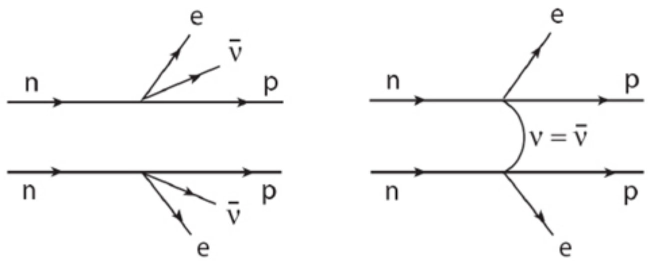

FIG. 1: Feynman diagram for a double beta decay. Typically there would be two anti-neutrinos emitted. But if the neutrino is a Majorana particle, it is the same as an antineutriono and the reaction could take place without any neutrinos emitted. Taken from: http://sites.uci.edu/energyobserver/2013/05/31/michael-moe-honored-for-the-discovery-of-double-beta-decay-at-uc-irvine/

Majorana Fermions

1. What are majorana fermions?
2. The Kitaev model
3. Majorana edge states in the Kitaev model
4. Majorana states on TI edges
5. Majorana in wires
6. Majroana and non-abelian statistics
7. Floquet topological insulators (?)

I. GENERAL INTRO

In 1937 Ettore Majorana speculated that there could be a particle that is its own antiparticle.

In the language of second quantization, this is simply the statement that creating the particle, and annihilating it should be done with the same operator. Therefore:

$$\gamma = \gamma^\dagger \tag{1}$$

Since then high energy crowd is trying feverishly to find such a particle. Perhaps the neutrino is a majoarana particle? If so one may find a neutrinoless double beta decay (see Fig. 1).

But in the realm of table-top physics, this idea made a splash with the paper by Alexei Kitaev which specified that Majorana fermions in 2d systems will, due to their remarkable statistics - non-abelian - and that their non-local hilbert space will, enable a completely protected storage of quantum information, and braiding them will allow realizing almost all quantum gates (with one important missing ingredient).

A point of concern: if you can't really find majoranas in billion dollar experiments. How would we be able to find them in electronics that you could buy in radio-shack? [History, however, indicates that this would not be the first time.. cf Higgs]

Good question. One thing we know about Alexei Kitaev, is that he always has an answer. In this case it was the Kitaev model.

Today I would like to start with explaining a bit my perspective on majoranas. Then move on to show a model - the Kitaev model - that allows the exposed formation of Majoranas. After that talk about how majoranas could be realized in other systems, with a favorite example for me: the Fu-Kane majoranas. This will all be in preparation for what we'll discuss tomorrow - Majoranas in quantum wires, their non-abelian nature, and using wire networks for quantum computation.

In the last day I will shift gears a bit and talk about periodically driven systems and how they can becme topological.

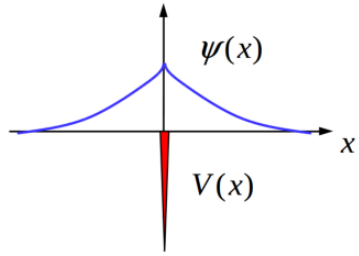

FIG. 2: A delta-function potential dip, and the bound state associated with it.

II. MAJORANA INTRO

How do we even start thinking about majoranas? start simple. Start wth a single bound state, say at a delta-function potential.

The hamiltonian is:

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} - V\delta(x) \tag{2}$$

There are a ton of positive energy states here. We don't care about them. There is only one negative energy state:

$$\psi(x) = \exp(-|x|/\xi) \tag{3}$$

What is the spectrum in this many-body hilbert space, which is only concerned with the negative energy part of the single particle spectrum? The state could be empty:

$$|0\rangle, E = 0 \tag{4}$$

and it could be full:

$$|1\rangle = c^\dagger |0\rangle, E = -\epsilon \tag{5}$$

We can see that also $|0\rangle = c|1\rangle$.

This may sound silly to you, but even here, there are Majoranas hiding. Nothing stops me from defining:

$$\gamma_+ = c + c^\dagger, \; \gamma_- = -i(c - c^\dagger) \tag{6}$$

both are real operators:

$$\gamma_\pm = \gamma_\pm^\dagger \tag{7}$$

both square to 1:

$$\gamma_\pm^2 = cc^\dagger + c^\dagger c = 1 \tag{8}$$

and they anticommute:

$$\{\gamma_+, \gamma_-\} = 0 \tag{9}$$

Furthermore, we can construct a number operator out of them:

$$n = c^\dagger c = 0.5(i\gamma_+\gamma_- + 1) \tag{10}$$

you can confirm that $i\gamma_+\gamma_- = 2c^\dagger c - 1$, which is the minus the parity. The hamiltonian is then:

$$H = -\epsilon c^\dagger c = -\epsilon/2 \cdot i\gamma_+\gamma_- + \text{const} \tag{11}$$

There you go! Majoranas! Majorana hamitonian, majorana states. so what's the big deal?

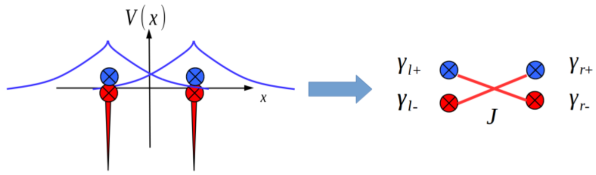

FIG. 3: The two-well system, and the four majoranas that describe it. In the majorana representation of the simple two-site hamiltonian (right) there are two crossing interaction lines.

Big problem here. Both operators are associated with a single wave function. they are like quarks. Inseparable. There is only one state here really. And one fermionic state is described by two majorana states. A basis rotation from the creation and annihilation. How do we separate them?

