From Critical Phenomena to Holographic Duality in Quantum Matter

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Lecture 3

“Holographic Approaches to Far From Equilibrium Dynamics”

Accompanying Slides
Outline

• Experimental motivation
• AdS/CMT and far from equilibrium dynamics
• Quenches and thermalization
• Holographic superfluids
• Heat flow between CFTs
• Potential for AdS/CFT to offer new insights
• Higher dimensions and non-equilibrium fluctuations
• Current status and future developments

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Experiment

Non-Equilibrium 1D Bose Gas

Integrability and Conservation Laws
Experiment

Experiment

Dicke quantum phase transition with a superfluid gas in an optical cavity

Kristian Baumann¹, Christine Guerlin¹†, Ferdinand Brennecke¹ & Tilman Esslinger¹


Self-Organisation

Domokos & Ritsch, PRL (2002)


⇓ = |Zero Momentum⟩  ‾↑ = |Finite Momentum⟩

“Spins” Coupled to Light
Observation of Dicke Transition

\[ H = \omega \psi^\dagger \psi + \omega_0 \sum_i S_i^z + g \sum_i (\psi^\dagger S_i^- + \psi S_i^+) \]
\[ + g' \sum_i (\psi^\dagger S_i^+ + \psi S_i^-) + U \sum_i S_i^z \psi^\dagger \psi \]


Driven Open System
Utility of Gauge-Gravity Duality

Quantum dynamics    Classical Einstein equations

Finite temperature  Black holes

Real time approach to finite temperature quantum dynamics in interacting systems, with the possibility of anchoring to 1+1 and generalizing to higher dimensions

Non-Equilibrium    Beyond linear response

Temporal dynamics in strongly correlated systems

Combine analytics with numerics

Dynamical phase diagrams

Organizing principles out of equilibrium
Progress on the CMT Side

Simple protocols and integrability

Methods of integrability and CFT have been invaluable in classifying equilibrium phases and phase transitions in 1+1.

Do these methods extend to non-equilibrium problems?

**Quantum quench** Parameter in $H$ abruptly changed

$$H(g) \rightarrow H(g')$$

System prepared in state $|\Psi_g\rangle$ but time evolves under $H(g')$

**Quantum quench to a CFT**

Calabrese & Cardy, PRL 96, 136801 (2006)

**Quantum quenches in quantum spin chains**

e.g. Calabrese, Essler & Fagotti, PRL 106, 227203 (2011)

**Quantum quench in BCS pairing Hamiltonian**

Andreev, Gurarie, Radzihovsky, Barankov, Levitov, Yuzbashyan, Caux...

**Thermalization** **Integrability** **Generalized Gibbs**
Quantum Quench to a CFT


\[ \langle \Phi(t) \rangle = A_b^{\Phi} \left( \frac{\pi}{4\tau_0} \frac{1}{\cosh[\pi t/2\tau_0]} \right)^x \sim e^{-\pi x t/2\tau_0} \]

Scaling dimension \( x \)  
Non-universal decay \( \tau_0 \sim m^{-1} \)

Ratios of observables exhibit universality

\[ \langle \Phi(r, t)\Phi(0, t) \rangle \sim e^{-\pi x r/2\tau_0} \quad t > r/2 \]

Emergent length scale or temperature scale

AdS/CFT gives access to higher dimensional interacting critical points and the possibility of universal results
Quenches in Transverse Field Ising

Recent work: Calabrese, Essler, Fagottini, PRL 106, 227203 (2011)

\[ H(h) = -\frac{1}{2} \sum_{l=-\infty}^{\infty} [\sigma^x_l \sigma^x_l + h \sigma^z_l] \]

QPT  Ferromagnetic \( h < 1 \)  Paramagnetic \( h > 1 \)

Quantum quench \( h_0 \to h \) within the ordered phase \( h_0, h \leq 1 \)

