S^1)}$, the value of Z on the cylinder with a full twist.

$\Rightarrow \lim_{\text{TubN}(\ast)} Z(S^1)$ is the subcategory of objects with trivial twist, i.e. an unframed line defect is an object L with $\theta_L = \text{id}_L$.

For L_1 and L_2 non-framed defects, $\left| \bigotimes_{L_1 \ L_2} \right|$ will in general have a framing dependence:

$$\bigotimes_{L_1 \ L_2} = \theta_{L_1 \otimes L_2} !$$

Indeed, by the balancing axiom $\theta_{L_1 \otimes L_2} = c_{L_2, L_1} \circ c_{L_1, L_2} \circ (\theta_{L_1} \otimes \theta_{L_2})$; with $\theta_{L_1} = \text{id}$ and $\theta_{L_2} = \text{id}$ this is the double braiding, which is not the identity in general. So the fusion of two unframed line defects need not be unframed.

For instance, for $X = BG$ (Example 21.1) the twist acts on the fibre $F(g)$ by the action of $g^{\pm 1}$ (depending on conventions); note g lies in its own centraliser. Wilson lines (F supported at $g = e$) and pure fluxes (trivial representations of the centralisers) are therefore unframed. But for G abelian, fusing the Wilson line of a character ρ with the flux at g gives the line $\mathbb{C}_\rho$ at g , on which the twist is $\rho(g)^{\pm 1}$ — the Aharonov–Bohm phase, which need not be 1.