Quantum matter
and
gauge-gravity duality

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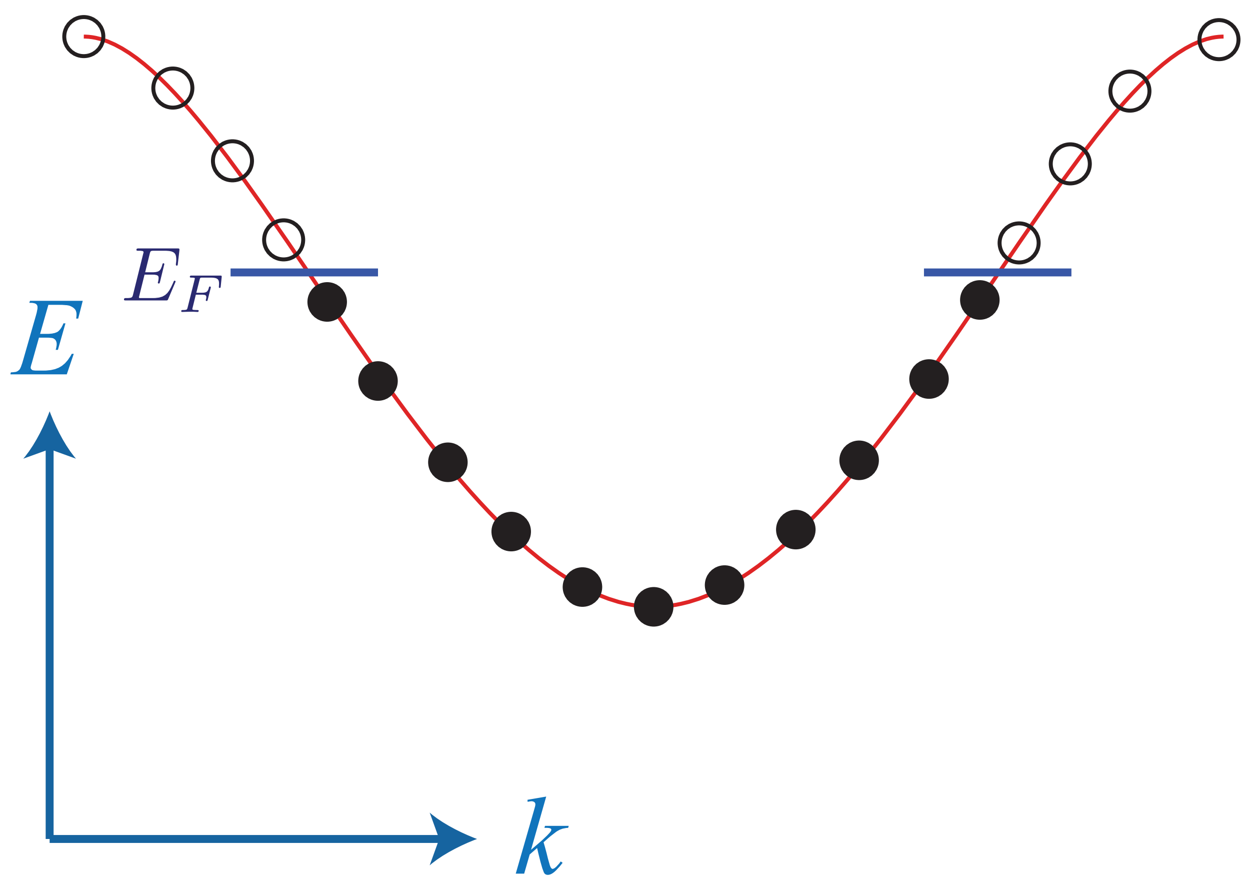
PHYSICS
HARVARD

Talk online at sachdev.physics.harvard.edu

Sommerfeld-Pauli-Bloch theory of metals, insulators, and superconductors:

many-electron quantum states are adiabatically connected to independent electron states

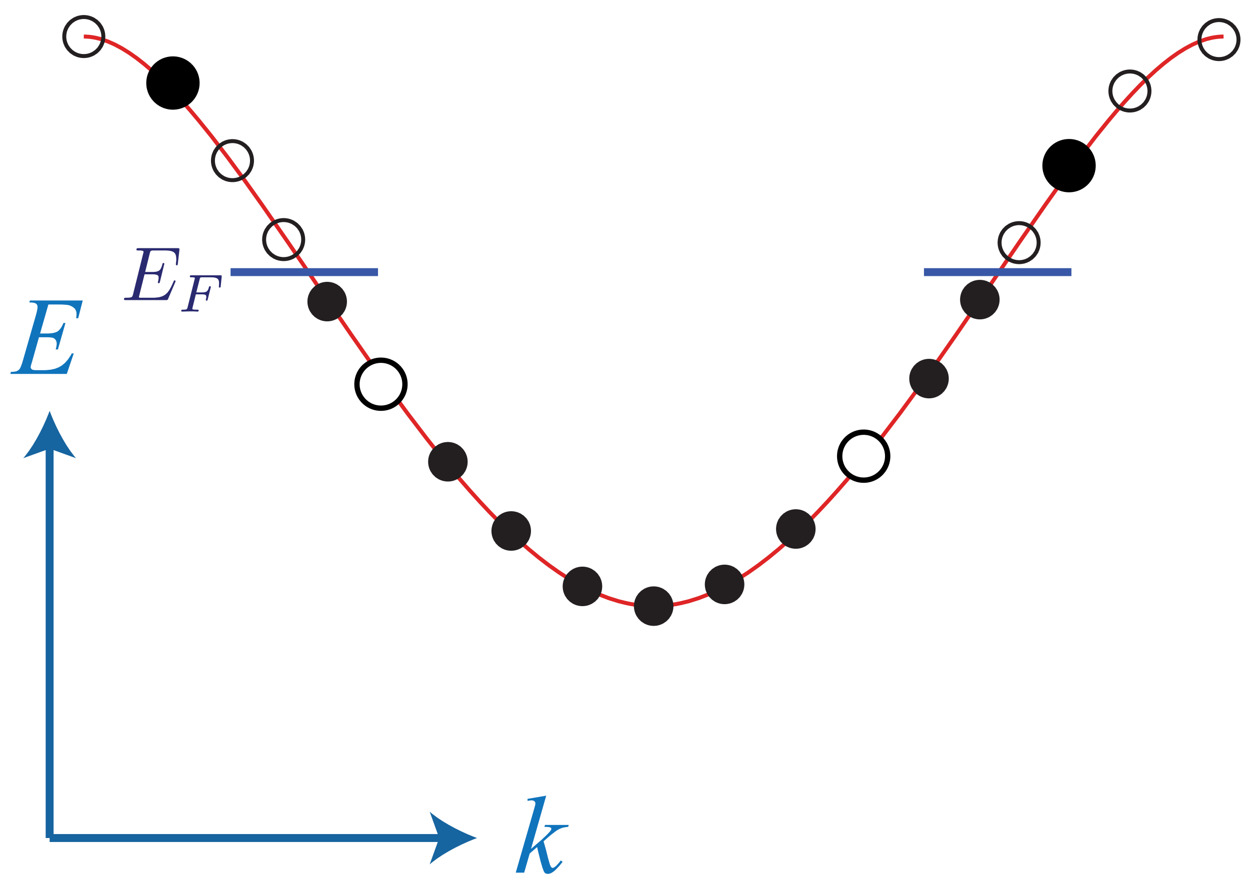
Metals



Boltzmann-Landau theory of dynamics of metals:

Long-lived ***quasiparticles*** (and ***quasiholes***) have weak interactions which can be described by a Boltzmann equation

Metals



Modern phases of quantum matter

Not adiabatically connected
to independent electron states:

1. Many-particle quantum entanglement

2. (a) Quasiparticles with quantum numbers different from those of the electron

(b) No quasiparticles

"Complex entangled" states of quantum matter, *not* adiabatically connected to independent particle states

Gapped quantum matter

Z_2 Spin liquids, quantum Hall states

Conformal quantum matter

Graphene, ultracold atoms, antiferromagnets

Compressible quantum matter

Strange metals, Bose metals

S. Sachdev, arXiv:1203.4565

“Complex entangled” states of quantum matter, *not* adiabatically connected to independent particle states

Gapped quantum matter

Z_2 Spin liquids, quantum Hall states

exotic quasiparticles, TQFT

Conformal quantum matter

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“Complex entangled” states of quantum matter, *not* adiabatically connected to independent particle states

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Graphene, ultracold atoms, antiferromagnets

no quasiparticles, CFT

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“Complex entangled” states of
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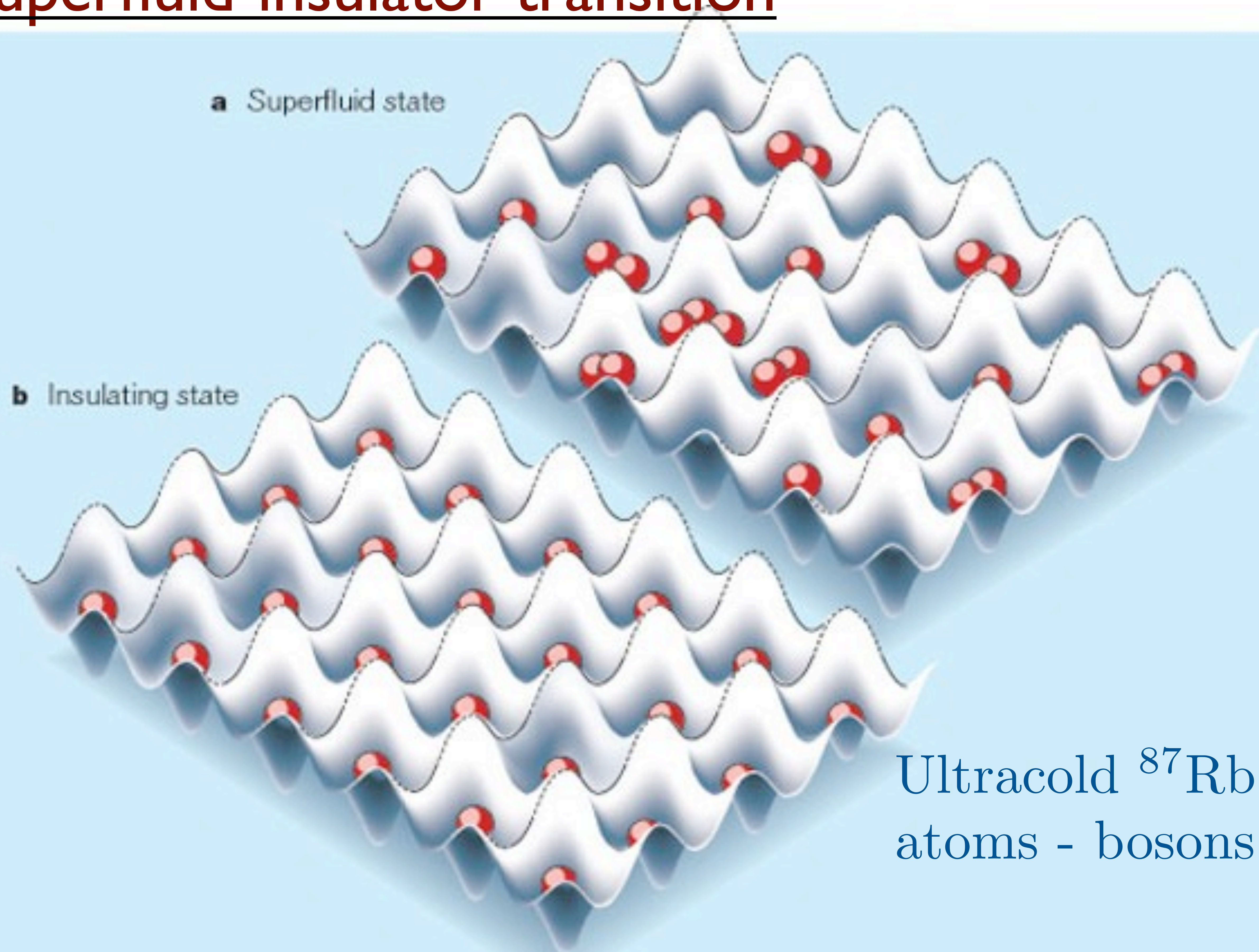
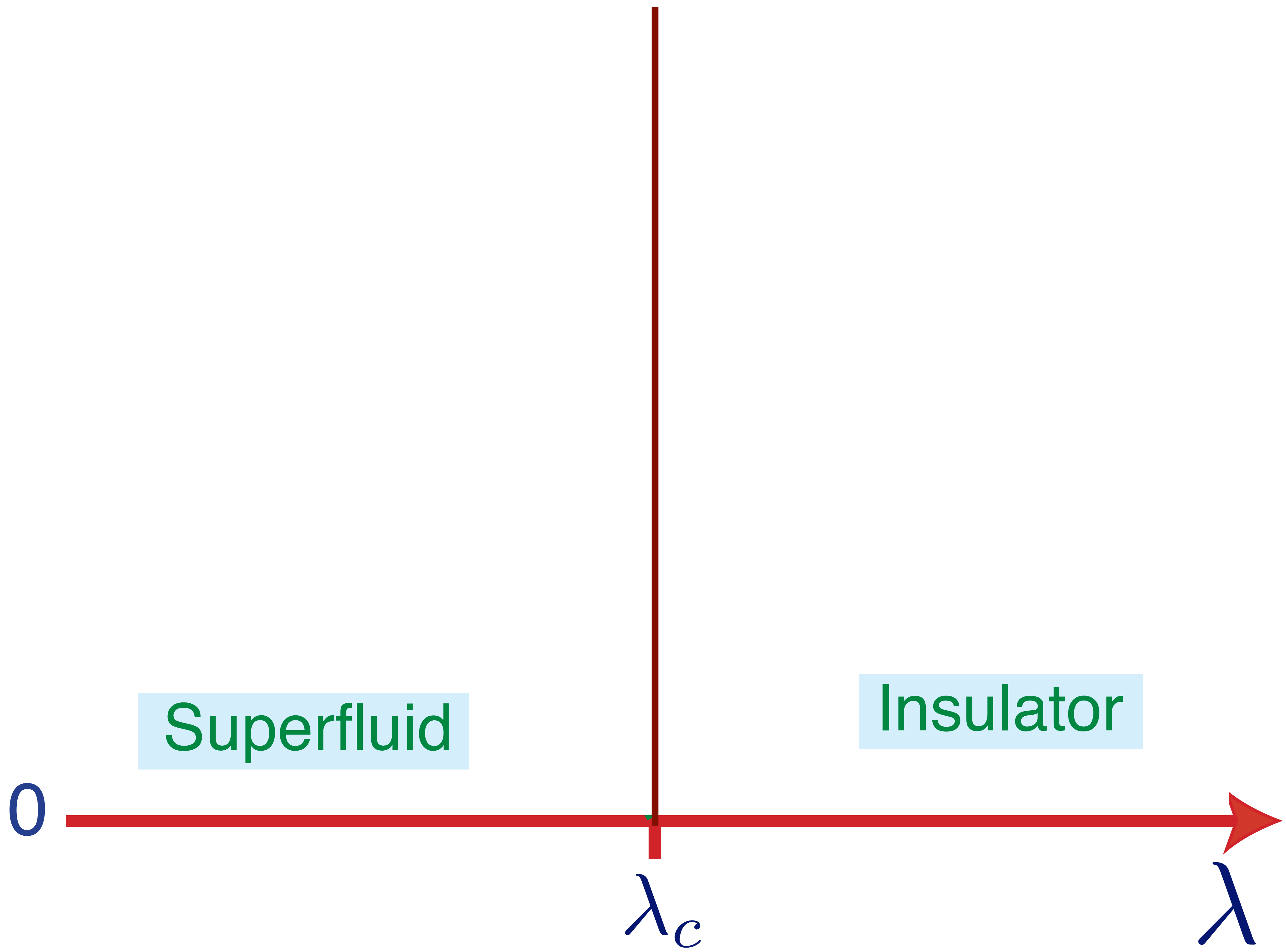
Superfluid-insulator transition

