'Holography' without gravity: Phases of matter which are characterized by their edge states

lecture notes for Arnold Sommerfeld Center School, August 2013

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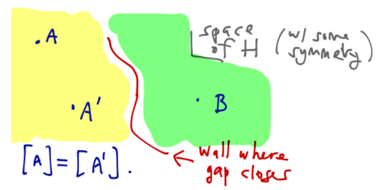
This lecture provides a low-tech quantum field theory point of view on symmetry-protected topological (SPT) states.

[Main references: Senthil and Vishwanath, 1209.3058 [1]; Senthil and Xu, 1301.6172 [2] and some talks by those authors. A useful, brief review is the second part of [3]. See also this Journal Club for Condensed Matter Physics commentary by Matthew Fisher.]

0.1 Introductory remarks

In the previous lecture we discussed the states of matter with the *most* gapless stuff – whole surfaces in momentum space. Now we are going to move to the opposite extreme – gapped states, where $E_1 - E_0$ is strictly nonzero.

A basic cond-mat question: how to distinguish different *phases*. (Starting right now restrict to gapped states for simplicity) Divide up the space of Hamiltonians (coupling constants are coordinates), into adiabatically connected regions – 'phases' [Do not confuse with 'phases' $e^{\mathbf{i}\phi}$!] – regions inside which we can move around by changing couplings without closing the gap.



A useful refinement: restrict attention to $\mathbf{H}$ preserving some symmetry group G.

1. A set of well-understood distinctions between phases: Landau, symmetry-breaking orders. What symmetries of $\mathbf{H}$ are broken by the groundstate? Representatives of these can be realized by product states – they are not very quantum mechanical.

2. A difficult set of distinctions: topological orders (TO).

Topological order: means deconfined, emergent gauge theory. For a 'gentle' review, see [4]. Consequences of this are:

1. Fractionalization of the quantum numbers of the microscopic particles: That is, emergent quasiparticle excitations carry quantum numbers (spin, charge) which are fractions of those of the constituents.

2. Groundstate degeneracy which depends on the topology of space: The fractional statistics of the quasiparticles (point 1) implies a groundstate degeneracy on *e.g.* the torus: Pair-create qp-antiqp pair, move them around a spatial cycle, then re-annihilate. This process $\mathcal{F}_x$ maps one gs to another. But $\mathcal{F}_x$ does not commute with $\mathcal{F}_y$, by the anyonic statistics. The space of groundstates must represent the algebra of these operators.

3. Long-ranged entanglement: correlations between regions of space which produce universal deviations from (in fact, deficits relative to) area law. This means that a state with topological order is far from a product state.[1]

Precise (gauge-invariant) definition is elusive in $D > 2+1$[2].

The poster examples of this are *fractional* quantum Hall states (experiments!), and (more theoretically) discrete gauge theory. Both have a description in terms of gauge fields, the former crucially involving Chern-Simons terms. The fractional charge and statistics come from holonomies of these gauge fields, and the groundstate degeneracy comes from their Wilson lines.

Abelian CS theory. (I want to explain an example of how properties 1 and 2 can be realized in QFT.)

Canonical examples of topologically-ordered states: (abelian) *fractional* quantum Hall states in $D = 2+1$.

The low-energy effective field theory is Chern-Simons-Witten gauge theory whose basic action is:

$$S_0[a_I] = \sum_{IJ} \frac{K_{IJ}}{4\pi} \int a_I \wedge \mathrm{d}a_J$$

a^I are a collection of abelian gauge fields. Notice that we wrote this action in a coordinate-invariant way without needing to mention a metric. This is a topological field theory.

Two more ingredients:

Notice that these gauge fields imply conserved currents $j^I_\mu = \epsilon_{\mu\nu\rho}\partial_\nu a^I_\rho$. This is automatically conserved by antisymmetry of $\epsilon_{\mu\nu\rho}$, as long as a is single-valued. In its realization as the EFT for a quantum Hall state, a linear combination of these currents is coupled to the external EM field $\mathcal{A}_\mu$:

$$S_{EM}[a_I, \mathcal{A}] = \int \mathcal{A}^\mu t_I j^I_\mu \ .$$

[1]Here is a more precise argument that $2 \implies 3$, due to [5]: Recall $S(A) \equiv -\mathrm{tr}\boldsymbol{\rho}_A \log \boldsymbol{\rho}_A$, the EE of the subregion A in the state in question. In such a state [6, 7], $S(A) = \Lambda\ell(\partial A) - \gamma$ ($\Lambda =$UV cutoff) $\gamma \equiv$ "topological entanglement entropy" $\propto \log(\#\text{torus groundstates}) \geq 0$. (Deficit relative to area law.) *c.f.*: For a state w/o LRE $S(A) = \oint_{\partial A} s d\ell$ (local at bdy) $= \oint \left(\Lambda + bK + cK^2 + \ldots\right) = \Lambda\ell(\partial A) + \tilde{b} + \frac{\tilde{c}}{\ell(\partial A)}$ Pure state: $S(A) = S(\bar{A}) \implies b = 0$.

