# From Critical Phenomena to Holographic Duality in Quantum Matter


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# Lecture 2

## “Relativisitic Quantum Transport in 2+1 Dimensions”

**Accompanying Slides**

# Useful Theory References

- M. Fisher, P. Weichman, G. Grinstein and D. Fisher
  "Boson localization and the superfluid-insulator transition",
  Phys. Rev. B, **40**, 546, (1989).

- K. Damle and S. Sachdev "Nonzero temperature transport
  near quantum critical points", Phys. Rev. B, **56**, 8714 (1997).

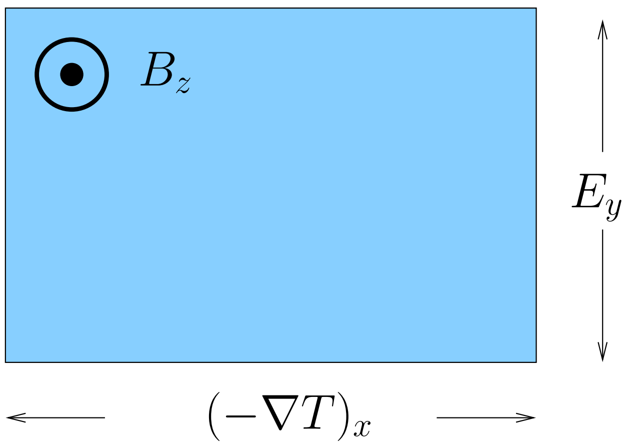
- S. Sachdev "Quantum Phase Transitions", Cambridge.
  Chapter 8 — Physics close to and above the UCD.
  Chapter 9 — Transport in $d = 2$.
  Chapter 10 — Boson Hubbard model.

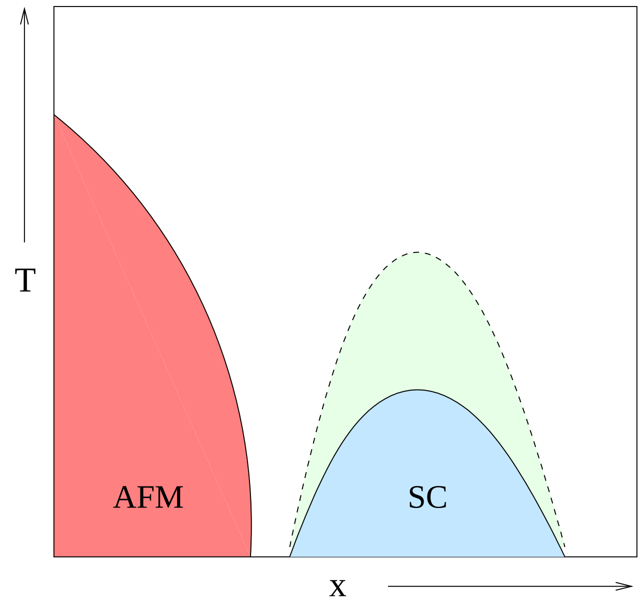
# Nernst Effect in the Cuprates

Xu, Ong, Wang, Kakeshita and Uchida, Nature **406**, 486 (2000)

