Lecture 2: What if the evolution is Unitary?

QFT + Classical GR $\Rightarrow$ loss of unitarity

Unacceptable, assume unitary evolution and follow the consequences

Imagine a collapse that starts from a pure state

$$|\Psi_{shell, RAD}\rangle \sim |shell\rangle \otimes |0_-\rangle$$

$$\hat{U}\, |shell\rangle \otimes |0_-\rangle = \sum_{m,n} A_{m,n}\, |Shell\rangle_n \otimes |RAD\rangle_m$$

$$|shell\rangle \Leftrightarrow |BH\rangle$$

Density matrix $\hat{\rho} = |\Psi\rangle\langle\Psi|$

The state is pure $\Rightarrow$ in some basis

$$\hat{\rho} = \begin{pmatrix} 1 & 0 & \\ & \ddots & \\ & & 0 \end{pmatrix}$$

but in the $|Shell\rangle_n \otimes |RAD\rangle_m$ basis

$\hat{\rho}$ looks completely random

So there exists some unitary matrix $\mathcal{U}$, $\mathcal{U}\mathcal{U}^{\dagger} = \mathbb{1}$

s.t. $\mathcal{U}\hat{\rho}\,\mathcal{U}^{\dagger} = \begin{pmatrix} 1 & 0 & \\ & \ddots & \\ & & 0 \end{pmatrix}$

(not to be confused with $\hat{U}$)

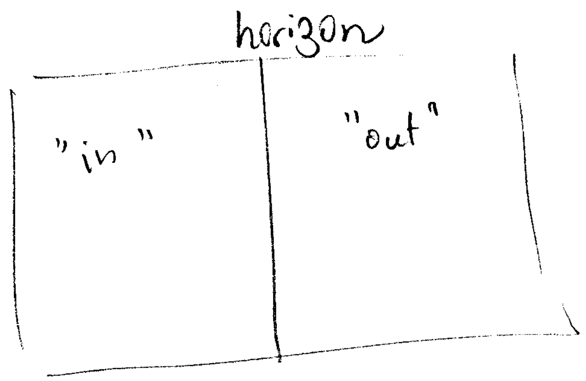
The Page model: Page '93

- Imagine that the horizon divides the "inside" and "out side" of the BH. Assume that both are characterized only by the size of their Hilbert spaces. Outside ~ |RAD⟩, inside ~ |BH⟩

Also, because the BH is evaporating the size of "in" and "out" can change, so size ~ time

- Treat U - the unitary matrix statistically
- Calculate statistical properties of the reduced density matrix $\hat{\rho}_{RAD}$ ($\hat{\rho}_{out}$) as a function of its size.

The actual calculations do not include specific dynamics, similar in spirit to quantum information problems.



Brief reminder about: density matrices, reduced density matrices entropy, Entanglement, Purity, Information

· Bipartite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$$|\psi\rangle = \sum_{i=1, j=1}^{n,m} A_{ij} |\alpha_i\rangle_1 |\beta_j\rangle_2$$

from BEY 0508217

Schmidt decomposition thm. (proof: rotate each state independently)

$$|\psi\rangle = \delta_{ij} A_i |\tilde{\alpha}_i\rangle_1 |\tilde{\beta}_j\rangle$$

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

$$\hat{\rho}_1 = \mathrm{Tr}_2 \hat{\rho} = \sum_k {}_2\langle\gamma_k| \hat{\rho} |\gamma_k\rangle_2 =$$

$$= \sum_k \langle\gamma_k| \delta_{ij} A_i |\tilde{\alpha}_i\rangle_1 |\tilde{\beta}_j\rangle_2 \cdot \delta_{rs} A_r^* {}_1\langle\tilde{\alpha}_r| {}_2\langle\tilde{\beta}_s|\gamma_k\rangle_2$$

$$= \sum_k \delta_{ki} \delta_{ij} A_i |\tilde{\alpha}_i\rangle_1 \quad \delta_{rs}\delta_{sk} A_r^* {}_1\langle\tilde{\alpha}_r|$$

$$\hat{\rho}_1 = \mathrm{Tr}_2 \hat{\rho} = \sum_k A_k A_k^* |\tilde{\alpha}_k\rangle_1 {}_1\langle\tilde{\alpha}_k|$$

Similarly for $\hat{\rho}_2 = \mathrm{Tr}_1 \hat{\rho}$

$\hat{\rho}_1$ is no longer pure!

$$\langle\psi| \hat{O}_1 |\psi\rangle = \mathrm{Tr}(\hat{\rho}_1 \hat{O}_1)$$

Matrix elements of restricted operators $\hat{O}_1$

Measures of purity

How many non-vanishing eigenvalues of $\hat{\rho}$?