[2]The best definition of topological order in general dimension is, I think, an interesting open question. In 2+1 dimensions, nontrivial transformation under adiabatic modular transformations seems to capture everything [4]; topology-dependent groundstate degeneracy is a corollary.

Finally, we must include information about the (gapped) quasiparticle excitations of the system. This is encoded by adding (conserved) currents minimally coupled to the CS gauge fields:

$$S_{qp} = \int a_I j^I_{qp}.$$

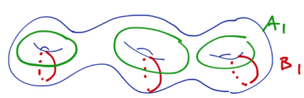
Let's focus on the case with a single such field (this describes *e.g.* the Laughlin state of electrons at $\nu = 1/3$ for $k = 3$).

In this case, the quasiparticles are anyons of charge e/k. The idea of how this is accomplished is called flux attachment. The CS equation of motion is $0 = \frac{\delta S}{\delta a} \sim -f_{\mu\nu}\frac{k}{2\pi} + j^{qp}_\mu$, where j^{qp} is a quasiparticle current, coupling minimally to the CS gauge field. The time component of this equation $\mu = t$ says $b = \frac{2\pi}{k}\rho$ – a charge gets $2\pi/k$ worth of magnetic flux attached to it. Then if we bring another quasiparticle in a loop C around it, its phase changes by

$$\Delta\varphi_{12} = q_1 \oint_C a = q_1 \int_{R, \partial R = C} b = q_1 \frac{2\pi}{k} q_2.$$

Hence, the quasiparticles have fractional braiding statistics.

# of groundstates = $|\det(K)|^{\text{genus}}$ Simplest case: $K = k$. $\mathcal{F}_x = e^{\mathbf{i}\oint_{C_x} a}$. According to the CS action, a_x is the canonical momentum of a_y. Canonical quantization then implies that

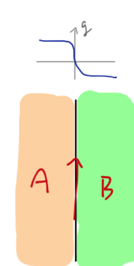
$\mathcal{F}_x\mathcal{F}_y = \mathcal{F}_y\mathcal{F}_x e^{2\pi i/k}$. 

$\longrightarrow k^g$ groundstates.

For more detail see the textbook by Wen [21].

Now we return to our list of possible ways to distinguish phases (without having to check every possible adiabatic path between them in the infinite-dimensional space of Hamiltonians).

3. A simple yet still-interesting question: how do we distinguish phases of matter which preserve G and don't have TO?



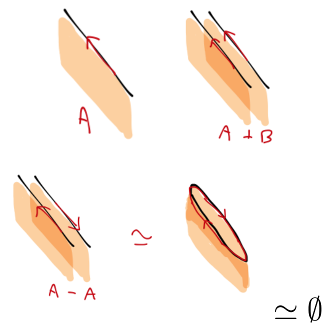
One answer: put them on a space with boundary, *i.e.* an interface with vacuum or with each other. Quantized (hence topology) properties of the surface states can be characteristic of distinct phases.

Rough idea: just like varying the Hamiltonian in time to another phase requires closing the gap $\mathbf{H} = \mathbf{H}_1 + g(t)\mathbf{H}_2$, so does varying the Hamiltonian in space $\mathbf{H} = \mathbf{H}_1 + g(x)\mathbf{H}_2$.

SO: this will be holography in the sense that we are using QFT in one lower dimension etc. but it's not a duality: the lower-dim'l QFT is a *label* here.

Def: a gapped groundstate of some Hamiltonian **H** preserving G (without TO) which is distinct from any trivial product state is called a SPT state wrt G.

These states form a group under composition. -A is the mirror image.



Note: with topological order, even if we can gap out the edge states, there is still stuff going on (e.g. fractional charges) in the bulk. Not a group.

How to characterize more precisely? Several possibilities:

1. if $G \supset U(1)$, EM response. quantized. [think: integer quantum hall]

Without a U(1), we need something else. (And two states with the same quantized EM response may be distinct for some other reason.)

2. What happens if you gauge G? In general this produces a new state, with TO (or gapless). That state is a label. (This works both ways: labelling TO or gapless states is the hard part.) I'm not going to say more about this here.

3. Weird quantized stuff at the surface. e.g. the surface can have fractionalization and TO.

Goals: SPT states of bosons (especially in $D = 3 + 1$). Interesting partly because it *requires* interactions. States of bosons also means states of spins by a mapping which I will not review here.

We'll maintain a vigorous focus on examples, and not worry about classification.

1 EM response of SPT states protected by $G \supset U(1)$

1.1 $D = 2 + 1$, $G = U(1)$

We assume we have a gapped state of matter made of some stuff, out of which we can construct a conserved U(1) current j_μ. This means we can couple this current to an external, background, non-dynamical gauge field $\mathcal{A}_\mu$ like so:

$$S[\text{the stuff}, \mathcal{A}] = S[\text{the stuff}] + \int j^\mu \mathcal{A}_\mu .$$

Here we'll treat $\mathcal{A}$ as a background field that we control[3]. Integrate out the stuff to see the EM response:

$$e^{\mathbf{i}S_{\text{eff}}[\mathcal{A}]} \equiv \int [D\text{stuff}] e^{\mathbf{i}S[\text{stuff},\mathcal{A}]}.$$

