# Black Hole Binary Dynamics from Classical and Quantum Gravitational Scattering

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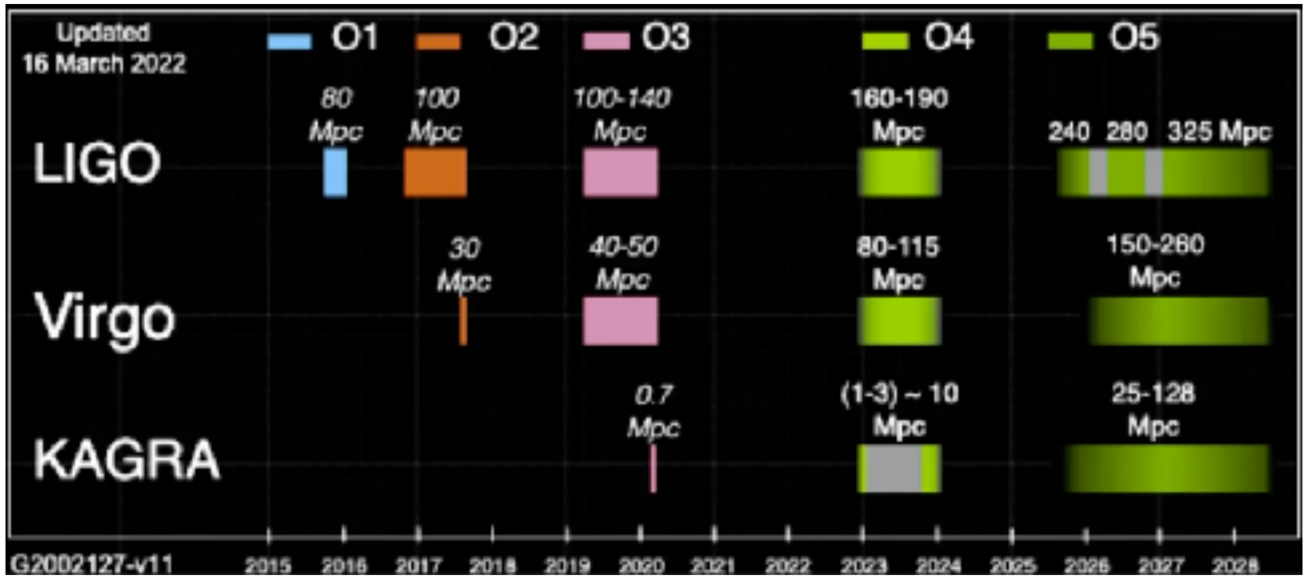
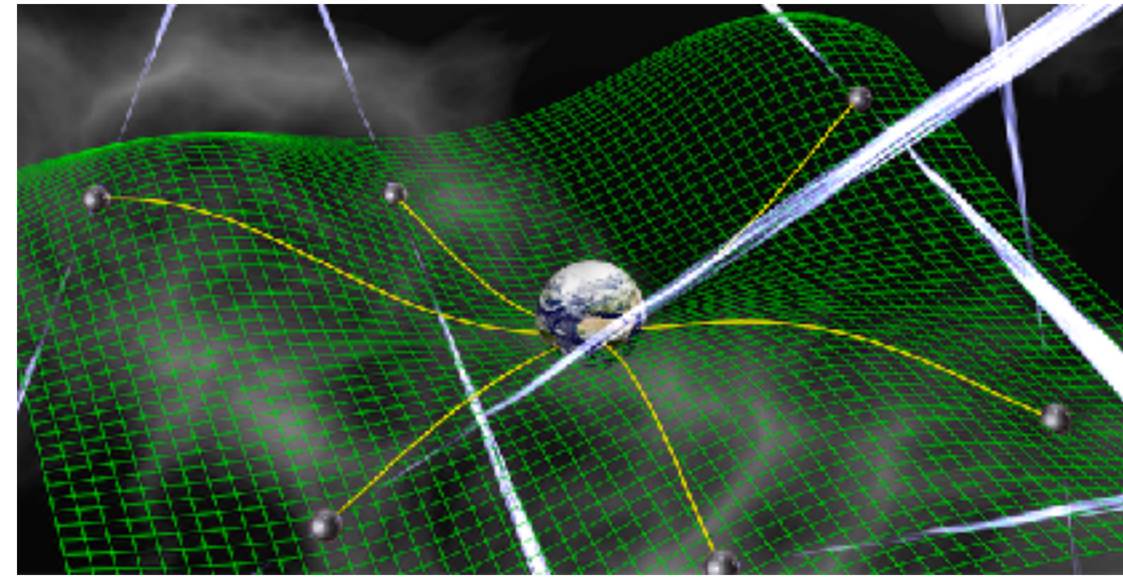
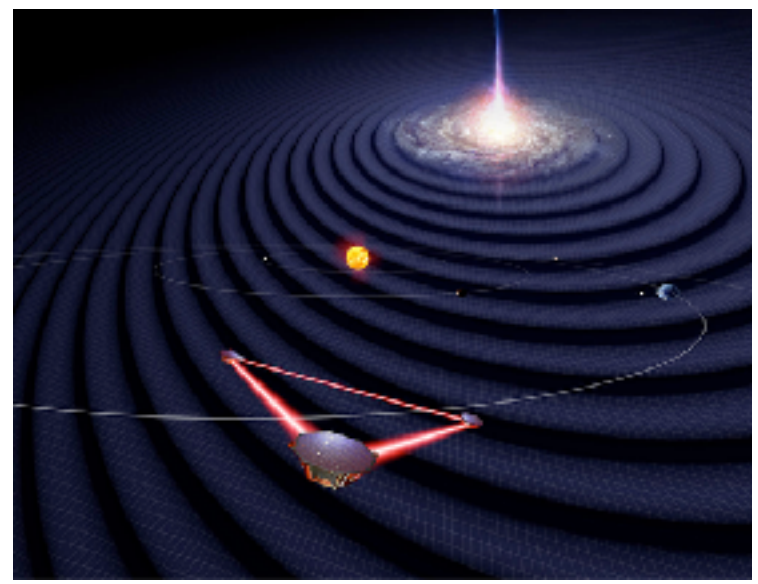
**Twenty-sixth Arnold Sommerfeld Lecture Series**
**Seminar, 12 May 2022**
**Ludwig Maximilians Universität, Munich**

