Black Holes are Quantum Complete

ASG - Lecture 2018

I Preliminaries

Concept (fixed point):

Let $G: X \to X$ be a map on a set X.
A point $x \in X$ for which $Gx = x$ is called a fixed point of G.

Concept (contraction):

Let (M, d) be a metric space.
A map $G: M \to M$ for which

$$d(Gx, Gy) \leq d(x, y) \quad \forall x, y \in M$$

is called a contraction.
If there is a $q < 1$ such that

$$d(Gx, Gy) \leq q\, d(x, y) \quad \forall x, y \in M,$$

then G is called a strict contraction.

<u>Statement</u> (contraction mapping principle): /
A strict contraction on a complete metric space has a unique fixed point. /

<u>Sketch:</u> /

<u>Uniqueness:</u> //

Assume $\boxed{Gx = x,\ Gy = y}$ $(x, y \in M)$.

Then $d(x,y) = d(Gx, Gy) \leq q\, d(x,y)$.

Since $0 \leq q < 1$ it follows that

$$d(x,y) = 0 \rightsquigarrow x = y. //$$

<u>Existence:</u> //

First note: G is automatically continuous since

$$d(x,y) < q^{-1}\varepsilon \text{ implies } d(Gx, Gy) < \varepsilon.$$

Next, let $x_0 \in M$ be arbitrary. Consider

$$\{x_n\}_{n\in\mathbb{N}} \text{ with } x_n = G^n x_0 \overset{\text{df}}{=} (\underbrace{G \circ \cdots \circ G}_{n \text{ times}})\, x_0$$

* We will show that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy:

$$d(x_n, x_{n+1}) = d(Gx_{n-1}, Gx_n) \leq q\, d(x_{n-1}, x_n)$$

$$d(x_{n-1}, x_n) = d(Gx_{n-2}, Gx_{n-1}) \leq q\, d(x_{n-2}, x_{n-1})$$

$$d(x_n, x_{n+1}) \leq q^2 d(x_{n-2}, x_{n-1})$$

$$\vdots$$

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1)$$

For $n > m$

$$d(x_n, x_m) \leq \sum_{a=m+1}^{n} d(x_a, x_{a-1})$$

$$\leq q^m (1-q)^{-1} d(x_0, x_1) \xrightarrow{m \to \infty} 0 .$$

Hence $\{\sigma^n x_0\}_{n \in \mathbb{N}}$ is Cauchy. Since M is complete there is a $x \in M$ so that

$$\sigma^n x_0 \xrightarrow{n \to \infty} x .$$

By continuity of σ,

$$\sigma x = \lim_{n \to \infty} \sigma x_n = \lim_{n \to \infty} x_{n+1} = x . \quad //$$

__Application:__

Let $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous.
We are interested in solving

$$\dot{y}(t) = f(t, y(t)), \quad y(0) = y_0 .$$

Contraction arguments give the existence of __local__ solutions, i.e. given y_0 we will find

$$y : (-\delta, \delta) \to \mathbb{R}^n$$

satisfying
1) $y(0) = y_0$
2) $\dot{y}(t) = f(t, y(t))$ for all $|t| < \delta$.

__Note:__ The question whether local solutions can be extended to global solutions is much touchier.

Differentiation makes functions less smooth, unless we consider only C^∞-functions.
This motivates to rewrite the equation of motion in integral form:

$$y(t) = y_0 + \int_0^t ds\, f(s, y(s)) . \qquad (*)$$

Given y_0 and δ, we consider the map

$$\mathcal{E} : C[-\delta,\delta] \longrightarrow C[-\delta,\delta]$$

cont. func. to $\mathbb{R}^n$

defined by

$$\Big(\mathcal{E}(t,0)\Big)(g) \overset{\text{df}}{=} y_0 + \int_0^t ds\, f(s, g(s)) .$$

Solving $(*)$ is equivalent to finding a fixed point of $\mathcal{E}$!

For simplicity we consider only the case where f is Lipschitz continuous, i.e.

given y_0,

there is a K, δ and ϵ so that

$$\|y - y_0\| < \epsilon, \ |t| < \delta$$

implies

$$\|f(t,y) - f(t,z)\| \leq K \|y - z\|$$

if $\|z - y_0\| < \epsilon$.