$$\boxed{\nu \equiv \frac{E_y}{(-\nabla T)_x B}}$$

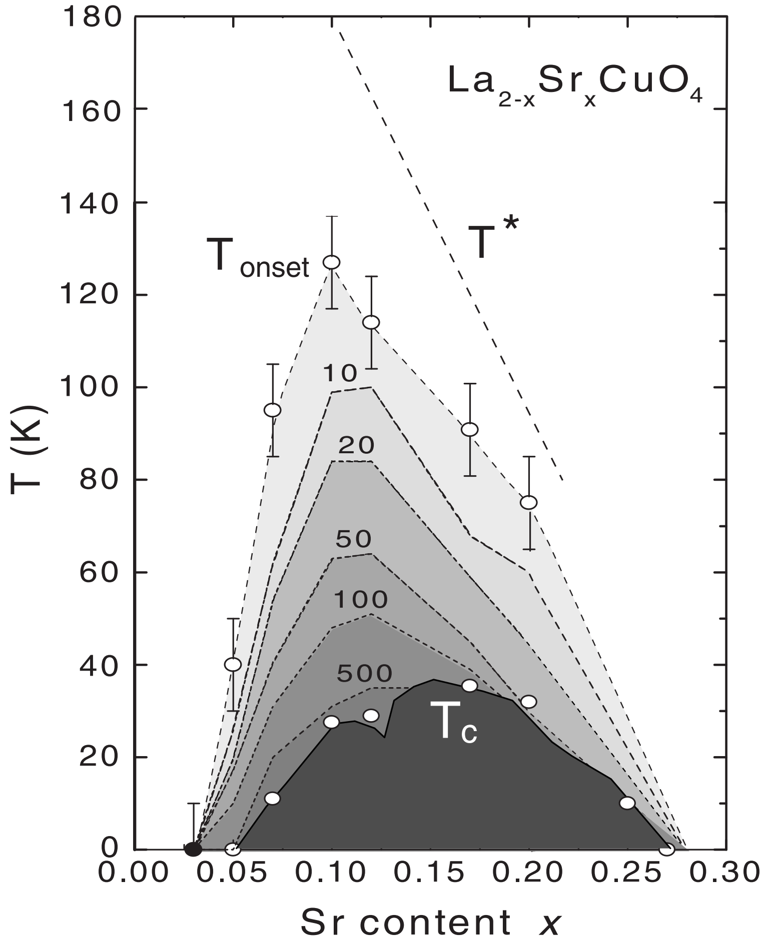
$$\boxed{\nu = \frac{1}{B}\frac{\alpha_{xy}\sigma_{xx} - \alpha_{xx}\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}}$$



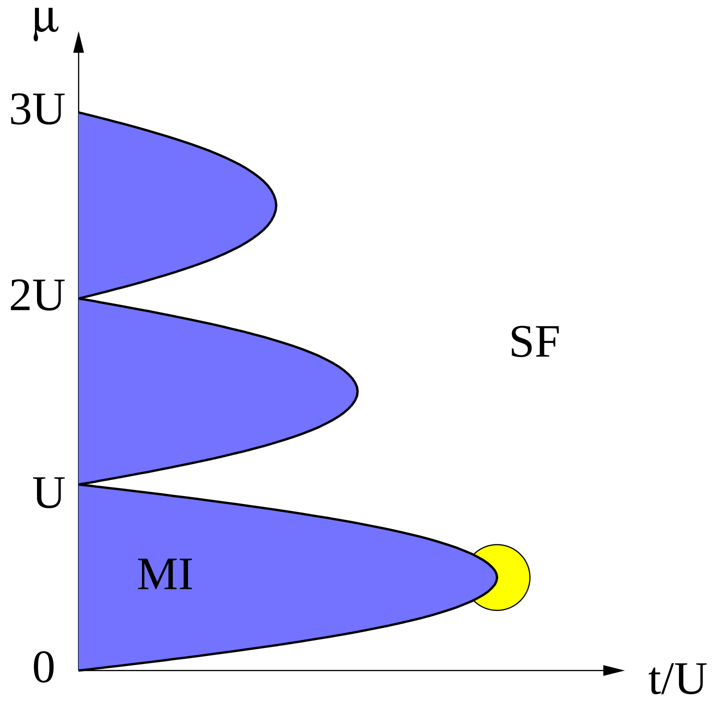
# Ong's Data

PRB **73**, 024510 (2010) Numbers indicate $\nu$ in $nV/KT$



# The Bose–Hubbard Model

$$H = -t\sum_{\langle ij\rangle}(a_i^\dagger a_j + \text{h.c.}) + \frac{U}{2}\sum_i n_i(n_i-1) - \mu\sum_i n_i$$