Let's think of two traps tehn. The hamiltonian is:

$$H = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} - V(\delta(x - L/2) + \delta(x + L/2)) \tag{12}$$

Roghly, there are now two low energy states:

$$\psi_{\ell,\, r} = \exp(-|x \mp L/2|/\xi) \tag{13}$$

Associated with the states on the left and the right. They make a good subspace for writing down a low energy hamiltonian in the space of $|\ell\rangle$ and $|r\rangle$:

$$H = \begin{pmatrix} -\epsilon & -J \\ -J & -\epsilon \end{pmatrix} \tag{14}$$

with $J = |\langle\ell|\, V(\delta(x - L/2)\, |r\rangle\, |$. This neglects the overlap between the two wave functions - but you could add that effect on your own later, and ask me for details if you can not. [Add here the inverted matrix]

Clearly there are two eigenstates - the symmetric and antisymmetric:

$$|\pm\rangle = |\ell\rangle \pm \langle r| \tag{15}$$

Easy. But we're not interested in these so much. We are more interested in the hamiltonian in a second quantized form:

$$H = -\epsilon \cdot (c_+^\dagger c_+ + c_-^\dagger c_-) - J(c_+^\dagger c_- + c_+^\dagger c_-) \tag{16}$$

and in Majorana language:

$$\begin{aligned} H &= -\epsilon/1 \cdot (i\gamma_{\ell+}^\dagger \gamma_{\ell-} + i\gamma_{r+}^\dagger \gamma_{r-}) - J/4\,((\gamma_{\ell+} + i\gamma_{\ell-})(\gamma_{r+} - i\gamma_{r-}) + (\gamma_{r+} + i\gamma_{r-})(\gamma_{\ell+} - i\gamma_{\ell-})) \\ &= -\epsilon/1 \cdot (i\gamma_{\ell+}^\dagger \gamma_{\ell-} + i\gamma_{r+}^\dagger \gamma_{r-}) - J/2\,(i\gamma_{\ell-}\gamma_{r+} - i\gamma_{\ell+}\gamma_{r-}) \end{aligned} \tag{17}$$

If ϵ were zero, we would have an easy time reading the solution. Take one of the combinations:

$$H = -iJ/2\,(\gamma_{\ell-}\gamma_{r+} - [\gamma_{\ell+}\gamma_{r-})] \tag{18}$$

we see that:

$$J/2[-i\gamma_{\ell-}\gamma_{r+}, \gamma_{\ell-}] = iJ\gamma_{r+} \tag{19}$$

and

$$J/2[-i\gamma_{\ell-}\gamma_{r+}, \gamma_{r+}] = -iJ\gamma_{\ell-} \tag{20}$$

so

$$J/2[-i\gamma_{\ell-}\gamma_{r+}, \gamma_{\ell-} \pm \gamma_{r+}] = \pm J(\gamma_{\ell-} \pm i\gamma_{r+}) \tag{21}$$

These are the ladder operators of the hamiltonian. By the same token, we also have the other set, with signs reversed. See Fig. 3.

FIG. 4: The necessary magic trick for getting a majorana

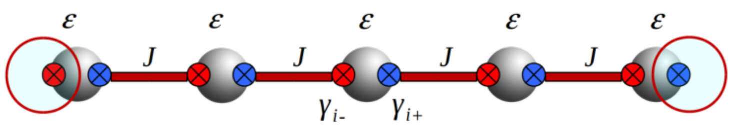

FIG. 5: The kitaev chain. As if one of the majorana interactions are missing in each bond. There are two edge majoranas that are decoupled if $\epsilon = 0$.

A. Kitaev's magic trick

What is our goal? We want to be able to take these majoranas - two for a state - and bring them apart. How can we break such an integral object? The two majoranas belong to a single state! We need something like Fig. 4...

We do habe a magician, though. Kitaev said - what about just removing one of the interaction lines in the two site problem? Let's make the model into a chain, and let's not have the two couplings between the majoranas - let's just have a linked chain.

This would be the following model:

$$H = \frac{1}{2}\sum_i \left(-i\epsilon\gamma_{i+}\gamma_{i-} - iJ\gamma_{i+}\gamma_{(i+1)-}\right) \tag{22}$$

Why is this magic? Check this out. set $\epsilon = 0$. The Majoranas are now disconnected. We can find the eigen creation and annihilation operators in each bond:

$$d_i = \gamma_{i+} + i\gamma_{(i+1)-},\, d_i^\dagger = \gamma_{i+} - i\gamma_{(i+1)-} \tag{23}$$

But this leaves a set of two zero modes on the two sides. left and right:

$$[H,\, \gamma_{0-}] = [H,\, \gamma_{L+}] = 0 \tag{24}$$

Two majorana edge modes! Thi is clearly not the case if we have $J = 0$, and $\epsilon > 0$. Then each site is left to its own, and there are no zero modes...These two points we discussed are simply speaking the example points of the topological and trivial pahse of the kitaev model.

B. Kitaev model for electrons

What is the model in terms of the original electrons? We need to key in the majorana definitions int he hamiltonain 26. not hard to do, with:

$$c_i = \frac{1}{2}\left(\gamma_{i+} + i\gamma_{i-}\right),\, c_i^\dagger = \frac{1}{2}\left(\gamma_{i+} - i\gamma_{i-}\right) \tag{25}$$

And we get:

$$H_K = \sum_i \left(-\epsilon c_i^\dagger c_i - \frac{J}{2}(c_{i+1} - c_{i+1}^\dagger)(c_i + c_i^\dagger)\right) = \sum_i \left(-\epsilon c_i^\dagger c_i - \frac{J}{2}(c_{i+1}^\dagger c_i + c_i^\dagger c_i) + \frac{\Delta}{2}(c_{i+1}c_i + c_i^\dagger c_{i+1}^\dagger)\right) \tag{26}$$

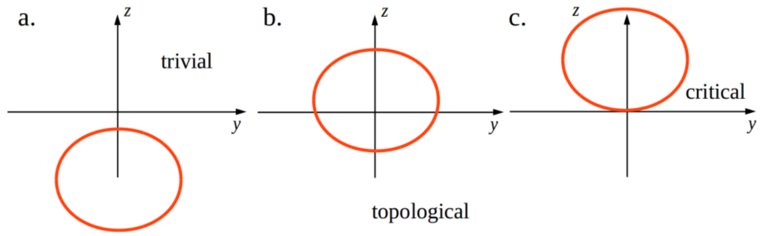

FIG. 6: The hamiltonian in tau space traces a circle (or ellipse when $J \neq \Delta$). (a) When $|\epsilon| > J$, the circle is notencircling the origin, and the phase is trivial. (b) When $|\epsilon| < J$ the circle surrounds the origin. (c) At criticality, the vector vanishes somewhere - this the gapless second order phase transitino between weak and strong pairing superconductivity

and I took the liberty to use Δ for the pairing term (for the Kitaev original case, we have $\Delta = J$). In principle, we could play with the relative strength of the two bond parameters here - and this is how we would get Δ and J to be different.

