

### 7.3. SYMMETRY CATEGORIES IN HIGHER DIMENSIONS <sup>(3-1)</sup>

The construction of the symmetry categories for higher dimensional QFTs is a relatively more recent endeavor and many aspects of it are topics of current research.

In 7.2 we have seen that for 2d QFTs with  $c = \bar{c} < 1$  we can say a lot about their symmetry categories when we have finite # of simple objects.

We have seen how to use Ward identities & symmetries mapping across different twisted sectors of the Hilbert space, as well as a simple constraint on

R/L flows.

Today I would like to discuss analogous effects in the case  $d+1 > 2$ .

First of all: continuous families of topological defects / operators can be constructed from current algebras.

- Consider the QFT<sub>dH</sub> of interest, look for a conserved current:

$$d * j^{(p+1)} = 0$$

up to contact terms

this gives rise to  $U(1)^{(p)}$  p-form

symmetry (Gaiotto, Kapustin, Seiberg, Vasiliev 14):

$$D(Z^{d-p}) = \exp i \theta \int_{Z^{d-p}} * j^{(p)}$$

$$U(1)^{(p)} \subset \text{End}(\underbrace{1, 1, \dots, 1}_p)$$

subcategories p-times

\* Lüscher's lectures for some examples

SSB: massless higher spin goldstones  
 confinement: unbroken higher sym. (plate)

- continuous 2-group symmetry

(Cordova, Dumitrescu, Intriligator 18)

$$U(1)_A^{(0)} \times_k U(1)_B^{(1)}$$

$$\begin{cases} d * j_B^{(2)} = 0 \\ d * j_A^{(1)} = \frac{k}{2\pi} F_A^{(2)} \wedge * j_B^{(2)} \end{cases}$$

2-group structure constant

the first example of such a higher sym.

Maxwell problems of SSB:

$$U(1)_A^{(0)} \times U(1)_B^{(1)} \rightarrow U(1)_B^{(1)}$$

$$U(1)_A^{(0)} \times U(1)_B^{(1)} \rightarrow \text{nothing}$$

NG Lagrangian:

$$v^2 \int (d\tau - A^{(1)}) \wedge * (d\tau - A^{(1)})$$

$$+ \frac{1}{2e^2} \int f^{(2)} \wedge * f^{(2)}$$

$$+ \frac{i}{2\pi} \int B^{(2)} - \frac{\kappa}{2\pi} \int F_A^{(2)} \wedge f^{(2)}$$

$A^{(1)}$  is background for  $U(1)_A^{(0)}$

$B^{(2)}$  is background for  $U(1)_B^{(1)}$

$$A^{(1)} \rightarrow A^{(1)} + d\lambda^{(0)}$$

$$B^{(2)} \rightarrow B^{(2)} + d\lambda_B^{(1)} + \frac{\lambda^{(0)}}{2\pi} F_A^{(2)}$$

(GSS mechanism)

hierarchy of scales

$$E^{uv}(j_B^{(2)}) \gg E^{uv}(j_A^{(1)})$$

1-form symmetry must  
emerge before the 0-form

examples we only constructed  
starting from free fermions  
and giving suitable  
perturbations in ABJ models  
 $U(1)$ .

- non-invertible ABJ  
symmetry:

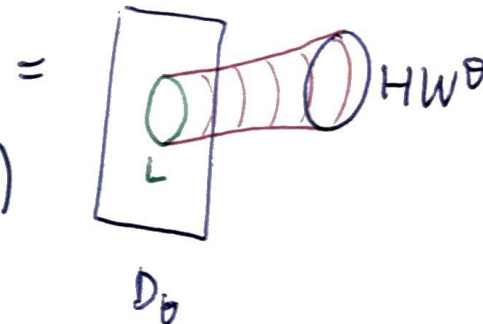
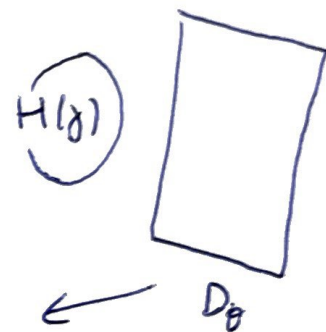
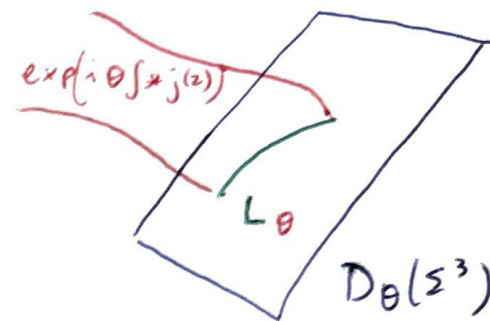
$$\star j_A^{(1)} = \frac{1}{2\pi} \star j_B^{(2)} \wedge \star j_B^{(2)}$$