Terms quadratic in $\mathcal{A}$ encode linear response:

$$\langle jj \rangle = \frac{\delta^2}{\delta\mathcal{A}\delta\mathcal{A}} \log Z|_{\mathcal{A}=0}$$

Because the stuff is gapped, S_{eff} is local. In a derivative expansion we can guess $S_{\text{eff}}[\mathcal{A}]$:

$\mathcal{A}$ is a gauge field, dimension 1.

$$S_{\text{eff}}[\mathcal{A}] = \int \Bigg(\underbrace{0 \cdot \mathcal{A}^2}_{\text{no symmetry breaking}} + \frac{\nu}{4\pi} \mathcal{A} \wedge \mathcal{F} + \frac{1}{g^2} \mathcal{F}_{..}\mathcal{F}^{..} \Bigg).$$

($\mathcal{F} = \mathrm{d}\mathcal{A}$.) With time-reversal symmetry, $\nu = 0$. Maxwell is irrelevant. (Actually, without Lorentz invarance we can have non-vacuum dielectric constant and magnetic permittivity ϵ, μ, but this won't affect our story.)

Kubo: $\sigma^{xy} = \lim_{\omega\to 0} \frac{\langle j^x j^y \rangle|_{k=0}}{\mathbf{i}\omega} = \nu \frac{e^2}{h} = \frac{\nu}{2\pi}$.

Next we'll show that under these assumptions ν is quantized. So different values of ν are distinct states, since an integer can't change continuously. (Note that there could be other distinctions – states with the same ν could be distinct.)

The following flux insertion argument implies it has exchange statistics $\pi\nu$.

Thread 2π worth of localized magnetic flux through some region of the sample (as in the $\otimes$ at right). This means

$$2\pi = \Delta\Phi = \int \mathrm{d}t \partial_t \left(\oint_{R|\partial R = C} \mathrm{d}\vec{a} \cdot \vec{B} \right) \overset{\text{Faraday}}{=} -\int \mathrm{d}t \oint_C \vec{E} \cdot \mathrm{d}\vec{\ell} \overset{j_r = \sigma_{xy} E_\varphi}{=} -\frac{1}{\sigma_{xy}} 2\pi \underbrace{\int \mathrm{d}t j_r}_{=\Delta Q}$$

which says that the inserted flux sucks in an amount of charge

$$\Delta Q = \sigma_{xy}.$$

[3]Notice that what we've done here is *not* gauging the U(1) symmetry. We are not changing the Hilbert space of the system. The gauge field is just like a collection of coupling constants

This object is a localized excitation of the system – it can move around, it's a particle. But from the Bohm-Aharonov effect, it has statistics angle $\pi\sigma_{xy}$ (the factor of two is confusing).

By assumption of no fractionalization, all particles including this one must have the same statistics as the microscopic constituents.

Conclusion : For a non-fractionalized state made from fermions, this means $\nu \in \mathbb{Z}$. For bosons, no fractionalization implies $\nu \in 2\mathbb{Z}$ [8].

1.2 $D = 3+1$ and $G = U(1) \ltimes \mathbb{Z}_2^T$

The effective field theory for any 3+1d insulator, below the energy gap, has the following form

$$S_{\text{eff}}[\vec{E}, \vec{B}] = \int \mathrm{d}^3x\mathrm{d}t \left(\epsilon \vec{E}^2 - \frac{1}{\mu} \vec{B}^2 + 2\alpha\theta \vec{E} \cdot \vec{B} \right) \tag{1}$$

where ϵ, μ are the dielectric constant and permittivity, and α is the fine structure constant. Flux quantization implies that

$$\frac{\alpha}{32\pi^2} \int_{M_4} \vec{E} \cdot \vec{B} = \frac{1}{16\pi^2} \int_{M_4} F \wedge F \in \mathbb{Z}$$

is an integer for any closed 4-manifold $M_4, \partial M_4 = \emptyset$. This means that the partition function is periodic

$$Z(\theta + 2\pi) = Z(\theta)$$

and hence the spectrum on a closed 3-manifold is periodic in θ. (As we will discuss, shifting θ by 2π is not so innocuous on a space with boundary or for the wavefunction.)

Time reversal acts by

$$\mathcal{T} : (\vec{E}, \vec{B}) \to (\vec{E}, -\vec{B})$$

which means $\theta \to -\theta$, which preserves the spectrum only for $\theta \in \pi\mathbb{Z}$. So time-reversal invariant insulators are labelled by a quantized 'magnetoelectric response' θ/π [9].

