

# Maxwell theory and its symmetries

In this lecture I will discuss Maxwell theory and its higher form symmetry structure.

It's a good example to understand how to study higher form symmetries in QFT, but more importantly: + Good testbed for math: symFT still wip!

+ Maxwell theory is rich enough to exhibit many of the phenomena we'd like to understand in more involved QFTs, but in a setting which is simple enough to understand fully. So if you are confused about some aspect of symmetries in QFT, see if you can formulate an analogous question in Maxwell.

+ Maxwell theory is rich because it's fairly universal: see for instance 2204.06495, where together with Michele we understood the 1-form symmetries of many  $N=2$  4d theories by reducing the problem to Maxwell theory.

$$S_{\text{Maxwell}} = \frac{1}{2e^2} \int F \wedge *F$$

Locally,  $F = dA$ , so

(I am setting  $\Theta = 0$  to keep things simple. Exercise: generalise everything I say to  $\Theta \neq 0$ )

$A \rightarrow A + \lambda$  is a symmetry as long as  $d\lambda = 0$ . Additionally, if  $\lambda$  is integrally quantised  $\lambda$  can always be written in the form  $\frac{1}{2\pi i} g^{-1} dg$  with  $g \in U(1) \rightarrow U(1)$  a gauge transformation, so the space of symmetry transformations is  $\frac{\text{closed 1-forms}}{\text{integrally quantised 1-forms}} =: \frac{\Omega^1_d(U(1))}{\Omega^1_{\mathbb{Z}}(U(1))} \leftarrow \text{closed too}$

Let's unpack this a bit:

Imagine that we have a circle  $S^1 \subset M^4$ , parameterised by  $\theta \in [0, 1)$ . We can choose  $g|_{S^1}(\theta) = e^{2\pi i n \theta}$ , with  $n \in \mathbb{Z}$ . (This is an example of "winding"). We then have

$$\frac{1}{2\pi i} \int_{S^1} g^{-1} dg = n \int_{S^1} d\theta = n.$$

[Note: In this argument I have implicitly used that any  $U(1)$  bundle with connection can be uniquely determined, up to gauge transformations, by holonomies and curvature (Cheeger-Simons)]

What is the Noether current for this symmetry?

We can run Noether's argument again. Integrating by parts (recall that we are ignoring boundary behaviour) we can write

$$S_{\text{Maxwell}} = \frac{1}{2e^2} \int A \wedge d * F$$

so the symmetry acts as  $S_{\text{Maxwell}} \rightarrow S_{\text{Maxwell}} + \frac{1}{e^2} \int \lambda \wedge d * F$ , and we find a Noether current (up to a factor):

$$2\pi i j_{(e)} = \frac{1}{e^2} * F$$

We call this current the "electric current". Note that it is a two-form, so we can integrate on 2-manifolds to generate

$$U_\alpha(M^2) := \exp\left(2 \int_{M^2} * F\right)$$

This operator is topological, at least away from charged operators. What are these?

## Wilson lines

Consider the insertion of a classical (non-fluctuating) <sup>very massive</sup> charged particle in the path integral. Assume that the particle has worldline  $\gamma$  (a 1-dimensional manifold).

Such an insertion of  $W_\gamma(x)$  (a "Wilson line on  $\gamma$ ") modifies the path integral as:

$$Z_{\text{Maxwell}} \rightarrow \langle W_\gamma(x) \rangle := \frac{1}{Z_{\text{Maxwell}}} \int [DA] e^{-S_{\text{Maxwell}}[A] + 2\pi i g \int_\gamma A}$$

or equivalently:

$$W_\gamma(x) := \exp\left(2\pi i g \int_\gamma A\right)$$

explain, from source modification to Maxwell's equations to  $d\star F = 2\pi i g \delta^3(\gamma)$

Note that under gauge transformations of the form  $A \rightarrow A + d\alpha$  this can only be gauge invariant if  $\partial\gamma = 0$ , due to Stokes theorem:

$$\int_\gamma A \rightarrow \int_\gamma A + \int_\gamma d\alpha = \int_\gamma A + \int_{\partial\gamma} \alpha$$

Even if we take  $\partial\gamma = 0$ , the Wilson line operator is not invariant under large gauge transformations: if  $A \rightarrow A + \lambda$  with  $\lambda \in \Omega_{\mathbb{Z}}(\mathcal{M}^4)$  then  $W_\gamma(x) \rightarrow W_\gamma(x) e^{2\pi i g \int_\gamma \lambda}$ . So only those Wilson loops with  $g \in \mathbb{Z}$  are invariant under large gauge transformations.