Let

$$\mathcal{G} \overset{\text{df}}{=} \left\{ g \in C[-\delta,\delta] : \|g(t) - y_0\| \leq \tfrac{1}{2} \varepsilon \ \forall t \in (-\delta,\delta) \right\}$$

What is the relevance of $\mathcal{G}$?

First // if $g \in \mathcal{G}$ then $\mathcal{E}(g) \in \mathcal{G}$ since

$$\left\| \mathcal{E}(g) - Y_0 \right\| = \left\| \int_0^\delta ds\, f(s, g(s)) \right\|$$

$$\leq \int_0^\delta ds \left\| f(s, g(s)) \right\|$$

$$\leq \delta \cdot \max_{\substack{|s| < \delta \\ \|g(s) - Y_0\| < \epsilon}} \left\| f(s, g(s)) \right\|$$

Shrinking δ we can be sure that

$$\left\| \mathcal{E}(g) - Y_0 \right\| \leq \frac{\epsilon}{2} \quad //$$

Second // $\mathcal{G}$ is a complete metric space under the supremum norm $\|\cdot\|_\infty$.

For $g_1, g_2 \in \mathcal{G}$ consider

$$\left\| \mathcal{E}(g_1) - \mathcal{E}(g_2) \right\|_\infty = \left\| \int_0^t ds \left[f(s, z) \right]_{g_2(s)}^{g_1(s)} \right\|_\infty$$

$$\leq \int_0^t ds \left\| \left[f(s, z) \right]_{g_2(s)}^{g_1(s)} \right\|_\infty \overset{\text{Lipschitz}}{\leq}$$

$$\leq \int_0^t ds\, C \left\| g_1 - g_2 \right\|_\infty \qquad (|t| < \delta)$$

$$\leq C \delta \left\| g_1 - g_2 \right\|_\infty$$

Shrinking δ we can be sure that

$$\left\| \mathcal{E}(g_1) - \mathcal{E}(g_2) \right\|_\infty < \frac{1}{2} \left\| g_1 - g_2 \right\|_\infty \quad //$$

Thus $\mathcal{E}$ is a strict contraction on $\mathcal{G}$.

The contraction mapping principle then grants a unique $g \in \mathcal{G}$ which satisfies $(*)$.

Conversely, any solution of $(*)$ must be in $\mathcal{G}$ for sufficiently small t and so must agree with the unique solution in $\mathcal{G}$ when t is small.

This proves local existence & uniqueness. /

Consider a Hamiltonian system $(\mathbb{P}, H)$ with phase space $\mathbb{P} = (0,\infty) \times \mathbb{R}$ and Hamilton function

$$H : \mathbb{P} \longrightarrow \mathbb{R},$$
$$(q,p) \longrightarrow H(q,p) \overset{\text{df}}{=} \frac{p^2}{2m} + V(q).$$

The equations of motion are

$$\dot{q}(t) = \frac{1}{m} p(t), \quad \dot{p}(t) = -\operatorname{grad} V(q(t))$$

For each $q_0 > 0$, p_0, $t_0 > 0$ the standard (by now) contraction argument gives a <u>unique</u> solution pair

$$(q(t), p(t)) \quad \text{for } t \in (t_0 - \delta, t_0 + \delta) \quad (\delta > 0)$$

satisfying $q(t_0) = q_0$, $p(t_0) = p_0$.

Statement: /

Suppose a global solution satisfying the initial conditions $q(t_0) = q_0 > 0$, $p(t_0) = p_0$ does not exist, i.e.

> the maximal interval on which the solution with these initial conditions exists is $[t_0, \tau)$, $\tau < \infty$.

Then either

$$\lim_{t \uparrow \tau} q(t) = 0 \quad \text{or} \quad \lim_{t \uparrow \tau} q(t) = \infty$$

/

Sketch: / By the construction of local solutions and the assumptions made on V:

for any compact $K \subset \mathbb{P} = (0, \infty) \times \mathbb{R}$ there is a T^K:

> $(q(t), p(t))$ is a unique solution for $t \in (t_1 - T^K, t_1 + T^K)$ with $q(t_1) = q_1$, $p(t_1) = p_1$
>
> specified

If we cannot extend the solution past $t = \tau$, then it cannot lie in K for any $t > \tau - T^K$.