Now consider what happens on a space with boundary. The interface with vacuum is a domain wall in θ, between a region where $\theta = \pi$ (TI) and a region where $\theta = 0$ (vacuum). The electromagnetic current derived from (1) is[4]

$$j^{\mu}_{EM} = \frac{e^2}{2\pi h} \epsilon^{\mu\cdots} \partial.\theta\partial.A. + \cdots \tag{2}$$

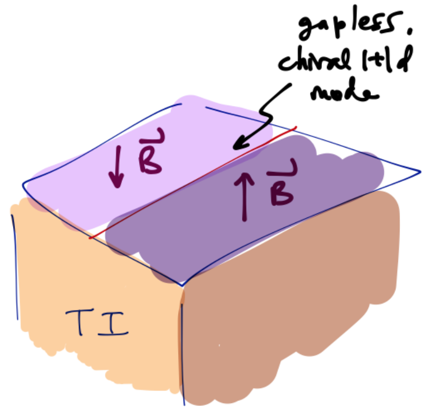
[4] A comment about notation: here and in many places below I refuse to assign names to dummy indices when they are not required. The $\cdot$s indicate the presence of indices which need to be contracted. If you must, imagine that I have given them names, but written them in a font which is too small to see.

where the $\cdots$ indicate contributions to the current coming from degrees of freedom at the surface which are not included in (1). If we may ignore the $\cdots$ (for example because the edge is gapped), then we find a surface Hall conductivity

$$\sigma_{xy} = \frac{e^2}{h}\frac{\Delta\theta}{2\pi} = \frac{e^2}{h}\left(\frac{1}{2} + n\right) \tag{3}$$

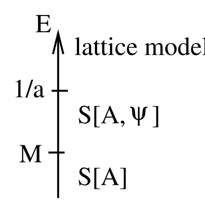
where $\Delta\theta$, the change in θ between the two sides of the interface, is a half-integer multiple of 2π. To be able to gap out the edge states, and thereby ignore the $\cdots$, it is sufficient to break $\mathcal{T}$ symmetry, for example by applying a magnetic field.

There are two different ways of breaking $\mathcal{T}$. The 1+1d domain wall between these on the surface supports a *chiral* edge mode.



The periodicity in $\theta \simeq \theta + 2\pi$ for the fermion TI can be understood from the ability to deposit an (intrinsically 2+1 dimensional) integer quantum Hall system on the surface. This changes the integer n in the surface Hall response (3). Following [8] we argued that a non-fractionalized system of bosons in 2+1d must have a Hall response which is an even integer; therefore a 3+1d boson TI has a θ parameter with period 4π.

Free fermion TIs exist and are a realization of this physics with $\theta = \pi$. The simplest short-distance completion of this model is a single massive Dirac fermion:



$$S[A,\psi] = \int \mathrm{d}^3x\mathrm{d}t\bar{\Psi}\left(\mathbf{i}\gamma^\mu D_\mu - m - \tilde{m}\gamma^5\right)\Psi.$$

It is convenient to denote $M \equiv m + \mathbf{i}\tilde{m}$.

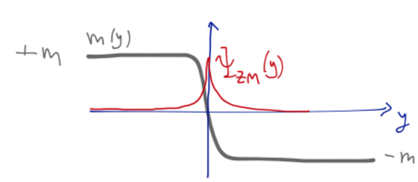
$$\mathcal{T} : M \to M^\star$$

so time reversal demands real M. Integrating out the massive Ψ produces an effective action for the background gauge field (and M) of the form above:

$$\log\int [D\Psi]e^{\mathbf{i}S_{3+1}[A,\psi]} = \frac{M}{|M|}\int \frac{\mathrm{d}^4x}{32\pi^2}\epsilon^{abcd}F_{ab}F_{cd} + \cdots$$

The sign of M determines the theta angle.

An interface between the TI and vacuum is a domain wall in M between a positive value and a negative value. Such a domain wall hosts a 2+1d massless Dirac fermion [10]. (The $\mathcal{T}$-breaking perturbation is just its mass, and the chiral edge mode in its mass domain wall has the same topology as the chiral fermion zeromode in the core of a vortex [11].)



A further short-distance completion of this massive Dirac fermion (as in the figure) comes from filling an integer number of bands with a nontrivial Chern-Simons invariant of the Berry curvature [12, 13, 9].

With interactions and disorder other edge states are possible within the same bulk phase, including gapped edge preserving $\mathcal{T}$ [14, 15, 16, 17, 18].

2 K-matrix construction of SPT states in D=2+1

[Lu-Vishwanath 1204]

Recall our TO example of abelian FQH

$$S[a_I] = \sum_{IJ} \frac{K_{IJ}}{4\pi} \int a_I \wedge \mathrm{d}a_J.$$

The real particle (electron or boson) current is $j_\mu = \epsilon_{\mu\nu\rho}\partial_\nu a^I_\rho t_I$. That is: coupling to ext gauge field: $j_{ext} = t^I \epsilon \partial a^I$.

$$\sigma^{xy} = \frac{1}{2\pi} t^{-1} K^{-1} t.$$

Number of gs = det K^g If we take det $K = 1$, no TO (probably).

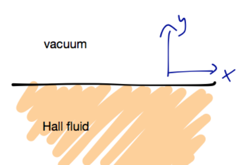
Actually one more set of data is required: quasiparticles are labelled by l^I couple to $l_I a^I$.
self (exchange) statistics: $\theta = \pi l^T K^{-1} l$.
mutual statistics: $\theta_{12} = 2\pi l_1^T K^{-1} l_2$.
external U(1) charge of qp l: $Q = t^T K^{-1} l$.

To make an SPT state, we must ensure that all these quantum numbers are multiples (not fractions!) of those of the microscopic constituents.

Boson IQH: $K = \sigma^x$. like two 'layers' or species of bosons. with $t = (1,1)$. (both species carry the charge) has $\nu = 2$. (we can take $l_1 = (1,0), l_2 = (0,1)$. These are self bosons and mutual bosons.)