The pairing, though, is between a single flavor of fermions - no spin up and down! The kitaev model is a p-wave superconductor. That's wjat it takes to make majoranas.

C. Kitaev model in Nambu representation

The easiest way to think about the kitaev model is to make it into a BdG equation. It is translationally invariant, so clearly things should work well also if we do Fourier transforms. Let's do the latter first. moving to c_k, $c_k^\dagger$ we can easily see that:

$$\begin{aligned} \sum_i c_i^\dagger c_i &= \sum_k c_k^\dagger c_k, \\ \sum_i c_i^\dagger c_{i+1} &= \sum_k e^{ik} c_k^\dagger c_k, \\ \sum_i c_i c_{i+1} = \sum_k (c_k c_{-k} e^{-ik} + c_{-k} c_k e^{ik}) &= -\sum_k (c_k c_{-k} \cdot 2i \sin k). \end{aligned} \tag{27}$$

For a BdG representation it is clear we get:

$$H_{BdG} = \tau^z(-\epsilon - J\cos k) - \Delta \tau^y \sin k \tag{28}$$

where the τ^α are the pauli matrices.

D. Topology of the Kitaev model

Here is how you must think of this. The τ^α are like unit vectors. The hamiltonian is a vector in a 2d space, which is a function of a one dimensional momentum, If you wish, it is a mapping from e^{ik} - the unit circle in the complex plane, to another 1d plane which we'll call the tau space. We kniow that this gives rise to topology. Indeed, in the tau space, the vector traces a circle. See Fig. 6.

The solutions as a funcion of momentum are obvious. What we care about the most are the edge states.

E. Edge states in the Kitaev Model

To get the edge states, let's consider what we want. We would like a Majorana edge state. It should be at $E = 0$. This is dictated by particle hole symmetry (**Exercise:** Construct p=h symmetry for this hamiltonian), and our wish to find a single state at the edge. $E \neq 0$ would imply two solutions at $\pm|E|$. Second, we would be really surprised if the form of the solution were not really closely related to:

$$\psi(x) \to \begin{pmatrix} u \\ v \end{pmatrix} e^{-qx}. \tag{29}$$

The boundary condition should be $\psi = 0$ at the edge. In lattice models this is essentially always the case for localized mid-gap states.

This makes our job somewhat easy. Step one: replace $e^{ik} \to e^{-q}$. Step two - solve for e^{-q}. Step three: see if you can fulfill boundary conditions. This gives:

$$H \to \tau^z(-\epsilon - \frac{J}{2}(e^q + q^{-q})) + i\frac{\Delta}{2}\tau_y(e^{-q} - e^q) \tag{30}$$

We want to find zero solutions of this, so some q where $H^2 = 0$. Actually, looking at the hamiltonian, I was going to suggest squaring it - the τ's either square to 1 or kill eachother due to anticommutation. We get:

$$\epsilon + J\cosh q = \pm\Delta\sinh q \tag{31}$$

This already looks like we are going to get more than one solution. Which solutions would we need?

Which is legit? The only way we can get $\psi = 0$ boundary condition satisfied, is if we had two solutions that had the same eigenvector in the kernel of H:

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}(e^{-q_1 x} - e^{-q_2 x}) \tag{32}$$

Looking back it seems that we can have two types of hamiltonians depending on the sign we choose for the first square root:

$$\tau^z \pm i\tau^y = \begin{pmatrix} 1 & \pm 1 \\ \pm(-1) & -1 \end{pmatrix} \tag{33}$$

Both possibilities are fine. They have the solutions (respectively):

$$\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} \tag{34}$$

as an eigenvector that makes them vanish. Good sign! We eant a majorana, and what are these amplitudes if not the coefficients of c and $c^\dagger$, and therefore add up to something reasonable and real (or pure imaginary, whch is easy to rectify). Then, for each choice of a sign, we want two solutions for e^-q that are smaller than 1. So let's make the choice of + in eq. (31). and:

$$e^{-2q}(J + \Delta) + 2e^{-q}\epsilon + (J - \Delta) = 0 \tag{35}$$

Easy to solve:

$$e^{-q} = \frac{-\epsilon \pm \sqrt{\epsilon^2 - (J^2 - \Delta^2)}}{(J + \Delta)} \tag{36}$$

We see that there are only two such solutions that are smaller than 1 when $\epsilon < J$. We can just dial in $\epsilon = J$, and immediately we get:

$$e^{-q} = \frac{-J \pm \Delta}{J + \Delta} \tag{37}$$

So one of the solutions is 1.

If you were in doubt regarding the boundary condition, here is the explanation. The reason for the homogeneous boundary conditions is really that we are trying to diagonalize a hamiltonian with no translation invariance using the eigenstates of a uniform problem. We can do it - as long as we find a combination that satisfies the right boundary conditions. The wave function should not venture to places where there is no lattice.

Exercises:

1. Find edge states for a domain wall between Δ and $\Delta e^{i\phi}$. These states are complicated and will not lie at $E = 0$, so assume $\epsilon = 0$ for this to makes things simple. What is $E(\phi)$?

2. Find an edge state of the kitaev model between two domains - one topological with $\epsilon = 0$ and the other with $\epsilon = 4J$. Assume $\Delta = 2J$.