Fisher, Weichman, Grinstein and Fisher, PRB **40**, 546 (1989)

$$L = \int d^D x\, |\partial_\mu \Phi|^2 - m^2|\Phi|^2 - \frac{u_0}{3}|\Phi|^4$$

# Effective Field Theory

Set up path integral representation for

$$Z = \mathrm{Tr}(e^{-\beta H})$$

Using a Hubbard Stratonovich transformation to decouple the hopping term

$$e^{-W \int_0^\beta d\tau \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_j^\dagger b_i)} \quad \rightarrow$$

$$\int \mathcal{D}\Phi^* \mathcal{D}\Phi \exp(-\int_0^\infty d\tau (-\Phi_i b_i^\dagger - \Phi_i^* b_i) + \Phi_i^* W_{ij}^{-1} \Phi_j)$$

and integrating over the $b_i$ one may ultimately obtain

$$Z = \int \mathcal{D}\Phi \mathcal{D}\Phi^* e^{-\int_0^\beta d\tau \int d^d x \mathcal{L}_B}$$

where

$$\mathcal{L}_B = K_1 \Phi^* \frac{\partial \Phi}{\partial \tau} + K_2 \left| \frac{\partial \Phi}{\partial \tau} \right|^2 + K_3 \left| \nabla \Phi \right|^2 + K_4 \left| \Phi \right|^2 + K_5 |\Phi|^4$$

# Currents and Normal Modes

Rather than imaginary time diagrams, include temperature via a transport equation for "normal modes" of complex bosonic field:

$$\begin{aligned}\Phi(\mathbf{x},t) &= \int \frac{d^dk}{(2\pi)^d}\frac{1}{\sqrt{2\varepsilon_{\mathbf{k}}}}\left[a_+(\mathbf{k},t)e^{i\mathbf{k}.\mathbf{x}} + a_-^\dagger(\mathbf{k},t)e^{-i\mathbf{k}.\mathbf{x}}\right]\\ \Pi(\mathbf{x},t) &= -i\int \frac{d^dk}{(2\pi)^d}\sqrt{\frac{\varepsilon_{\mathbf{k}}}{2}}\left[a_-(\mathbf{k},t)e^{i\mathbf{k}.\mathbf{x}} - a_+^\dagger(\mathbf{k},t)e^{-i\mathbf{k}.\mathbf{x}}\right]\end{aligned}$$

where $\varepsilon_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$. The (uniform) electric current reads

$$\boxed{\langle \mathbf{J}(t)\rangle = Q\int \tfrac{d^dk}{(2\pi)^d}\,\mathbf{v}_{\mathbf{k}}\left[f_+(\mathbf{k},t) - f_-(\mathbf{k},t)\right]}$$

where $Q = 2e$, $\mathbf{v}_{\mathbf{k}} = \mathbf{k}/\epsilon_{\mathbf{k}}$ and

$$f_\pm(\mathbf{k},t) = \langle a_\pm^\dagger(\mathbf{k},t)a_\pm(\mathbf{k},t)\rangle$$

Dropped "anomalous" terms $\langle aa\rangle$ and $\langle a^\dagger a^\dagger\rangle$ for $\omega < 2m$.

# Quantum Boltzmann Equation

**Real time & finite T** Damle & Sachdev, PRB **56**, 8714 ('97)

Opposite charge particles + applied field + interactions $|\Phi|^4$

$$\left(\frac{\partial}{\partial t} \pm Q\mathbf{E}(t).\frac{\partial}{\partial \mathbf{k}}\right) f_{\pm}(\mathbf{k},t) = -\frac{2u_*^2}{9}\int d\mu\, \mathcal{F}_{\pm}$$

$$\begin{aligned} d\mu \quad &\equiv \quad \frac{d^d k_1}{(2\pi)^d}\frac{d^d k_2}{(2\pi)^d}\frac{d^d k_3}{(2\pi)^d}\frac{1}{16\,\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{k}_1}\varepsilon_{\mathbf{k}_2}\varepsilon_{\mathbf{k}_3}} \times \\ &(2\pi)^d\delta^d(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)(2\pi)\delta(\varepsilon_{\mathbf{k}}+\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}-\varepsilon_{\mathbf{k}_3}). \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{\pm}(\mathbf{k},\mathbf{k_1},\mathbf{k}_2,\mathbf{k}_3) \quad \equiv \quad & 2f_{\pm}(\mathbf{k})f_{\mp}(\mathbf{k}_1)[1+f_{\pm}(\mathbf{k}_2)][1+f_{\mp}(\mathbf{k}_3)] \\ & +f_{\pm}(\mathbf{k})f_{\pm}(\mathbf{k}_1)[1+f_{\pm}(\mathbf{k}_2)][1+f_{\pm}(\mathbf{k}_3)] \\ & -2[1+f_{\pm}(\mathbf{k})][1+f_{\mp}(\mathbf{k}_1)]f_{\pm}(\mathbf{k}_2)f_{\mp}(\mathbf{k}_3) \\ & -[1+f_{\pm}(\mathbf{k})][1+f_{\pm}(\mathbf{k}_1)]f_{\pm}(\mathbf{k}_2)f_{\pm}(\mathbf{k}_3) \end{aligned}$$