Consider abelian CS theory on the LHP:

$$S = \frac{k}{4\pi} \int_{\mathbb{R}\times\mathrm{LHP}} a \wedge \mathrm{d}a$$



EoM for a_0: $0 = f$
$\implies a = \mathbf{i} g^{-1}\mathrm{d}g = \mathrm{d}\phi$, $\phi \simeq \phi + 2\pi$. [19, 20, 21, 22].

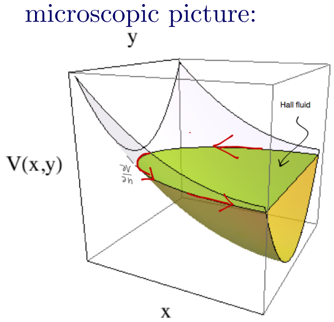
Only gauge transfs which approach $\mathbb{1}$ at the bdy preserve $S_{CS} \implies \phi$ is dynamical.

Boundary condition: $0 = a - v(\star_2 a)$ i.e. $a_t = v a_x$. v is UV data.

$$S_{CS}[a = \mathrm{d}\phi] = \frac{k}{2\pi} \int dt dx \left(\partial_t \phi \partial_x \phi + v \left(\partial_x \phi \right)^2 \right) .$$

Conclusion: ϕ is a chiral boson. $kv > 0$ required for stability. The sign of k determines the chirality.

microscopic picture:



For free fermions in a magnetic field, the velocity of the edge states is determined by the slope of the potential which is holding the electrons together. It is clearly not universal information.

Note: The Hamiltonian depends on the boundary conditions; the $\mathcal{H}$ does not.

Put back indices:

$$S = \frac{K^{IJ}}{4\pi} \int_{\mathbb{R} \times \mathrm{LHP}} a_I \wedge \mathrm{d} a_J$$

$$S_{CS}[a^I = \mathrm{d}\phi^I] = \frac{1}{4\pi} \int dt dx \left(K^{IJ} \partial_t \phi^I \partial_x \phi^J + v_{IJ} \partial_x \phi^I \partial_x \phi^J \right) .$$

(v is a positive matrix, non-universal.)

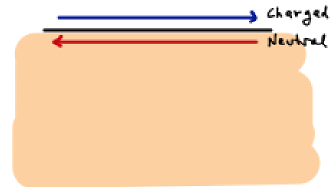
This is a collection of chiral bosons. The number of left-/right-movers is the number of positive/negative eigenvalues of K.

For $\nu = 2$ boson IQH [Senthil-Levin 1206]: $K = \boldsymbol{\sigma}^x$.

$$S_{CS}[a^I = \mathrm{d}\phi^I] = \frac{1}{4\pi} \int dt dx \left(\partial_t \phi^+ \partial_x \phi^+ - \partial_t \phi^- \partial_x \phi^- + v \left(\partial_x \phi^\pm \right)^2 \right)$$

$\implies \phi^\pm \equiv \frac{1}{\sqrt{2}} \left(\phi^1 \pm \phi^2 \right)$ are left/rightmoving.

(Beware my factors of two please.)



Conclusion: it's just a non-chiral free boson (at the SU(2) radius). Ordinary, at least for string theorists.

BUT: ONLY the rightmover is charged! The difference arises in the coupling to the external gauge field: since $t = (1, 1)$,

$$L \ni \mathcal{A}^\mu \partial_\mu \left(\phi^1 + \phi^2\right) \propto \mathcal{A}^\mu \partial_\mu \phi^+ \ .$$

Specifically, although $c_L = c_R$, only the left mover ϕ_+ carries the U(1) charge. This means that preserving U(1), we can't backscatter, that is we can't add to the action (local) terms like $\Delta S = g_\pm \cos\left(\phi^+ \pm \phi^- + \alpha\right)$ (α is a constant) which would lift the edge states. (Such terms made from just ϕ^+ could not be local.) This means the U(1) *protects* the edge states.

Comment: in its realization e.g. as the gapless low-energy theory of the spin-$\frac{1}{2}$ AF Heisenberg chain, only the $\mathrm{SU}(2)_{\mathrm{diag}}$ is manifest, not the chiral symmetries, which are emergent.

3 BF theory for 3+1d boson SPTs

Consider the following $D = 3 + 1$ analog of CS theory [Senthil-Vishwanath 1209]:

$$S[B, a] = \sum_I \frac{1}{2\pi} \epsilon^{\cdots\cdot} B^I_{\cdot\cdot} \partial_\cdot a^I_\cdot + \vartheta \sum_{IJ} \frac{K_{IJ}}{4\pi} \partial_\cdot a^I_\cdot \partial_\cdot a^J_\cdot \epsilon^{\cdots\cdot}.$$

Note that the theta angle ϑ here is not the same as the θ in the magnetoelectric resonse.

It is topological, like CS theory, in that we didn't need to introduce the metric to integrate the action covariantly. In $D = 3 + 1$ we need the form degrees to add up $2 + 1 + 1 = 4$. We can add analogs of Maxwell terms, but just like in $D = 2 + 1$ they are irrelevant, *i.e.* they merely introduce new UV physics, they don't change the IR.