An alternative way of understanding this statement is by writing  $W_\gamma(x) = \exp\left(2\pi i g \int_D F\right)$  with  $\partial D = \gamma$ . (It's not always possible to choose such a  $D$ .)

Then for two choices  $D, D'$  with  $\partial D = \partial D' = \gamma$ :

$$W_\gamma(D) W_\gamma^+(D') = \exp\left(2\pi i g \int_{D-D'} F\right).$$

We have that  $\partial(D - D') = \partial D - \partial D' = 0$ , so if

$\int_{\Sigma^2} F \in \mathbb{Z}$  (the usual quantisation of  $U(1)$  theory, but more on this below) then  $W_g(D) = W_g(D') \quad \forall D, D' \text{ iff } g \in \mathbb{Z}$ .

So whenever  $g \notin \mathbb{Z}$  we should really view the Wilson line as the boundary of an open operator  $M_g(\Sigma^2) = \exp(2\pi i g \int_{\Sigma^2} F)$ .

Note:  $dF = d(dA) = d^2A = 0$ , so  $M_g(\Sigma^2)$  is also deformation invariant, so it's a symmetry. What does it measure?  
 → More on this later.

Under a symmetry transformation  $A \mapsto A + \lambda$  we have

$$W_g(\gamma) \rightarrow W_g e^{2\pi i g \int_{\gamma} \lambda}$$

so our electric symmetry generator is charged under

$$E_\alpha(\Sigma^2) = \exp\left(\alpha \int_{\Sigma^2} \underbrace{*F}_{\text{recall that this was the Noether current for the symmetry.}}\right)$$

relation between  
 $\lambda$  and  $\alpha \Sigma^2$ :  $\text{hol}(\text{link } \Sigma^2) = \alpha \text{ mod } 1$

which in the Euclidean path integral formulation is

$$\oint W_g(\gamma) E_\alpha(\Sigma^2) = e^{2\pi i g \alpha L(\Sigma^2, \gamma)} W_g(\gamma)$$

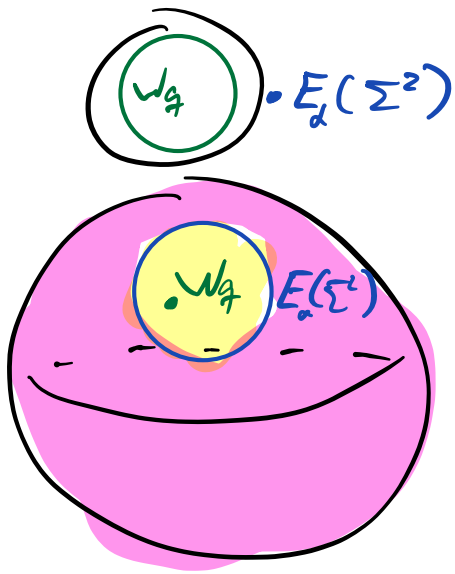
with the linking  $L(\Sigma^2, \gamma)$  defined as

Notes  $D^3, \gamma$  with  $\partial D^3$

① In general there can be multiple choices for  $D^3$  that give different linking numbers.

Take for instance a spacetime of the form  $S^3 \times S^1$ , with  $\gamma = \text{pt} \times S^1$ .

We can construct a  $\Sigma^2$  by taking a  $S^2 \subset S^3$ , and  $pt \subset S^1$ . Schematically:



We have two choices for  $D^3$ , indicated in yellow ( $D$ ) and pink ( $D'$ ), both with  $\partial D = \partial D' = \gamma$  (with right choices of orientation).

But  $\gamma \cdot D = 1$  while  $\gamma \cdot D' = 0$ .

(Schematically, two ways of contracting  $\gamma \rightarrow \emptyset$ ).