Exact results for long time asymptotics of order parameter

\[ \langle \sigma^x_l(t) \rangle \propto \exp \left[ t \int_0^\pi \frac{dk}{\pi} \epsilon'_h(k) \ln(\cos \Delta_k) \right] \]

\[ \epsilon_h(k) = \sqrt{h^2 - 2h \cos k + 1} \]

\[ \cos \Delta_k = \frac{hh_0 - (h + h_0) \cos k + 1}{\epsilon_h(k) \epsilon_{h_0}(k)} \]

Exact results for asymptotics of two-point functions

\[ \langle \sigma^x_l(t) \sigma^x_{1+l}(t) \rangle \propto \exp \left[ l \int_0^\pi \frac{dk}{\pi} \ln(\cos \Delta_k) \theta(2\epsilon'_h(k)t - l) \right] \]

\[ \times \exp \left[ 2t \int_0^\pi \frac{dk}{\pi} \epsilon'_h(k) \ln(\cos \Delta_k) \theta(l - 2\epsilon'_h(k)t) \right] \]
Quenches in Transverse Field Ising

Recent work: Calabrese, Essler, Fagottini, PRL 106, 227203 (2011)

Good agreement between numerics and analytics

Determinants  Form Factors

Integrable quenches in interacting field theories
BCS Quench Dynamics


Time dependent BCS Hamiltonian

\[
H = \sum_{p,\sigma} \epsilon_p a_{p,\sigma}^\dagger a_{p,\sigma} - \frac{\lambda(t)}{2} \sum_{p,q} a_{p,\uparrow}^\dagger a_{-p,\downarrow}^\dagger a_{-q,\downarrow} a_{q,\uparrow}
\]

Pairing interactions turned on abruptly

\[
\lambda(t) = \lambda \theta(t)
\]

Generalized time-dependent many-body BCS state

\[
|\Psi(t)\rangle = \prod_p \left[ u_p(t) + v_p(t) a_{p,\uparrow}^\dagger a_{-p,\downarrow}^\dagger \right] |0\rangle
\]

Integrable
Collective Oscillations

Barankov, Levitov and Spivak, PRL 93, 160401 (2004)

Time-dependent pairing amplitude

\[ \Delta(t) \equiv \lambda \sum_p u_p v_p^*(t) = e^{-i\omega t} \Omega(t) \]

Equation of Motion

\[ \dot{\Omega}^2 + (\Omega^2 - \Delta_-^2)(\Omega^2 - \Delta_+^2) = 0 \]

Oscillations between \( \Delta_- \) and \( \Delta_+ \)
Regimes of BCS Quench Dynamics


Initial pairing gap $\Delta_s$  Final pairing gap $\Delta_0$

(A) Oscillations between $\Delta_{\pm}$  (B) Underdamped approach to $\Delta_a$

(C) Overdamped approach to $\Delta = 0$

Emergent temperature $T^*$ and gap $\Delta(T^*)$
**Bloch Dynamics**

**Anderson Pseudo-Spins**

\[ H = - \sum_p 2\epsilon_p s^z_p - \lambda(t) \sum_{pq} s^-_p s^+_q \]

\[
\begin{align*}
\Delta(t) &= \Delta_a + A(t) \sin(2\Delta_a t + \alpha) \\
A(t) &\propto t^{-1/2} \\
\Delta(t) &\propto (\Delta_s t)^{-1/2} e^{-2\Delta_s t}
\end{align*}
\]

**Non-Equilibrium Dynamical Phase Diagram**

*Yuzbashyan et al*  *Andreev et al*
AdS Quenches


\[ ds^2 = -dt^2 + e^{B_0(t)}d\mathbf{x}^2_\perp + e^{-2B_0(t)}d\mathbf{x}^2_\parallel \]

The dependent shear of the geometry \( B_0(t) = \frac{1}{2} c [1 - \tanh(t/\tau)] \)

Jan de Boer & Esko Keski-Vakkuri et al, “Thermalization of Strongly Coupled Field Theories”, PRL (2011)

\[ ds^2 = \frac{1}{z^2} \left[-(1 - m(v))z^d dv^2 - 2dzdv + d\mathbf{x}^2\right] \]