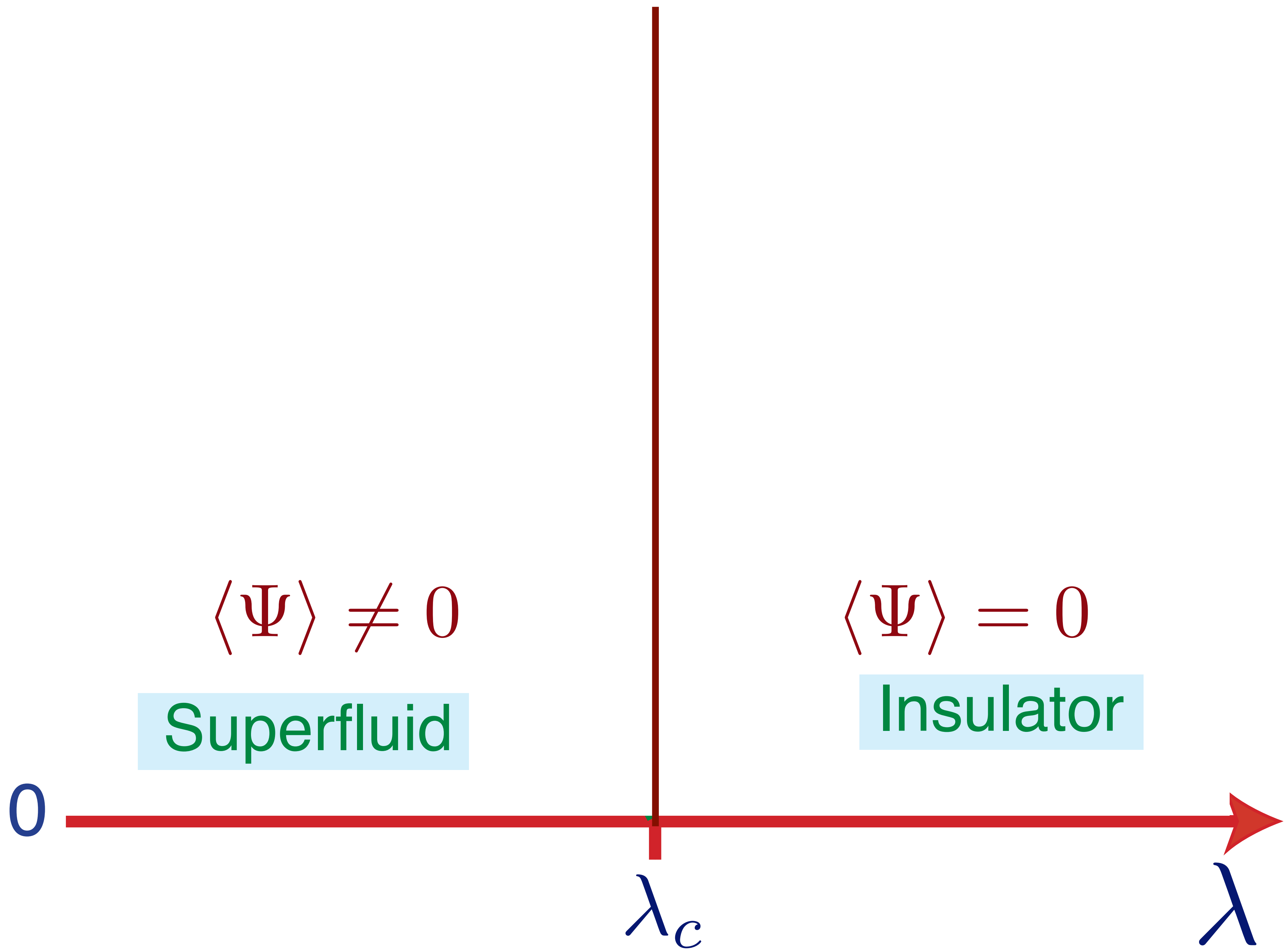


Ultracold ^{87}Rb atoms - bosons

M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

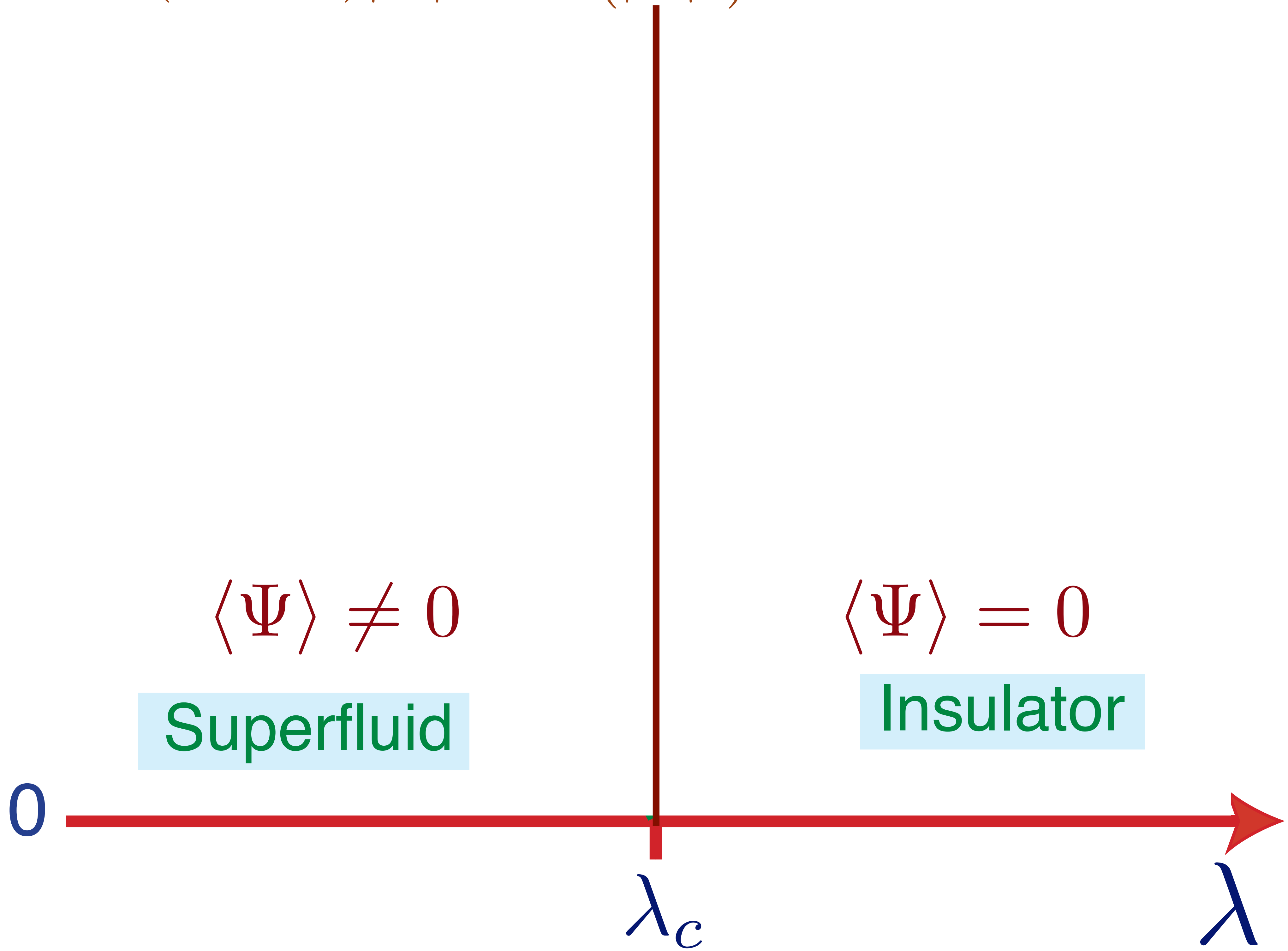

Superfluid
Insulator
0
λ_c
λ

$\Psi \rightarrow$ a complex field representing the Bose-Einstein condensate of the superfluid



$$\mathcal{S} = \int d^2r dt \left[|\partial_t \Psi|^2 - c^2 |\nabla_r \Psi|^2 - V(\Psi) \right]$$

$$V(\Psi) = (\lambda - \lambda_c)|\Psi|^2 + u\left(|\Psi|^2\right)^2$$



M.P. A. Fisher, P.B. Weichmann, G. Grinstein, and D.S. Fisher, *Phys. Rev. B* **40**, 546 (1989).

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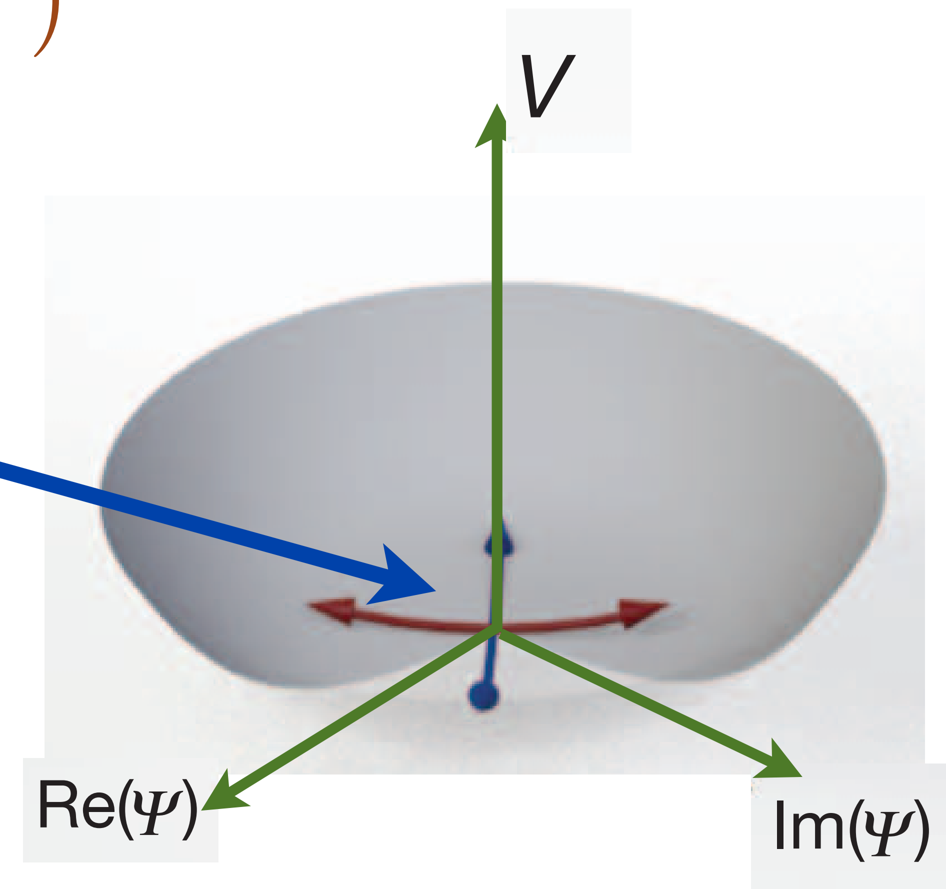
Particles and holes correspond to the 2 normal modes in the oscillation of Ψ about $\Psi = 0$.



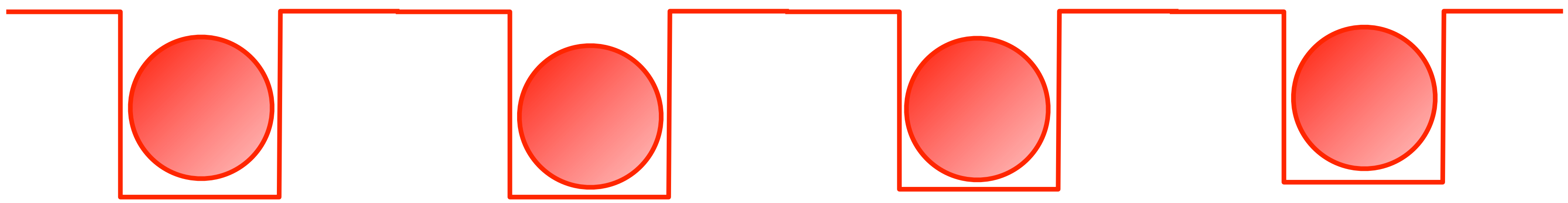
$\langle \Psi \rangle \neq 0$

Superfluid

$\langle \Psi \rangle = 0$

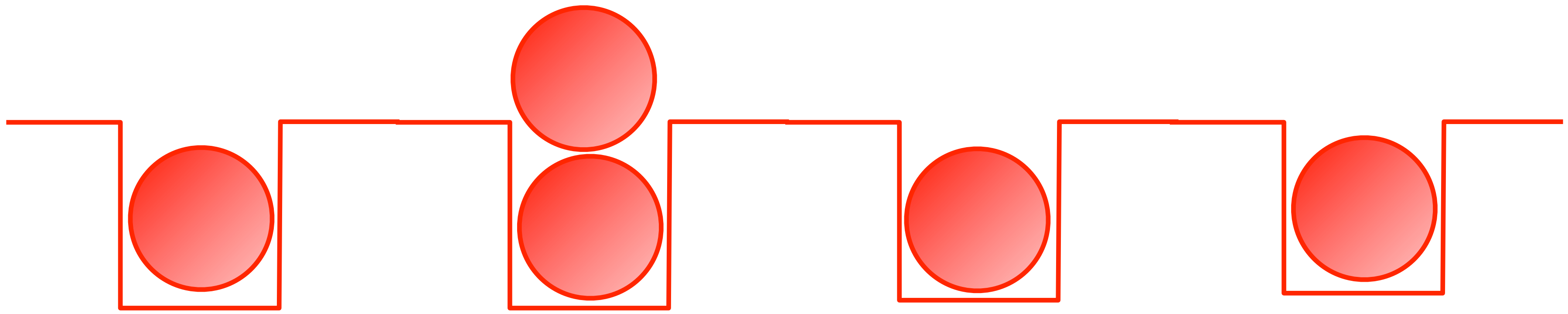
Insulator

0 $\qquad \lambda_c \qquad \lambda$



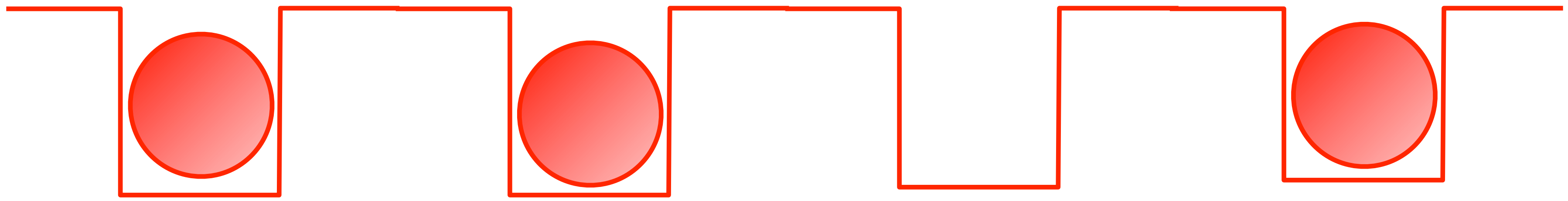
Insulator (the vacuum)
at large repulsion between bosons

Excitations of the insulator:



$$\text{Particles} \sim \Psi^\dagger$$

Excitations of the insulator:



$$\text{Holes} \sim \Psi$$

$$\mathcal{S} = \int d^2r dt \left[|\partial_t \Psi|^2 - c^2 |\nabla_r \Psi|^2 - V(\Psi) \right]$$

