

A nice example of a theory with discrete gauge symmetries is $N=4$ $su(2)$ SYM in $d=4$.

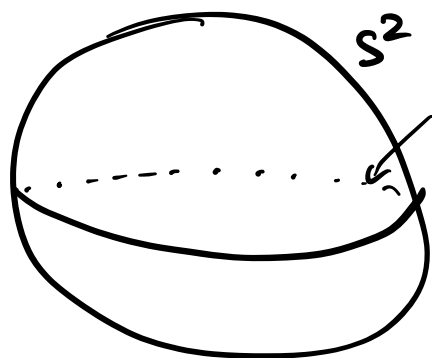
The choice of algebra fully fixes the structure of local correlators in the theory, but the full theory requires the specification of extra data.

$SU(2)$ vs. $SO(3)$

There are two Lie groups with $su(2)$ Lie algebra: $SU(2)$ and $SO(3) = SU(2)/\mathbb{Z}_2$. They differ (for example) in their irreps: $SO(3)$ only contains irreps of integral spin, while $SU(2)$ allows for half-integral spin.

This can be translated into the language of bundles: there are

- $SU(2)$ bundles (these are those bundles to which we can consistently couple dynamical particles in half-integral spin irreps).
- $SO(3)$ bundles (we only require that we couple integral spin particles)



gauge transform: $S^1 \rightarrow G$

up to deformation of the image of the map. This is $\pi_1(G)$.

Exercise:

Show $SU(2) \cong S^3$

We have $\pi_1(SU(2)) = \pi_1(S^3) = 0$

$\pi_1(SO(3)) = \pi_1(SU(2)/\mathbb{Z}_2) = \pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$.

So for $SU(2)$, every bundle on S^2 is trivial.

But we can put non-trivial $SO(3)$ bundles on S^2 , and this is measured by an element in $\mathbb{Z}_2$.

Concretely, the map in the second case, when lifted to $SU(2)$, sends $1 \rightarrow -1$ as we go around the equator. So we cannot define a ^{consistent} coupling of a half-integral spin particle to such a bundle.

For general manifolds, the obstruction to coupling $\frac{1}{2}$ spin particles to a given $SO(3)$ bundle is measured by

$$w_2(E) \in H^2(X; \mathbb{Z}_2) \cong \text{Hom}(H_2(X; \mathbb{Z}_2), \mathbb{Z}_2)$$

(Roughly, this measures when is the bundle trivial when restricted to 2-cycles in X .)

The difference between $SU(2)$ and $SO(3)$ at the level of the path integral is then:

$$\mathcal{Z}_{SU(2)}[\mu^4; z] = \sum_{\substack{w_2(E) \\ = 0}} \int \underset{\substack{\uparrow \\ \text{connections} \\ \text{in } E}}{[DA]} e^{-S_{SU(2)}^{N=4}[\tau, A]} \quad \swarrow \text{ignoring fermions and scalars}$$

$$\mathcal{Z}_{SO(3)}[\mu^4; z] = \sum_E \int [DA] e^{-S_{SU(2)}^{N=4}[\tau, A]}$$

In fact, because in the second term we are including bundles with $w_2(E) \neq 0$, we can introduce a variant:

$$\mathcal{Z}_{SO(3)\pm}[\mu^4; z] = \sum_E \int [DA] e^{-S_{SU(2)}^{N=4}[\tau, A]} (\pm 1)^{P(w_2(E))/2}$$

where $P: H^2(X; \mathbb{Z}_2) \rightarrow H^2(X; \mathbb{Z}_4)$ is the "Pontryagin square".

In simple cases $P(x) = \bar{x}^2 \bmod 4$, with $\bar{x} \in H^2(X; \mathbb{Z})$ such that $x = \bar{x} \bmod 2$. (In general $P(x) = \bar{x} \cup \bar{x} + \bar{x} \cup \delta \bar{x} \bmod 4$)

Montonen-Olive duality

It is believed (supported by a lot of very convincing evidence) that $d=4$ $N=4$ SYM has Montonen-Olive duality, which is the statement

$$N=4 \text{ with } G \text{ and } \tau \stackrel{\frac{\theta}{2\pi} + \frac{4\pi i}{g^2}}{\longleftrightarrow} N=4 \text{ with } {}^L G \text{ and } -\frac{1}{\tau}$$

with ${}^L G$ the Langlands dual of G :

G	$U(1)$	$SU(2)$	$SU(N)$	$USp(2N)$	...
${}^L G$	$U(1)$	$SO(3)_+$	$SU(N)_{\mathbb{Z}_N}$	$SO(2N+1)$	...