The remaining task is to show that the statement "$(q(t), p(t))$ leaves K eventually" implies that the point mass eventually leaves any compact subset $C \subset (0, \infty)$.

At this stage energy conservation is very useful:

$$H(q(t), p(t)) = H(q_0, p_0) =: E_0 .$$

If $q(t) \in C \subset (0,\infty)$, then

$(q(t), p(t)) \in$

$$\Big\{ p(t) : |p(t)| \leqq \sqrt{2m\big(E_0 - \inf_{q \in C} V(q)\big)} \Big\}$$

Thus for all $n \in \mathbb{N}$ there is a t_n such that

$$q(t) \notin \left(\tfrac{1}{n}, n\right)$$

for $t > t_n$.

<u>Concept</u> (complete):

The classical motion generated by V is called <u>complete</u> at zero (infinity) if there is no $(q_0, p_0) \in \mathbb{P} = (0,\infty) \times \mathbb{R}$ so that $q(t)$ runs off to zero (infinity) in a finite time.

<u>Statement:</u>

The classical motion generated by V is <u>not</u> complete at zero if and only if $V(q)$ is bounded above near zero.

<u>Sketch:</u> V is not bounded above at zero if and only if there is a sequence $\{q_n\}_{n\in\mathbb{N}}$ with $q_n \to 0$ so that $V(q_n) \to \infty$.

1) Assume: V is not bounded above at zero.
Show: V is complete

By conservation of energy

$$\frac{p^2(t)}{2m} + V(q(t)) = \frac{p_0^{\,2}}{2m} + V(q_0) =: E_0$$

Hence $V(q(t)) \leq E_0$.

Thus $q(t)$ can never equal q_n for n sufficiently large.
So $q(t)$ can never go to zero.
Thus V is complete.

2) Assume: $V(q) \leq M$ on $(0,1)$.
Show: V is incomplete.

Let $q(t_0) = q_0 = 1$ and choose $p_0 < 0$ and so that

$$E_0 = M + 1.$$

By energy conservation

$$\frac{p^2(t)}{2m} \geq 1$$

for all t. So the point particle reaches zero in a finite time. Hence V is incomplete.

Remark: Geodesic completeness /

Let $(\mathcal{M}, g)$ be a spacetime.

A curve in $\mathcal{M}$ is a smooth mapping

$$\gamma : \underset{\substack{\uparrow \\ \text{open}}}{I} \subset \mathbb{R} \longrightarrow \mathcal{M}$$

I has a coordinate system consisting of the identity map u of I. At each $t \in \mathbb{R}$ we can picture the coordinate vector $\frac{d}{du}\big|_t \in T_t\mathbb{R}$ as the unit vector at t in the positive u-direction.

Concept (Velocity vector) : //

Let $\gamma : I \longrightarrow \mathcal{M}$ be a curve.

The velocity vector of γ at $t \in I$ is

$$\dot\gamma(t) \overset{\text{df}}{=} d\gamma\left(\frac{d}{du}\Big|_t\right) \in T_{\gamma(t)}\mathcal{M}. \quad //$$

Note that $\dot\gamma$ does not involve geometry!

The acceleration $\ddot\gamma$ does involve geometry:

If $\ddot\gamma = 0$, then $\dot\gamma$ is said to be parallel.

A geodesic in a spacetime $\mathcal{M}$ is a curve

$$\gamma : I \longrightarrow \mathcal{M}$$

whose vector field $\dot\gamma$ is parallel. Equivalently, geodesics are the curves of vanishing acceleration.

Statement: // Given any tangent vector $v \in T_p\mathcal{M}$ there is a unique geodesic γ_v in $\mathcal{M}$ such that

a) $\dot\gamma_v(0) = v$

b) $\mathrm{dom}(\gamma_v)$ is the largest possible: If $\alpha : J \longrightarrow \mathcal{M}$ is a geodesic with $\dot\alpha(0) = v$, then $J \subset I$ and $\alpha = \gamma_v|J$.