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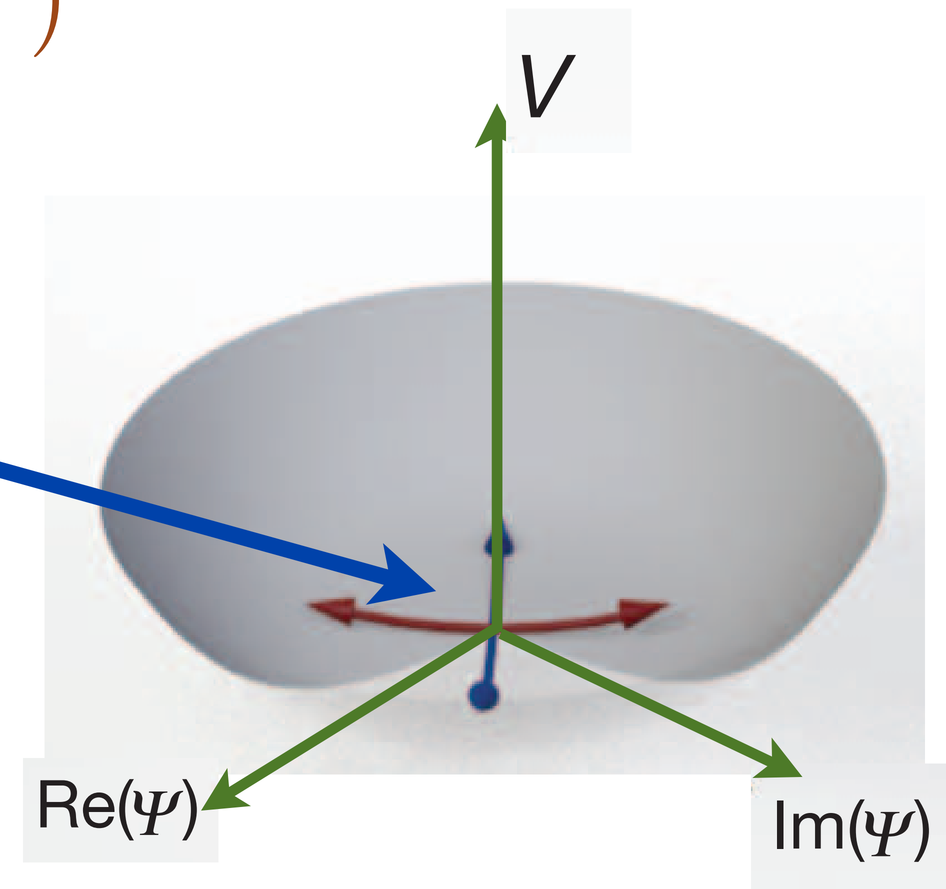

$\langle \Psi \rangle \neq 0$

Superfluid

$\langle \Psi \rangle = 0$

Insulator

0

λ_c

λ

$$\mathcal{S} = \int d^2r dt \left[|\partial_t \Psi|^2 - c^2 |\nabla_r \Psi|^2 - V(\Psi) \right]$$

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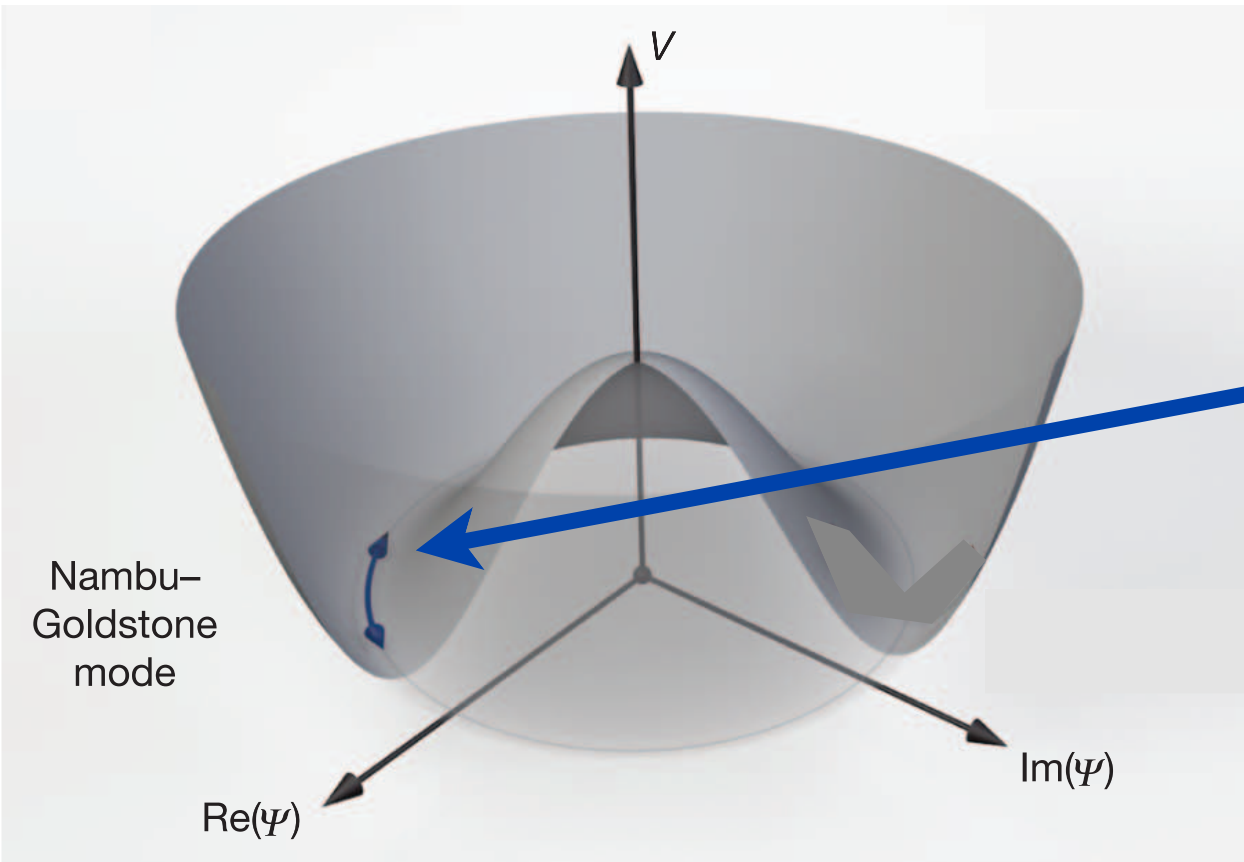

Nambu-Goldstone mode is the oscillation in the phase Ψ at a constant non-zero $|\Psi|$.

$\langle \Psi \rangle \neq 0$

Superfluid

$\langle \Psi \rangle = 0$

Insulator

0 λ_c λ

$$\mathcal{S} = \int d^2r dt \left[|\partial_t \Psi|^2 - c^2 |\nabla_r \Psi|^2 - V(\Psi) \right]$$

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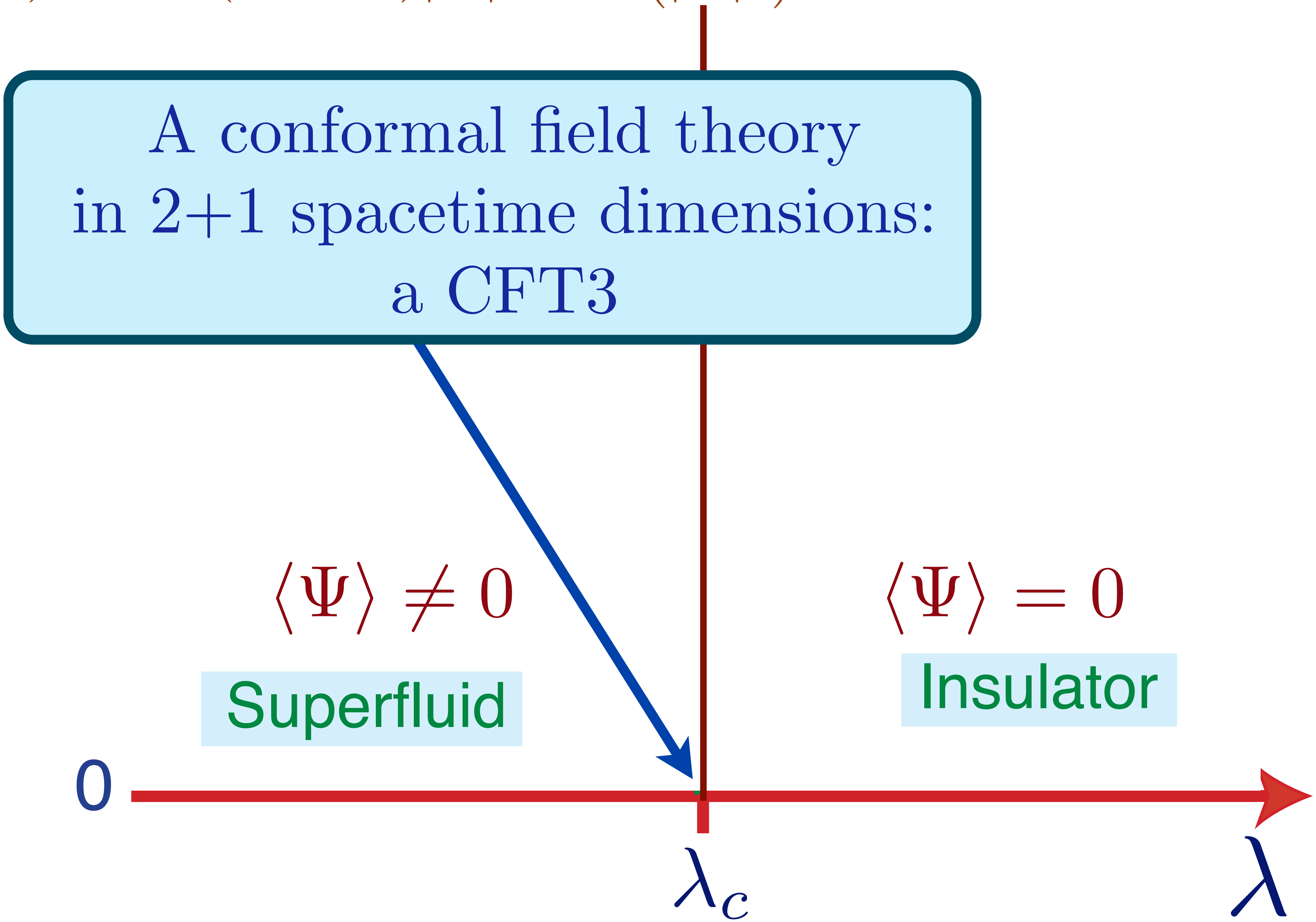

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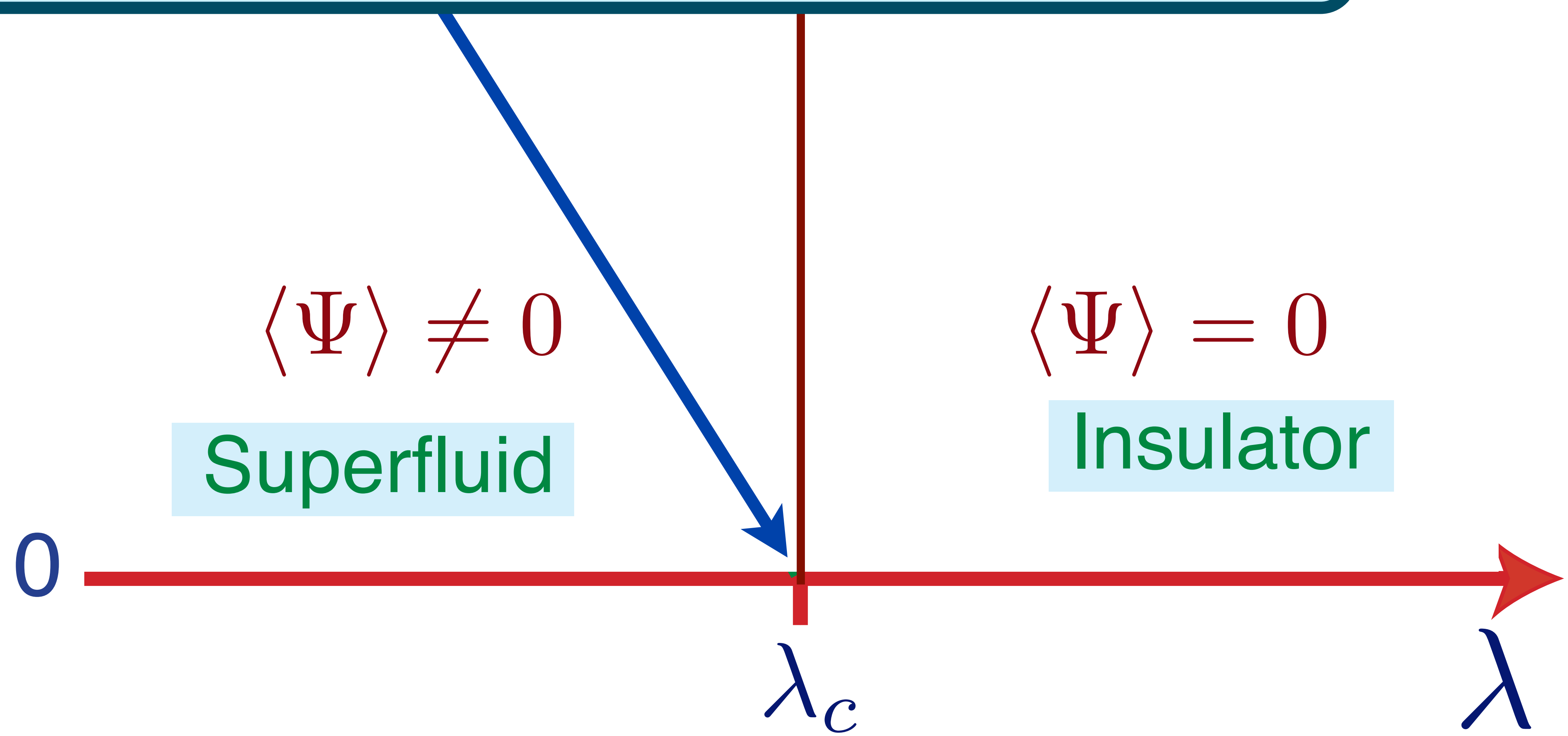
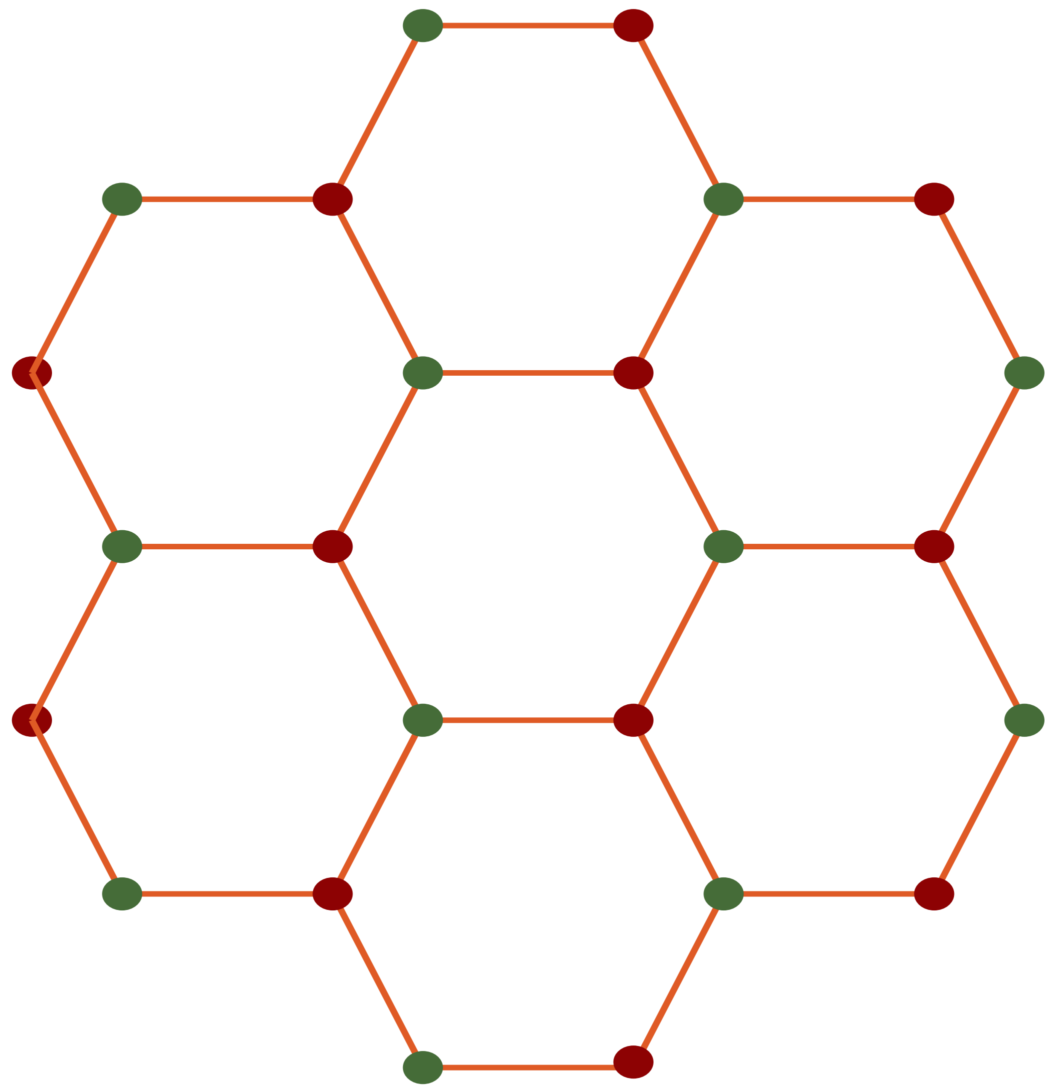
CFT3: The simplest class of theories with many-body entanglement and no quasiparticles