Vaidya metric quenches \( m(v) = \frac{1}{2} M[1 + \tanh(v/v_0)] \)


Holographic Superconductor


Abelian Higgs Coupled to Einstein–Hilbert Gravity

\[ S = \int d^Dx \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4} F_{ab} F^{ab} - |D_a \psi|^2 - m^2 |\psi|^2 \right] \]

\[ F_{ab} = \partial_a A_b - \partial_b A_a \quad D_a = \partial_a - iqA_a \quad \Lambda = -\frac{D(D-1)}{4L^2} \]

Finite temperature  Black hole

Below critical temperature black hole with \( \psi = 0 \) unstable

Black hole with charged scalar hair \( \psi \neq 0 \) becomes stable

Spontaneous U(1) symmetry breaking
Non-Equilibrium AdS Superconductors


Start in unstable normal state described by a Reissner–Nordström black hole and perturb

Temporal dynamics from supercooled to superconducting
\[ \psi(t, z) = \psi_1(t)z + \psi_2(t)z^2 + \psi_3(t)z^3 + \cdots \equiv \psi_1(t)z + \tilde{\psi}(t, z)z^2 \]

Gaussian perturbation of Reissner–Nordström–AdS

\[ \tilde{\psi}(t = 0, z) = \frac{A}{\sqrt{2\pi}\delta} \exp \left[ -\frac{(z-z_m)^2}{2\delta^2} \right] \]

\[ A = 0.01, \quad \delta = 0.05, \quad z_m = 0.3 \]
Time Evolution of the Order Parameter

Start in normal state (Reissner–Nordström) & perturb

Numerical solution of Einstein equations

\[ \langle O_2(t) \rangle \equiv \sqrt{2} \psi_2(t) \]

\[ T/T_c = 1.1, 1.2, 1.4 \]

\[ T > T_c \quad \text{Relaxation to N} \]

\[ T < T_c \quad \text{Evolution to SC} \]

\[ |\langle O_2(t) \rangle| = C \exp(-t/t_{\text{relax}}) \]

\[ |\langle O_2(t) \rangle| = C_1 \exp(-t/t_{\text{relax}}) + C_2 \]
Questions

Where are the oscillations?

• Is there always damping or are there different regimes?
• Are the dynamics of holographic superconductors related to condensed matter systems or intrinsically different?
• What is the role of coupling to a large number of critical degrees of freedom?
• What is the role of large $N$ and strong coupling?
• What is their influence on the emergent timescales?
• Do thermal fluctuations reduce the amplitude of oscillations?
• Beyond BCS and mean field dynamics using AdS/CFT?
• What is the dynamics at short, intermediate and long times?
• What happens if one quenches from a charged black hole?

AdS/CMT far from equilibrium?
Homogeneous isotropic dynamics of CFT from Einstein Eqs
Dynamical Phase Diagram

Conjugate field pulse

$\psi(t, z) = \psi_1(t)z + \psi_2(t)z^2 + \ldots$

$\psi_1(t) = \bar{\delta} e^{-(t/\bar{\tau})^2}$

$\delta = \bar{\delta}/\mu_i$  \quad $\tau = \bar{\tau}/\mu_i$  \quad $\tau = 0.5$  \quad $T_i = 0.5T_c$
Three Dynamical Regimes


(I) Damped Oscillatory to SC
\[ \langle \mathcal{O}(t) \rangle \sim a + be^{-kt} \cos[l(t - t_0)] \]

(II) Over Damped to SC
\[ \langle \mathcal{O}(t) \rangle \sim a + be^{-kt} \]

(III) Over Damped to N
\[ \langle \mathcal{O}(t) \rangle \sim be^{-kt} \]

Asymptotics described by black hole quasi normal modes

Far from equilibrium → close to equilibrium
Approach to Thermal Equilibrium

Collapse on to Equilibrium Phase Diagram

Emergent temperature scale $T_*$ within superfluid phase
Quasi Normal Modes

\[ |\langle O(t) \rangle| \approx |\langle O \rangle_f + A e^{-i\omega t}| \]