Recall that this should all be understood inside a correlator, so we have shown:

$$\langle W_q(\gamma) \rangle \stackrel{\substack{\uparrow \\ \text{contracting } E_q(\Sigma^2) \text{ from outside}}}{=} \langle E_q(\Sigma^2) W_q(\gamma) \rangle \stackrel{\substack{\downarrow \\ \text{contracting inside}}}{=} e^{2\pi i q \alpha} \langle W_q(\gamma) \rangle$$

which for  $q \neq 0$  and generic  $\alpha$  implies  $\langle W_q(\gamma) \rangle = 0$ .

This is a version for higher dimensional operators of the familiar statement that expectation values of charged operators necessarily vanish on a compact space.

[Note: for spontaneous breaking we would be interested in non-compact spacetimes so the picture above would get modified. What we are computing here is closer to  $\text{Tr}(e^{-\frac{1}{T}H})$ , which is indeed invariant under the symm.]

## ② Magnetic symmetry

Quite by accident, we found a new symmetry:  $M_\alpha(\Sigma^2) = \exp(2\pi i \alpha \int_{\Sigma^2} F)$ . What does this measure?

# Maxwell's equations in vacuum

$$dF = d \star F = 0$$

(forgetting about quantisation)

have a symmetry  $F \leftrightarrow \star F$  (or, in fact, any  $GL(2, \mathbb{R})$  action on the  $(F, \star F)$  doublet). Quantum mechanically the situation is more interesting: there's a parameter  $\tau$  (valued in the upper half plane) that measures instanton  <sup>$\sim \theta/2\pi + \frac{4\pi i}{g^2}$</sup>  contributions, and the transformation above is not (generically) a symmetry, but a duality:  $Z_{\text{Maxwell}}[\tau] = Z_{\text{Maxwell}}[-1/\tau]$  (up to some prefactors I'll ignore  <sup>$\sim 1/g^2$</sup> , see hep-th/9505186 by Witten for a determination of these factors, and some other subtleties I'll gloss over, and also Hsieh, Tachikawa, Yonekura 2003.11550)

We start with

$$S_{\text{Maxwell}} = \frac{1}{2e^2} \int F \wedge \star F$$

Here  $F = dA$ , with  $A$  a connection on a  $U(1)$  bundle.

We introduce a new dynamical field  $B$ , a connection 2-form on a  $U(1)$  bundle, with field strength  $H = dB$ .

We also introduce a 1-form connection  $A^D$ , with field strength  $F^D = dA^D$ , also dynamical. Consider this set of fields with action:

$$S_D = \frac{1}{2e^2} \int (F - B) \wedge \star (F - B) + 2\pi \int B \wedge F^D \quad (*)$$

The gauge transformations in this theory are  
(or more generally a large gauge transform)

$$A_D \rightarrow A_D + d\alpha \quad A, B \text{ invariant}$$

and

$$A_0 \text{ invariant, } \left. \begin{array}{l} A \rightarrow A + \lambda \\ B \rightarrow B + d\lambda \end{array} \right\} \begin{array}{l} \text{comment on gauge} \\ \text{invariance of second term} \end{array}$$

$\hookrightarrow A$  1-form connection

Claim: This theory is equivalent to Maxwell theory.

Proof: Integrating by parts the second term we have

$$-\frac{1}{2\pi i} \int H \wedge A^D$$

so  $A^D$  appears as a Lagrange multiplier, setting  $H=0$ .

Integrating over  $F_D$  (the topological sectors) instead,

imposes  $\int B \in \mathbb{Z}$ . So  $B$  is an integrally quantised (closed) form, so it can be fully removed by a large gauge transformation, and we are left with the standard Maxwell action, subject to the remaining gauge transforms of the form

$$\lambda = d\beta. \quad [\text{Note: we didn't really use } F \text{ for anything here, so } \frac{1}{2e^2} \int B \wedge *B + \frac{1}{2\pi i} \int B \wedge F^D \text{ with } B \text{ a 2-form is an equally good presentation of Maxwell theory.}]$$

Magnetic theory

Once we have written the action as above, there's a second thing we can do: if we fully gauge  $A$  away using the gauge invariance, we have

$$S_D = \frac{1}{2e^2} \int B \wedge *B + 2\pi i \int B \wedge F^D$$

This is a quadratic action on  $B$ , so we can do the path integral explicitly. Up to a 1-loop determinant factor, we have:

$$e^D := \frac{1}{2\pi i e}$$

$$\int [DB] e^{-S_D} = \exp\left(-\frac{4\pi^2 e^2}{2} \int F^D \wedge *F^D\right) \quad [\text{Exercise}]$$