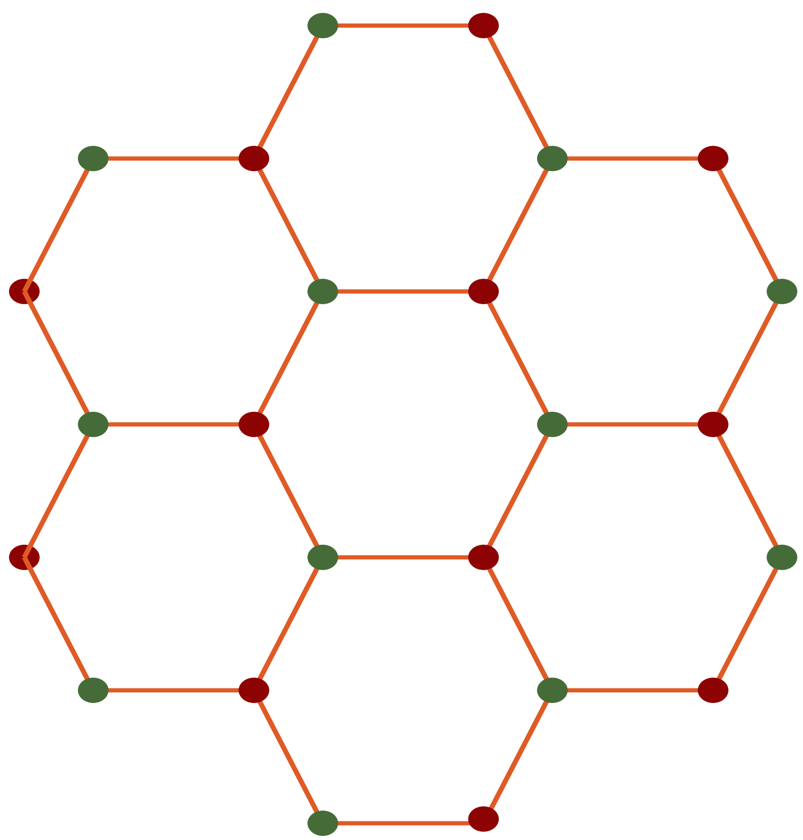


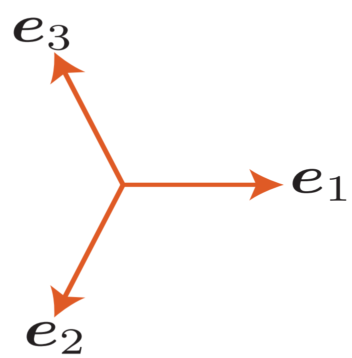
Honeycomb lattice

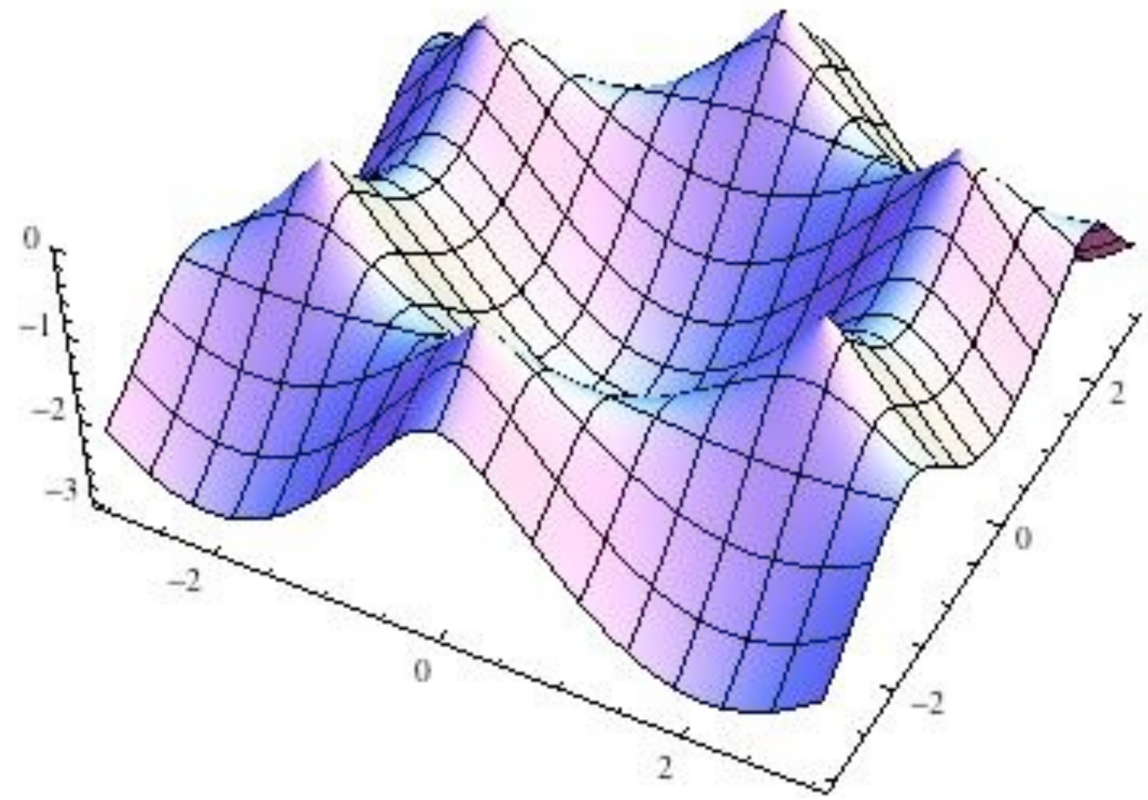
(describes graphene after adding long-range Coulomb interactions)

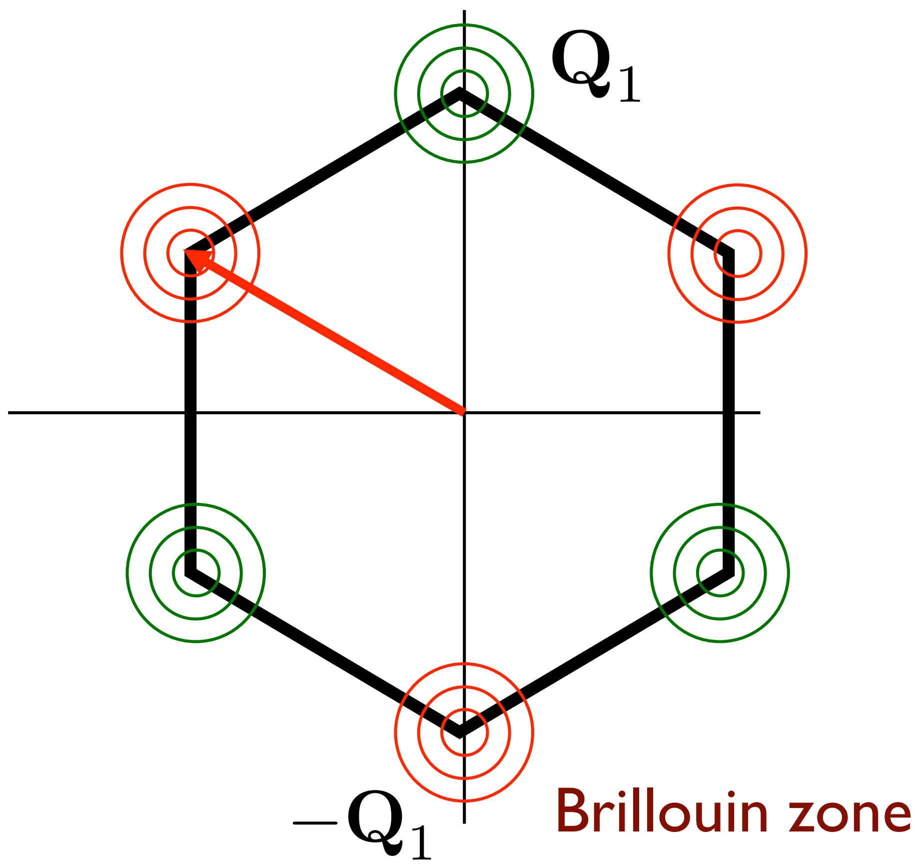


$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$









Semi-metal with
massless Dirac fermions
at small *U*/*t*

We define the Fourier transform of the fermions by

$$c_A(\mathbf{k}) = \sum_{\mathbf{r}} c_A(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{4}$$

and similarly for c_B. **A and B are sublattice indices.**
The hopping Hamiltonian is

$$H_0 = -t \sum_{\langle ij \rangle} \left(c^\dagger_{Ai\alpha} c_{Bj\alpha} + c^\dagger_{Bj\alpha} c_{Ai\alpha} \right) \tag{5}$$

where α is a spin index. If we introduce Pauli matrices τ^a in sublattice space ($a = x, y, z$), this Hamiltonian can be written as

$$\begin{aligned} H_0 \quad = \quad \int \frac{d^2k}{4\pi^2} c^\dagger(\mathbf{k}) \Big[-t \Big(\cos(\mathbf{k}\cdot\mathbf{e}_1) + \cos(\mathbf{k}\cdot\mathbf{e}_2) + \cos(\mathbf{k}\cdot\mathbf{e}_3) \Big) \tau^x \\ + t \Big(\sin(\mathbf{k}\cdot\mathbf{e}_1) + \sin(\mathbf{k}\cdot\mathbf{e}_2) + \sin(\mathbf{k}\cdot\mathbf{e}_3) \Big) \tau^y \Big] c(\mathbf{k}) \end{aligned} \tag{6}$$

The low energy excitations of this Hamiltonian are near $\mathbf{k} \approx \pm \mathbf{Q}_1$.

In terms of the fields near $\mathbf{Q}_1$ and $-\mathbf{Q}_1$, we define

$$\begin{aligned}
\Psi_{A1\alpha}(\mathbf{k}) &= c_{A\alpha}(\mathbf{Q}_1 + \mathbf{k}) \\
\Psi_{A2\alpha}(\mathbf{k}) &= c_{A\alpha}(-\mathbf{Q}_1 + \mathbf{k}) \\
\Psi_{B1\alpha}(\mathbf{k}) &= c_{B\alpha}(\mathbf{Q}_1 + \mathbf{k}) \\
\Psi_{B2\alpha}(\mathbf{k}) &= c_{B\alpha}(-\mathbf{Q}_1 + \mathbf{k})
\end{aligned} \tag{7}$$

We consider Ψ to be a 8 component vector, and introduce Pauli matrices ρ^a which act in the $1,2$ valley space. Then the Hamiltonian is

$$H_0 = \int \frac{d^2k}{4\pi^2} \Psi^\dagger(\mathbf{k}) \Big(v\tau^y k_x + v\tau^x \rho^z k_y \Big) \Psi(\mathbf{k}), \tag{8}$$

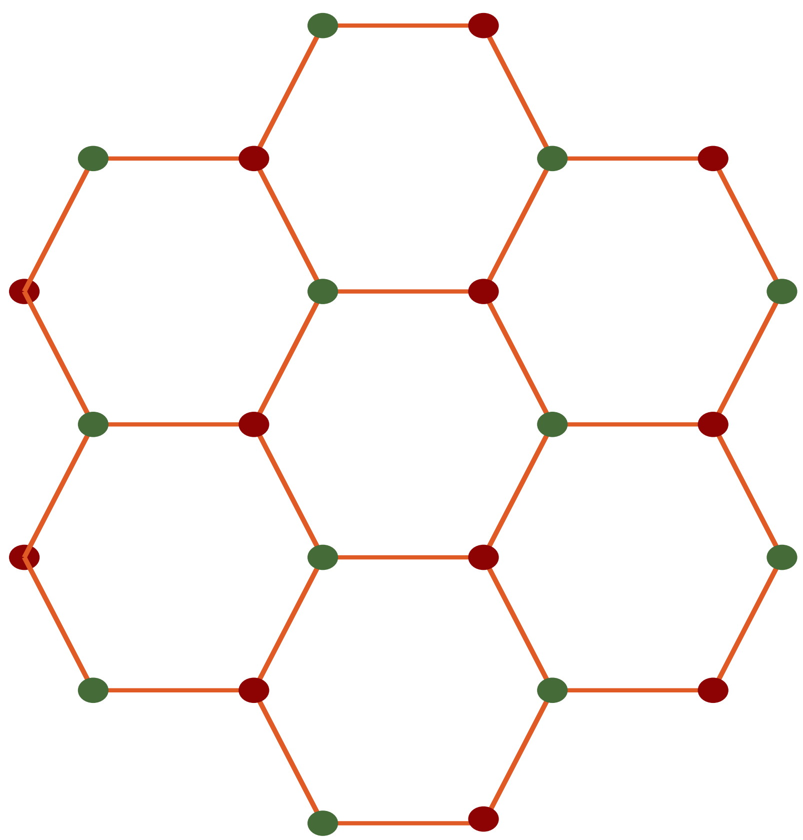
where $v = 3t/2$; below we set $v = 1$. Now define $\overline{\Psi} = \Psi^\dagger \rho^z \tau^z$. Then we can write the imaginary time Lagrangian as

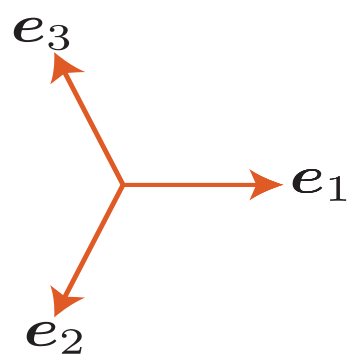
$$\mathcal{L}_0 = -i\overline{\Psi} \left(\omega\gamma_0 + k_x\gamma_1 + k_y\gamma_2 \right) \Psi \tag{9}$$