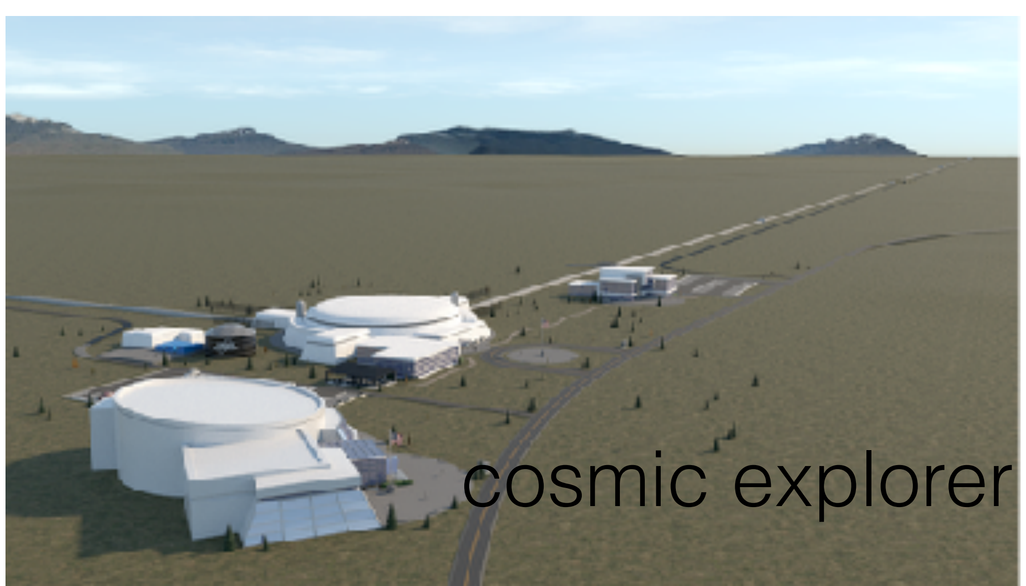
# Future Gravitational Wave Detectors

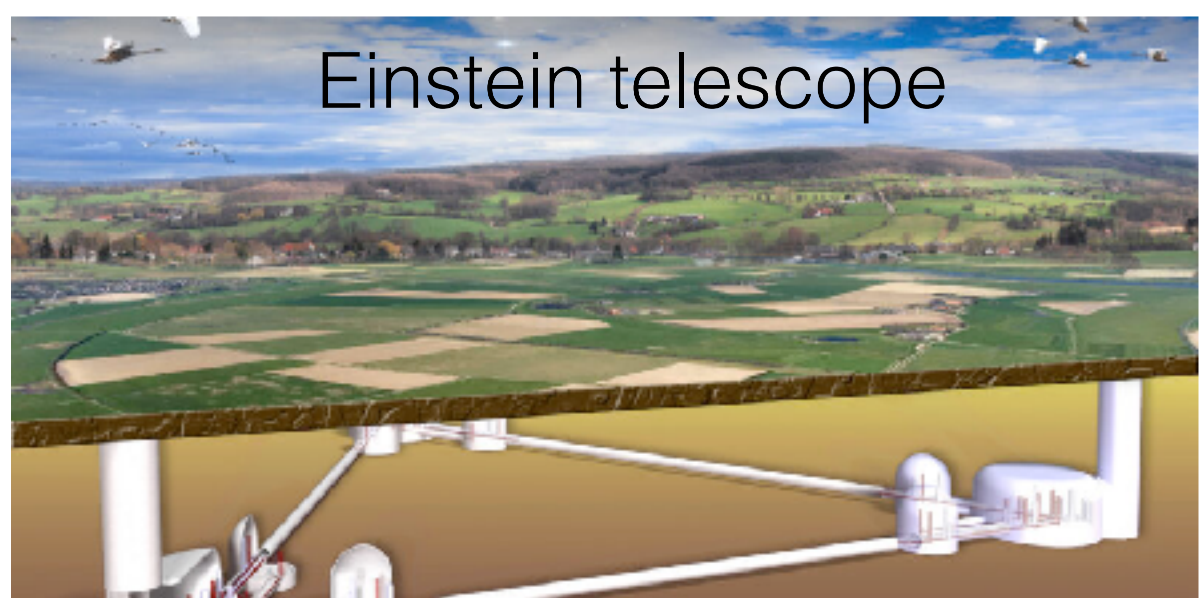


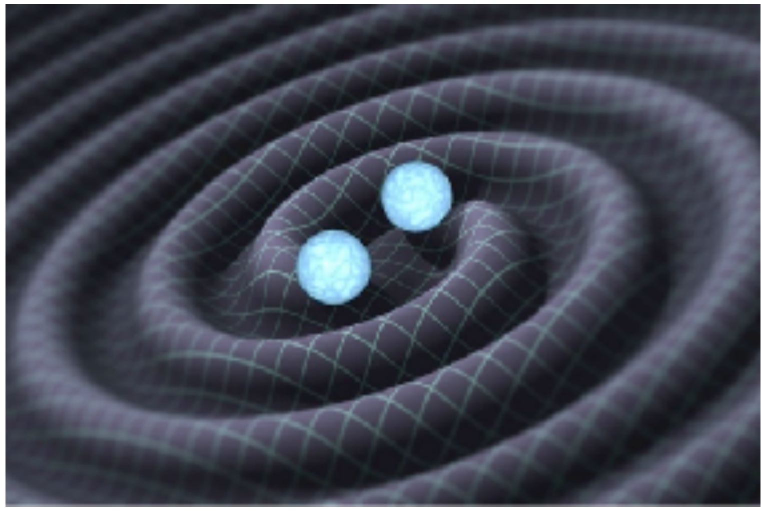


pulsar timing array


ligo india

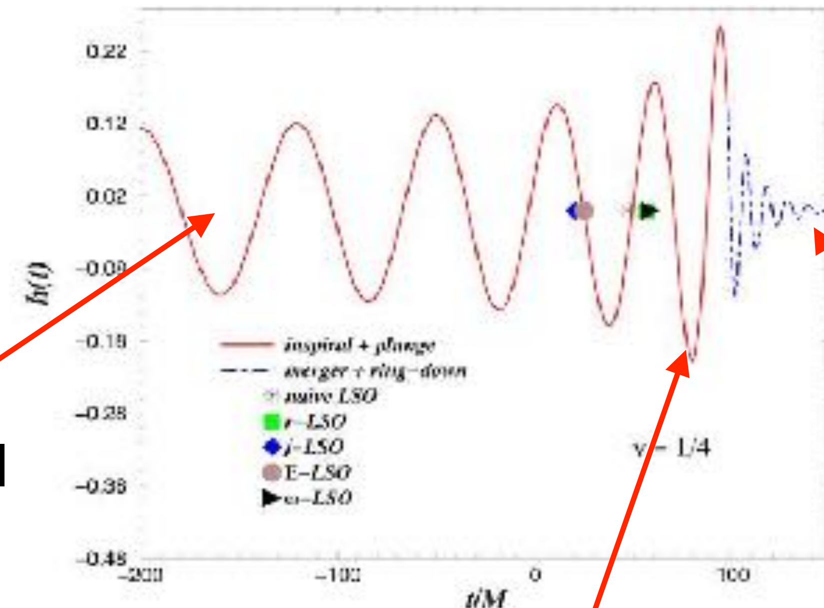

lisa

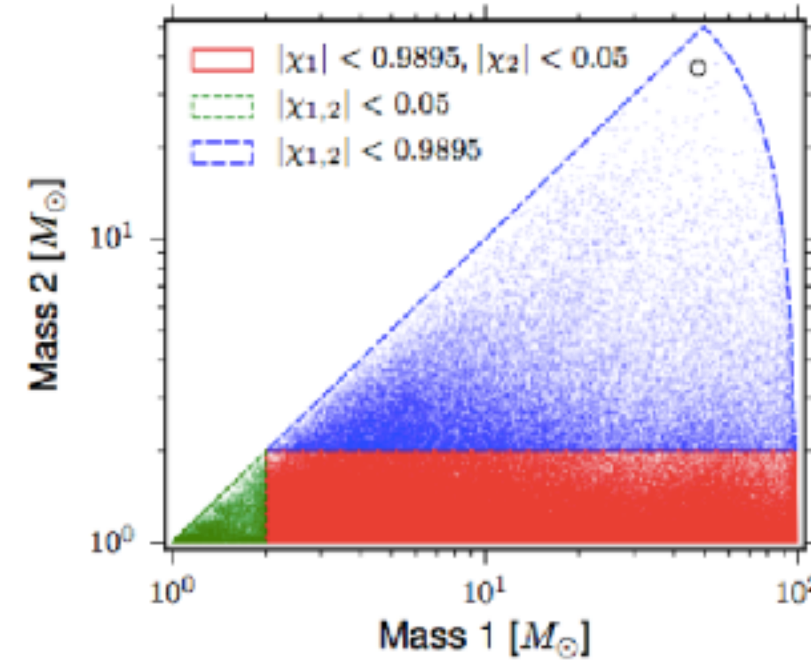