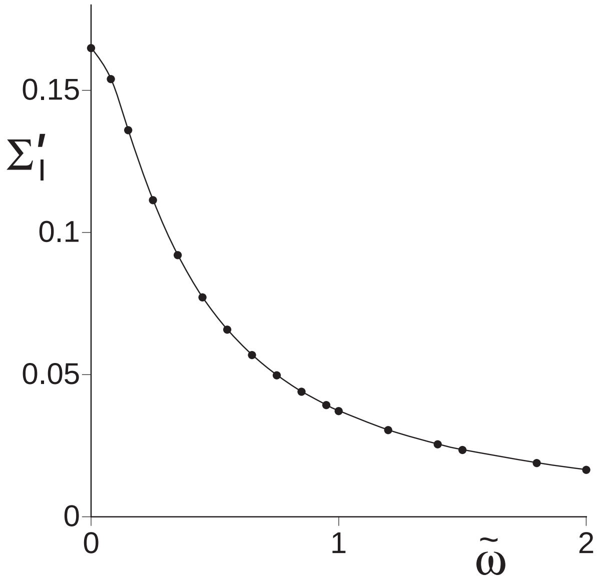
We have suppressed the $t$ dependence

$\varepsilon_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$ $\qquad \epsilon = 3-d$ $\qquad u_* = \frac{24\pi^2\epsilon}{5}$ $\qquad m^2 = \frac{4\epsilon T^2}{15}$

# Universal Transport

Damle and Sachdev, PRB **56**, 8714 (1997)

$$\boxed{\sigma(\omega) = \frac{(2e)^2 T^{d-2}}{\epsilon^2} \Sigma\left(\frac{\omega}{\epsilon^2 T}\right)} \qquad d = 3 - \epsilon$$




Two spatial dimensions $\boxed{\sigma(0) \approx 1.037\left(\frac{4e^2}{h}\right)}$

**Crossover between hydrodynamic and collisionless regimes**

# Transport near Quantum Critical Points

**Linear response for SF-MI transition in Bose–Hubbard**

Damle and Sachdev, PRB **56**, 8714 (1997)

$$\boxed{\sigma(\omega) = \frac{(2e)^2 T^{d-2}}{\epsilon^2} \Sigma\left(\frac{\omega}{\epsilon^2 T}\right)} \qquad d = 3 - \epsilon \qquad \boxed{\sigma(0) \approx 1.037 \left(\frac{4e^2}{h}\right)}$$

Bhaseen, Green and Sondhi, PRL **98**, 166801 (2007)

$$\boxed{\alpha_{xy} = \frac{\mathcal{S}}{B}} \qquad \boxed{\bar{\kappa}_{xx} \simeq g\epsilon^2 \frac{T^{d+3}}{(2e)^2 B^2}} \qquad g \approx 5.5$$

Hartnoll, Kovtun, Müller and Sachdev, PRB **76**, 144502 (2007)

$$\boxed{\bar{\kappa}_{xx}(B) = \frac{TS^2}{B^2 \sigma_{xx}(0)}}$$

**Relativistic hydrodynamics & AdS/CFT link all coeffs**

QBE Müller *et al*, PRB (2008) Bhaseen *et al*, PRB (2009)