Three regimes and an emergent \( T_* \)
Recent Experiments


Time-dependent experiments can probe excitations

What is the pole structure in other correlated systems?
Thermalization

Why not connect two strongly correlated systems together and see what happens?
Non-Equilibrium CFT


Two critical 1D systems (central charge $c$) at temperatures $T_L$ & $T_R$

Join the two systems together

Alternatively, take one critical system and impose a step profile

Local Quench
Steady State Heat Flow


If systems are very large \((L \gg vt)\) they act like heat baths

For times \(t \ll L/v\) a steady heat current flows

\[
J = \frac{c \pi^2 k_B^2}{6h} (T_L^2 - T_R^2)
\]

Non-equilibrium steady state

Universal result out of equilibrium

Direct way to measure central charge; velocity doesn’t enter

Linear Response


\[ J = \frac{c\pi^2 k_B^2}{6h} (T_L^2 - T_R^2) \]

\[ T_L = T + \Delta T/2 \quad T_R = T - \Delta T/2 \quad \Delta T \equiv T_L - T_R \]

Quantum of Thermal Conductance

\[ g_0 = \frac{\pi^2 k_B^2 T}{3h} \approx (9.456 \times 10^{-13} \text{WK}^{-2}) T \]

Free Fermions


\[ \frac{\kappa}{\sigma T} = \frac{\pi^2}{3e^2} \quad \sigma_0 = \frac{e^2}{h} \quad \kappa_0 = \frac{\pi^2 k_B^2 T}{3h} \]

Conformal Anomaly

Experiment

Heuristic Interpretation of CFT Result

\[ J = \sum_m \int \frac{dk}{2\pi} \hbar \omega_m(k) v_m(k) [n_m(T_L) - n_m(T_R)] \mathcal{T}_m(k) \]

\[ v_m(k) = \partial \omega_m / \partial k \quad n_m(T) = \frac{1}{e^{\beta \hbar \omega_m} - 1} \]

\[ J = f(T_L) - f(T_R) \]

Consider just a single mode with \( \omega = vk \) and \( \mathcal{T} = 1 \)

\[ f(T) = \int_0^\infty \frac{dk}{2\pi} \frac{\hbar v k^2}{e^{\beta \hbar v k} - 1} = \frac{k_B^2 T^2}{\hbar} \int_0^\infty dx \frac{x}{e^x - 1} = \frac{k_B^2 T^2}{\hbar} \frac{\pi^2}{6} \quad x \equiv \frac{\hbar v k}{k_B T} \]

Velocity cancels out

\[ J = \frac{\pi^2 k_B^2}{6\hbar} (T_L^2 - T_R^2) \]

For a 1+1 critical theory with central charge \( c \)

\[ J = \frac{c\pi^2 k_B^2}{6\hbar} (T_L^2 - T_R^2) \]
Stefan–Boltzmann

Cardy, *The Ubiquitous ‘c’: from the Stefan-Boltzmann Law to Quantum Information*, arXiv:1008.2331

**Black Body Radiation in 3 + 1 dimensions**

\[ dU = TdS - PdV \]

\[ \left( \frac{\partial U}{\partial V} \right)_T = T \left( \frac{\partial S}{\partial V} \right)_T - P = T \left( \frac{\partial P}{\partial T} \right)_V - P \]

\[ u = T \left( \frac{\partial P}{\partial T} \right)_V - P \]

For black body radiation \( P = u/3 \)

\[ \frac{4u}{3} = \frac{T}{3} \left( \frac{\partial u}{\partial T} \right)_V \quad \frac{du}{4u} = \frac{dT}{T} \quad \frac{1}{4} \ln u = \ln T + \text{const.} \]

\[ u \propto T^4 \]
Stefan–Boltzmann and CFT

Cardy, The Ubiquitous ‘c’: from the Stefan-Boltzmann Law to Quantum Information, arXiv:1008.2331