cosmic explorer


Einstein telescope



# Theoretical need for improved templates




**1. conservative dynamics**
**2. radiation reaction force**
**3. (resummed) waveform**

inspiral

merger
numerical
relativity

ringdown

BH perturbation

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}},$$

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r},$$

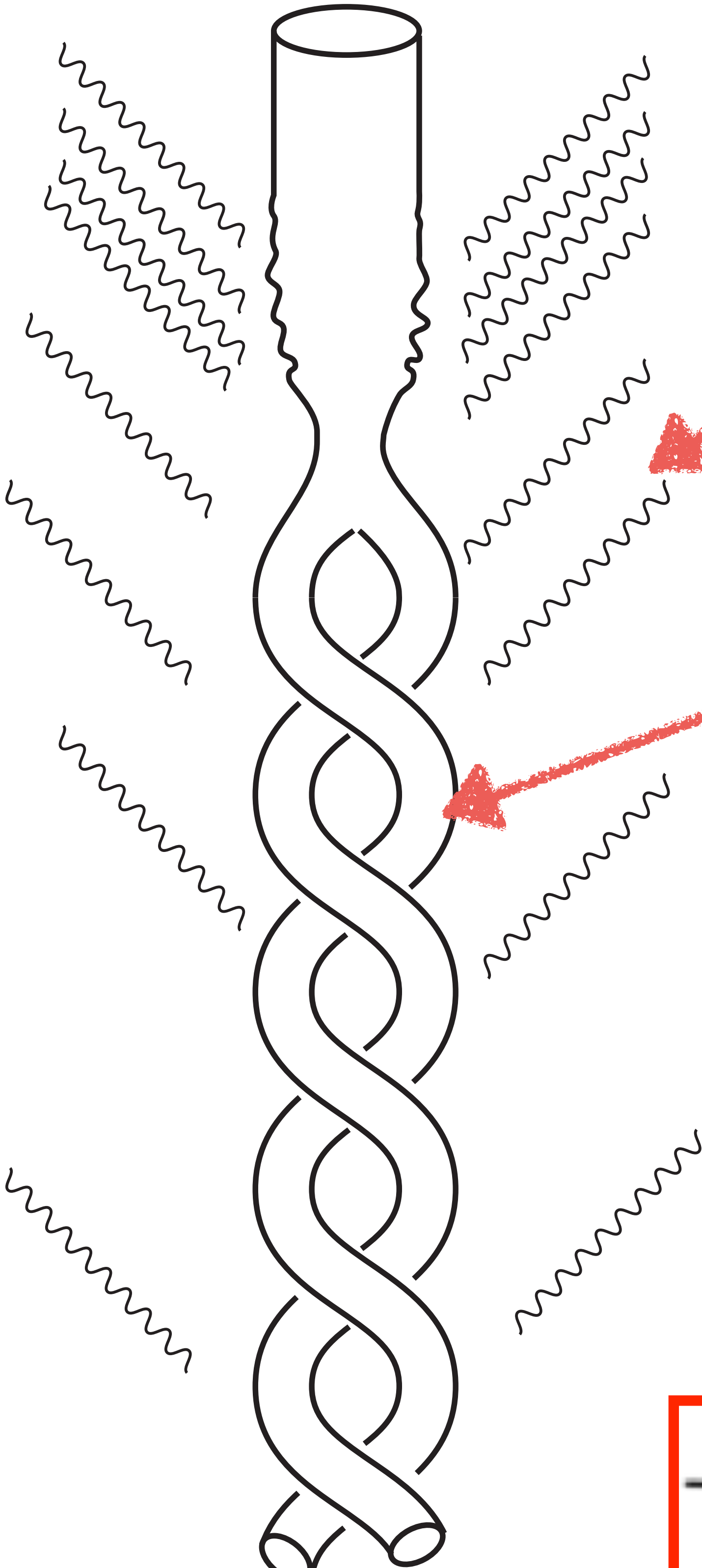
$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi},$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi .$$