**Viscosity/Entropy** **Quark Gluon Plasma** **Graphene**

# Linearized Equations

In the absence of **E**

$$f_{\pm}(k,t) = n(\varepsilon_k) = 1/(e^{\beta\varepsilon_k} - 1)$$

In the presence of **E** one may parameterise

$$f_{\pm}(\mathbf{k},\omega) = 2\pi\delta(\omega)n(\varepsilon_k) \pm \mathbf{k}.\mathbf{E}(\omega)\psi(k,\omega)$$

To linear order in **E**, and after three angular integrals and two radial (in $d = 3$ to leading order) one **eventually** obtains...

$$-i\omega\psi(k,\omega) + \frac{1}{k}\frac{\partial n(k)}{\partial k} = -\epsilon^2 \int_0^\infty dk_1 \left[\mathrm{F}_1(k,k_1)\psi(k,\omega) + \mathrm{F}_2(k,k_1)\psi(k_1,\omega)\right]$$

**Integral equation for the departure from equilibrium**

# Kernels

After a considerable slog, the kernels $\mathrm{F}_i(k, k_1)$ can be explicitly evaluated. For example,

$$\mathrm{F}_1 = \frac{6\pi}{25} \frac{n(k_1)n(k-k_1)}{k^2\, n(k)} \left[\Theta(k-k_1)\mu_2(k,k_1) - \Theta(k_1-k)\mu_2(k_1,k)\right]$$

where

$$\mu_n(x,y) \equiv \beta^{-n} \left[\mathrm{Li}_n(1) + \mathrm{Li}_n(e^{-\beta x}) - \mathrm{Li}_n(e^{-\beta y}) - \mathrm{Li}_n(e^{-\beta(x-y)})\right].$$

$\mathrm{Li}_p(z)$ is the polylogarithm of order $p$, $n(k)$ is the Bose distribution:

$$\mathrm{Li}_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}, \quad n(k) = \frac{1}{e^{\beta k} - 1}.$$

Note that the mass has been set equal to zero in these expressions.

$$\mathrm{F}_2 \equiv \mathrm{F}_2^a + \mathrm{F}_2^b$$

$$\begin{aligned}
\mathrm{F}_2^a &= \frac{2\pi}{75}\,\frac{[1+n(k)]n(k+k_1)}{k^4\,n(k_1)}\,\mathrm{L}_2^a(k,k_1) \\
\mathrm{F}_2^b &= \frac{4\pi}{75}\,\frac{n(k)n(k_1-k)}{k^4\,n(k_1)}\,\left[\Theta(k-k_1)\mathrm{L}_2^b(k,k_1)-(k\leftrightarrow k_1)\right]
\end{aligned}$$

$$\begin{aligned}
\mathrm{L}_2^a &= 24\lambda_4^- + 12[\,k\,\eta_3 + (k\leftrightarrow k_1)\,] - 6kk_1\lambda_2^+ \\
\mathrm{L}_2^b &= -3\,[4\mu_4 + 2(k-k_1)\mu_3 - kk_1\mu_2 + 4k_1\nu_3 + 2kk_1\nu_2] \\
\lambda_n^\pm &\equiv \beta^{-n}\left[\mathrm{Li}_n(e^{-\beta x}) + \mathrm{Li}_n(e^{-\beta y}) \pm \mathrm{Li}_n(e^{-\beta(x+y)}) \pm \mathrm{Li}_n(1)\right] \\
\eta_n &\equiv \beta^{-n}\left[\mathrm{Li}_n(e^{-\beta x}) - \mathrm{Li}_n(e^{-\beta y}) - \mathrm{Li}_n(e^{-\beta(x+y)}) + \mathrm{Li}_n(1)\right] \\
\nu_n &\equiv \beta^{-n}\left[\mathrm{Li}_n(e^{-\beta x}) - \mathrm{Li}_n(e^{-\beta y})\right]
\end{aligned}$$

Note that the kernels have integrable singularities when $k = k_1$.