Energy-Momentum Tensor in \( d + 1 \) Dimensions

\[
T_{\mu \nu} = \begin{pmatrix}
  u \\
  P \\
  P \\
  \vdots
\end{pmatrix}
\]

Traceless \( P = u/d \)

Thermodynamics

\[
u = T \left( \frac{\partial P}{\partial T} \right)_V - P \quad u \propto T^{d+1}
\]

For 1 + 1 Dimensional CFT

\[
u = \frac{\pi ck_B^2 T^2}{6\hbar v} \equiv \mathcal{A}T^2
\]

\[
J = \frac{\mathcal{A}v}{2} (T_L^2 - T_R^2)
\]
Stefan–Boltzmann and AdS/CFT


Entropy of $SU(N)$ SYM = Bekenstein–Hawking $S_{BH}$ of geometry

\[ S_{BH} = \frac{\pi^2}{2} N^2 V_3 T^3 \]

Entropy at Weak Coupling = $8N^2$ free massless bosons & fermions

\[ S_0 = \frac{2\pi^2}{3} N^2 V_3 T^3 \]

Relationship between strong and weak coupling

\[ S_{BH} = \frac{3}{4} S_0 \]

Energy Current Fluctuations


Generating function for all moments

\[
F(\lambda) \equiv \lim_{t \to \infty} t^{-1} \ln \langle e^{i\lambda \Delta t Q} \rangle
\]

**Exact Result**

\[
F(\lambda) = \frac{c\pi^2}{6h} \left( \frac{i\lambda}{\beta_l (\beta_l - i\lambda)} - \frac{i\lambda}{\beta_r (\beta_r + i\lambda)} \right)
\]

Denote \( z \equiv i\lambda \)

\[
F(z) = \frac{c\pi^2}{6h} \left[ z \left( \frac{1}{\beta_l^2} - \frac{1}{\beta_r^2} \right) + z^2 \left( \frac{1}{\beta_l^3} + \frac{1}{\beta_r^3} \right) + \ldots \right]
\]

\[
\langle J \rangle = \frac{c\pi^2}{6h} k_B^2 (T_L^2 - T_R^2)
\]

\[
\langle \delta J^2 \rangle \propto \frac{c\pi^2}{6h} k_B^3 (T_L^3 + T_R^3)
\]

**Poisson Process**

\[
\int_0^\infty e^{-\beta \epsilon} (e^{i\lambda \epsilon} - 1) d\epsilon = \frac{i\lambda}{\beta (\beta - i\lambda)}
\]
Full Counting Statistics

A large body of results in the mesoscopic literature

Free Fermions


Luttinger Liquids and Quantum Hall Edge States


Quantum Impurity Problems

Experimental Setup

Saminadayar et al, PRL 79, 2526 (1997)
Shot Noise in the Quantum Hall Effect


Non-Equilibrium Fluctuation Relation


\[ F(\lambda) \equiv \lim_{t \to \infty} t^{-1} \ln \langle e^{i\lambda \Delta_t Q} \rangle = \frac{c\pi^2}{6h} \left( \frac{i\lambda}{\beta_l(\beta_l - i\lambda)} - \frac{i\lambda}{\beta_r(\beta_r + i\lambda)} \right) \]

\[ F(i(\beta_r - \beta_l) - \lambda) = F(\lambda) \]

Irreversible work fluctuations in isolated driven systems

**Crooks relation**

\[ \frac{P(W)}{P(-W)} = e^{\beta(W - \Delta F)} \]

**Jarzynski relation**

\[ \langle e^{-\beta W} \rangle = e^{-\beta \Delta F} \]

Entropy production in non-equilibrium steady states

\[ \frac{P(S)}{P(-S)} = e^S \]

Esposito et al, “Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems”, RMP 81, 1665 (2009)
Lattice Models

Quantum Ising Model

\[ H = J \sum_{\langle i,j \rangle} S_i^z S_j^z + \Gamma \sum_i S_i^x \]

\[ \Gamma = J/2 \quad \text{Critical} \quad c = 1/2 \]