Additionally $SO(3)_{-[\tau]} = SO(3)_{-[-1/\tau]}$

(For an impressive test of these relations on manifolds such as $K3$, see Vafa, Witten [th/9408074](#), where they compute $\sum_{K3} [SU(2), \tau] = (\dots) \sum_{K3} [SO(3)_+, -1/\tau].$)

For these lectures, the moral is that there is no "canonical" choice of global form for $N=4$ YM theory, given just its algebra.

A similar story holds for $SU(N)$: we can choose $SU(N)$ or $PSU(N) := \frac{SU(N)}{\mathbb{Z}_N}$ (or more generally $\frac{SU(N)}{\mathbb{Z}_k}$ with $k|N$), together with some choices of SPTs to couple to, and they are related in involved ways

→ Aharony, Seiberg, Tachikawa 1705.0318

So there is no canonical choice.

AdS/CFT

I will start with the basic form of AdS/CFT:

$$\begin{array}{c} su(N) \mathcal{N}=4 \text{ SYM} \\ \text{on } S^4 \end{array} = \begin{array}{c} \text{IIB string theory} \\ \text{on } \text{AdS}^5 \times S^5 \\ \text{with } \int_{S^5} F_5 = N \end{array}$$

The basic question in our context: the algebra for the SCFT side is $su(N)$, but which global form should we choose? $[su(N), su(N)/\mathbb{Z}_N, \dots]$

well, generalising a bit from $su(2)$ to $su(N)$
In fact, in this basic case there is no choice to make: recall that $w_2(E) \in H^2(M^4; \mathbb{Z}_N)$, but

$$H^2(S^4; \mathbb{Z}_N) = 0$$

so every $PSU(N) = \frac{SU(N)}{\mathbb{Z}_N}$ bundle lifts to a $SU(N)$ bundle, and there is no choice to be made.

So we replace $S^4 \rightarrow M^4$, for M^4 generic. We expect that the holographic dual will still look like $M^4 \times \mathbb{R}$, at least close to the boundary (the UV from the point of view of the CFT, so locally the same as S^4). Let us denote the bulk geometry X^5 . (In fact X^5 might stop being geometric far enough from the boundary, but this does not matter for the argument.)

$$\left. \begin{array}{c} \mathbb{R} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right| M^4 \Big\} X^5 \text{ close to the boundary}$$

Now the question becomes interesting: what is the difference between the holographic duals of $SUC(N)$, $PSUC(N)$... on M^4 ? Answered by Witten: hep-th/9812012

We will assume that the bulk supergravity is the same in all cases. (Reasonable, hard to imagine otherwise since the local CFT data is independent of the choice of global form.)

We focus on the following coupling on X^5 surm:

$$2\pi i \int_{S^1 \times X^5} F_5 \wedge C_2 \wedge dB_2 \rightarrow 2\pi i \int_{X^5} C_2 \wedge dB_2$$

[It is not obvious that one can neglect the kinetic terms for C_2 and B_2 in this analysis, as pointed out in papers by Belov and Moore (04-06).

Only addressed recently in 2510.03392 by Barber, Dymarsky, Shapere]

The result found by Witten is that the different global forms for the theory are in 1-to-1 correspondence with boundary conditions for the BF theory above.

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states in the Hilbert space of the BF theory on M^4 .

[Recall that in QFT a full definition of $Z^{\mathbb{C}}$ requires to specify states on boundary components]

Quantising 5d BF theory

$$U(\Sigma^2) = \exp\left(2\pi i \int_{\Sigma^2} B_2\right)$$

We take

$$\partial \Sigma^2 = \partial \Theta^2 = 0$$

$$V(\Theta^1) = \exp\left(2\pi i \int_{(\Theta^1)^1} C_2\right)$$

for gauge invariance.