# Electrical Conductivity

$$\boxed{\mathbf{J}(t) = Q \int \frac{d^d k}{(2\pi)^d} \mathbf{v}_k [f_+(k,t) - f_-(k,t)]}$$

Substituting in the parameterization of $f_\pm(k,\omega)$ into the current

$$\mathbf{J}(\omega) = 2Q^2 \int \frac{d^d k}{(2\pi)^d} \mathbf{v}_k \mathbf{k}.\mathbf{E}(\omega)\psi(k,\omega)$$

It follows that electrical conductivty is given by

$$\boxed{\sigma_{xx}(\omega) = 2Q^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{k_x^2}{\varepsilon_k}\right) \psi(k,\omega)}$$

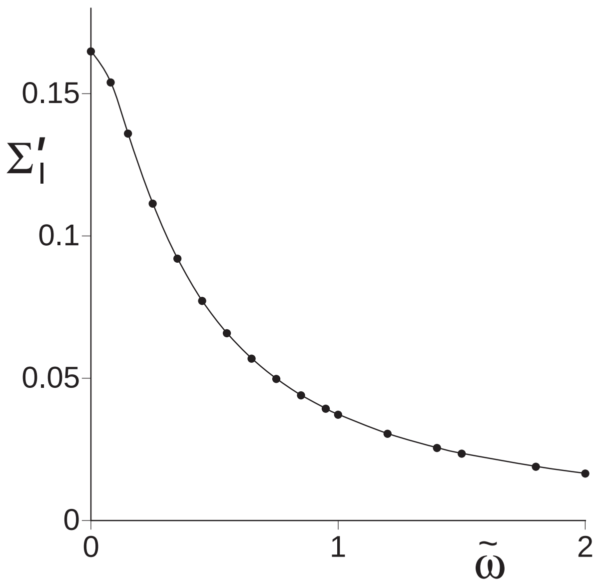
Introducing rescaled variables $\bar{k} \equiv \frac{k}{T}$ $\quad \tilde{\omega} = \frac{\omega}{\epsilon^2 T}$ $\quad \psi(k,\omega) = \frac{\Psi(\bar{k},\tilde{\omega})}{\epsilon^2 T^3}$

$$\boxed{\sigma(\omega) = \frac{Q^2 T^{d-2}}{\epsilon^2} \Sigma(\tilde{\omega})} \qquad \boxed{\Sigma(\tilde{\omega}) = \frac{1}{(3\pi)^2} \int_0^\infty d\bar{k}\bar{k}^3 \Psi(\bar{k},\tilde{\omega})}$$

# Universal Transport

Damle and Sachdev, PRB **56**, 8714 (1997)

$$\sigma(\omega) = \frac{(2e)^2 T^{d-2}}{\epsilon^2} \Sigma\left(\frac{\omega}{\epsilon^2 T}\right) \qquad d = 3 - \epsilon$$




Two spatial dimensions $\boxed{\sigma(0) \approx 1.037\left(\frac{4e^2}{h}\right)}$

# Nernst Coefficient

Ong, Ussishkin, Sondhi, Oganesyan, Huse,...

$$\nu \equiv \frac{1}{B} \frac{E_y}{(-\nabla T)_x}$$

Defining relations of the transport coefficients

$$\begin{pmatrix} \mathbf{J}_e^{\mathrm{tr}} \\ \mathbf{J}_h^{tr} \end{pmatrix} = \begin{pmatrix} \sigma & \alpha \\ T\alpha & \bar{\kappa} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ -\nabla T \end{pmatrix}$$

Set $\mathrm{J}^y = 0$ $\quad \nu = \frac{1}{B} \frac{\alpha_{xy}\sigma_{xx} - \alpha_{xx}\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}$

Particle-hole symmetry $\sigma_{xy} = 0$ $\quad \nu = \frac{1}{B} \frac{\alpha_{xy}}{\sigma_{xx}}$

Need $\alpha_{xy}$: Thermal response to electric field with no temp gradient

# QBE in a Magnetic Field

Bhaseen, Green and Sondhi, *Magnetothermolectric Response at a Superfluid–Mott-insulator Transition*, PRL **98**, 166801 (2007)

To compute $T\alpha_{xy}$ we may equivalently consider the transverse heat current which flows in response to an electric field

$$\frac{\partial f_\pm}{\partial t} \pm Q(\mathbf{E}(t) + \mathbf{v}_\mathbf{k} \times \mathbf{B}).\frac{\partial f_\pm}{\partial \mathbf{k}} = \mathrm{I}^*[f_+, f_-]$$