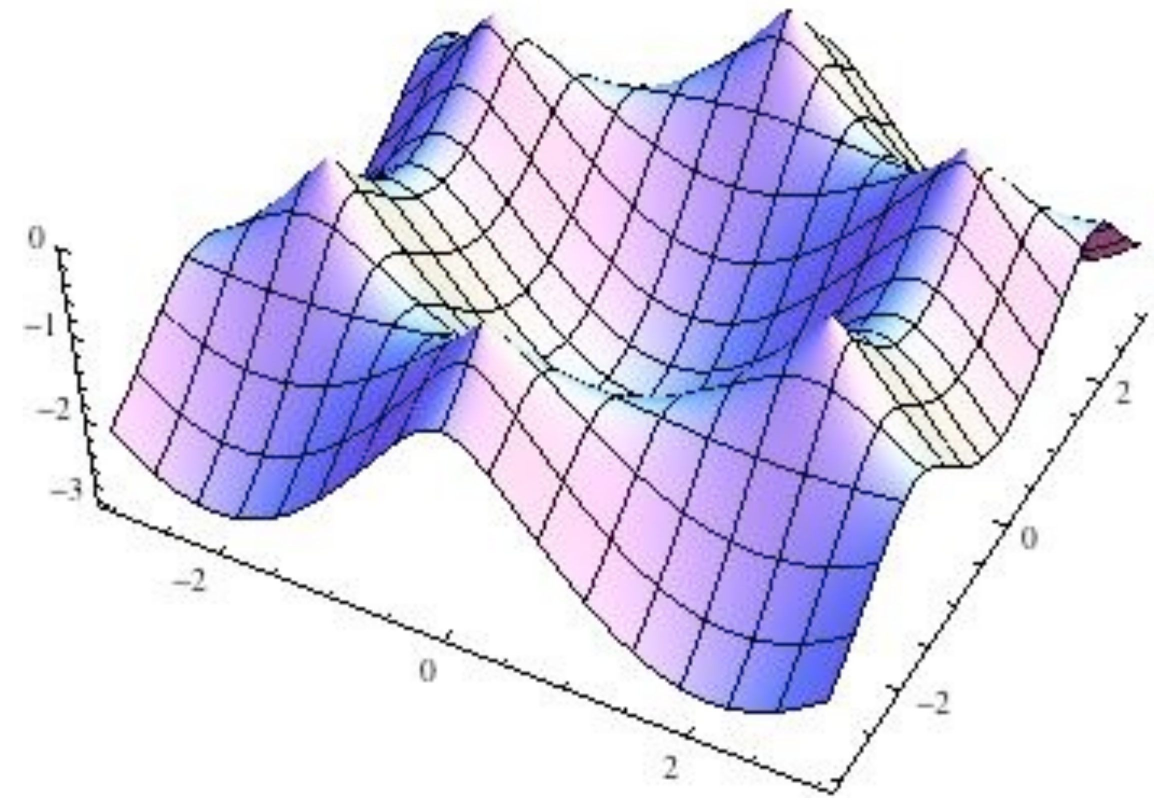
where

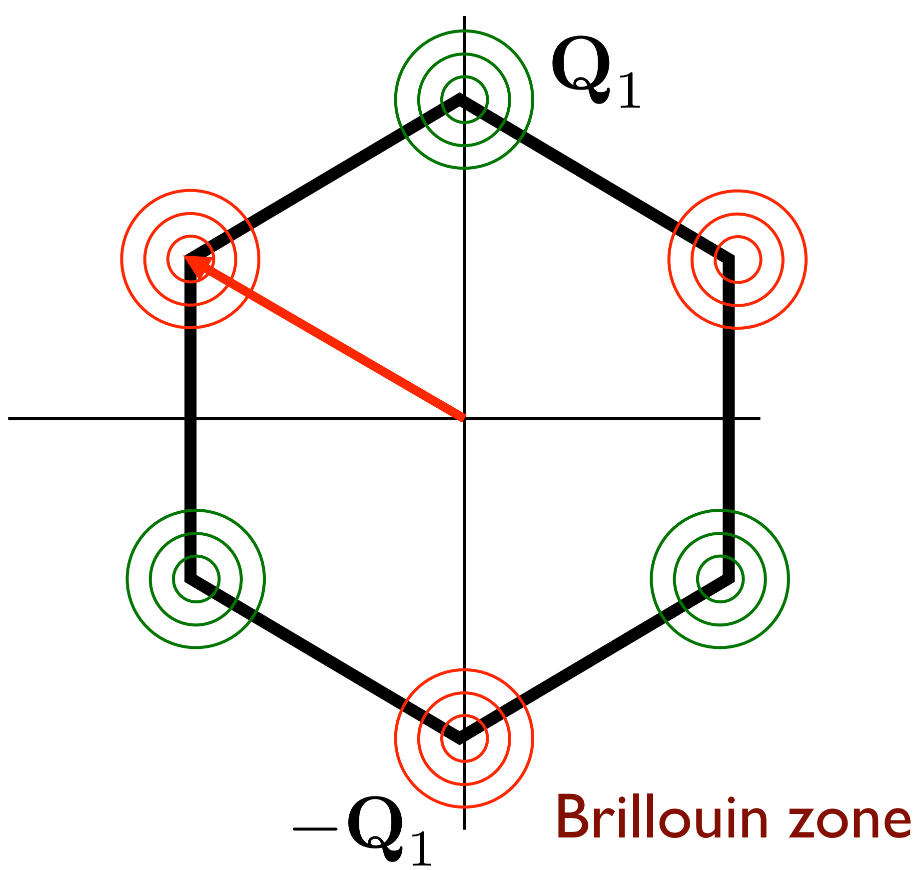
$$\gamma_0 = -\rho^z \tau^z \quad \gamma_1 = \rho^z \tau^x \quad \gamma_2 = -\tau^y \tag{10}$$

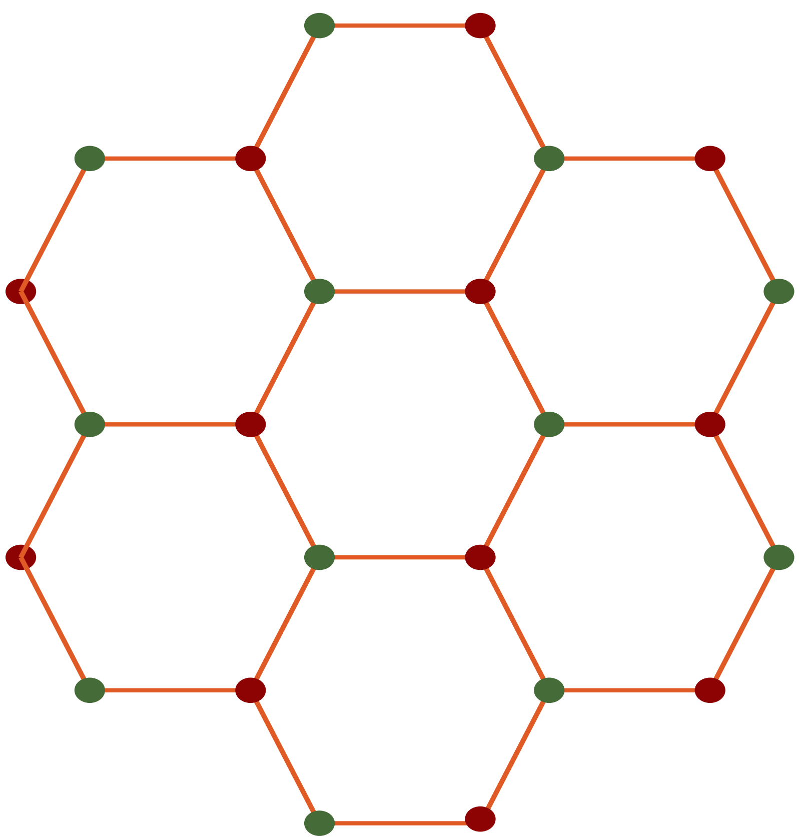
Exercise: Observe that $\mathcal{L}_0$ is invariant under the scaling transformation $x' = xe^{-\ell}$ and $\tau' = \tau e^{-\ell}$. Write the Hubbard interaction U in terms of the Dirac fermions, and show that it has the tree-level scaling transformation $U' = Ue^{-\ell}$. So argue that all short-range interactions are *irrelevant* in the Dirac semi-metal phase.

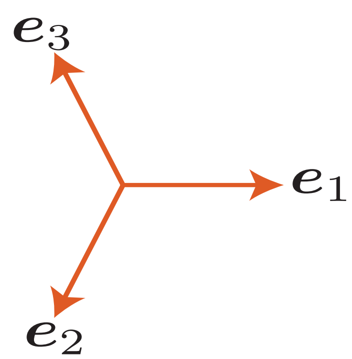


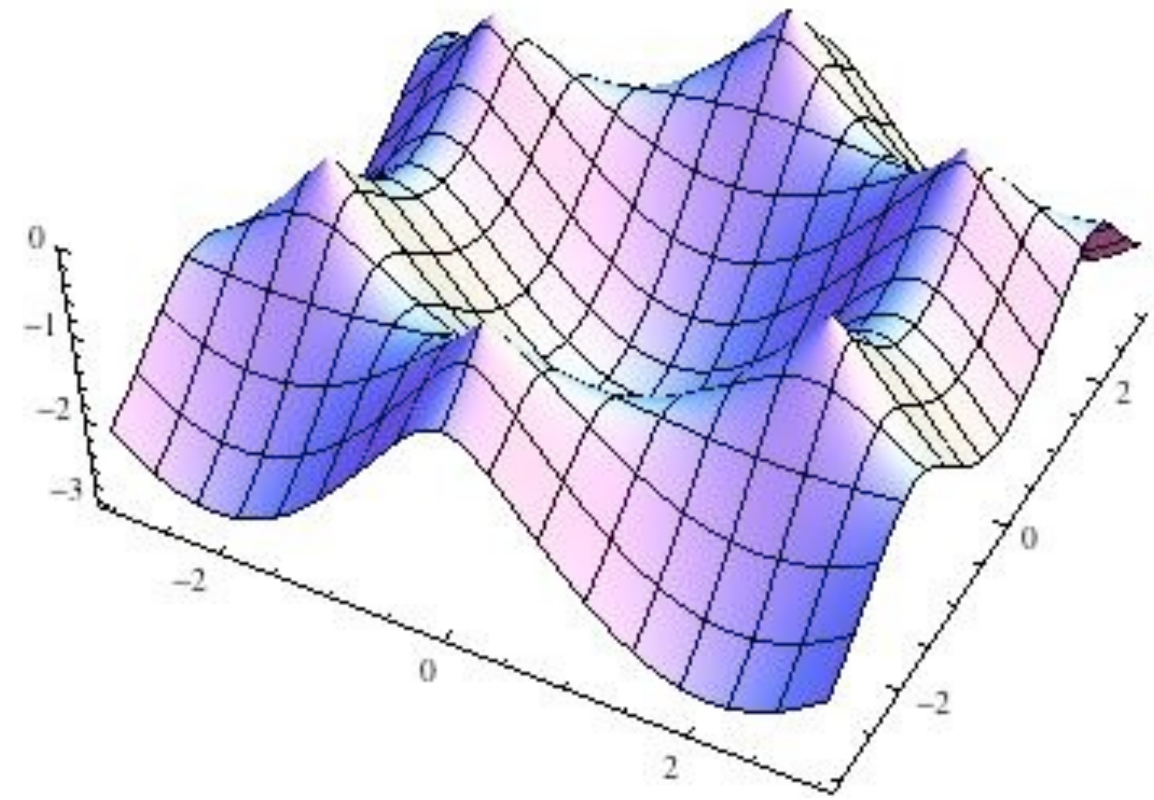

e_3
e_1
e_2

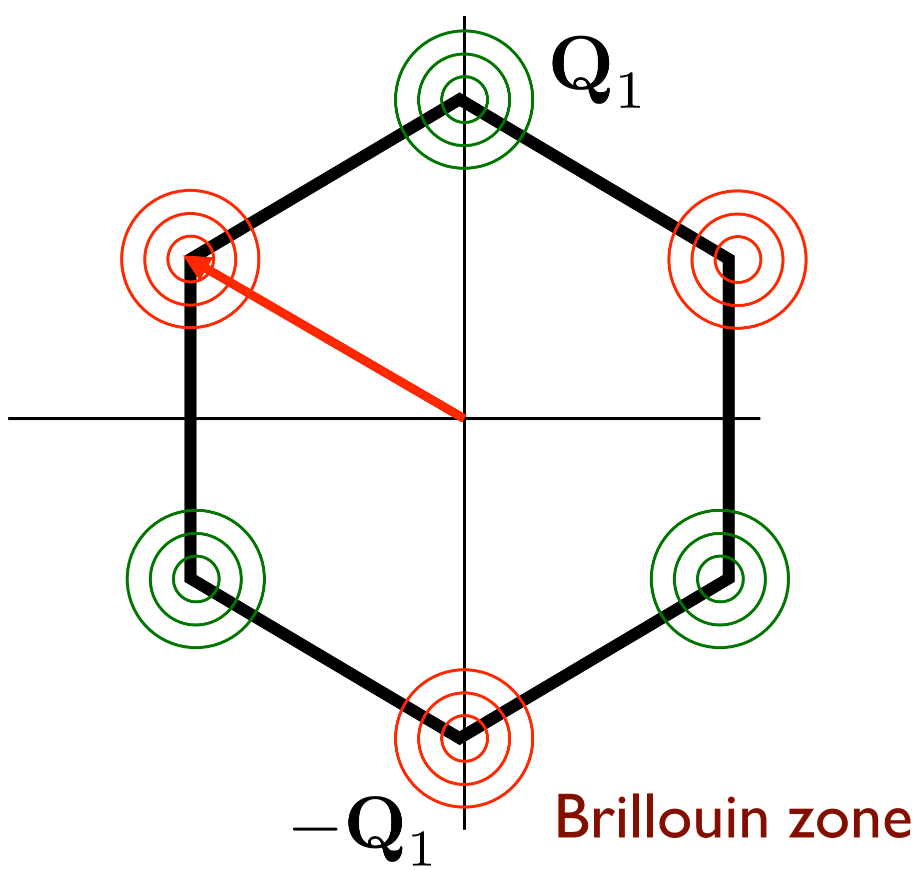

0
-1
-2
-3
-2
0
2
2
0
-2


$\mathbf{Q}_1$
$-\mathbf{Q}_1$
Brillouin zone









The theory of free Dirac fermions is invariant under conformal transformations of spacetime. This is a realization of a simple conformal field theory in 2+1 dimensions: a <u>CFT3</u>

The Hubbard Model at large U

$$H = -\sum_{i,j} t_{ij} c^\dagger_{i\alpha} c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2}\right)\left(n_{i\downarrow} - \frac{1}{2}\right) - \mu \sum_i c^\dagger_{i\alpha} c_{i\alpha}$$

In the limit of large U, and at a density of one particle per site, this maps onto the Heisenberg antiferromagnet

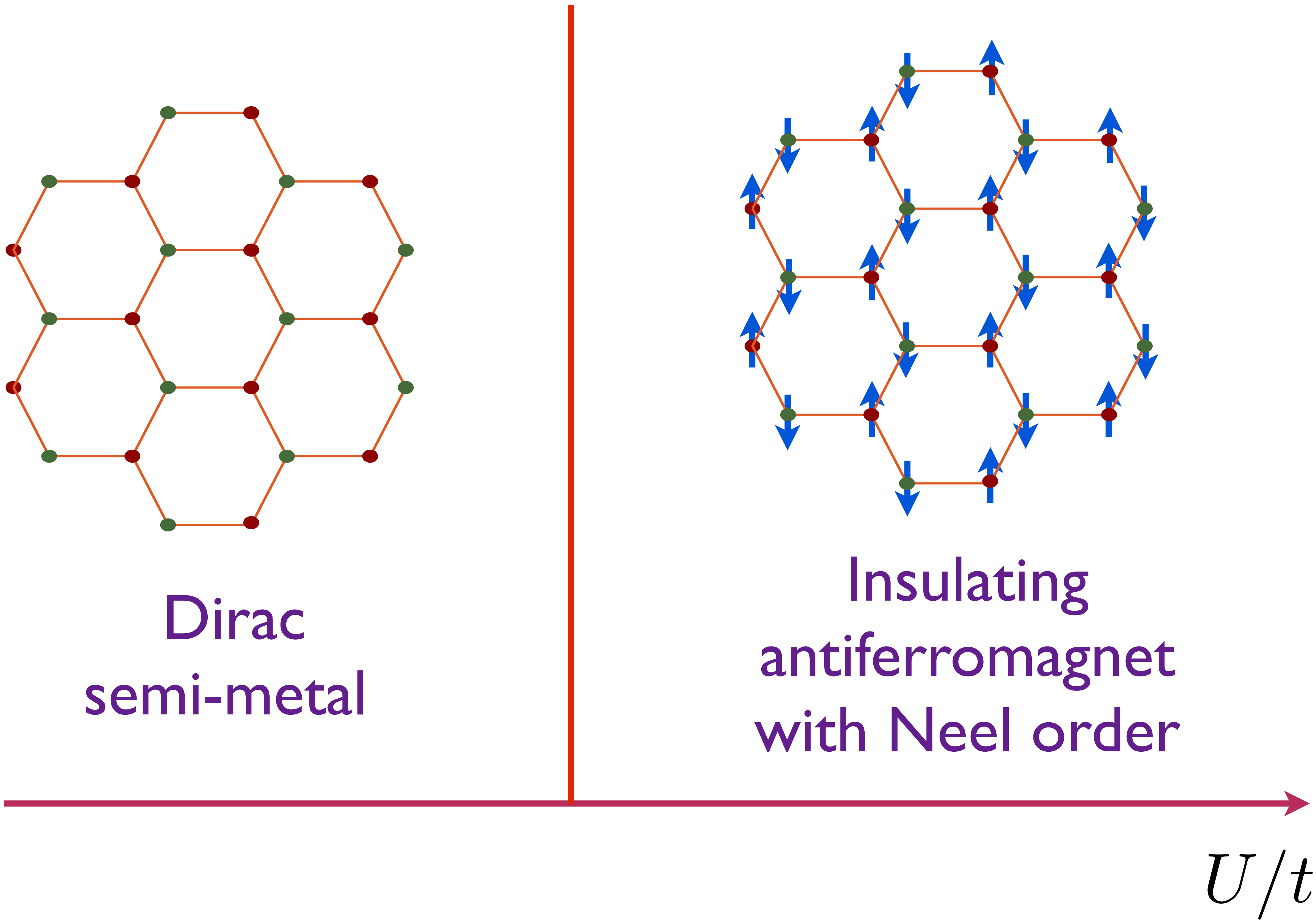
$$H_{AF} = \sum_{i<j} J_{ij} S^a_i S^a_j$$

where $a = x, y, z,$

$$S^a_i = \frac{1}{2} c^{a\dagger}_{i\alpha} \sigma^a_{\alpha\beta} c_{i\beta},$$

with σ^a the Pauli matrices and

$$J_{ij} = \frac{4t^2_{ij}}{U}$$


Dirac
semi-metal
Insulating
antiferromagnet
with Neel order
U/t

Antiferromagnetism

We use the operator equation (valid on each site i):

$$U\left(n_\uparrow - \frac{1}{2}\right)\left(n_\downarrow - \frac{1}{2}\right) = -\frac{2U}{3}S^{a2} + \frac{U}{4} \qquad (11)$$