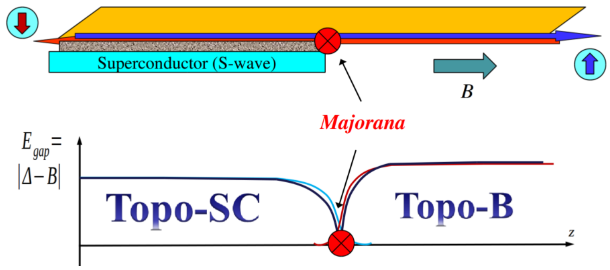

FIG. 7: (a) Proximity and magnetic field induced gapped domains. (b) In a domain wall between the B and Δ phases, there is a point of vanishing gap, which supports a majorana

III. P-WAVE SUPERCONDUCTOR PLEASE...

Where can we find a Kitaev model? Kitaev says in his paper from 2001 that he doesn't think that solid state will give a possible model. He says that spin-orbit coupling might be helpful, but will not eb sufficient. Luckily, he was wrong on that.

A. Edges of topological insulators

As you by now heard, topological insulators in 2d possess a helical edge mode. It has right-moving up electrons, left-moving down electrons, and an hamiltonian that could be gapped with a magnetic field perpendicular to the spin-orbit direction. In momentum space:

$$H_{FK} = v\sigma^z p + \vec{B}\cdot\vec{\sigma} \tag{38}$$

But such a model can also be gapped by superconductivity. When we take superconductivity into account, the (BdG) hamiltonian for this becomes:

$$H_{FK} = v\sigma^z\tau^z + B_x\sigma^x + B_y\sigma^y + \mathrm{Re}\Delta\tau^x + \mathrm{Im}\ \Delta\tau^y \tag{39}$$

Let's simplfy to real Δ and $\vec{B} = B\hat{x}$. Then we have:

$$H_{FK} = vp\sigma^z\tau^z + B\sigma^x + \Delta\tau^x \tag{40}$$

This is the Fu Kane Hamiltonian. They suggested the construction for localizing individual majoranas. This may look confusing - why no nambu τ's on the magnetic field term? The second quantize hamiltonian is:

$$\hat{H} = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\downarrow} \\ -c^\dagger_{-k\uparrow} \end{pmatrix}^\dagger H_{FK} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\downarrow} \\ -c^\dagger_{-k\uparrow} \end{pmatrix} \tag{41}$$

As an exercise, you can confirm that H_{FK} is the correct form for the hamiltonian with σ^α operating in the 12-12 and 34-34 subspaces, and τ^α connecting the 12 and 34 sectors.

As it turns out, this model, too, has a phase transition when $|\Delta| = |B|$. And if we have a domain wall between these two phases, it, too has a majorana.

To see this, start by squaring the hamiltonian:

$$H^2 = (vp)^2 + B^2 + \Delta^2 + \sigma^x\tau^x 2B\Delta \tag{42}$$

This would be easy, except for the $\tau^x\sigma^x$. Well, we are in luck! This commutes with the hamiltonian. All we need to do is choose a value for it, and stick with it. choose $\sigma^x\tau^x = -1$. Then we have:

$$H^2 = (vp)^2 + (B-\Delta)^2 \tag{43}$$

Indeed gaplessness at the point of crossing. And a gap $E_{gap} = |B - \Delta|$ in general. But how to see the majorana? Here is a neat way. Let's have:$B = \Delta + bx$. Also, note that if we know that $\sigma^x \tau^x = -1$ we also know that $\tau^x = -\sigma^x$. So:

$$H_{FK} \to pv\sigma^z\tau^z + (B(x) - \Delta)\sigma^x = pv\sigma^z\tau^z + bx\sigma^x \tag{44}$$

To some of you this will look familiar. It has the same structure as Landau levels in a Dirac cone. What do we know about LL's in a Dirac cone? They have a zero mode. Square the hamiltonian, and what do you get?

$$H_{FK}^2 = v^2p^2 + b^2x^2 + vbi\sigma^y\tau^z[p, x] \tag{45}$$

The eigenvalues of the first operator are known: $(n + 1/2)\omega = (n + 1/2)\sqrt{2v^2 \cdot 2b^2} = (2n + 1)vb$. The second term has an operator on it. No fear - it commutes with $\sigma^x\tau^x$, so we can diagonalize it too. Guess what - it is either plus or minus 1.So:

$$E^2 = (2n + 1)vb \pm vb \tag{46}$$

$n = 0$ and the minus choice gives a zero mode. It is a majorana mode - easy to see. Not hard to figure out that all other states are the $E = \pm\sqrt{(2m)vb}$, with m integer.

B. Majorana wires

You would think that topological edges are good enough. You are almost right. But they are hard to handle, hard to fabricate outside Leiden, and maybe also Rice, and they are very hard to put in proximity to a superconductor. You could think of 2d systems in general, but ultimately there will always be complications. The was one insight from two dimensions that came from Jay Say, Roman Lutchyn and Sankar Das Sarma, and also from Jason Alicea: Use spin orbit in regular wells. With α describing Rashba coupling, and β describing the Dresselhaus coupling, we have:

$$H^2 = p^2/2m + (p_x\sigma^y - p_y\sigma^x)\alpha + \beta p_x\sigma^z \tag{47}$$

This could be gapped with a B field perpendicular to the plane in the case of just Rashba, and parallel to the plane in the case of a dominant Dresselhaus interaction in [110] growth direction, and also with a superconductor. The magnetic field essentially eliminates one spin flavor, and the superconductor makes for a p-wave superconductor. The core of vortex in the superconductor will have a Majorana.

But moving vortices is hard. We want wires.

To realize Majoranas in quantum wires we need semiconducting wires with strong spin orbit: InAs or SbTe were the examples that came to our minds, then we almost get the edge states of the topolgoical insulators:

$$H = p^2/2m + \sigma^z\lambda p \tag{48}$$

At low momenta, this is nothing but the topological insulator helical edge. Further away, at high momenta, there are additinal states near energy $E = 0$ which we need to care about. Problem. But this problem has a very simple solution. Gap the whole thing with a superconductor. See Fig. 8.