Anisotropic Heisenberg Model (XXZ)

\[ H = J \sum_{\langle i,j \rangle} \left( S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z \right) \]

\[ -1 < \Delta < 1 \quad \text{Critical} \quad c = 1 \]
Time-Dependent DMRG


Dimerization \( J_n = \begin{cases} 1 & n \text{ odd} \\ \lambda & n \text{ even} \end{cases} \)

\( \Delta_n = \Delta \)  \quad \text{Staggered}  \quad b_n = \frac{(-1)^n b}{2}
Time-Dependent DMRG


\[
\lim_{t \to \infty} \langle J_E(n, t) \rangle = f(T_L) - f(T_R)
\]

\[
f(T) \sim \begin{cases} 
T^2 & T \ll 1 \\
T^{-1} & T \gg 1 
\end{cases}
\]

Beyond CFT to massive integrable models (Doyon)
Energy Current Correlation Function


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Beyond Integrability

Importance of CFT for pushing numerics and analytics
AdS/CFT

Heat flow may be studied within pure Einstein gravity

\[ g_{\mu\nu} \leftrightarrow T_{\mu\nu} \]
Possible Setups

Local Quench     Driven Steady State     Spontaneous
General Considerations

$$\partial_\mu T^{\mu\nu} = 0 \quad \partial_0 T^{00} = -\partial_x T^{x0} \quad \partial_0 T^{0x} = -\partial_x T^{xx}$$

Stationary heat flow $\implies$ Constant pressure

$$\partial_0 T^{0x} = 0 \implies \partial_x T^{xx} = 0$$

In a CFT

$$P = u/d \implies \partial_x u = 0$$

No energy/temperature gradient

Stationary homogeneous solutions without local equilibrium
AdS/CFT

Pure Einstein Gravity \[ S = \int d^3x \sqrt{-g} (R - 2\Lambda) \]

Unique homogeneous solution in AdS$_3$ is boosted BTZ

\[
ds^2 = \frac{r^2}{L^2} \left[ -\left(1 - \frac{r_0^2}{r^2} \cosh^2 \eta \right) dt^2 + \frac{r_0^2}{r^2} \sinh(2\eta) dt dx \\
+ \frac{L^4}{(1 - \frac{r_0^2}{r^2})} dr^2 + \left(1 - \frac{r_0^2}{r^2} \sinh^2 \eta \right) dx^2 \right]
\]

(neglect rotating black hole solutions)

\[ r_0 = 2\pi T \text{ unboosted temp} \quad L \text{ AdS radius} \quad \eta \text{ boost param} \]

\[
\langle T_{ij} \rangle = \frac{L^3}{8\pi G_N} g_{ij}^{(0)} \\
\langle T_{tx} \rangle = \frac{L^3}{8\pi G_N} \frac{r_0^2}{2L^2} \sinh(2\eta) = \frac{\pi L}{4G_N} T^2 \sinh(2\eta)
\]

\[ c = 3L/2G_N \]

\[
\langle T_{tx} \rangle = \frac{\pi c}{6} T^2 \sinh(2\eta) = \frac{\pi c}{12} (T^2 e^{2\eta} - T^2 e^{-2\eta})
\]

\[ T_L = T e^\eta \quad T_R = T e^{-\eta} \quad \langle T_{tx} \rangle = \frac{c\pi^2 k_B^2}{6h} (T_L^2 - T_R^2) \]
Work In Progress & Future Directions

Firmly establish 1D result for average energy flow

Uniqueness in AdS$_3$ and identification of $T_L$ and $T_R$

Energy current fluctuations

Exact results in $1 + 1$ suggests simplifications in AdS$_3$

Higher dimensions

Conjectures for average heat flow and fluctuations

Absence of left-right factorization at level of CFT

Free theories  Poisson process

Generalizations

Other types of charge noise  Non-Lorentz invariant situations

Different central charges  Fluctuation theorems  Numerical GR