Then we decouple the interaction via

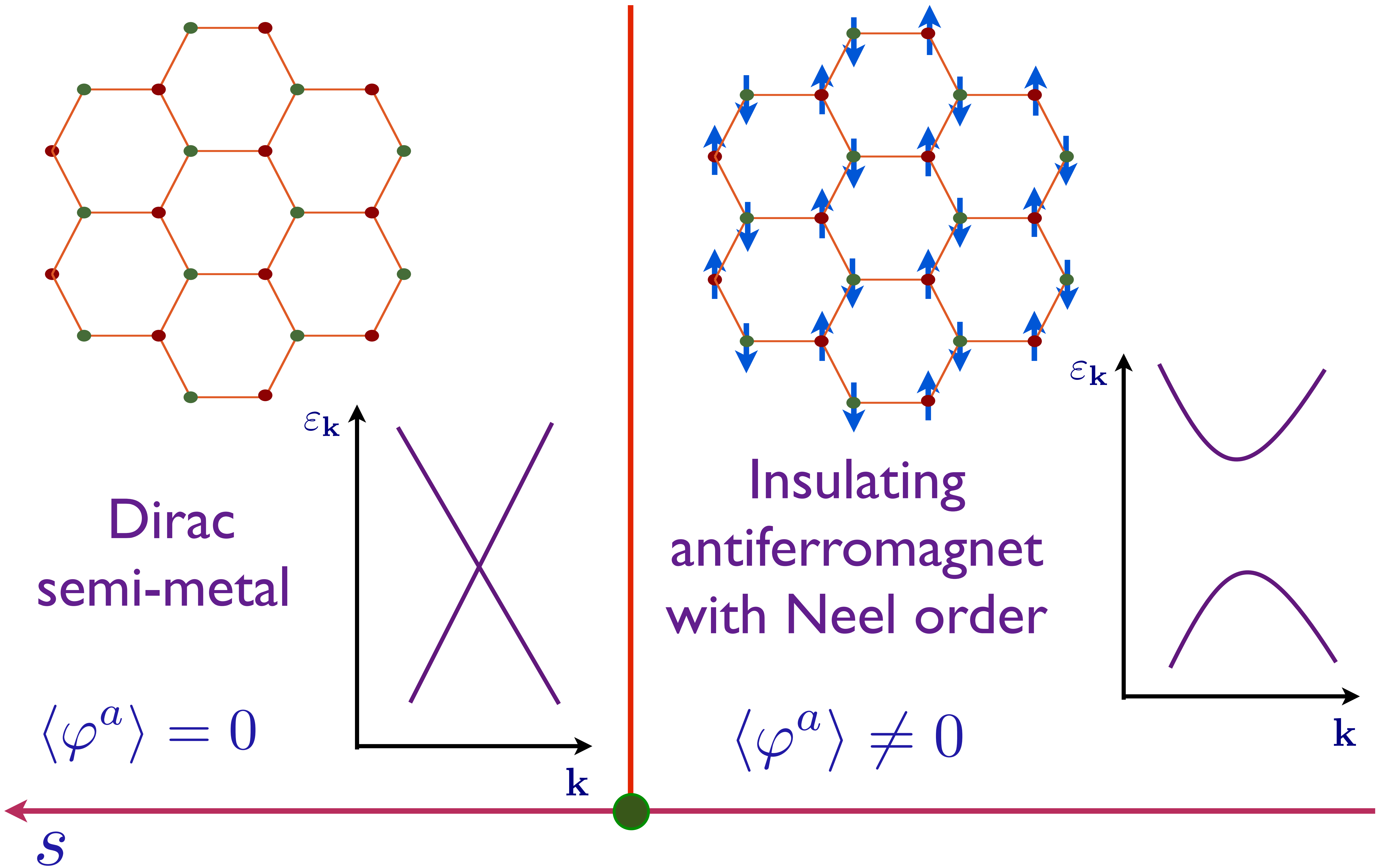
$$\exp\left(\frac{2U}{3}\sum_i \int d\tau S_i^{a2}\right) = \int \mathcal{D}J_i^a(\tau) \exp\left(-\sum_i \int d\tau \left[\frac{3}{8U}J_i^{a2} - J_i^a S_i^a\right]\right) \qquad (12)$$

We now integrate out the fermions, and look for the saddle point of the resulting effective action for J_i^a.

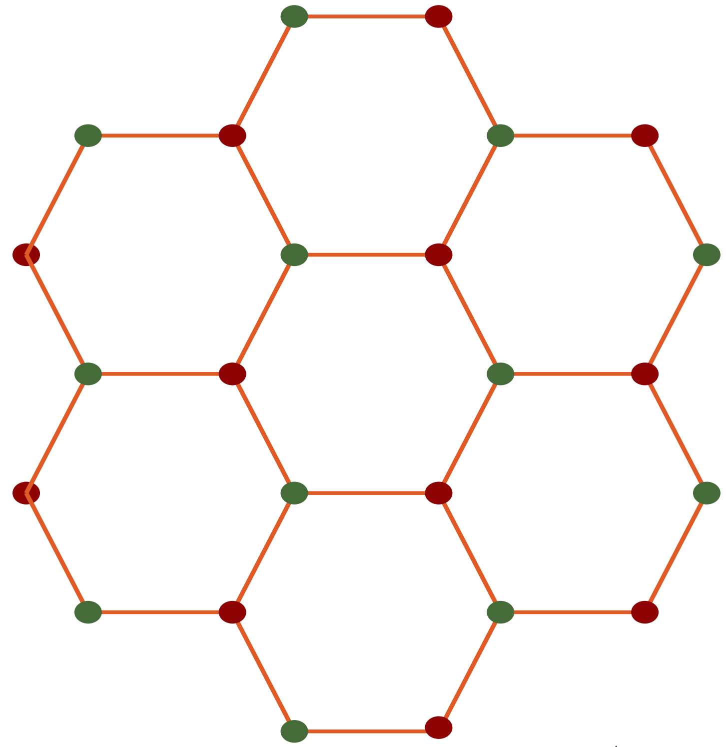
Long wavelength fluctuations about this saddle point are described by a field theory of the Néel order parameter, φ^a, coupled to the Dirac fermions in the **Gross-Neveu** model.

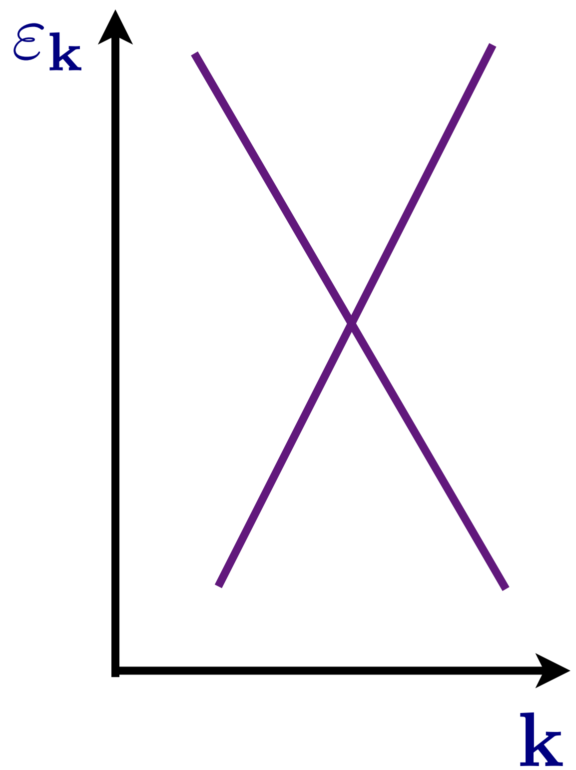
$$\boxed{\mathcal{L} = \overline{\Psi}\gamma_\mu\partial_\mu\Psi + \frac{1}{2}\left[(\partial_\mu\varphi^a)^2 + s\varphi^{a2}\right] + \frac{u}{24}\left(\varphi^{a2}\right)^2 - \lambda\varphi^a\overline{\Psi}\rho^z\sigma^a\Psi}$$

I.F. Herbut, V. Juricic, and B. Roy, *Phys. Rev. B* **79**, 085116 (2009)



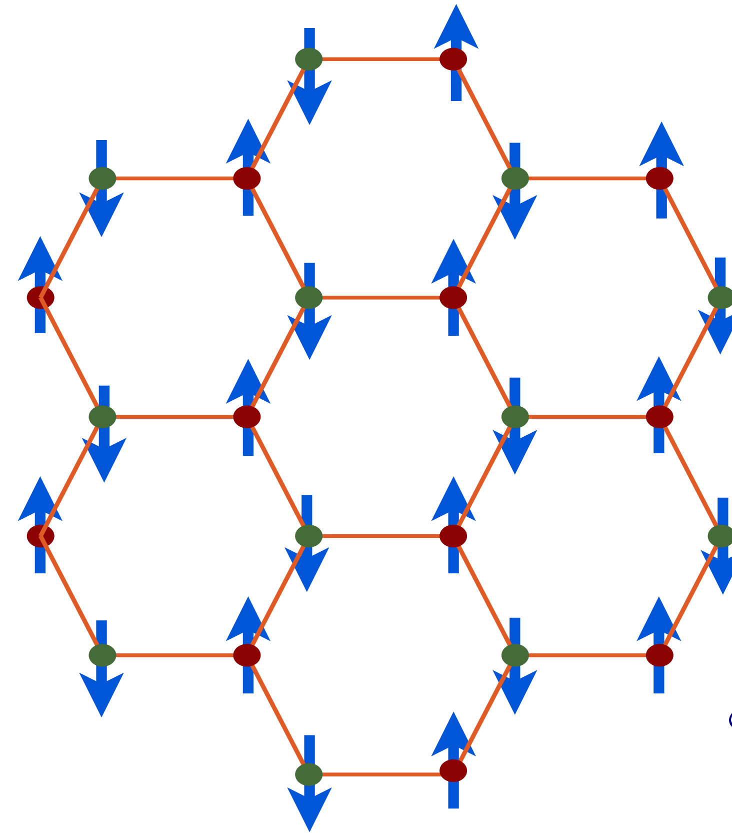
Dirac
semi-metal
$\langle \varphi^a \rangle = 0$
$\varepsilon_{\mathbf{k}}$
$\mathbf{k}$
Insulating
antiferromagnet
with Neel order
$\langle \varphi^a \rangle \neq 0$
$\varepsilon_{\mathbf{k}}$
$\mathbf{k}$
s



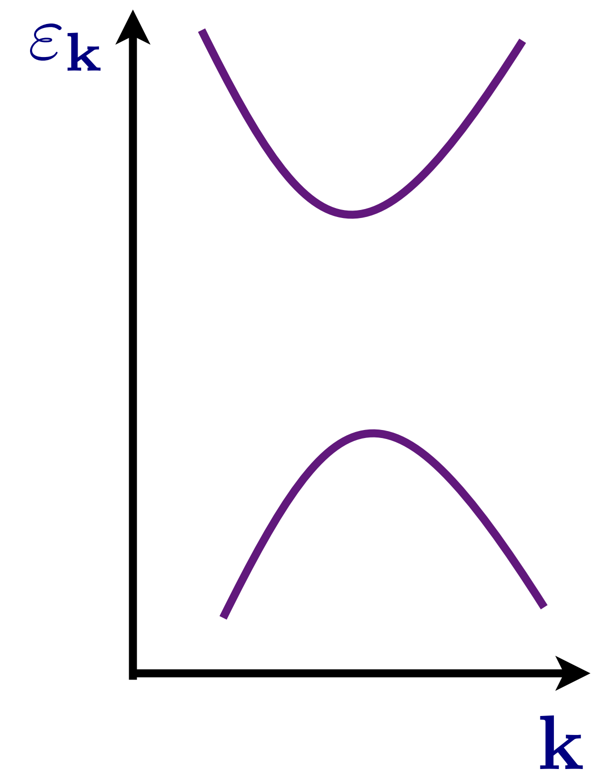


Dirac semi-metal

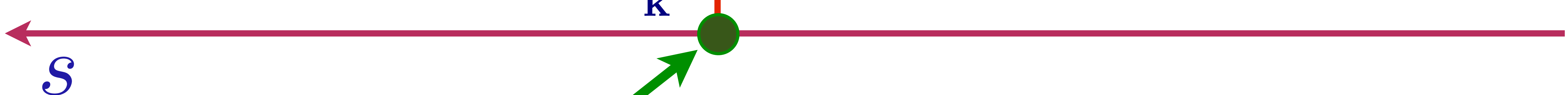
$$\langle \varphi^a \rangle = 0$$



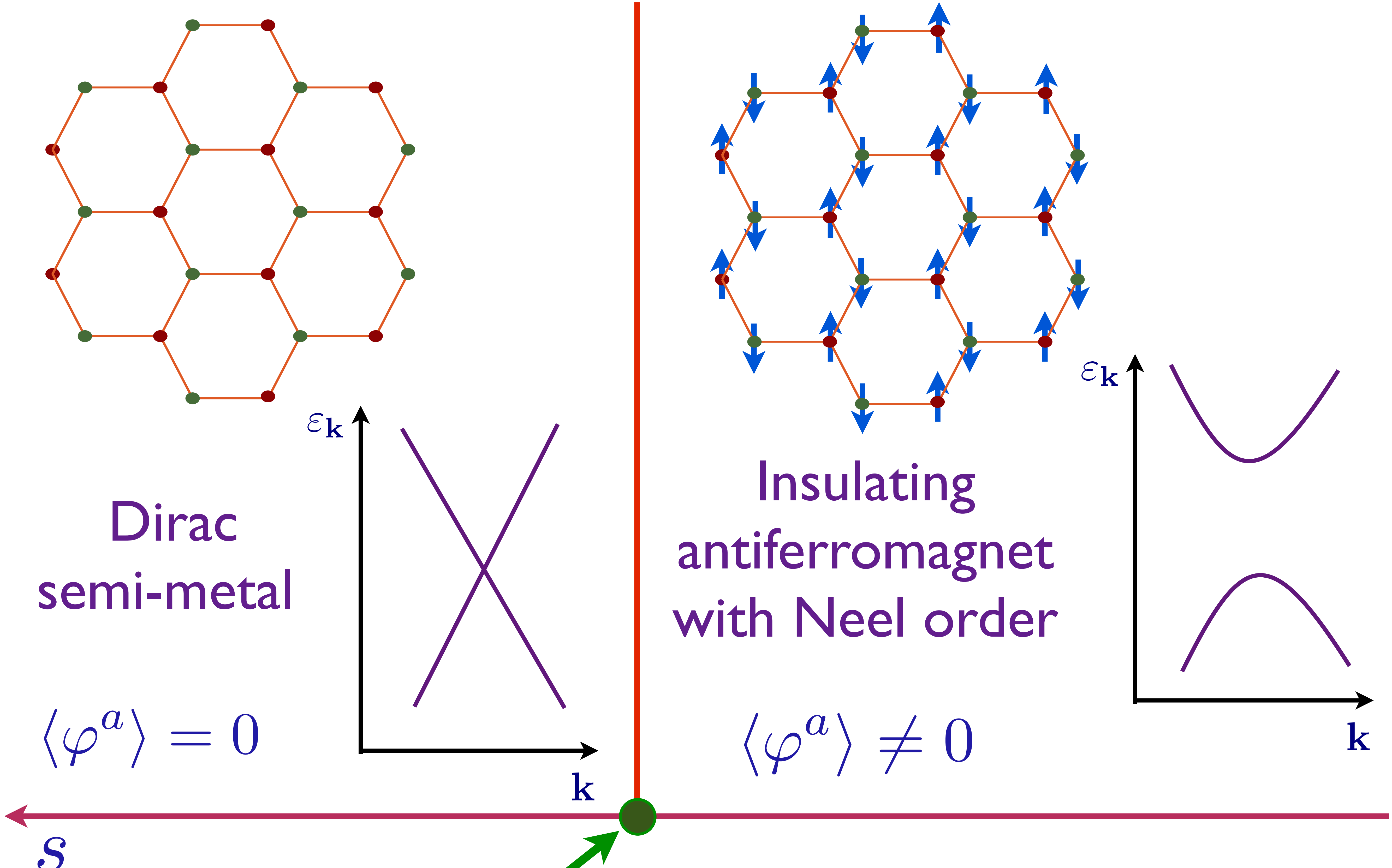
Insulating antiferromagnet with Neel order