//

Because of b) γ_v is said to be <u>maximal</u>.

A spacetime for which every maximal geodesic is defined on the entire real line is said to be <u>geodesically complete</u>.

<u>Concept</u> (quantum-mechanically complete):

The potential $V(q)$ is called <u>qm-complete</u> if $H = \frac{1}{2m} p \cdot p + V(q)$ is essentially self-adjoint on $C_0^\infty(0,\infty)$.

II Formal Constructions

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime.

Choose a time function t and a vector field v on $\mathcal{M}$ so that $\nabla_v t = -1$ and the spacelike surfaces $(\Sigma_t)_{t \in I \subset \mathbb{R}}$ are Cauchy hypersurfaces. In most cases $I =]0, t_{in}]$.

Let $\mathcal{C}(\Sigma_t)$ be the set of instantaneous field configurations $\phi : \mathcal{M} \longrightarrow \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Consider $(\mathcal{C}(\Sigma_t), \mathcal{D}\phi)$ to be a formal measure space.

Let $\mathcal{L}^2(\mathcal{C}(\Sigma_t), \mathcal{D}\phi)$ be the $\mathbb{C}$-vector space of wave functionals

$$\Psi_t : \mathcal{C}(\Sigma_t) \longrightarrow \mathbb{C}$$

which are measurable and $\int \mathcal{D}\phi \, |\Psi_t[\phi]|^2$ exists.

$\mathcal{L}^2(\mathcal{C}(\Sigma_t), \mathcal{D}\phi)$ is equipped with the semi norm

$$\| \Psi_t[\phi] \| = \left\{ \int \mathcal{D}\phi \, |\Psi_t[\phi]|^2 \right\}^{1/2}.$$

In order to promote this to a norm introduce

$$\mathcal{N}(\mathcal{C}(\Sigma_t), \mathcal{D}\phi) \stackrel{\text{df}}{=} \left\{ \Psi_t \in \mathcal{L}^2(\mathcal{C}(\Sigma_t), \mathcal{D}\phi) : \Psi_t[\phi] = 0 \ \mathcal{D}\phi\text{-almost everywhere} \right\}$$

The physical state space is the quotient space

$$L^2(\mathcal{C}(\Sigma_t), \mathcal{D}\phi) = \mathcal{L}^2(\mathcal{C}(\Sigma_t), \mathcal{D}\phi) / \mathcal{N}(\mathcal{C}(\Sigma_t), \mathcal{D}\phi).$$

Interpretation: If $\mathcal{U}$ is a measurable subset of $\mathcal{C}(\Sigma_t)$ and $\chi_{\mathcal{U}}$ its indicator functional, then

$$\| \chi_{\mathcal{U}} \Psi_t[\phi] \|^2$$

is the probability for the field configuration on Σ_t to be given by some $\phi \in \mathcal{U}$.

Quantisation:

1) Smeared configuration field operator $\Phi[f]$ is just the operator for multiplication with $\phi[f]$. Its domain is characterized by demanding

$$E_{\Psi_t}(\Phi[f]) = \| \sqrt{\phi[f]}\, \Psi_t[\phi] \|^2$$

to be well defined.

2) Conjugated momentum field operator $\Pi[f]$ is characterized by Heisenberg's fundamental uncertainty relation

$$[\Phi[f_1], \Pi[f_2]] = i\,(f_1, f_2).$$

III QFT - Complete

Concept (strongly continuous semigroup of evolution operators) /

A family $\{\mathcal{E}(t,t_0) : t, t_0 \in I \subset \mathbb{R}^+\}$ of evolution operators

$$\mathcal{E}(t,t_0) : L^2(\mathcal{C}(\Sigma_{t_0}), \mathcal{D}\phi) \to L^2(\mathcal{C}(\Sigma_t), \mathcal{D}\phi),$$
$$\Psi_{t_0} \to \Psi_t \overset{\mathrm{df}}{=} \mathcal{E}(t,t_0)\, \Psi_{t_0}$$

is a strongly continuous semigroup if

(1) $\mathcal{E}(t_0,t_0) = \mathrm{id}_{L^2}$

(2) $\mathcal{E}(t,s)\,\mathcal{E}(s,t_0) = \mathcal{E}(t,t_0)$

(3) $I \to L^2(\mathcal{C}(\Sigma_t), \mathcal{D}\phi)$, $t \to \mathcal{E}(t,t_0)\, \Psi_{t_0}$ is continuous for each $\Psi_{t_0} \in L^2(\mathcal{C}(\Sigma_{t_0}), \mathcal{D}\phi)$. /