Most familiar: Von-Neumann entropy

$$S = -\operatorname{Tr} \rho \ln \rho = \sum_{i=1}^{N} p_i(-\log p_i)$$

$\hat{\rho}$ is pure $\Rightarrow$ $S = 0$

$\hat{\rho}$ is maximally mixed (thermal) $\quad p_i = \frac{1}{N} \Rightarrow S = \log N$

Most useful estimates involve $\operatorname{Tr}\rho^2 = \sum_{i=1}^{N} p_i^2$

Purity: $P = \dfrac{\operatorname{Tr}\rho^2}{(\operatorname{Tr}\rho)^2}$ $\quad P = \operatorname{Tr}\rho^2$ if $\operatorname{Tr}\rho = 1$

If $\hat{\rho}$ is pure $P = 1$.

$\hat{\rho}$ is maximally mixed $\quad P = \sum_{i=1}^{N} \frac{1}{N^2} = \frac{1}{N}$

Participation ratio:
$$PR = \frac{1}{P} = \begin{cases} 1 & \\ N & \text{Max. Mixed} \end{cases}$$

Participation ratio is the most direct measure of the effective number of non-vanishing eigenvalues.

Renyi entropy

$$H_2 = \ln PR = -\ln P = -\ln\left(\sum_{i=1}^{N} p_i^2\right)$$

$\hat{\rho}$ pure $H_2 = 0$

$\hat{\rho}$ Max. mixed $H_2 = \ln N$

Much easier to calculate the S, has essentially same content.

In QFT for generic highly excited states

$N \sim \langle \hat{n} \rangle !$ $\langle \hat{n} \rangle$ - average number of particles

"size of Hilbert space"

So $S, H_2 \propto \langle \hat{n} \rangle$ "extensive"

Information: $I = S_{thermal} - S$

$(H_2)_{thermal} - H_2$

How far is $\hat{\rho}$ from a maximally mixed state?

$I = 0$ for a thermal state

I - maximal for a pure state.

Thermofield double (TFD)

$$|\psi\rangle = \sum_i e^{-\frac{\beta E_i}{2}} |E_i\rangle_1 |E_i\rangle_2 \quad \text{pure state.}$$

$$\hat{\rho} = |\psi\rangle\langle\psi| = \sum_{i,j} e^{-\frac{\beta(E_i+E_j)}{2}} |E_i\rangle_1 |E_i\rangle_2 \, {}_1\langle E_j| \, {}_2\langle E_j|$$

$$\hat{\rho}_1 = \mathrm{Tr}_2 \hat{\rho} = \sum_m {}_2\langle m| \hat{\rho} |m\rangle_2 = \sum_{i,j} \delta_{im}\delta_{jm} e^{-\frac{\beta(E_i+E_j)}{2}} |E_i\rangle_1 {}_1\langle E_j|$$

$$= \sum_i e^{-\beta E_i} |E_i\rangle_1 \, {}_1\langle E_i| \quad . \text{ TFD "purifier" of } \hat{\rho}_1 .$$

$S(\rho_1)$ - Entropy of entanglement (or just entanglement)

TFD - maximal entanglement

$|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ product state

vanishing entanglement

Problem 1:

For a density matrix of size $N \times N$, with K eigenvalues of magnitude x and $N-K$ of magnitude y, calculate

S, P, PR, H_2, I

Problem 2 : 2-spin system

a. Calculate the density matrix for

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \right)$$

in the $|\ \rangle_1 \otimes |\ \rangle_2$ basis.

b. Calculate $\hat{\rho}_1, \hat{\rho}_2$

c. Calculate $S_1, S_2, P_1, P_2, (H_2)_1, (H_2)_2$

d. Show that $\langle 0,0 | \hat{O}_1 | 0,0 \rangle = \mathrm{Tr}(\rho_1 O)$.

Back to the Page model

$$\hat{\rho} = |\psi_{BH,RAD}\rangle\langle\psi_{BH,RAD}| = \sum \rho_{A,i,B,j} \, |A,i\rangle\langle B,j|$$

$$i,j = 1,\dots,n \, , \quad n = e^{S_{BH}}$$

$$A,B = 1,\dots,m \, , \quad m = e^{S_{RAD}}$$

$$m,n \gg 1$$

$$\rho_{A,i,B,j} = V_{Ai} V^{\dagger}_{jB} \, , \quad V_{Ai} = \langle A,i|\psi_{BH,RAD}\rangle$$

$$\sum V_{Ai} V^{\dagger}_{Ai} = 1 = \sum |V_{Ai}|^2$$

$$(\hat{\rho}_{RAD})_{AB} = \mathrm{Tr}_{BH}\,\hat{\rho} = \sum_i V_{Ai} V^{\dagger}_{iB}$$

Follow Lubkin '78 ! Treat V_{Ai} statistically

V_{Ai} random unit vector;
a column of a unitary matrix

Uniform probability w.r.t. to "Haar measure" - Volume of sphere

Quick and dirty method to get Lubkin results

(Details, precision
- see paper)