Just as before, we can dress this with a pairing, and upgrade to a nambu representation:

$$H = \tau^z(p^2/2m - \mu + pv\sigma^z) + B\sigma^x + \Delta\tau^x \tag{49}$$

And nothing much is changed frm the case of the topological insulator. When $B = \Delta$ the k=0 piece has a gapless point, and a phase transition, while the high memontum regions are still gapped. Interestingly, hre we can tune a pahse transition in terms of chemical potential. At $p = 0$, we see that

$$H = -\mu\tau^z + B\sigma^x + \Delta\tau_x \tag{50}$$

The eigenvalues of this hamiltonian are clear:

$$E = \pm B \pm (\mu^2 + \Delta^2) \tag{51}$$

so now the phase transition occurs at $\mu^2 + \Delta^2 = B$, by and large.

Majorana states appear either at the edge of a wire in the topological phases, or in domain walls, as in Fig. 9.

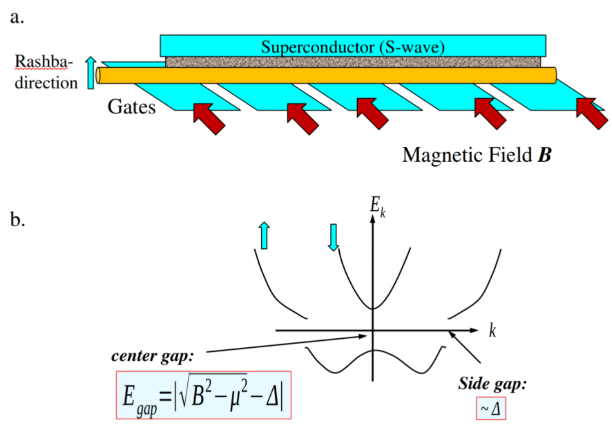

FIG. 8: (a) A wire with proximity to a superconductor, a magnetic field, and a series of gates. (b) Dispersion of a spin-orbit coupled wire.

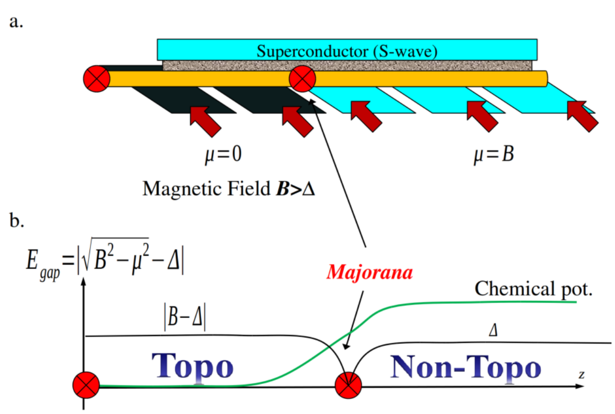

FIG. 9: A domain wall between the two phases of the majorana-supporting quantum wires could be tuned using a space-dependent chemical potentials. This could be achieved by an array of individually controlled gates, and a chemical potential that depends on space.

In fact, this model becomes the Kitaev model, pretty much at the limit of high magnetic field. Think of the limit of $B \sim \mu \gg m\lambda^2$. In this case only the down spins survive. Without superconductivity we have:

$$H = \frac{p^2}{2m} + vp\vec{e}\cdot\vec{\sigma} + B\sigma^z \tag{52}$$

assuming that B is large, the wave function in the lower band would be:

$$\Psi(x) = \begin{pmatrix} \alpha \\ 1 \end{pmatrix} e^{ipx} \tag{53}$$

Doing first order perturbation theory we see that

$$\alpha \sim \frac{vp(e_x - ie_y)}{2B} \tag{54}$$

Substituting this into the Hamiltonian with pairing we have:

$$H = (\frac{p^2}{2m} - \mu - B)c_p^\dagger c_p + \Delta\frac{vp(e_x - ie_y)}{2B}c_p c_{-p} + H.C. \tag{55}$$

This is the continuoum limit of the Kitaev model.

Generally speaking, the penetration length of the majorana states will be given as $v/\hbar E_{gap}$.

The leaders in realizing majoranas in such wires are Leo Kouwenhoven, Moti Heiblum, and Charlie Marcus. It is interesting to note what the relevant parameters are for the wires. Marcus managed to obtain hard gaps of the

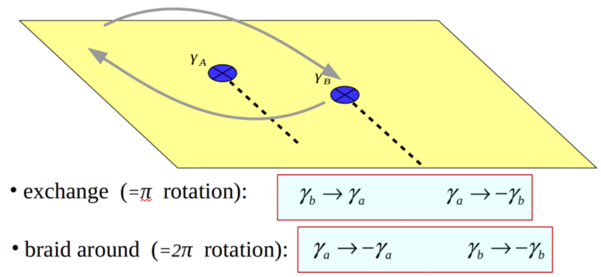

FIG. 10: Each majorana fermion has a branch cut, and when other majorana cross it, their operator gets a minus sign. Upon exchange this happens only to one majorana, but a full braid will make both majoranas have a minus sign. Contrast this with regular fermions, for which an exchange produces a minus sign, and a braid returns the state to its original phase.

order of 2K in InAs wires. Kouwenhoven measured spin orbit coupling of upto $1eV\mathring{A}$, which correspond to $10^5 m/s$ spin-orbit velocity in InSb. The g factors of the wires vary depending on wire thickness and can rage from 2 to 30, presumably.

C. Majorana in iron chains

A recent development, which I had not fully digested yet, is the realization of the same model as the wires by placing a chain of iron atoms on lead superconductors. The lead is heavy - hence spin orbit. The iron is magnetic - hence the Zeeman field. It is sitting atop a superconductor. Seems like all ingredients are there. So far reported are gaps of order $0.1meV$ and a remarkably short decay length. Authors to follow: Yazdani and Bernevig's group for the discovery and initial picture, and von Oppen and Glazman for the more detailed explanation.