(Note that the more general seeming thing with a more general matrix coupling F and B can be removed by an integer-valued field redefinition which changes nothing.)

Focus on the case $K = \boldsymbol{\sigma}^x$. One virtue of this effective action is that it reproduces the EM response we expect of a topological insulator. If we couple to an external U(1) gauge field $\mathcal{A}$ by

$$\Delta\mathcal{L} = \mathcal{A}_\mu \left(j_1^\mu + j_2^\mu\right)$$

then

$$\log \int [DaDB] e^{\mathbf{i}S[a,B,\mathcal{A}]} = \int \frac{2\vartheta}{16\pi^2} \mathrm{d}\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \cdots$$

that is, the magneto-electric response is $\theta_{EM} = 2\vartheta$. So $\vartheta = \pi$ will be a nontrivial boson TI.

Briefly, who are these variables? in 2+1: the flux of the CS gauge field was some charge density. Here, each copy is the 3+1d version of charge-vortex duality, where for each boson current

$$j_\mu^{I=1,2} = \frac{1}{2\pi} \epsilon_{\mu\cdots} \partial . B^I_{..}$$

which has $\partial^\cdot j^I_\cdot = 0$ as long as B is single-valued.

The point of a is to say that B is flat. The magnetic field lines of a_μ are the vortex lines of the original bosons b_I.

4 Ways of slicing the path integral

Now let's think about the path integral for a QFT with a theta term. Examples include the BF theory above, and many non-linear sigma models which arise by coherent-state quantization of spin systems. In general what I mean by a theta term is a term in the action which is a total derivative, and where the object multiplied by theta evaluates to an integer on closed manifolds. The following point of view has been vigorously emphasized by Cenke Xu [23, 2].

When spacetime is closed $Z(\theta + 2\pi) = Z(\theta)$. on a closed spacetime manifold M_D

$$Z_\theta(M_D) \equiv \int [D\text{stuff}] e^{-S} = \sum_{n \in \pi_2(S^2)} e^{\mathbf{i}\theta n} Z_n$$

and $Z_\theta(M_D) = Z_{\theta+2\pi}(M_D)$. In particular, we can take $M_D = S^1 \times N_{D-1}$ to compute the partition function on any spatial manifold N_{D-1}. This means the bulk spectrum is periodic in θ with period 2π.

With boundaries, it not so in general. A boundary in space produces edge states. We'll come back to these.

A boundary in time in the path integral means we are computing wavefunctions. For quantum mechanics of a single variable $q(t)$, this is manifested in the Feynman-Kac formula

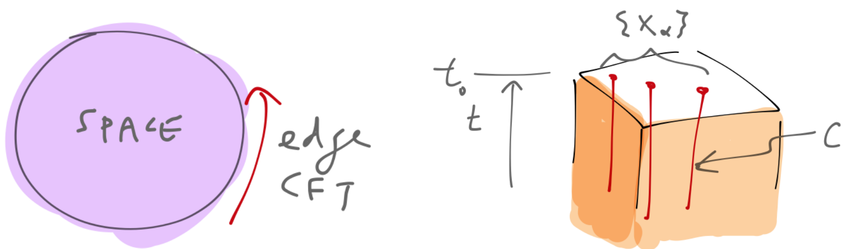
$$\psi(q) = \int_{q(t_0)=q} \prod_{t \in (-\infty, t_0)} \mathrm{d}q(t) e^{-S_{\text{euclidean}}[q]} .$$

For a field theory, 'position-space wavefunction' means a wavefunctional $\Psi[\phi(x)]$, in

$$|\Psi\rangle = \int [D\phi(x)] \Psi[\phi(x)] |\phi(x)\rangle$$

where x labels *spatial* positions, and $|\phi(x)\rangle$ are coherent states for the field operator $\hat{\phi}(x)$. Which wavefunction? If the path integral is over a large euclidean time T before reaching the boundary, this is a *groundstate* wavefunction, since the euclidean time propagator $e^{-T\mathbf{H}}$ is a (un-normalized) projector onto lowest-energy states.

Semi-philosophical digression: An important guiding concept in the study of interesting gapped states is that it is the *same stuff* living at a spatial boundary (edge modes) as at a temporal boundary (the wavefunction) [19, 20, 24, 25]. This perspective first arose (I think) in the context of quantum Hall states where famously [24] one can write groundstate and several-quasiparticle wavefunctions as correlation functions of certain operators in a 1+1d CFT, which is the same CFT that arises at a spatial edge. Why should this be true? It's because the bulk can be described by a path integral for a Chern-Simons gauge theory which has a certain WZW model living at its boundaries, wherever they are. For a spatial boundary, it produces a copy of that CFT at the boundary (roughly the group-valued CFT field g is related to the CS gauge field by $A = g^{-1}\mathrm{d}g$).