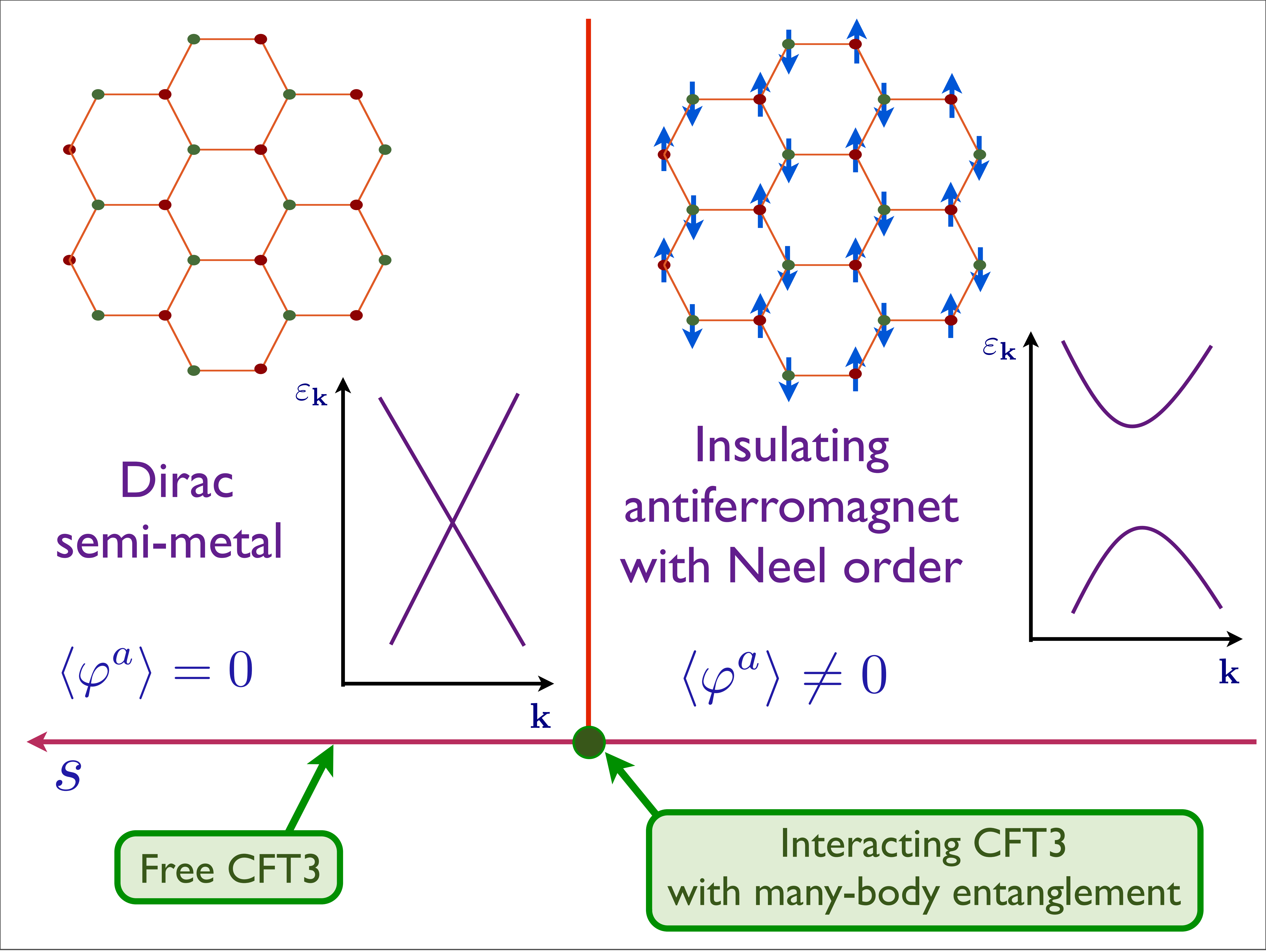
$$\langle \varphi^a \rangle \neq 0$$



$$\mathcal{L} = \overline{\Psi}\gamma_\mu\partial_\mu\Psi + \frac{1}{2}\left[(\partial_\mu\varphi^a)^2 + s\varphi^{a2}\right] + \frac{u}{24}\left(\varphi^{a2}\right)^2 - \lambda\varphi^a\overline{\Psi}\rho^z\sigma^a\Psi$$



At the quantum critical point, the non-linear couplings λ and u in the Gross-Neveu model reach non-zero fixed-point values under the renormalization group flow. The critical theory is an *interacting* CFT3


$\varepsilon_{\mathbf{k}}$
$\mathbf{k}$
Dirac
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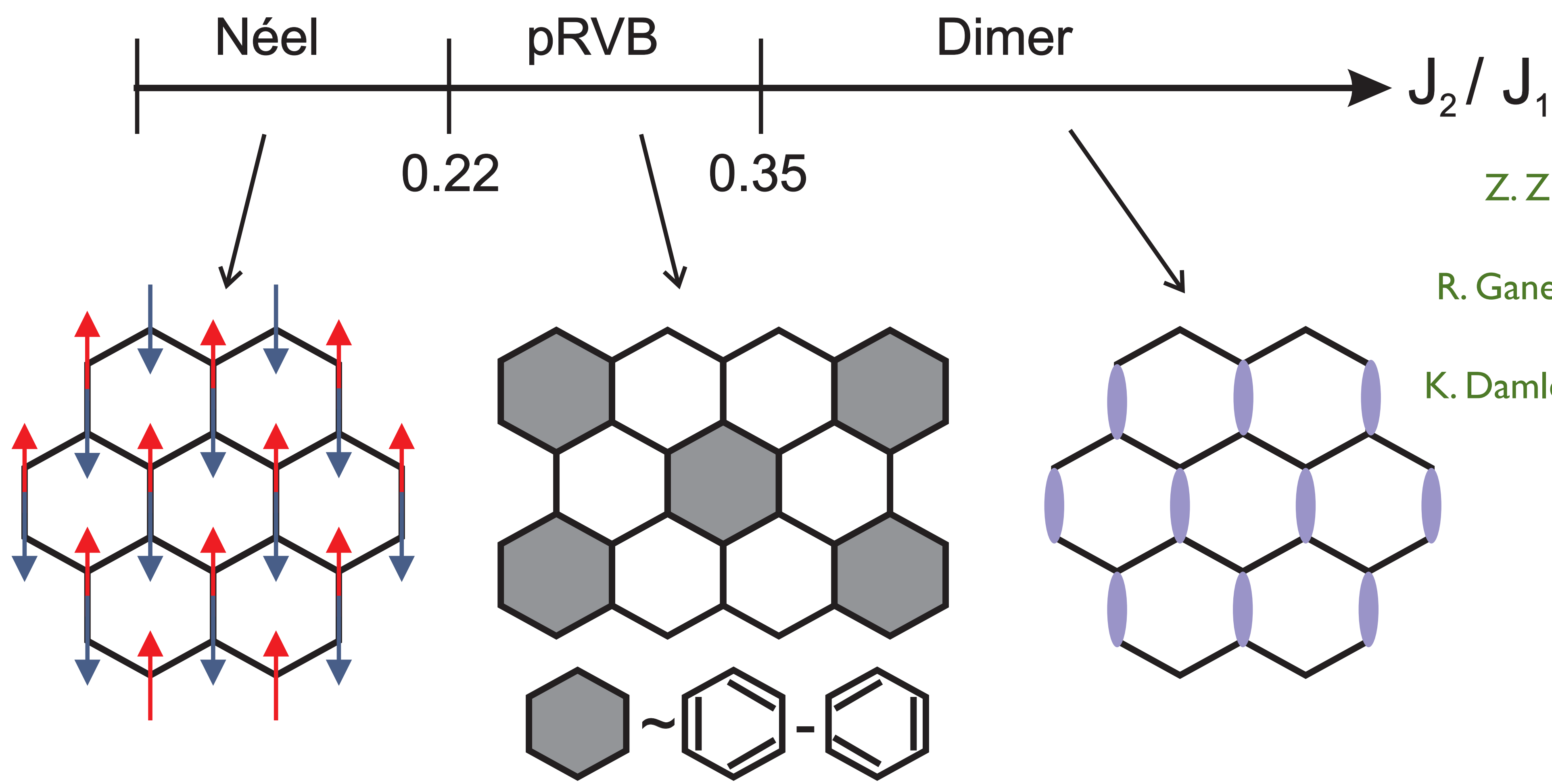
$\langle \varphi^a \rangle = 0$
Insulating
antiferromagnet
with Neel order
$\varepsilon_{\mathbf{k}}$
$\mathbf{k}$
$\langle \varphi^a \rangle \neq 0$
s
Free CFT3
Interacting CFT3
with many-body entanglement

Numerical studies of the *S=1/2* antiferromagnet on the honeycomb lattice with second-neighbor exchange



Z. Zhu, D.A. Huse, and S. R.White, arXiv:1212.6322

R. Ganesh, J. van den Brink, S. Nishimoto, arXiv:1301.0853

K. Damle, F.Alet, and S. Pujari, (J-Q model) arXiv:1302.1408

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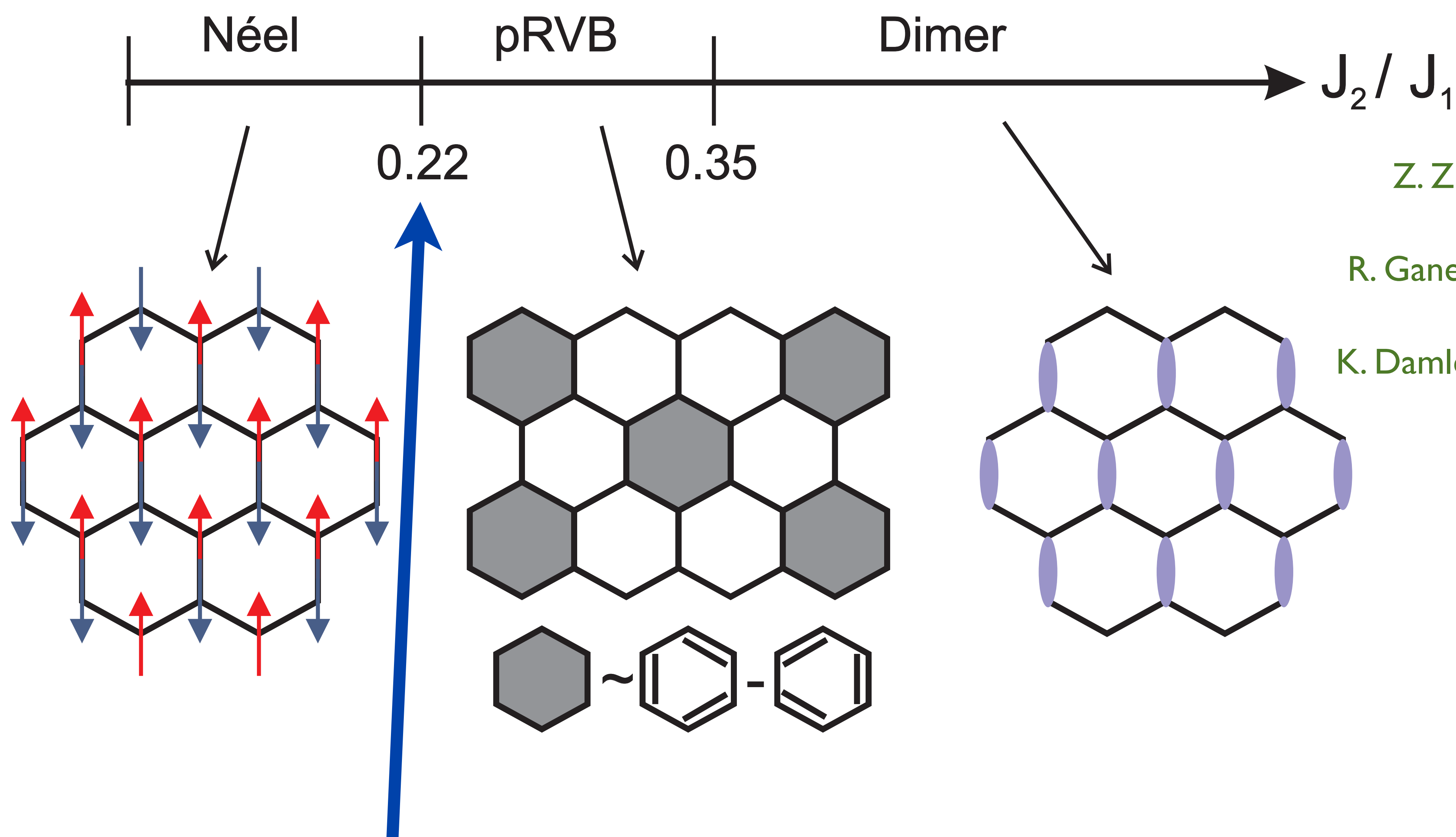

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CFT3 with fractionalization and emergent gauge fields: Write Néel order $\varphi^a = z_\alpha^* \sigma^a_{\alpha\beta} z_\beta$, and CFT3 is

$$\mathcal{L} = \left|(\partial_\mu - i a_\mu) z_\alpha\right|^2 + s|z_\alpha|^2 + u\left(|z_\alpha|^2\right)^2$$

Electrical transport

The conserved electrical current is

$$J_\mu = -i\overline{\Psi}\gamma_\mu\Psi. \tag{1}$$

Let us compute its two-point correlator, $K_{\mu\nu}(k)$ at a spacetime momentum k_μ at $T = 0$. At leading order, this is given by a one fermion loop diagram which evaluates to

$$\begin{aligned} K_{\mu\nu}(k) &= \int \frac{d^3 p}{8\pi^3} \frac{\mathrm{Tr}\left[\gamma_\mu (i\gamma_\lambda p_\lambda + m\rho^z\sigma^z)\gamma_\nu(i\gamma_\delta(k_\delta + p_\delta) + m\rho^z\sigma^z)\right]}{(p^2+m^2)((p+k)^2+m^2)} \\ &= -\frac{2}{\pi}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\int_0^1 dx \frac{k^2 x(1-x)}{\sqrt{m^2 + k^2 x(1-x)}}, \end{aligned} \tag{2}$$

where the mass $m = 0$ in the semi-metal and at the quantum critical point, while $m = |\lambda N_0|$ in the insulator. Note that the current correlation is purely transverse, and this follows from the requirement of current conservation

$$k_\mu K_{\mu\nu} = 0. \tag{3}$$

Of particular interest to us is the K_{00} component, after analytic continuation to Minkowski space where the spacetime momentum k_μ is replaced by (ω, k). The conductivity is obtained from this correlator via the Kubo formula

$$\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} K_{00}(\omega, k). \tag{4}$$

In the insulator, where $m > 0$, analysis of the integrand in Eq. (2) shows that that the spectral weight of the density correlator has a gap of $2m$ at $k = 0$, and the conductivity in Eq. (4) vanishes. These properties are as expected in any insulator.
In the metal, and at the critical point, where $m = 0$, the fermionic spectrum is gapless, and so is that of the charge correlator. The density correlator in Eq. (2) and the conductivity in Eq. (4) evaluate to the simple universal results