$\int dV = 1$ - normalization

Average on sphere (integration over angles)

$$\int du \; V_{Ai} V^*_{Bj} \simeq \frac{1}{N} \delta_{AB} \delta_{ij}$$

Normalization $\quad \mathrm{Tr}(\hat{\rho}_{RAD})_{AB} = 1$

$$\Rightarrow \int dV \; V_{Ai} V^*_{Ai} = 1 \quad \Rightarrow \quad \frac{1}{N} \sum_A \delta_{AA} \sum_i \delta_{ii} = \frac{mn}{N} \Rightarrow N = mn$$

$$V \to V/\sqrt{mn}$$

$$\int dV \quad V_{Ai} V^*_{Bj} V_{Cl} V^*_{Dl} \propto \left(\delta_{AB} \delta_{ij} \delta_{CD} \delta_{kl} + \delta_{AD} \delta_{il} \delta_{BC} \delta_{jk} \right)$$

$$\mathrm{Tr}\, \hat{\rho}_{RAD}^2 \simeq \int dV \sum_{i,j,A,B} V_{Ai} V^*_{iB} V_{Bj} V^*_{jA}$$

$$\simeq \frac{1}{(mn)^2} \sum_{i,j,A,B} \left(\delta_{ij} \delta_{ji} \delta_{AA} \delta_{BB} + \delta_{ii} \delta_{jj} \delta_{AB} \delta_{BA} \right)$$

$$= \frac{m^2 n + n^2 m}{(mn)^2} = \frac{m+n}{mn}$$

$$P_{RAD} = \frac{e^{S_{BH}} + e^{S_{RAD}}}{e^{S_{BH}} e^{S_{RAD}}}$$

Recall: $e^{S_{RAD}}$ ~ size of RAD Hilbert space

$e^{S_{BH}}$ ~ size of BH Hilbert space

Use N_{RAD} - average number of emitted Hawking particles as a "time-variable"

So $S_{RAD} \sim N_{RAD}$

$S_{BH} \approx S_{\text{Bekenstein-Hawking}} \sim S_{BH}(0) - N_{RAD}$

Define "the Page time" when $m=n$ $S_{BH} = S_{RAD}$

mid-point of evaporation (in terms of entropy)

$N_{RAD} < N_{page}$ $\quad P_{RAD} \sim \dfrac{e^{S_{BH}}}{e^{S_{BH}} e^{S_{RAD}}} = e^{-N_{RAD}}$

$$(H_2)_{RAD} \approx N_{RAD}$$

so, the state of the radiation is essentially thermal.

$N_{RAD} > N_{Page}$

$$P_{RAD} \simeq \frac{e^{S_{RAD}}}{e^{S_{BH}} e^{S_{RAD}}} = e^{-S_{BH}}$$
$$= e^{-(S_{BH}(0) - N_{RAD})}$$

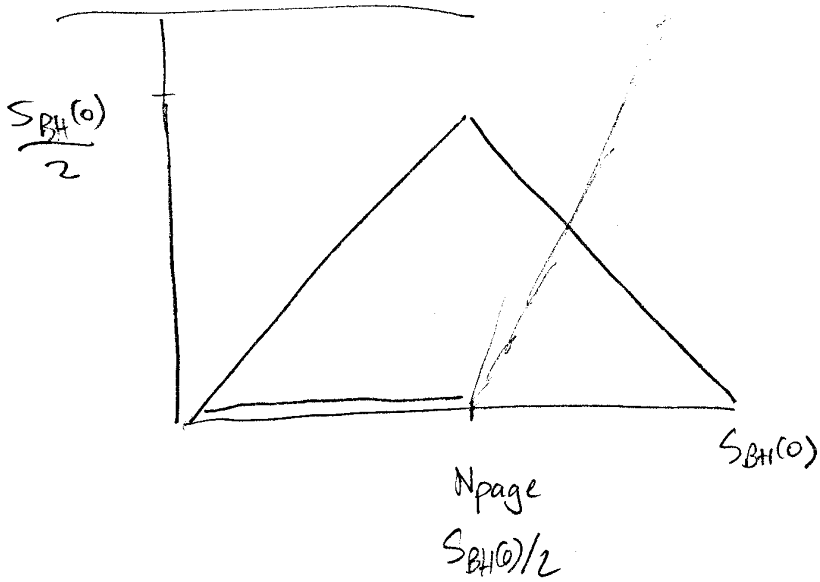
$$(H_2)_{RAD} \simeq S_{BH}(0) - N_{RAD}$$

The radiation starts to purify with rate of order 1.

$$I_{RAD} = N_{RAD} - (H_2)_{RAD} = \begin{cases} 0 & N_{RAD} < N_{page} \\ 2N_{RAD} - S_{BH}(0) & N_{RAD} > N_{page} \end{cases}$$

$S_{BH}(0)$ –

Page curve



Hayden-Preskill $\frac{dI}{dN} \sim \begin{cases} 0 & N < N_{page} \\ 1 & N > N_{page} \end{cases}$

Throw a quantum system into a BH $\leftrightarrow$ Information mirror

Problem 3: Calculate numerically the Page curve for unitary random matrices

Using Mathematica (or any other program) generate random Unitary matrices.
Then calculate the Renyi and Von-Newmann entropies of the reduced density matrix as a function of it's size.