For a temporal boundary, the path integral expression for the wavefunctional (with some Wilson line insertions related to quasiparticles) takes the form

$$\Psi[g(x)] = \int_{A(t_0,x)=g^{-1}\mathrm{d}g} e^{\mathbf{i}S[A]} W[C] = \langle \prod_\alpha V_\alpha(x_\alpha) \rangle_{WZW}. \tag{4}$$

A too-brief explanation of this rich formula: the wilson line insertion is $W[C] = \mathrm{tr}_R \mathcal{P} e^{\mathbf{i}\oint_C A}$ where R is a representation of the gauge group G and $\mathcal{P}$ is path ordering. In (4), x_α are the locations where the curve C intersects the fixed-$t = t_0$ surface, and V_α are some operators in the CFT the appropriate representations R of G.

For spin chains, this point of view is used in [26] to construct spin chains whose continuum limit is the $\mathrm{SU}(2)_k$ WZW model with $k > 1$.

For 3+1d boson SPT states, the analogous bulk EFT is, instead of CS gauge theory, some weird BF theory or strongly-coupled sigma model, both of which we'll discuss below. At a spatial edge, we have some vortex excitations in $D = 2 + 1$. Correspondingly, the bulk wavefunctions will turn out to have a nice representation in a basis of states labelled by vortex loop configurations in $D = 3 + 1$.

Side remark: the canonical application of this story is to the *Haldane chain* – a chain where each site carries a representation of SO(3). At low energies, such chains are described by an

NLSM with a theta term. $\theta = 0$ is trivial and gapped. $\theta = 2\pi$ is gapped and trivial in the bulk but the edge states are spin $\frac{1}{2}$s – a projective representation of SO(3).

Apply to BF theory: (The basic manipulation we are doing here is from [Xu-Senthil 1301 [2]])

In contrast to the case of a closed manifold, if we compute the path integral on an (infinite) cylinder (*i.e.* with two boundaries, at $\tau = \pm\infty$), then θ does matter, not just mod 2π.

Choose $A_0 = 0$ gauge. Since A and B are conjugate variables, the analog of position space here is $|\vec{A}(x)\rangle$. For the same reason, we can only specify BCs on one or the other:

$$\int_{\substack{\vec{A}(x,\tau=\infty)=\vec{A}(x)\\ \vec{A}(x,\tau=-\infty)=\vec{A}'(x)}} [D\vec{A}(x,\tau)DB(x,\tau)]e^{-S[\vec{A}(x,\tau),B]} = \langle\vec{A}(x)|0\rangle\langle 0|\vec{A}'(x)\rangle \tag{5}$$

Notice that in expressions for functionals like $S[A(x,\tau)]$ I am writing the arguments of the function A to emphasize whether it is a function at fixed euclidean time or not. The fact that the theta term is a total derivative

$$F^I \wedge F^J = \mathrm{d}\left(A^I \wedge F^J\right) \equiv \mathrm{d}w(A)$$

means that the euclidean action here is

$$S[\vec{A}(x,\tau)] = \int_{M_D} \frac{1}{4\pi} B \wedge F + \mathbf{i}\theta \int_{\partial M_D} \mathrm{d}^{D-1}x \left(w(\vec{A}(x)) - w(\vec{A}'(x))\right).$$

The θ term only depends on the boundary values, and comes out of the integral in (5).

The integral over B^I is

$$\int [DB] e^{\mathbf{i}\frac{1}{4\pi}\int B^I \wedge F^I} = \delta[F^I].$$

The delta functional on the RHS here sets to zero the flux of the gauge field for points in the interior of the cylinder.

After doing the integral over B, there is nothing left in the integral and we can factorize the expression (5) to determine:

$$\Psi[\vec{A}_I(x)] = \underbrace{\exp \mathbf{i}\frac{\vartheta}{8\pi^2}\int_{\text{space}} A^I_{\cdot}\partial_{\cdot}A^J_{\cdot}\epsilon^{\cdots}K^{IJ}}_{\overset{K=\boldsymbol{\sigma}^x}{=} \mathbf{i}\frac{\vartheta}{2}(\text{linking \# of } 2\pi \text{ magnetic flux lines})} \tag{6}$$

What does this mean? Label configurations of A by the flux loops (i.e. the field lines of the vector field). This wavefunction is $(-1)^{\text{linking number of the 1-loops and the 2-loops}}$.

If we break the U(1) × U(1) symmetry, the flux lines of 1 and 2 will collimate (by the Meissner effect).

Claim: in the presence of an edge, these flux lines can end. The ends of these flux lines are fermions.

(warning: requires a framing of the flux lines – i.e. they shouldn't collide.)