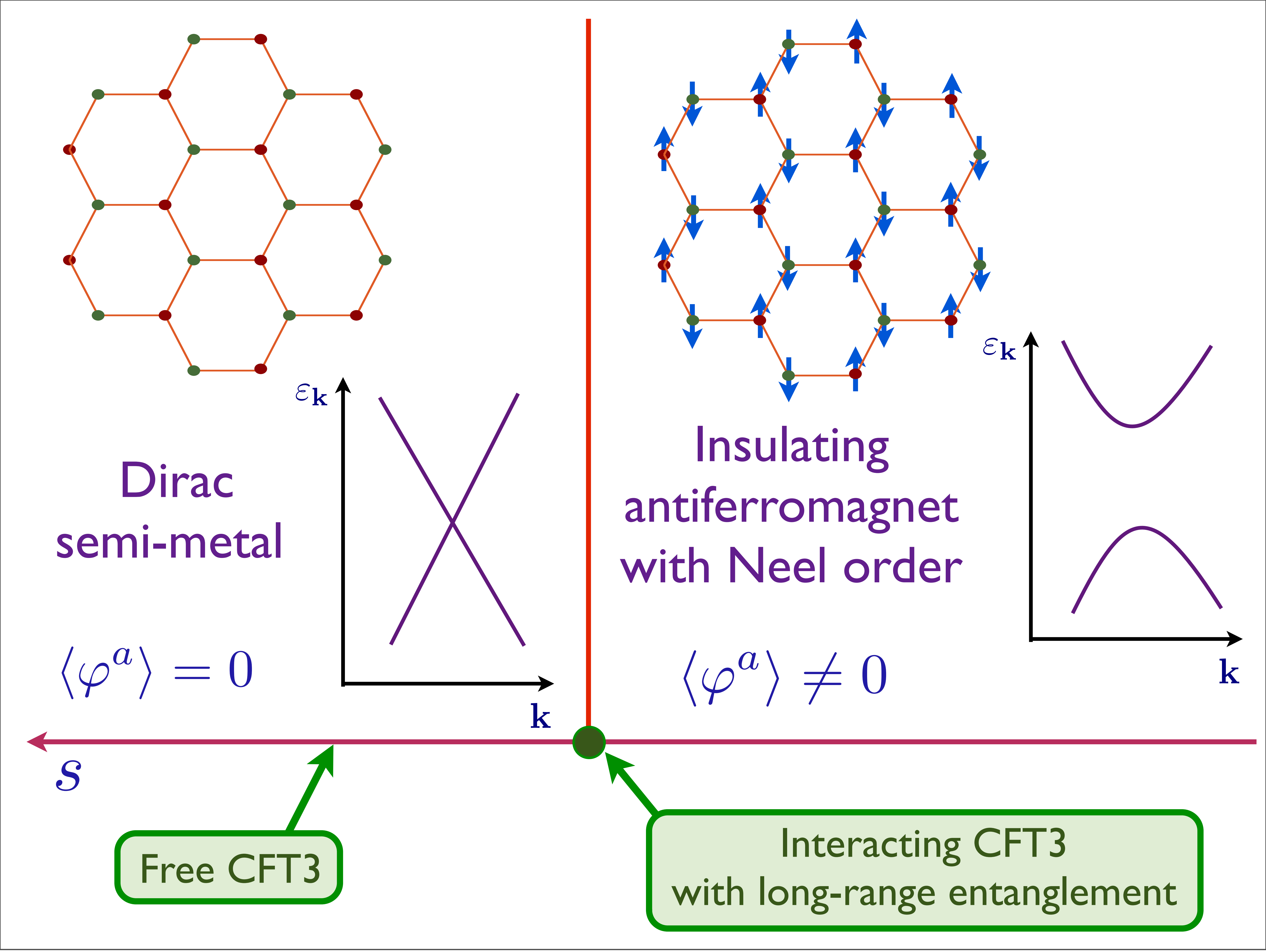
$$\begin{aligned} K_{00}(\omega, k) &= \frac{1}{4} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\ \sigma(\omega) &= 1/4. \end{aligned} \tag{5}$$

Going beyond one-loop, we find *no change* in these results in the semi-metal to all orders in perturbation theory. At the quantum critical point, there are no anomalous dimensions for the conserved current, but the amplitude does change yielding

$$\begin{aligned} K_{00}(\omega, k) &= \mathcal{K} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\ \sigma(\omega) &= \mathcal{K}, \end{aligned} \tag{6}$$

where $\mathcal{K}$ is a universal number dependent only upon the universality class of the quantum critical point. The value of the $\mathcal{K}$ for the Gross-Neveu model is not known exactly, but can be estimated by computations in the $(3-d)$ or $1/N$ expansions.

Also note $K_{\mu\nu} = \mathcal{K}|k| \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$



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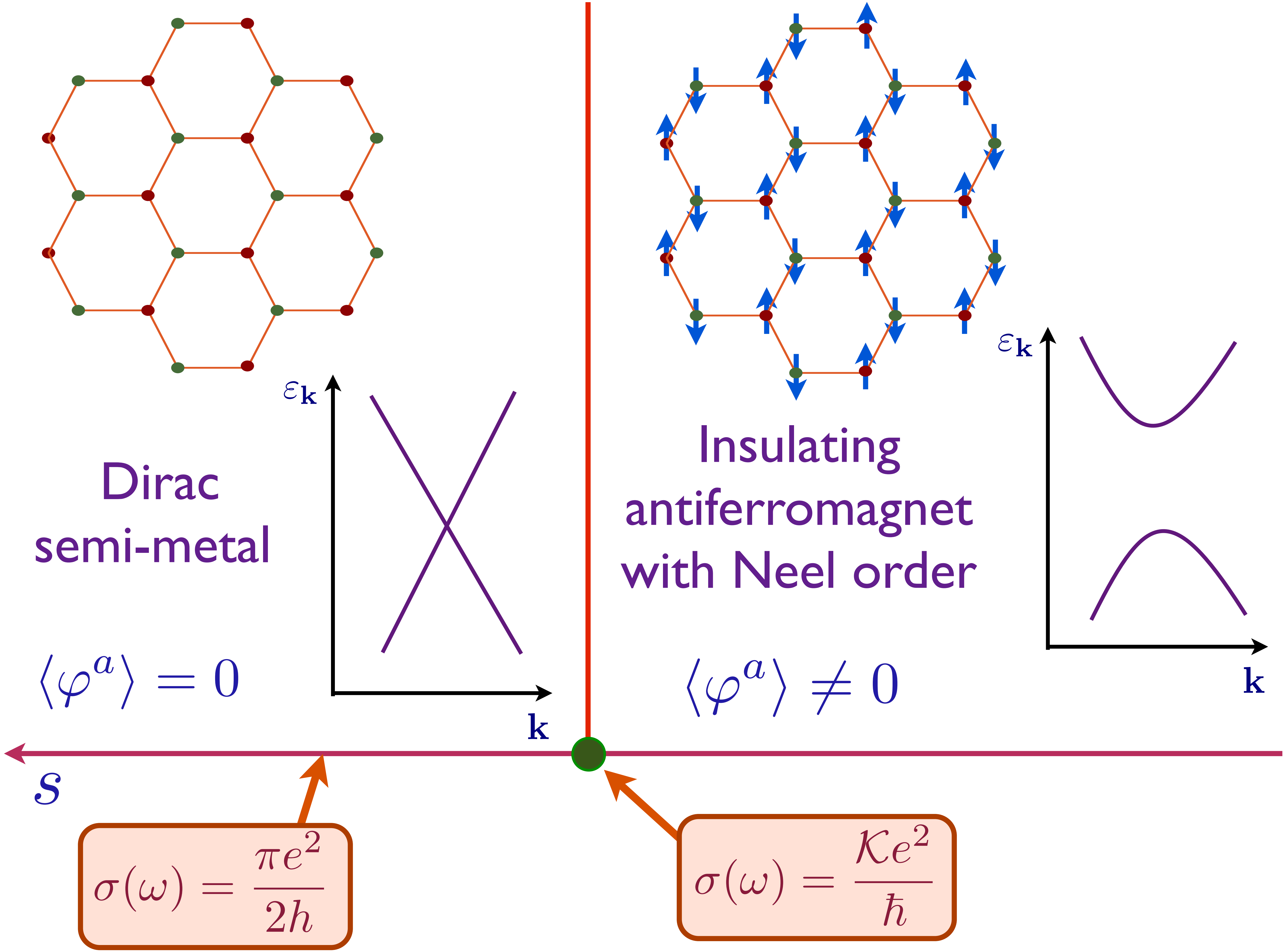
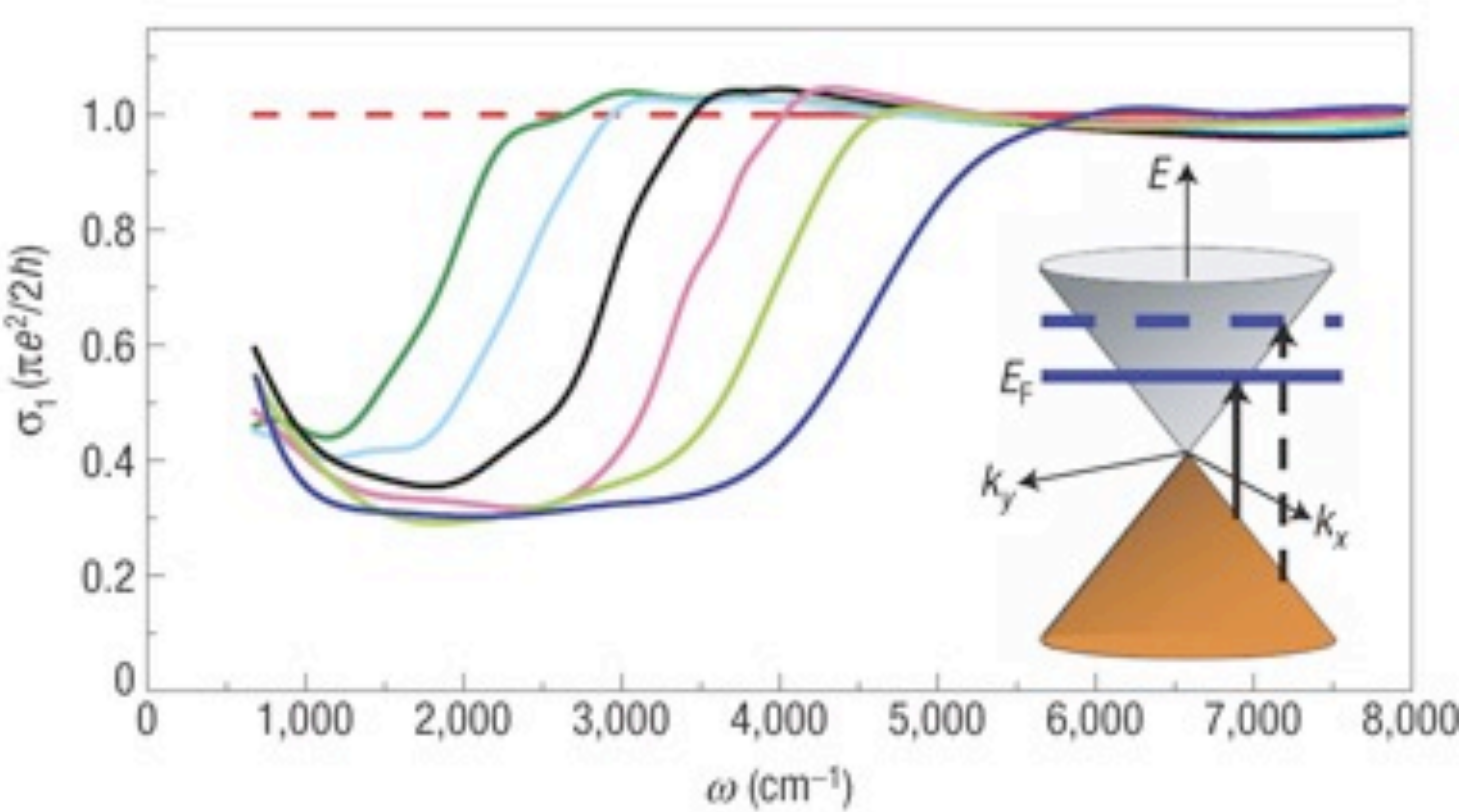


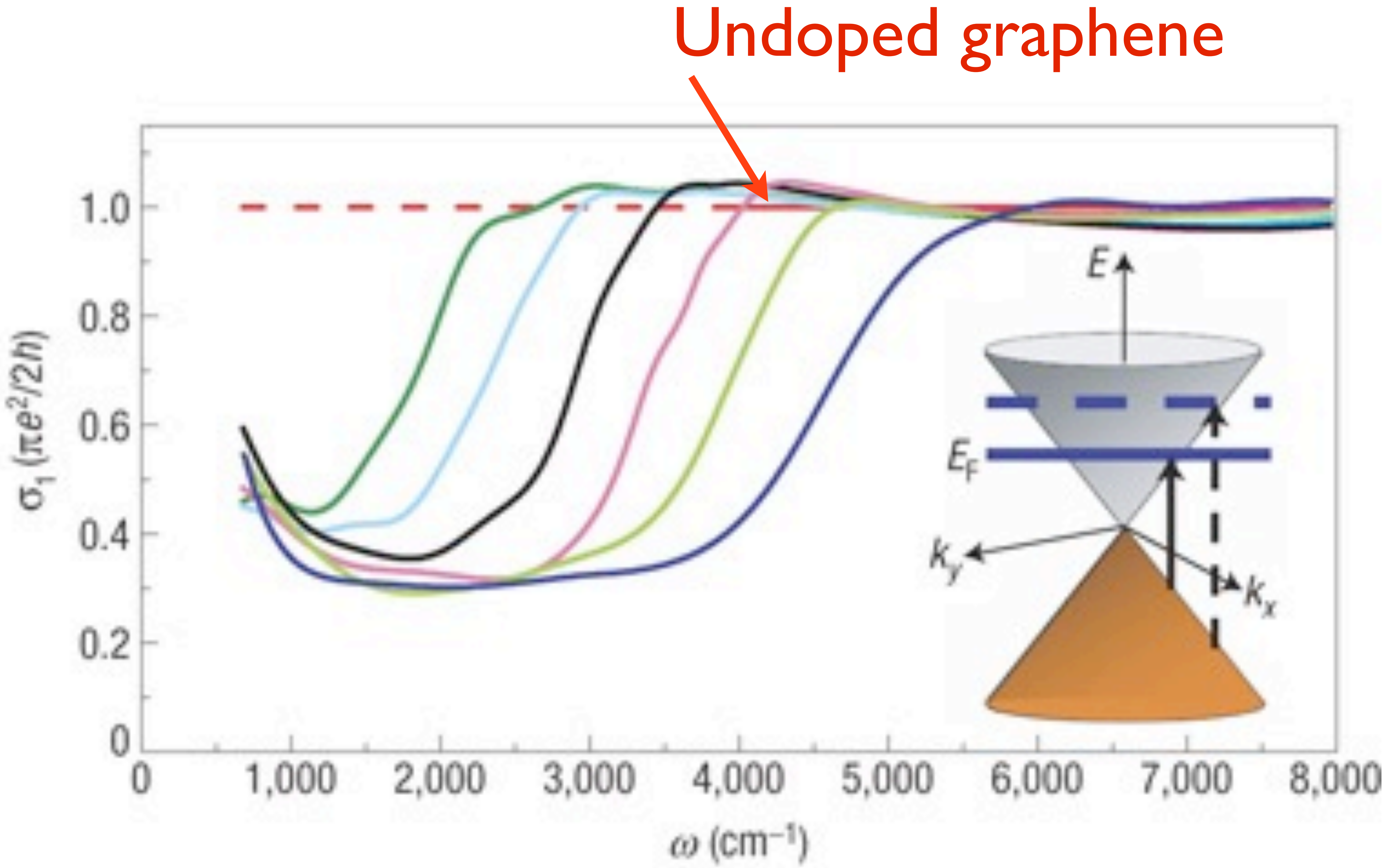
Dirac
semi-metal
$\langle \varphi^a \rangle = 0$
$\varepsilon_{\mathbf{k}}$
$\mathbf{k}$
Insulating
antiferromagnet
with Neel order
$\langle \varphi^a \rangle \neq 0$
$\varepsilon_{\mathbf{k}}$
$\mathbf{k}$
s
$\sigma(\omega) = \frac{\pi e^2}{2h}$
$\sigma(\omega) = \frac{\mathcal{K} e^2}{\hbar}$

Optical conductivity of graphene



Z. Q. Li, E. A. Henriksen, Z. Jiang, Z. Hao, M. C. Martin, P. Kim, H. L. Stormer, and D. N. Basov, Nature Physics 4, 532 (2008).

Optical conductivity of graphene



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