$$h_{\ell m} \equiv h^{(N,\epsilon)}_{\ell m} \hat{h}^{(\epsilon)}_{\ell m} \hat{h}^{\rm NQC}_{\ell m}$$

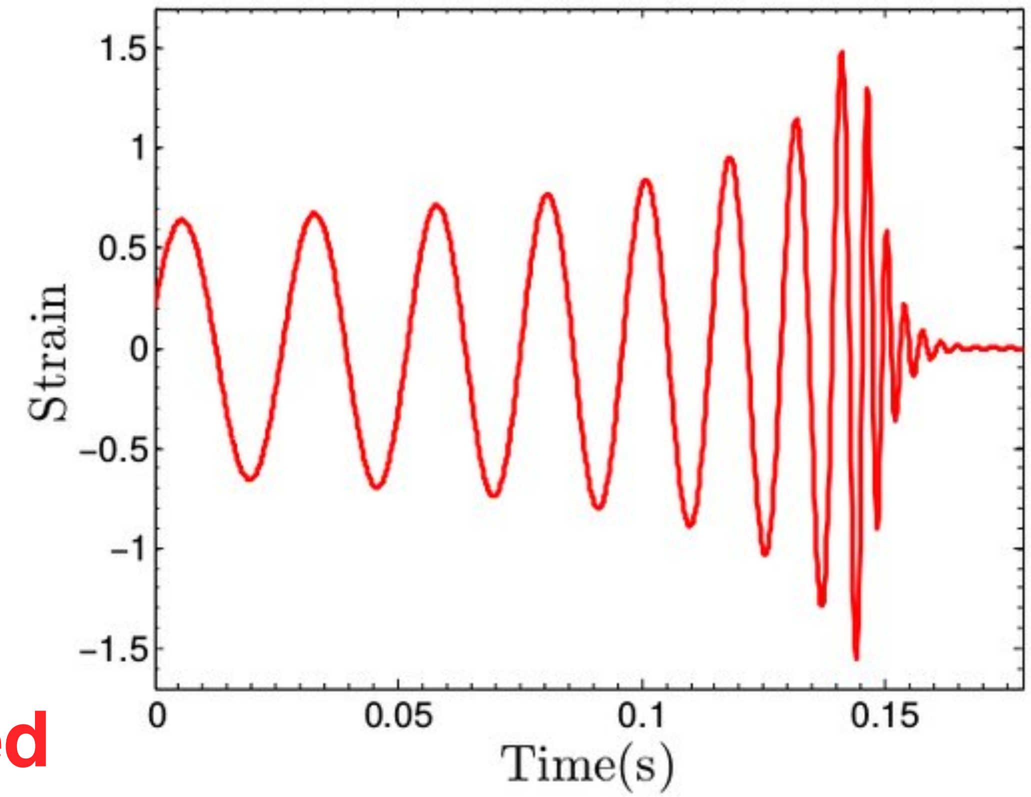
$$\hat{h}^{(\epsilon)}_{\ell m} = \hat{S}^{(\epsilon)}_{\rm eff} T_{\ell m} e^{i\delta_{\ell m}} \rho^{\ell}_{\ell m}$$