In linear response we may consider

$$\mathbf{J}_h(\omega) = \int \frac{d^dk}{(2\pi)^d}\, \epsilon_k \mathbf{v}_k \, [f_+(\mathbf{k},\omega) + f_-(\mathbf{k},\omega)]$$

**We need to solve QBE for distribution functions**

Turns out to be useful to recall the relativistic kinematics of a charged particle in crossed $E$ and $B$ fields

# Solution in Drift Regime $E < cB$

Move to frame moving with drift velocity where $E'$ vanishes

$$\mathbf{V}_{\mathrm{D}} = \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{B}|^2}$$

Particle in pure magnetic field which doesn't affect its energy so we expect a Bose-Distribution in moving frame. Solution in lab frame:

$$f_{\pm}(\mathbf{k}) = f_0(\varepsilon'_{\mathbf{k}}) = f_0\left(\frac{\varepsilon_k - \mathbf{v}_{\mathrm{D}}.\mathbf{k}}{\sqrt{1 - v_{\mathrm{D}}^2/c^2}}\right)$$

**Solution of QBE even with leading order collision term**

$$\alpha_{xy} = \frac{2c^2}{dBT} \int \frac{d^d k}{(2\pi\hbar)^d} \, k^2 \left(-\frac{\partial f_0}{\partial \varepsilon_k}\right)$$

$\boxed{\alpha_{xy} = 2\frac{\mathcal{S}}{B}}$ $\mathcal{S}$ is the entropy density of a scalar field

Heat flow by entropy drift: $J_h = 2(T\mathcal{S})v_D = 2T\mathcal{S}\frac{E}{B}$ and $J_h = T\alpha E$

# Temperature Gradient

$$\boxed{\frac{\partial f_\pm}{\partial t} + \mathbf{v}_k . \frac{\partial f_\pm}{\partial \mathbf{x}} \pm Q(\mathbf{v}_k \times \mathbf{B}) . \frac{\partial f_\pm}{\partial \mathbf{k}} = I_\pm[f_+, f_-]}$$

Absence of any material inhomogenity

$$\frac{\partial f_\pm}{\partial \mathbf{x}} = \nabla T \left( \frac{\partial f_\pm}{\partial T} \right) = \nabla T \left( -\frac{\varepsilon_k}{T} \frac{\partial f_0}{\partial \varepsilon_k} \right)$$

Longitudinal and transverse shifts of distribution function

$$\boxed{f_\pm(\mathbf{k}) = f_0(\varepsilon_k) + \mathbf{k}.\mathbf{U}\psi(k) \pm Q\mathbf{k}.(\mathbf{U} \times \mathbf{B})\psi_\perp(k)}$$

$$\mathbf{U} \equiv \frac{-\nabla T}{T}$$

To lowest order in epsilon expansion

$$\psi(k) = 0 \quad \psi_\perp(k) = \frac{\varepsilon_k}{Q^2 B^2} \left( -\frac{\partial f_0}{\partial \varepsilon_k} \right)$$

**Reproduces previous result**

**Onsager satisfied without magnetization subtractions**

# Thermal Transport Coefficient

Bhaseen, Green and Sondhi, PRL **98**, 166801 (2007)

To lowest order in epsilon

$$\boxed{\psi(k) = 0} \qquad \boxed{\psi_\perp(k) = \frac{\varepsilon_k}{Q^2B^2}\left(-\frac{\partial f_0}{\partial \varepsilon_k}\right)}$$

To next order in the epsilon expansion

$$\psi(k) = \epsilon^2 \varepsilon_k \int dk_1 \left[\psi_\perp(k)\mathrm{F}_1(k,k_1) + \psi_\perp(k_1)\mathrm{F}_2(k,k_1)\right]$$

Yields **finite** thermal transport coefficient

$$\boxed{\bar{\kappa} = g\epsilon^2 \frac{T^{d+3}}{Q^2B^2}} \qquad \boxed{g \approx 5.5} \; (\hbar = c = k_B = 1)$$