A probabilistic interpretation is only possible for a special class of evolution semigroups:

Concept (contractive evolution semigroup): /

A contractive evolution semigroup is a strongly continuous evolution semigroup and moreover

$$\inf \{ C \geq 0 : \| \mathcal{E}(t,t_0)\Psi_{t_0} \| \leq C \, \|\Psi_{t_0}\| \text{ for all } \Psi_{t_0} \in L^2(\mathcal{C}(\Sigma_{t_0}), \mathcal{D}\phi) \}$$

for all $t \in I$. /

We still have

$$\mathcal{E}(t,t_0) = \exp\left\{ -i \int_{t_0}^{t} ds \; H[\Phi, \pi; g] \right\},$$

$$\underset{\substack{\uparrow \\ \text{generator}}}{H[\Phi,\pi;g]} = \int_{\Sigma_s} d\mu \; \mathcal{H}(\Phi, \pi; g_{\Sigma_s}).$$

Question: Is there an analogue

$$(\text{unitary}, \text{self-adjoint}) \mathrel{\widehat{=}} (\text{contractive}, \,\cdot\,) \;?$$

Answer: Yes:

Consider $S \in [L^2(C(\Sigma), D\phi)]^*$ with

$$\|S\| = \|\Psi_t\|^2 .$$

Formally, Hahn-Banach guarantees existence.

Concept (accretive): The generator H is called <u>accretive</u> if

$$\mathrm{Im}\big(S(H\Psi_t)\big) \leqq 0$$

for any $\Psi_t \in \mathrm{dom}(H)$ and for all $t \in I$.

$$(\text{unitary}, \text{self-adjoint}) \mathrel{\widehat{=}} (\text{contractive}, \text{accretive})$$

Concept (qft-complete):

Let Σ_0 be a spacelike geodesic boundary of (M, g) located at $t \to 0$.

The spacetime (M, g) is called <u>qft-complete</u> if the evolution semigroup is contractive in (M, g) and if

$$\|\mathcal{E}(t, t_0)\|_{\inf} \xrightarrow{t \to 0} 0 .$$

Kernel method:

Consider a bilinear functional

$$\mathcal{K}_t : \mathcal{C}(\Sigma_t) \times \mathcal{C}(\Sigma_t) \longrightarrow \mathbb{C},$$

$$\mathcal{K}_t[\phi_1, \phi_2] \overset{\mathrm{df}}{=} \int_{\Sigma_t} d\mu(x) d\mu(y)\, \phi_1(x)\, \mathcal{K}_t(x,y)\, \phi(y) .$$

If the BKL-conjecture holds then

$$\mathcal{K}_t(x,y) = k(t)\, \delta(x,y)$$

in the vicinity of the geodesic boundary.

Ground state:

$$\Psi_t[\phi] = \mathcal{N}_t\, \mathcal{G}_t[\phi]$$

$$\mathcal{N}_t = \mathcal{N}_{t_0} \exp\left\{ \frac{i}{2} \int_{t_{in}}^{t} ds \int_{\Sigma_s} d\mu\, \frac{1}{2} \pi^2 \mathcal{K}_s[\phi,\phi] \right\}$$

$$\mathcal{G}_t[\phi] = \exp\left\{ -\frac{1}{2} \mathcal{K}_t[\phi,\phi] \right\}$$

IV Black Holes

The kernel is of BKL-type

$$\mathcal{K}_t(x,y) \approx \frac{i}{t^2 |\ln(t/t_{in})|}\, \delta(x,y) + \text{less sing.}$$

in the vicinity of the geodesic border located at $t \to 0$.

Hence,

$$\Psi_t \approx 0 .$$

So qft grants no probabilistic support to the geodesic border. Events cannot populate Σ_0 and therefore measurement cannot probe the singularity.