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\rm max}} \sum_{m=1}^{\ell} (m\Omega)^2 |R h^{(\epsilon)}_{\ell m}|^2$$



GW emission

from **radiation-reacted** binary dynamics




$$ds^2 = g_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu$$

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} = 0$$

$$-g^{\mu\nu} g_{\alpha\beta,\mu\nu} + g^{\mu\nu} g^{\rho\sigma} \big(g_{\alpha\mu,\rho} g_{\beta\nu,\sigma} - g_{\alpha\mu,\rho} g_{\beta\sigma,\nu} + g_{\alpha\mu,\rho} g_{\nu\sigma,\beta} + g_{\beta\mu,\rho} g_{\nu\sigma,\alpha} - \frac{1}{2} g_{\mu\rho,\alpha} g_{\nu\sigma,\beta}\big) = 0$$

# Tools used for the GR 2-body pb

**Post-Newtonian** (PN) approximation (**expansion in 1/c; ie v^2/c^2 and GM/(c^2r)**)

**Post-Minkowskian** (PM) approximation (**expansion in G; ie in GM/(c^2b)**)
and its recent **Worldline EFT avatars**

**Multipolar post-Minkowskian** (MPM) approximation
theory to the GW emission of binary systems

**Matched Asymptotic Expansions** useful both for the motion of strongly self-gravitating bodies, and for the nearzone-wavezone matching

**Gravitational Self-Force** (SF): expansion in m1/m2, with « first law of BH mechanics » (LeTiec-Blanchet-Whiting'12,…)

**Effective One-Body** (EOB) Approach

**Numerical Relativity** (NR)

**Effective Field Theory** (EFT)

**Quantum scattering amplitude** aided by Double-Copy, Generalized Unitarity,« Feynman-integral Calculus » (IBP, DE, regions, reverse unitarity,…), Kosower-Maybee-O'Connell
**+ Worldline QFT**

**Tutti Frutti** method

# New Angle of Attack on Two-Body Dynamics: Classical and/or Quantum Two-Body Scattering

**TD 2016, 2017:**

**Gravitational scattering, post-Minkowskian approximation, and effective-one-body theory**

**High-energy gravitational scattering and the general relativistic two-body problem**

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

one-loop
G^2

two-loop
G^3+G^4

**Cheung-Rothstein-Solon 2018**

**From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion**

We combine tools from effective field theory and generalized unitarity to construct a map between on-shell scattering amplitudes and the classical potential for interacting spinless particles. For general relativity, we obtain analytic expressions for the classical potential of a binary black hole system at second order in the gravitational constant and all orders in velocity. Our results exactly match all known results up to fourth post-Newtonian order, and offer a simple check of future higher order calculations. By design, these methods should extend to higher orders in perturbation theory.