$$d \star j_A^{(1)} = \kappa (\star j_B^{(2)} \wedge \star j_B^{(2)})$$

$$\theta \in \mathbb{Q}/\mathbb{Z}$$

$$D_\theta(\Sigma^3) = \exp(i\theta \int_{\Sigma^3} \star j^{(1)}) \otimes Z_\theta(\Sigma_3; B^{(2)} = \star j^{(2)})$$

where  $Z$  is a 3d TQFT  
such that it has a  
1-form symmetry which  
is non-invertible and such  
that it compensates the  
non-invertible RHS of the  
Wick algebra



effect that is the  
higher dimensional  
analogue to the

$$\sigma \downarrow = \downarrow \quad \text{in the Ising model}$$

RNk (interplay w/ Rk) (3.2)

this same symmetry  
category arises in the  
 $CP^1$   $\sigma$ -model in 4d

Chen T-matrix 2210

$$\pi_1 CP^1 \cong 0$$

$$\pi_2 CP^1 \cong \mathbb{Z} \rightarrow \text{vertex string charge}$$

$$\pi_3 CP^1 \cong \mathbb{Z} \rightarrow \text{Hopfion charge}$$

$$\pi_4 CP^1 \cong \mathbb{Z}_2$$

this is an EFT and one  
can ask whether these  
solitons survive in UV  
completions.

This is traced back to  
the matching of the  
higher symmetry in  
the Rk flow

for applications go  
can see Oz, Meier,  
Myhrh, Oheri  
2408  
+ in progress

~~Moduli space structure~~  
~~we can consider~~

Of course the goal is one  
of the advantages of  
topological defect / operators  
is that we can also  
discuss finite symmetries  
& their numbers  
from this perspective.

What is in memory?

From my perspective  
numbers are obstructed  
to gggg.

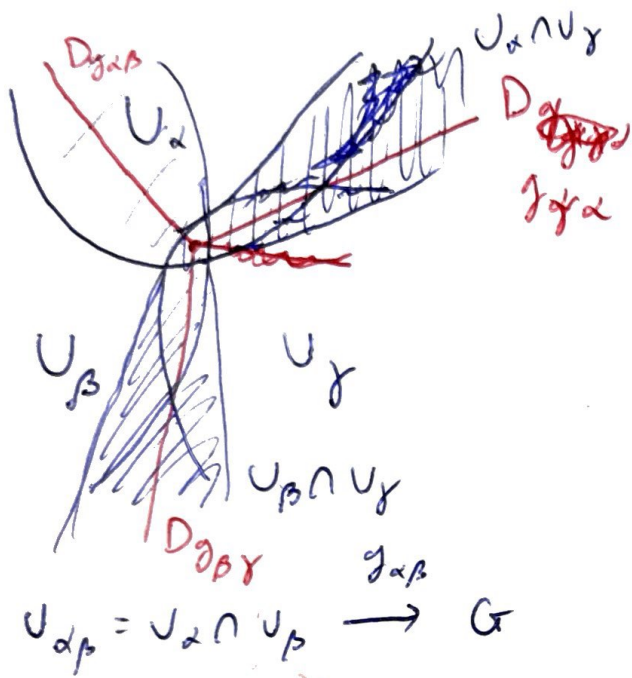
in the case of  
invertible symmetries  
there are also  
obstructions to line  
& trivially gapped  
vacuum -

For non-invertible  
symmetries the two  
characterizations are  
not equivalent.