Note that if we insert $U^N(\Sigma^2)$ in the path integral:

$$\begin{aligned} & \int \mathcal{D}B \mathcal{D}C \, e^{2\pi i N \int B_2 \wedge dC_2} U^N(\Sigma^2) \\ &= \int \mathcal{D}B \mathcal{D}C \, e^{2\pi i N \int B_2 \wedge (dC_2 + PD[\Sigma^2])} \quad (*) \end{aligned}$$

But $PD[\Sigma^2]$ is closed (since $\partial \Sigma^2 = 0$) and integrally quantised (since $\int_{\pi^3} PD[\Sigma^2] = \pi^3 \cdot \Sigma^2 \in \mathbb{Z}$),
so $PD[\Sigma^2] = dD_2$ for some D^2 :

$$(*) = \int \mathcal{D}B \mathcal{D}C \, e^{2\pi i N \int B_2 \wedge d(C_2 + D_2)}$$

and we can redefine D_2 away. So we learn that $U^N = 1$, and similarly $V^N = 1$.

We are interested in constructing states, so we switch to a Hamiltonian picture, quantizing towards the boundary ($X^5 = M^4 \times \mathbb{R}_+$).

From the Lagrangian

$$\mathcal{L} = 2\pi i N \int B_2 \wedge dC_2$$

we learn that B_2 and C_2 are canonically conjugate:

$$[B_2(x), C_2(y)] = -\frac{1}{2\pi i N} \delta^{(4)}(x-y)$$

So:

$$U(\Sigma^2) V(\Theta^1) = e^{-2\pi i \overbrace{\Sigma^2 \cdot \Theta^1}^{\text{intersection product on } M^4} / N} V(\Theta^1) U(\Sigma^2)$$

So U and V operators do not commute in general.

In order to define states we can specify eigenvalues for a maximal set of mutually commuting operators.

(This is like choosing the $\hat{x}$ operators and giving a wavefunction $\psi(x)$. Once I do this the effect of momentum $\hat{p}$ on $|\psi\rangle = \int dx \psi(x) |x\rangle$ is fully fixed.)

Example: $N=2$:

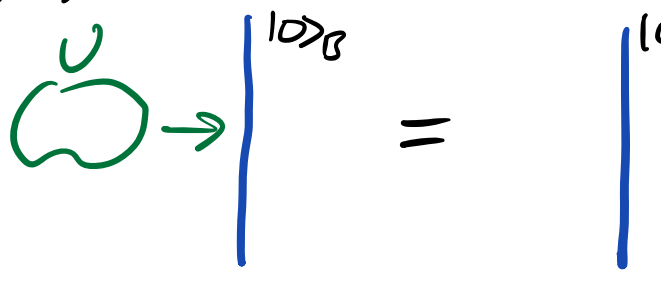
The algebra of operators is ^{generated by} U, V, UV . (Recall $U^2 = V^2 = 1$.)

There are three choices of maximal commuting sets of operators: $\{U\}$, $\{V\}$, $\{UV\}$.

Saying that

$$U(\Sigma^2) |0_B\rangle = |0_B\rangle \quad \forall \Sigma^2$$

is then saying that $B|_0 = 0$.

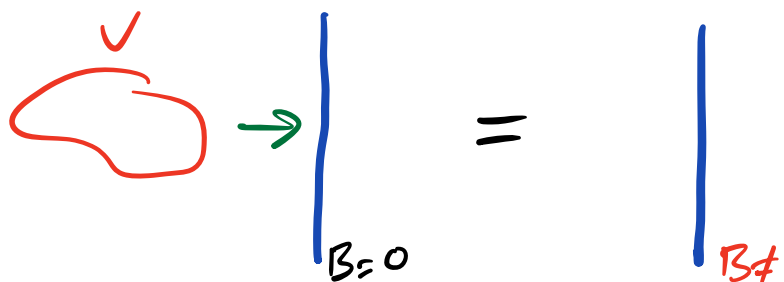


$|0\rangle_B$ (U becomes trivial when pushed to the boundary.)