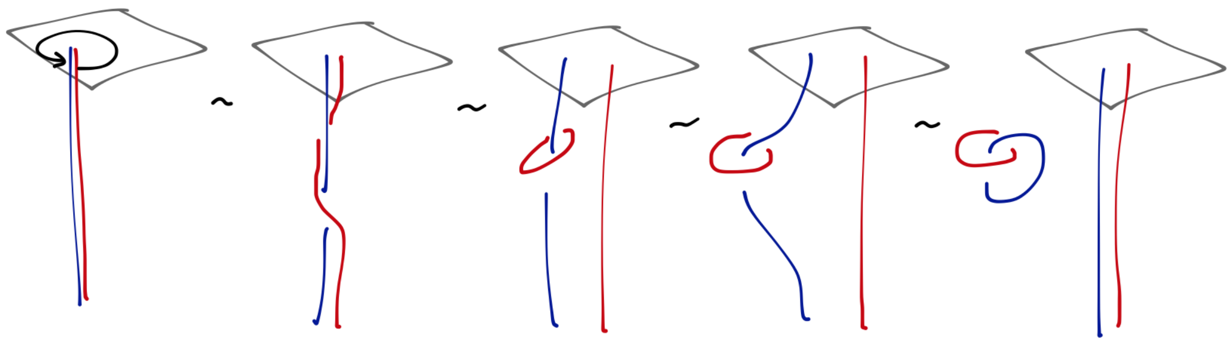


Figure 1: The end of the ribbon is a fermion, from [2]. In the first step, we rotate the red string around the blue one. The squiggles mean that the states associated with these configurations have the same amplitude in the groundstate, according to (6).

Conclusion: on the surface of this SPT state of bosons there are *fermionic* vortices.

Note that the BF theory describes a very strongly confined abelian gauge theory: the flux gets set to zero by the B term. (with a string source for B the flux gets localized to the source.)

Comment on kodama state of gravity. This wavefunction (6) actually solves the Schrödinger equation for quantum Maxwell theory at *finite* coupling. There is even a non-Abelian version of it for which this is also true. There is even an analog for gravity called the 'Kodama state'! What's the catch? It's not normalizable as a wavefunction for photon fields [27]; attempting to quantize the model about this groundstate gives negative energy for one of the two circular polarization states. But as a wavefunction for the confining phase of the gauge theory it's fine.

Note that [2] does an analogous thing for very strongly coupled sigma models with theta terms; they just set the kinetic term to zero (!) and find wavefunctions closely analogous to (6). They would have the same problem as Witten points out if they thought of their wavefunctions as wavefunctions for gapless magnons. But for the disordered phase of the sigma model (gapped and analogous to confinement) it is just fine.

Speaking of confining gauge theories in $D = 3 + 1$: for a long time it was believed (more defensible: I believed) that only deconfined phases of parton gauge theories were interesting for condensed matter – why would you want to introduce a gauge field if it is just going to be confined? Parton constructions for quantum Hall states are an easy and lucrative exception to this, in that the gauge field acquires a mass via a Chern-Simons term and so is massive but not confined. A more dramatic exception appears in [28, 29, 30, 31] where confined states of parton gauge theories are used to describe superfluid phases in $D = 2 + 1$ in terms of variables that allow access to nearby fractionalized phases (and the degrees of freedom that become light at the intervening phase transitions).

More recently, confined states of parton gauge theories have been used to discuss $D = 3+1$ SPT states in [32, 33, 34, 16].

5 Concluding remarks

1. An important omission is the 'coupled-layer' construction of SPT states, which allows a construction of the bulk state directly using many copies of the boundary excitations, as in the picture. This is quite holographic in spirit. The way the edge excitations

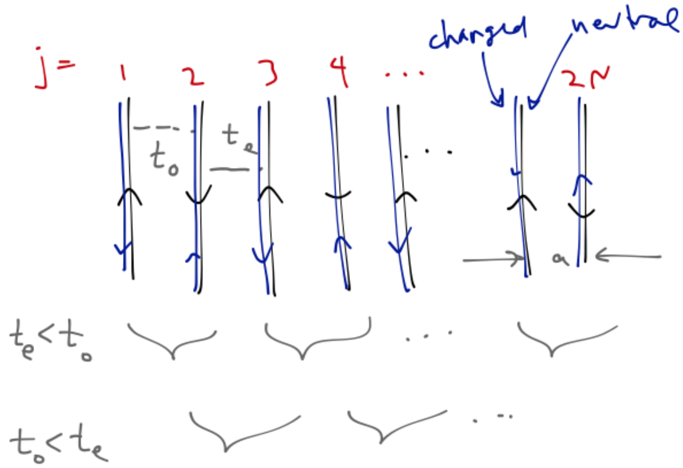


Figure 2: Coupled layer construction of the boson IQHE [1]

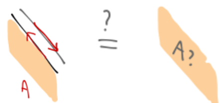
   emerge then is just like in the classic picture of edge charge from polarization of an insulator; the role of the polarization angle is played by arctan $\left(\frac{t_e}{t_0}\right)$.

2. Another important omission: spin systems and neel sigma models.

3. Classification

4. **Obstructions to symmetric regulators**

   Notice that the possibility of characterizing a state by its edge implies that the $D-1$-dimensional models involved must not be realizable intrinsically in $D-1$ dimensions, consistent with the symmetries involved. If they could be so realized, preserving G, we could paint them on the surface of any bulk state, and they therefore could not be characteristic. This suggests a useful strategy for identifying obstructions to symmetry-preserving regulators of quantum field theory (QFT), which has been exploited so far in examples in [1, 35, 33, 36]. An attempt to systematize these obstructions is here [37].



The edge states of SPTs are QFTs which are NOT like sod (at least in how they realize symmetries). They are more like the leaves of a tree – they need the tree to live! The fact that can reverse this logic and find obstructions generalizing the Nielsen-Ninomiya theorem [36] is a rare real-physics reason to study models in 4+1 dimensions!

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