(Check and generalize to  $\Theta \neq 0$ , see paper)

For the theory with action  $S_D$ , the equation of motion for  $B$  is  $F^D = \frac{1}{2\pi i e^2} \star(F-B)$ , so we identify Wilson lines of the electric theory with magnetic lines of the dual theory, and vice versa.

$$\oint \star F = 1 \rightarrow \oint \star(F-B) = 1 \rightarrow \oint F^D = 1$$

In the original

In the intermediate

In the dual

Using the properties of the Hodge star  $\star^2 = 1$  we obtain that the object charged under  $\int F$  is a Wilson line for  $A^D$ . Such objects are known as 't Hooft lines, and can be thought of as insertions of classical (= very massive = rigid) monopole worldlines in the path integral.

(By definition, a <sup>fundamental</sup> monopole has  $\int F = 1$  around it.)

Summary In summary, we have found that  $U(1)$  Maxwell theory in four dimensions has (at least) two  $U(1)$  1-form symmetries, with generators

$$E_\alpha(\Sigma^2) = \exp\left(\alpha \int_{\Sigma^2} \star F\right)$$

$$M_\alpha(\Sigma^2) = \exp\left(2\pi i \alpha \int_{\Sigma^2} F\right)$$

and charged objects

$$W_\alpha(\gamma)$$

$$H_n(\gamma) = \exp\left(iq \int_\gamma A^D\right)$$

$n = \int_{\Sigma^2} F$ , where  $\Sigma^2$  links  $\gamma$ .

Open "surface" where we integrate  $B-F$  in the intermediate formulation.

$\Rightarrow (B-F) \wedge \delta(\Sigma) + B \wedge F^D \Rightarrow F^D = -\delta(\Sigma) \Rightarrow dF^D \neq 0$ , so we need to change the nature of  $A^D$  to include disorder on  $\gamma$  (as you might expect)

$\Rightarrow$  Exercise: write down  $E_\alpha, M_\alpha, W_\alpha$  and  $H_n$  and compute their linking in the intermediate form.

# Goldstone's theorem for long range symmetries

[Polyakov '80, Kounner, Rosenstein, Eliezer '91, Kounner, Rosenstein '92]

[Gaiotto, Kapustin, Seiberg, Willett]

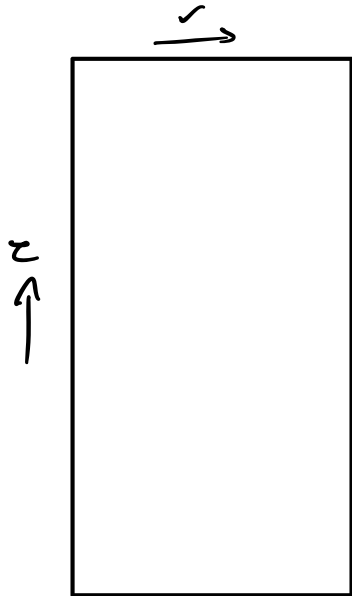
[Lake 1802.07747] [Iqbal, Hofman 1802.09512]†

Consider an extended operator, such as  $W_g(\gamma)$ , which has perimeter law

$$\langle W_g(\gamma) \rangle \underset{\text{large perimeter}}{\sim} e^{-T \text{length}(\gamma)} \quad (\text{"perimeter law"})$$

[Note: here  $\gamma$  is taken to be topologically trivial, so  $W_g(\gamma)$  has net 0 charge] [Computation in chapter 82 of Srednicki]

Consider a Wilson loop of the form



with  $z \gg r \gg a$ . <sup>a UV cutoff</sup>  
Then the expectation value of the loop is dominated by  $e^{-zH}$  in the presence of charged i-sections. In QED, we have a potential  $V(r) \sim \frac{1}{r}$ , so the contribution to the energy is

$$H \sim \frac{1}{r} + \frac{1}{a}, \text{ with 'a' some cut-off}$$

dependent self-energy. So  $\langle W(\gamma) \rangle \sim \exp(-(\frac{1}{r} + \frac{1}{a})z)$ .