We also have

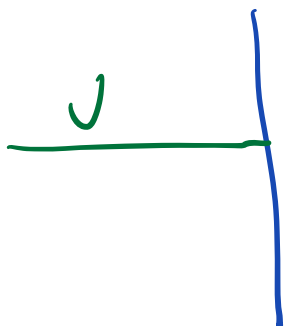
$$\begin{aligned} & U(\Sigma^2) U(\Theta^2) |0_B\rangle \\ &= e^{-2\pi i \Sigma^2 \cdot \Theta^2 / 2} U(\Theta^2) U(\Sigma^2) |0_B\rangle \\ &= e^{-2\pi i \Sigma^2 \cdot \Theta^2 / 2} U(\Theta^2) |0_B\rangle = (-1)^{\Sigma^2 \cdot \Theta^2} U(\Theta^2) |0_B\rangle \end{aligned}$$

So $U(\Theta^2)$ has the effect of turning background B_2 flux on any cycle intersecting Θ^2 an even number of times.



$B=0$ $B \neq 0$ (So V remains as a non-trivial topological operator.)

There is one additional important observation: U operators can end on the boundary:



Since we are fixing $B=0$ at the boundary, so open surface operators are OK (we only need to worry about

gauge symmetry for fields we integrate over).

This does not work for V :

Start with $|0_c\rangle$, which satisfies

$$V|0_c\rangle = |0_c\rangle$$

Now note:

$$\frac{1}{\#} \sum_{\Sigma} U(\Sigma) |0_c\rangle = |0_B\rangle$$

So $|0_B\rangle$ can be interpreted as an average over all U backgrounds. A V defect would be charged under these backgrounds, so when doing the sum it would lead to a vanishing correlator. (This is basically a discrete version of the Gauss law.)

There are three versions of this picture, depending on whether we choose Dirichlet boundary conditions for U , V , or UV .

For some choice of duality frame, we can identify these theories with $SU(2)$, $SO(3)_+$ and $SO(3)_-$.

Check: Under $SL(2, \mathbb{Z})$: $\begin{pmatrix} B \\ C \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix}$

so:

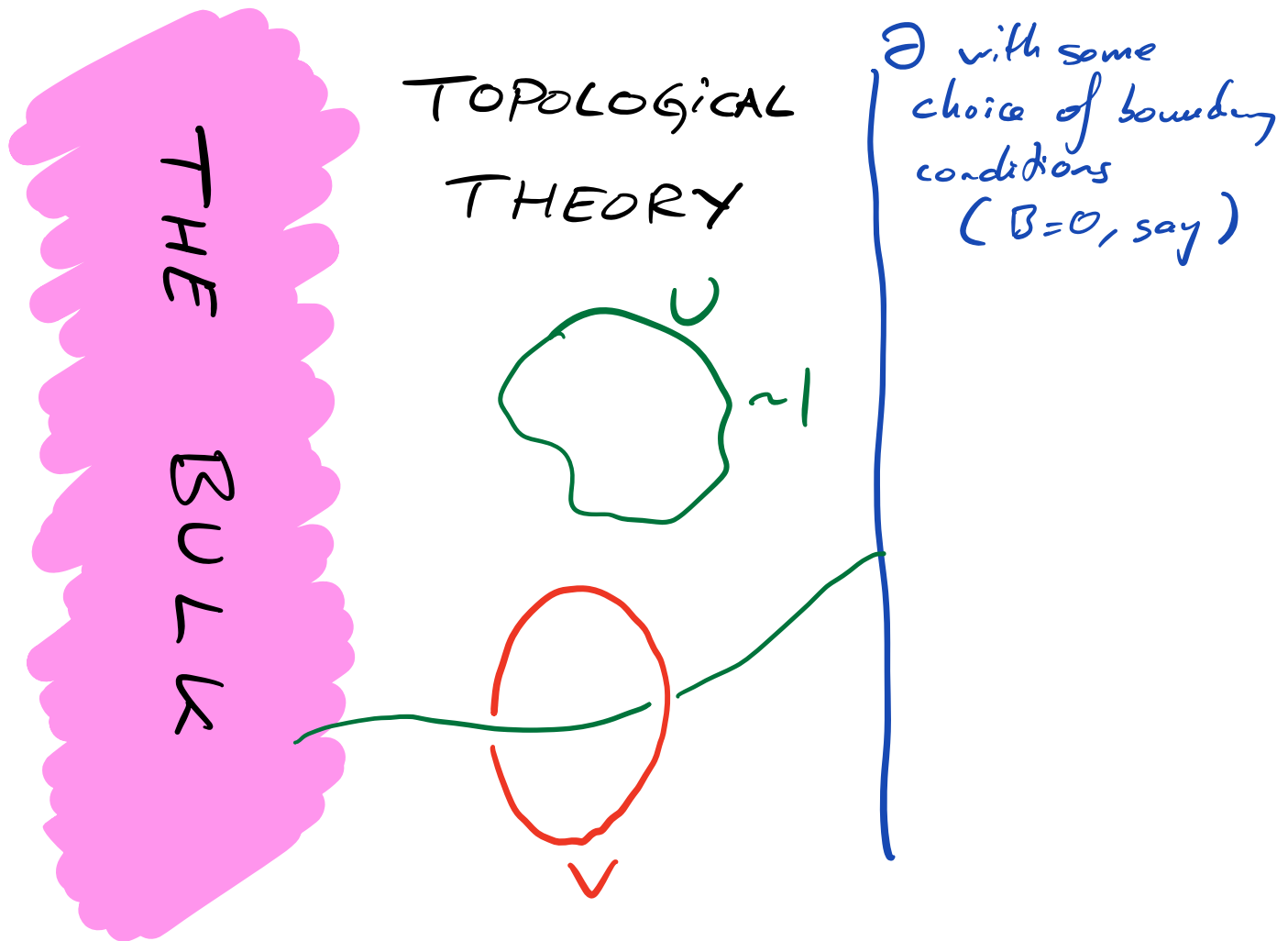
$$\begin{array}{ccccc} SU(2) & \xleftrightarrow[S]{S} & SO(3)_+ & \xleftrightarrow[T]{T} & SO(3)_- \\ \begin{pmatrix} 1 & \\ 0 & \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix} \end{array}$$