IV. MAJORANAS AND QUANTUM COMPUTATION

A. Majorana statistics

What makes maojranas so coveted? They are nonabelian anyons. Let's cosider two separated majoranas swimming in an ocean of p-wave superconductivity. call them γ_A and γ_B. First off, you can store information in them in a really non-local way. The parity of the two majoranas can encode information (although not yet quantum). The hilbert space, as you may recall is:

$$|0\rangle\,,\ |1\rangle = \frac{1}{2}(\gamma_A + i\gamma_B)\,|0\rangle \tag{56}$$

This is amazing - since the information about the parity is not in one place or the other. It is everywhere. If the two majoranas are far from eachother, then nothing local can actually detect their state. This is the point that Kitaev made. Remember?

Each majorana operator is made out of fermions. And fermions have to get a -1 upon exchange. This -1 could be facilitated by thinking of a branch cut - a string that stretches from each fermionic entity, and when other fermionic cross it, they get slapped with the -1. If we have a full braid - then each fermion will get a -1, and overall, the wave function will remain invariant.

With majoranas it becomes more complex. Literally. Now instead of electrons think of the majoranas. Upon exchange, γ_B goes through the string of γ_A, but not vice versa. Then they replace each other: $\gamma_A \rightarrow -\gamma_B$ and $\gamma_B \rightarrow \gamma_A$. For the Hilbert space above, the operator $\gamma_A + i\gamma_B$ is of extreme importance:

$$\gamma_A + i\gamma_B \rightarrow -\gamma_B + i\gamma_A = i(\gamma_A + i\gamma_B) \tag{57}$$

hmm. . . the state just got slapped a i^n factor, with n the occupation, 0 or 1. Do it again, and one gets $(-1)^n$. This way of doing this follows from Ivanov's "Non-abelian statistics of half-quantum vortices in p-wave superconductors", D. A. Ivanov, Phys. Rev. Lett. 86, 268 (2001).

B. Nonabelianess and qubits

For real quantum information perspective, we need to store the info in something that really could have an honest to god superposition. This is not the case with parity, since a system can not be at a superposition of two parity states. So let's add two other majoranas. This makes a qubit. The four majoranas, say A $\pm$ and B $\pm$ could be label with an even parity state:

$$|0_A 0_B\rangle\,,\;|1_A 1_B\rangle \tag{58}$$

And similarly for the odd parity. The superconducting sea gives us any number of cooper pairs we want, so no problem there. It is this doublet of same-parity states that allow us to protect and process quantum information. Protection: due to the non-locality of the parity information in the majorana doublets. Process: through braiding.

now suppose we take Majoranas A_+ and B_+ and exchange them. What happens? To figure it out We need to know how to change bases. Let's go back to the creation operators (allow me to drop factors of 1/2):

$$c_A = \frac{1}{2}\left(\gamma_{A+} + i\gamma_{A-}\right),\; c_B = \frac{1}{2}\left(\gamma_{B+} + i\gamma_{B-}\right) \tag{59}$$

Would be nice to know how to move to a basis where the parity states belonged to the + and − pairs. Then we would be able to know the results of braiding with no problem whatsoever. We can suppose that:

$$|0_A 0_B\rangle = \zeta(|0_+ 0_0\rangle + \alpha\,|1_+ 1_-\rangle),\; |1_A 1_B\rangle = \eta(|0_+ 0_0\rangle + \beta\,|1_+ 1_-\rangle), \tag{60}$$

We don't care about ζ and η, but α and β are crucial. What are they? For that let's also think of the operators. So let's define:

$$c_+ = \frac{1}{2}\left(\gamma_{A+} + i\gamma_{B+}\right),\; c_- = \frac{1}{2}\left(\gamma_{B-} + i\gamma_{A-}\right), \tag{61}$$

Note the reversal in the second definition. We can't help but have it there. We'll see later that we could have made a different choice, but it is a confusing one to make. Also, note that $|1_+1_-\rangle = c_-^\dagger c_+^\dagger\,|0_+0_-\rangle$. The order matters.

Next, let's work backwards: What are the $c_{A,B}$ in terms of the new $\pm$ basis? Easy.

$$c_A = \frac{1}{2}\left(c_+ + c_+^\dagger + i\frac{1}{i}(c_- - c_-^\dagger)\right),\; c_B = \frac{1}{2}\left(\frac{1}{i}(c_+ - c_+^\dagger) + (c_- + c_-^\dagger)\right) \tag{62}$$

And here is the trick:

$$c_A\,|0_A 0_B\rangle = c_B\,|0_A 0_B\rangle = 0,\; c_A^\dagger\,|1_A 1_B\rangle = c_B^\dagger\,|1_A 1_B\rangle = 0. \tag{63}$$

By using 62 in conjunction with 60, we get:

$$(c_+ + c_+^\dagger + (c_- - c_-^\dagger))(|0_+ 0_0\rangle + \alpha\,|1_+ 1_-\rangle) = |1_+ 0_-\rangle - |0_+ 1_-\rangle + \alpha(|1_+ 0_-\rangle - |0_+ 1_-\rangle) \tag{64}$$

and $\alpha = -1$. In exactly the same way, we find that $\beta = 1$. Put in terms of the basis:

$$|0_A 0_B\rangle = \frac{1}{\sqrt{2}}(|0_+ 0_0\rangle - |1_+ 1_-\rangle),\; |1_A 1_B\rangle = \frac{1}{\sqrt{2}}(|0_+ 0_0\rangle + |1_+ 1_-\rangle). \tag{65}$$

Note now that a different choice of signs in (61) would have led to a contradiction (Try it out!) unless we had assumed that the parity denoted in the $\pm$ basis is different, and that (60) would have different apparent parities on the two sides. Ultimately, it is a gauge choice, and not more. But one must be aware of this.