Let's discuss gggg -  
~~back~~

gggg: couple to  
to a background  
(+) average over  
all possible  
backgrounds.

what is a background  
for a finite symmetry?  
consider 0-form symmetry



$$U_\alpha U_\beta = U_\alpha \cap U_\beta \rightarrow G$$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

cocycle condition

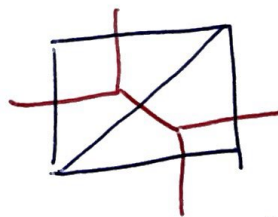
equivalent to prescribing a collection of codimension 1 defects (implying the transition functions)

gauge: sum over all possible decorations

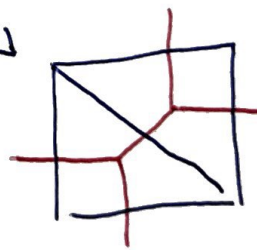
compatible w/ transition functions -

combinatorially we have a simplicial decomposition of spacetime not a dual decoration w/ defects.

$$e.g. d = 1 + 1$$



2 → 2 patcher



1 → 3



patcher

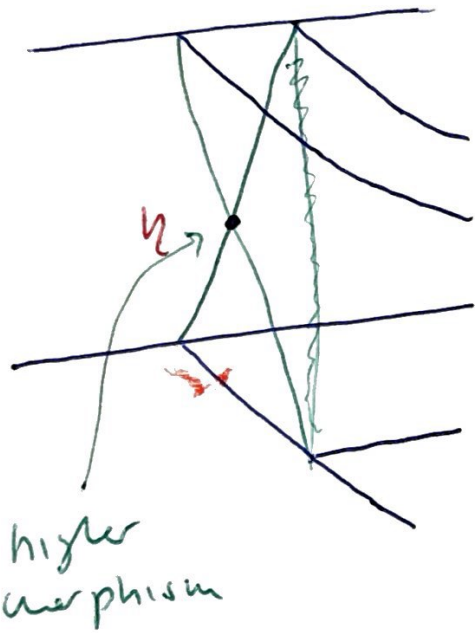
In the 2 → 2 patcher move you can recognize the F symbol -

In the case of invertible defects, if  $w \in H^3(BG, U(1))$  is nontrivial then, becomes an obstruction to gauge.

you can see it also at the level of the  $\tau^2$  patcher function:

$$\square = F \square$$

For the case of (3-3)  
topological defects of  
higher dimensionality  
most of these patcher  
never can be associated  
by topology:



simplest example is  
provided by higher  
group one

but one can consider  
this construction more  
generally -

... Cargenille,  
Rouhet,  
Schumann 1905...

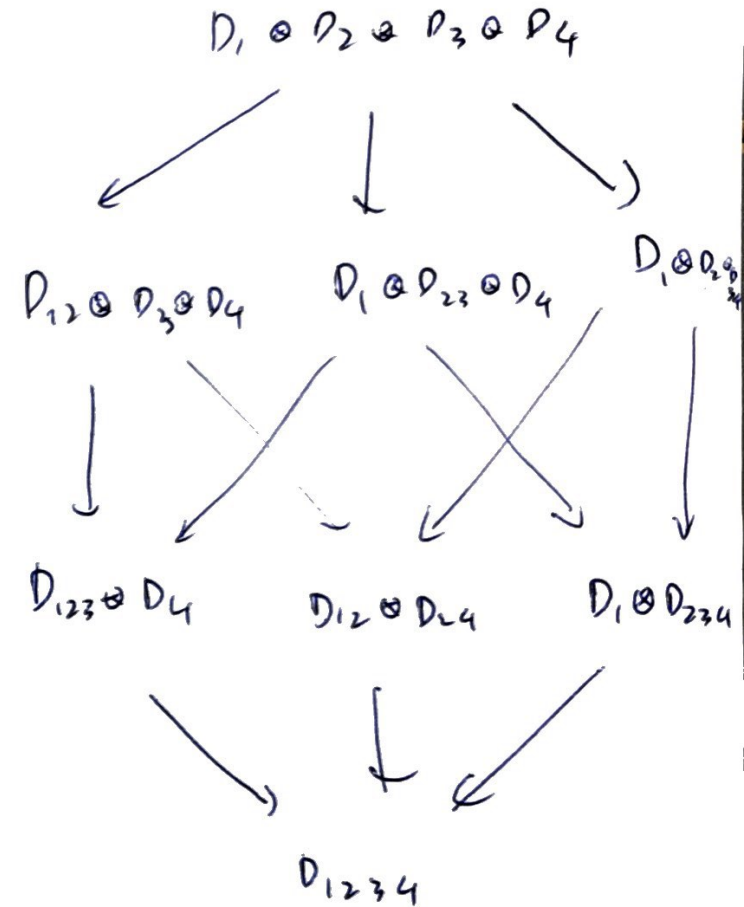
Carelli, DZ, Ohner, Wang  
2005...