which agrees with the expectation from field theory.

In fact the set of boundary conditions and the action of duality on them, agrees with the classification of $SU(N)$ theories in Aharony, Seiberg, Tachikawa 1705.0318 for all N .

Summary



This is of course the usual sandwich picture for the SymTFT! (Normally in the SymTFT we would replace the gravitational bulk with the gapless modes dictated by the CFT dynamics, but according to AdS/CFT both descriptions should be equivalent.)

Other tests

By now there are many tests that this derivation of the SymTFT (by looking to the topological features of the bulk "near infinity")

Let me highlight some interesting cases:

- In holography, a global $U(1)$ symmetry in the bulk is expected to be associated with a $U(1)$ gauge theory on the bulk. Recall that Maxwell could be described as

$$\frac{1}{2e^2} \int B \wedge B + 2\pi \int B \wedge A$$

The topological limit is described by $\frac{1}{2e^2} \rightarrow 0$, which leaves a $\int B \wedge A$ term, where B is a form ($\sim \mathbb{R}$ valued connection) and A a $U(1)$ connection.

This is indeed a valid model proposed for the SymTFT for continuous $U(1)$ symmetries.

[Brennan, Sun '24, Antinucci, Benini '24]

- Non-abelian also works

[...] + [Bonetti, Del Zotto, Minasian '24], [Antinucci, Benini, Rizzi '24], ...

- Gravity (more specifically, 4d gravity) also seems to fit the pattern.

[Apruzzi, Dondi, IGE, Lam, Schäfer-Nameki '25]

- Non-invertible symmetries: they are roughly TFTs over a submanifold

$$\int D\phi e^{-S[\dots]}$$

Many examples of these have been recovered from the physics of D-branes. [Apruzzi, Bah, Bonetti, Schäfer-Nameki '25] [IGE'25]

In fact, recently discovered phenomena in categorical symmetries get translated into classical brane physics:

- Twisted~untwisted transitions $\rightarrow$ Hanany-Witten effect
- Non-invertible selection rules $\rightarrow$ Freed-Witten anomalies.

Big OPEN QUESTIONS

We just saw that the SymTFT seems to live near i-finity in holography. There are two questions:

① Can we turn this into a precise mathematical procedure?

② Can we "push towards the bulk", by deforming the SymTFT in a well defined way, to recover the super bulk dynamics?

All the fields seem to be present in the SymTFT, so the bulk super is somehow built out of symmetries!