In contrast to zero field case where response is **infinite**

Dependence on epsilon is **inverse** to zero field conductivity

# Hydrodynamics and AdS/CFT

Hartnoll, Kovtun, Müller and Sachdev, *Theory of the Nernst effect near quantum phase transitions in condensed matter and in dyonic black holes*, Phys. Rev. B **76**, 144502 (2007)

$$\bar{\kappa}_{xx}(B) = \frac{TS^2}{B^2\sigma_{xx}(0)}$$

**"Wiedemann–Franz like"**

Thermal conductivity *inversely* related to conductivity

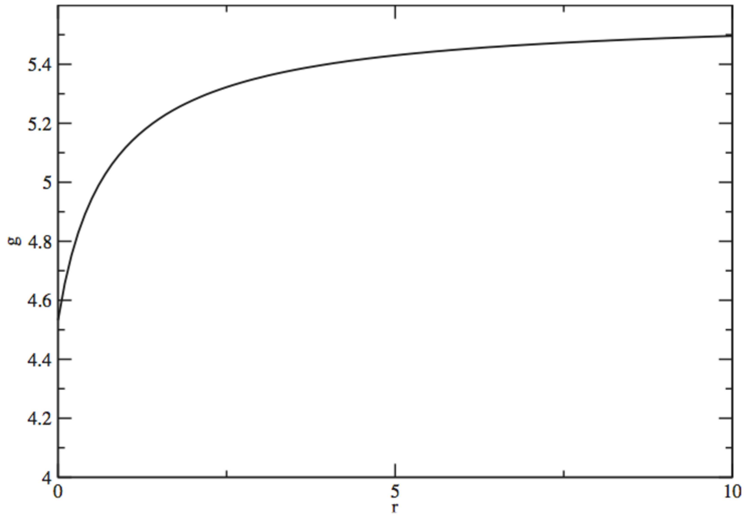
**All transport coefficients are related to electrical conductivity and a thermodynamic variables**

Exact duality relation may be obtained by QBE and $\epsilon$-expansion

Bhaseen, Green and Sondhi, *Magnetothermoelectric Response near Quantum Critical Points*, PRB **79**, 094502 (2009)

# Crossover of Thermal Coefficient

Bhaseen, Green and Sondhi, *Magnetothermoelectric Response near Quantum Critical Points*, PRB **79**, 094502 (2009)




$$\boxed{r \propto B^2/(\epsilon T)^4}$$

Within accuracy of 3D Monte Carlo integrations

$$g_\infty \approx 5.55 \qquad g_0 = 8\pi(2\pi^2/45)^2/(1.037) \approx 4.66$$

$$\boxed{\bar{\kappa}_{xx}(B) = \frac{TS^2}{B^2\sigma_{xx}(0)}}$$

**Can also be extracted analytically from Boltzmann**

# Implications for Nernst

Hartnoll, Kovtun, Müller and Sachdev, PRB **76**, 144502 (2007)

Impurities and chemical potential **Divergences regulated**

$$\alpha_{xy} = \left(\frac{2ek_B}{h}\right)\left(\frac{S/k_B}{B/\phi_0}\right)\left[\frac{\gamma^2 + \omega_c^2 + \gamma/\tau(1-\mu\rho/TS)}{(\gamma+1/\tau)^2+\omega_c^2}\right]$$

$$\phi_0 = \frac{h}{2e} \qquad \omega_c \equiv \frac{2eB\rho}{\varepsilon+P} \qquad \gamma \equiv \frac{\sigma_Q B^2}{\varepsilon+P}$$

Generalizations exist for *all* other transport coefficients

$$\nu = \frac{1}{B}\left(\frac{k_B}{2e}\right)\left(\frac{\varepsilon+P}{k_B T\rho}\right)\left[\frac{\omega_c/\tau}{(\omega_c^2/\gamma+1/\tau)^2+\omega_c^2}\right]$$

Diverges in clean PH limit: $\nu \to \tau/T$

Graphene Müller, Fritz and Sachdev, PRB **78**, 115406 (2008)