$D_1 \otimes D_2 \rightarrow D_{12}$  fusion  
morphisms  
are ~~not~~ forget of  
the possible  
inner channels

$D_1 \otimes D_2 \otimes D_3 \rightarrow D_{12} \otimes D_3$   
 $\downarrow$   $\swarrow$   $\searrow$   
 $D_1 \otimes D_{23} \rightarrow D_{123}$

associative higher  
morphisms

we can combine and  
consider higher morphisms



each face is a 2-morphism  
the cube is a 3-morphism  
we increase cohesion  
at each step.

until we reach a point.  
After that the next  
assocater becomes a  
constraint.

in the case of ordinary  
symmetries this gives  
rise to a group  
coherency problem.

eg. in quantum  
mechanics



$$\alpha \rightarrow \theta(g) + \theta(g') - \theta(gg') + \alpha \text{ shift } \alpha \text{ by } \text{coboundary}$$

$$U_g U_{g'} = e^{i\alpha(g,g')} U_{gg'}$$

$$U_g (U_{g'} U_{g''}) = (U_g U_{g'}) U_{g''}$$

$$\Leftrightarrow \alpha \in H^2(\mathcal{G}, U(1))$$

$$\alpha(g, g'g'') + \alpha(g', g'') =$$

$$2\text{-cocycle} = \alpha(g, g') + \alpha(gg', g'')$$

in QM machinery itself is  
the constraint.

$$1 - d = 1 + 1$$

$$\leadsto F \in H^3(BG, U(1))$$

similarly for finite p-form  
symmetry in d-dimensions  
we get to

$$H^{d+1}(B^{p+1}G, U(1))$$

this is a naive  
characterization of  
manifolds.

there is however a  
refinement which  $\neq$  is  
~~exact~~ ~~about~~ well  
known to experts and  
was proposed by

Gaiotto and Johansen  
Freed

1712...

we can consider  
projecting the  
higher cohomology?

describe the junction  
morphisms by  
suitable SPTs  
obtain a generalized  
group coherency  
problem

$$H^{d+1}(B^{p+1}G, \text{SPT}^{\bullet})$$

Structure  
suitable  
SPT spectrum

is this the same as

$$\Omega_{d+1}^{\text{structure}}(BG) ?$$

There is also  
further data  
which comes from  
gauging over  
submanifolds

→ "condensates"

better "higher  
gauge  
defects"

~~Add the FOAM of~~  
can be constant  
network of topological  
defects invariant  
wrt picture changes  
or FOAM, or  
Algebraic object

(3.4)

you can consider  
filling in submanifolds  
using SDC form -

There are very  
explicit

$$e.g. - \mathbb{Z}_k^{(1)} \subset U(1)^{(1)}$$

$$\exp i \frac{2\pi}{k} \int_{\Sigma^2} * j^{(1)}$$

$$C_k^{(1)}(\Sigma^2) = \sum_{\varphi \in H^0(\Sigma, \mathbb{Z}_k)} \exp \frac{2\pi i}{k} \int \varphi U * j^{(1)}$$

$$C_{k, \alpha}^{(0)}(\Sigma^1) = \sum_{\alpha \in H^1(\Sigma^1, \mathbb{Z}_k)} \exp \frac{2\pi i}{k} \int \alpha U * j^{(0)} + B(\alpha)$$

ALL THIS DATA IS DISCRETE  
DATA OF  $\mathbb{C}$  THAT MUST MATCH  
ALONG RG : WIDE GENERALIZATION  
OF + HODGE AND ONLY MATCHES

there are also  
consistent subtypes  
of gauge defects -

You see that we  
started populating  
our symmetry category.

Quest / task : develop

QFT tools  
to compute  
these structures

SPT  
for  $\mathbb{Z}_k^{(1)}$   
data.

how  
does  
this  
generalize  
or the  
issue?

explore further  
connections on  $Rb$   
flow arising from  
the higher structure?

Symmetry transmutation  
...

How does the symmetry  
(showing up on the  
Hilbert space?)

...

Work in  
progress!

Stay tuned!!