In the limit of very large  $r$  the term depending on  $a$  dominates.

We can define a renormalised operator  $\bar{W}_g(\gamma) \sim W_g(\gamma) e^{\frac{1}{a} \text{length}(\gamma)}$

so that for the geometry above

$$\langle \bar{W}_g(\gamma) \rangle \sim \exp(-\frac{z}{r} + \frac{z}{a}) \sim 1 \quad (\text{With some appropriate choices})$$

Note: any finite nonzero  $\nu$  in the  $r, z \rightarrow \infty$  limit will do.

Consider now our spacetime to be  $\mathbb{R}^4$ , and  $x = \mathbb{R}$ . The equation above still holds, and additionally  $\bar{W}_q(x)$  is charged under  $E_\alpha(S_r^2)$  with  $S_r^2$  a sphere around the origin in the "spatial  $\mathbb{R}^3$ " <sup>of radius  $r$</sup> . We have:

$$\langle E_\alpha(S_r^2) \bar{W}_q(x) \rangle = \langle e^{iq\alpha} \bar{W}_q(x) \rangle = e^{iq\alpha}$$

Taking the derivative with respect to  $\alpha$  on both sides, this implies:

$$\int_{S_r^2} \langle *F(x) \bar{W}_q(x) \rangle d\Omega = iq$$

So the left hand side is constant. Due to the rotational symmetry of the configuration in  $\mathbb{R}^3$ , we must have

$$\langle *F(x) \bar{W}_q(x) \rangle \sim \rho(r)$$

Since  $\int_{S_r^2} d\Omega = 4\pi r^2$ , it must be that

$$\int_{S_r^2} \langle *F(x) \bar{W}_q(x) \rangle = \frac{iq}{4\pi r^2}$$

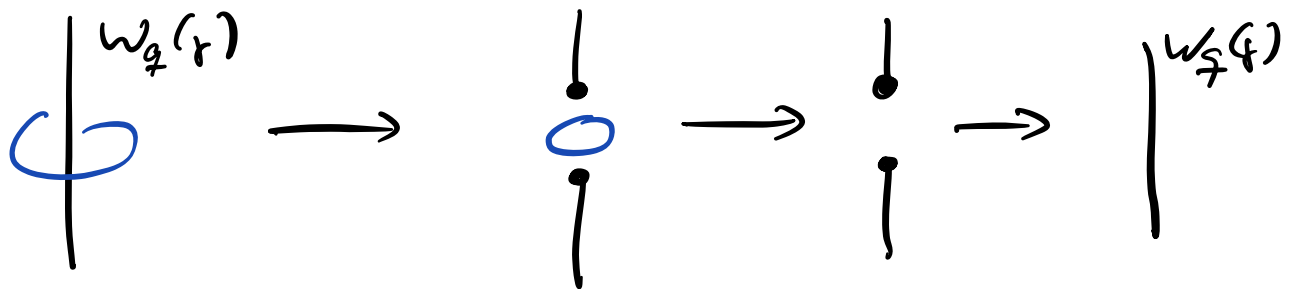
This is a power law correlation, which indicates the presence of a massless mode.  $\square$

So there is a massless mode, which we identify with the Goldstone mode. What is it? We can identify the Goldstone mode as the mode that shifts linearly under the spontaneously broken symmetry. By construction,  $E_\alpha(\Sigma^2)$  shifts  $A$  linearly, so this is the Goldstone mode!

QED In QED (Maxwell + charged matter) we no longer have the electric  $U(1)$  symmetry  $E_\alpha(\Sigma^2) = e^{\alpha \int_{\Sigma^2} *F}$ , because  $d*F \propto j_e$ , with  $j_e \neq 0$  the electric current.

Alternatively, heavy probe particles can decay into dynamic excitations, so the Wilson line can break, so it cannot carry conserved charges:

[Say something about particles of charge  $g > 1$ ]



But, as long as we don't have dynamical monopoles,  $M_2(\Sigma^2)$  is still a spontaneously broken  $U(1)$  symmetry, so the argument above mostly applies. The only difficulty is that we naturally get  $A^D$  from the shifting argument, but this object is not well defined in the usual presentation.

Compute  $\langle 0 | F | \text{photon} \rangle$  to see that the photon is the desired Goldstone. [Exercise]