Now we are ready to do braiding!

$$T_{A_+,B_+}\,|0_A 0_B\rangle = \frac{1}{\sqrt{2}}(T_+\,|0_+ 0_-\rangle - T_+\,|1_+ 1_-\rangle) = \frac{1}{\sqrt{2}}(|0_+ 0_-\rangle + |1_+ 1_-\rangle) = |1_A 1_B\rangle \tag{66}$$

so we have a not gate. Similarly: $T_{A_+,B_+}\,|1_A 1_B\rangle = |0_A 0_B\rangle$.

What about exchange? That's slightly more complicated. But not outrageously so:

$$E_+\,|0_A 0_B\rangle = \frac{1}{\sqrt{2}}(E_+\,|0_+ 0_0\rangle - E_+\,|1_+ 1_-\rangle) = \frac{1}{\sqrt{2}}(|0_+ 0_-\rangle - i\,|1_+ 1_-\rangle) \tag{67}$$

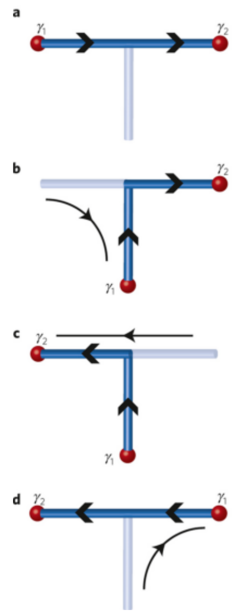

FIG. 11: Sequence of operations to achieve in order to perform an exhange. Taken from Alicea et al. Nature Physics.

What is that? probably some superposition. Indeed,

$$= \frac{1}{2}\left((1+i)\left|0_+0_-\right\rangle + (1-i)\left|1_+1_-\right\rangle\right) \tag{68}$$

This is essentially an implementation of the Hadamard gate.

What else is needed to do quantum computation? According to Kitaev and Bravyi (quant-phys/0403025), we are missing the magic state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left|0_+0_-\right\rangle + e^{i\pi/4}\left|1_+1_-\right\rangle\right) \tag{69}$$

But this requires a $\sqrt{i}^n$ operation on the majoranas. Doing this in a protected way is still and outstanding problem.

Where is the nonabelianness? The two matrices that were constructed above are essentially like a σ^x for braiding, and a Hadamard gate ($\sigma^y + \sigma^z$ or so). These don't commute. The order of operations is important.

V. BRAIDING EXCHANGE OPERATIONS WITH WIRES

A. T-junctions

One may ask, how can we achieve exchange and braiding with wires? Not so many years back, it was important to show that braiding is possible in principle. After all, wires are not a 2d object.

The ubiquitous trick is T-junctions. Depending on your inclination, you may want to call them Y junction though. If we have a topological wire segment, and we would like to exchange the two majoranas, we will need something extra. A place for one of the majornaas to wait while the other is pushing across. Add another wire. And then we can bring the domain wall on the right tot he T intersection, make a portion of the second wire topological, and then bring the left majorana over to the junction, and beyond. Now we can safely bring down what used to be the right majorana and bring it to the place where the left majorana used to be. See Fig. 11

How do we know that this was braiding? We must look into the details. But instead of proving this in full generality, let's do it for a Kitaev model. In particular, in the Kitaev model we had the pairing term be:

$$-\Delta c_j c_{j+1} = \Delta c_{j+1} c_j \tag{70}$$

so Delta pointed from $j+1$ to j. Let's add arrows int he diagram. Now, as the topological phase spills over to the second wire, the direction it dictates is partiuclar. But then when we get back to the original configuration, we see that we had to reverse the sign of the Δ.

If we hadn't, we would run into a situation of Δ changing signs mid line. This leads to two degenerate Marjoanas, that can destroy the quantum information in the system.

This is not the original configuration, is it? If we want to compare the braiding properly, we need to do a guage transformation that will return the physics to what it was. To do it let's invoke our knowledge of the BdG Hamiltonian, where Δ appears as:

$$\Delta\tau^x \tag{71}$$

So if Δ changed sign, and this became $-\Delta\tau^x$, we need to do the following gauge transformation:

$$\mathcal{H} \to e^{i\frac{\tau^z}{2}\pi}\mathcal{H}e^{-i\frac{\tau^z}{2}\pi} \tag{72}$$

The transformation used is simple:

$$e^{i\frac{\tau^z}{2}\pi} = \cos\frac{\pi}{2} + i\tau^z \sin\frac{\pi}{2} = i\tau^z \tag{73}$$

This transformation we can also apply to the MArjoanas. Recall from our calculation, that the left Majorana in a Kitaev wire has the form:

$$\gamma_L = \frac{1}{i}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \gamma_R = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{74}$$

These majoranas replace lable by the T-junction manipulation, which ends up with a $\Delta \to -\Delta$. But the gauge transformation will fix that. Now comes the moment of truth! What do we get at the end, after the gauge transformation back?

$$\gamma_R \to U\gamma_R = i\tau^z \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\gamma_L \tag{75}$$

and:

$$\gamma_L \to U\gamma_L = \tau^z \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \gamma_R \tag{76}$$

Just as with Ivanov. So this works.

B. The y-setup for braiding

From a practical perspective, it might be best to have a stationary configuration where we only change fluxes. Such a configuration was suggested by Beenakker's group. By putting together three wires, all in the topological phase, we achieve a hamiltonian of coupling that is:

$$H = -i\Delta_x\gamma_0\gamma_x - i\Delta_y\gamma_0\gamma_y - i\Delta_z\gamma_0\gamma_z \tag{77}$$

For reasons I will not explain now, but you are welcome to read Beenakker's paper ... the couplings Δ_α were exponentially controlled by a flux that controlled the proximity of the wire to a superconductor. Roughly:

$$\Delta_\alpha \sim e^{-c\sqrt{|\cos\Phi_\alpha|}} \tag{78}$$

This allowed tuning each of hte arms between zero coupling and a coupling much bigger than the others with great ease, even if precision of the flux control is lacking.

This is a cool setup for exchange. If we start with $\Delta_x = 1, \Delta_y = \Delta_z = 0$, then we have two zero modes clearly lodged at the y and z sides of the y. Now doing clock moves, and making $\Delta_y \to 1$, $\Delta_x \to 0$, will shift $\gamma_y \to \gamma_x$. Then doing $\Delta_y \to 0$, $\Delta_z \to 1$, shifts $\gamma_z \to \gamma_y$. Lastly, $\Delta_z \to 0$, $\Delta_x \to 1$, shifts $\gamma_z \to \gamma_y$, brings what used to be γ_y to γ_z. That's it - we facilitated an exchange. See Fig. 12

When engaging with such an hamiltonian, all we want, really, is to find ladder operators of the form $\Gamma = x\gamma_x + y\gamma_y + z\gamma_z + w\gamma_0$ Commuting Γ with H, seeking $[H, \Gamma] = \epsilon\Gamma$, we find that this maps to the Schrödinger equation:

$$E\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = i\begin{pmatrix} 0 & -\Delta_x & -\Delta_y & -\Delta_z \\ \Delta_x & 0 & 0 & 0 \\ \Delta_y & 0 & 0 & 0 \\ \Delta_z & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \tag{79}$$

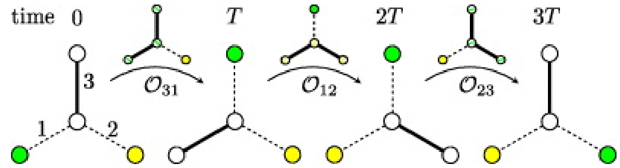

FIG. 12: The y-exchange sequence as described in the text. Notation is a bit different $1, 2, 3 \to y, z, x$. Taken from "Coulomb-assisted braiding of Majorana fermions in a Josephson junction array", van Heck, Akhmerov, Hassler, Burrello, Beenakker. NJP.

These have two solutions though that are zero modes, and two gapped modes.

Here we can solve everything using geometry. Think about polar coordinates, and appreciate that the row vector at the top of the matrix is simply $\vec{r}_\Delta = (\Delta_x, \Delta_y, \Delta_z)$. And also write:

$$\vec{\Delta} = |\vec{r}_\Delta| \cdot (\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta) \tag{80}$$

with θ adn ϕ the polar angles for the 3d coordinate system.

Let us also think of the Majorana operators as:

$$\Gamma = w\gamma_0 + \vec{v} \cdot \vec{\gamma}. \tag{81}$$

Now it is easy to see the solutions. Try out:

$$\vec{v} = \vec{r_\Delta} \tag{82}$$

Then we get:

$$E \begin{pmatrix} w \\ \vec{r}_\Delta \end{pmatrix} = \begin{pmatrix} r_\Delta^2 \\ w\vec{r}_\Delta \end{pmatrix} \to Ew = r_\Delta^2,\, E = w \tag{83}$$

so

$$E = w \pm |\vec{r}_\Delta| \tag{84}$$

Two solution which make up the gapped mode - the two majoranas that make up a finite energy fermionic state.

By the same token the majoranas that make the zero modes are:

$$w = 0,\, \vec{v} = \hat{e}^\theta_\Delta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \tag{85}$$

and:

$$w = 0,\, \vec{v} = \hat{e}^\phi_\Delta = (-\sin\phi, \cos\phi, 0) \tag{86}$$

Clearly $\Gamma_{\theta,\,\phi} = \vec{\gamma} \cdot \hat{e}^{\theta,\phi}_\Delta$ are zero modes of the y hamiltonian.

The thing that is left to do is figure out the berry phase associated with the two states that $\Gamma_{\theta,\,\phi}$ give rise to. We can parametrize this subspace as:

$$|\psi_0\rangle\,,\; \frac{1}{2}(\Gamma_\phi + i\Gamma_\theta)\,|\psi_0\rangle \tag{87}$$

Our next goal is to calculate the *relative* Berry phase α between these two states as the parameters of the model are changed adiabatically. I'll leave it to you to prove that:

$$\frac{d\alpha}{dt} = i\,\langle\psi_0|\left\{\frac{1}{2}(\Gamma_\phi - i\Gamma_\theta), \frac{1}{2}\frac{d}{dt}(\Gamma_\phi + i\Gamma_\theta)\right\}|\psi_0\rangle \tag{88}$$

The time derivative on the right hand side is like the derivative of unit vectors in polar coordinates. We can quickly get:

$$\frac{d}{dt}\Gamma_\phi = \dot{\phi} \cdot (-\hat{e}^r_\Delta \sin\theta \cdot \vec{\gamma} - \cos\theta\Gamma_\theta), \quad \frac{d}{dt}\Gamma_\theta = \dot{\phi} \cdot (\cos\theta\Gamma_\phi) - \dot{\theta}\hat{e}^r_\Delta \sin\theta \cdot \vec{\gamma} \tag{89}$$

From that we see that:

$$\left\{(\Gamma_\phi - i\Gamma_\theta), \tfrac{d}{dt}(\Gamma_\phi + i\Gamma_\theta)\right\} = \left\{(\Gamma_\phi - i\Gamma_\theta), -\Gamma_\theta\dot{\phi}\cos\theta + i\dot{\phi}\cdot\cos\theta\Gamma_\phi\right\} \\ = 2i\dot{\phi}\cos\theta. \tag{90}$$

And finally, we get:

$$\frac{d\alpha}{dt} = -\dot{\phi}\cos\theta \tag{91}$$

just like a spin-1/2. up to a missing factor of a half.

Doing the clock moves from above, then, cover an octant - so an eighth - of the sphere, and the phase associated with that is:

$$\alpha_{exchange} = \frac{1}{4}2\pi = \frac{1}{2}\pi,\ e^{i\alpha_{exchange}} = i \tag{92}$$

As expected.

Why am I teaching you this? It is a neat example of how Majorana fermions will be manipulated in practice, it is also the key for making the Kitaev-Bravyi magic states, I believe. But for that, stay tuned!