one-loop
G^2

# Quantum Scattering Amplitudes and 2-body Dynamics

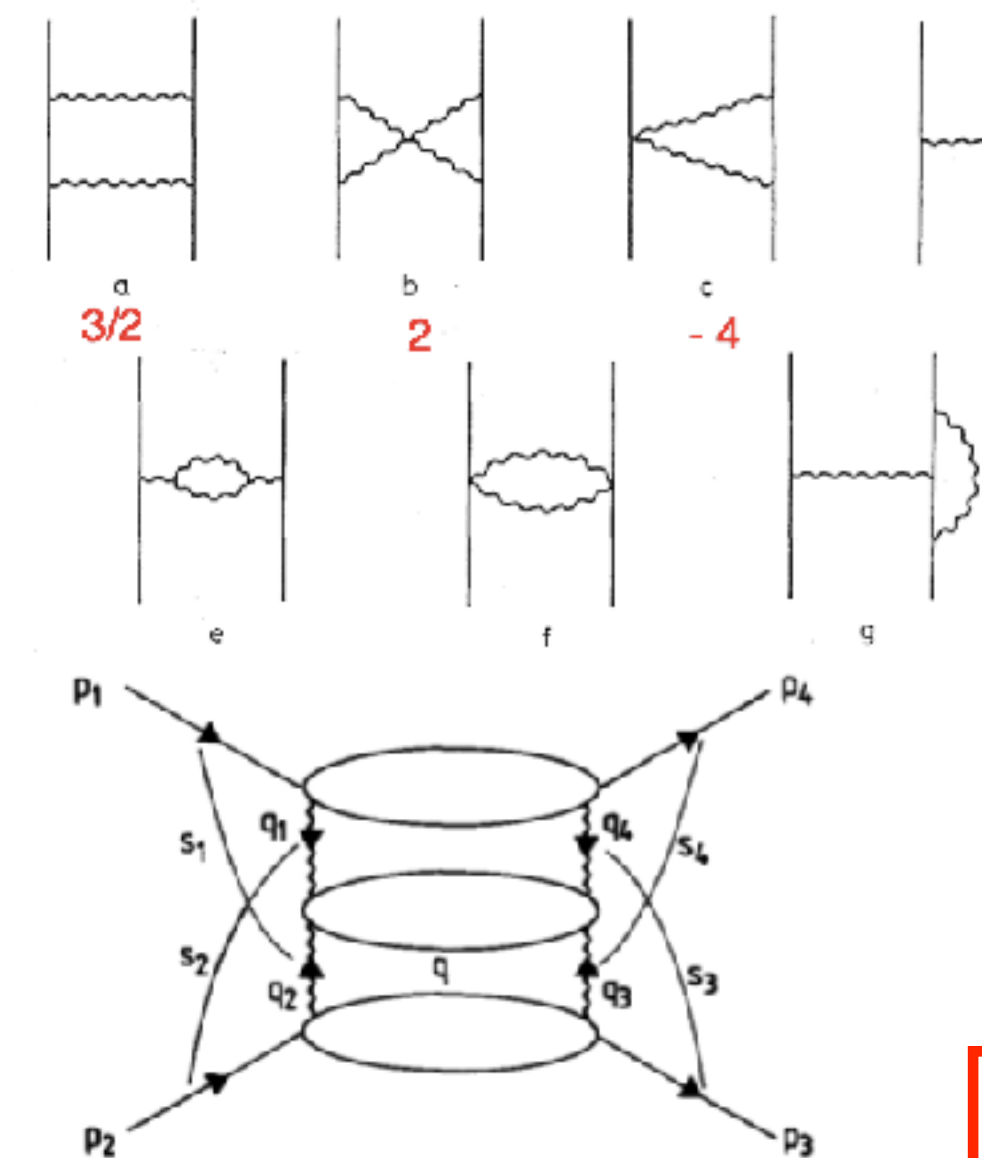



Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

- Quantum Scattering Amplitudes —> Potential

one-graviton exchange :
Corinaldesi '56 '71,
Barker-Gupta-Haracz 66,
Barker-O'Connell 70, Hiida-Okamura72

Nonlinear: Iwasaki 71 [First post-Newtonian approx.],
Okamura-Ohta-Kimura-Hiida 73[2 PN]

## Amati-Ciafaloni-Veneziano 1987-2008

**Ultra-High-Energy (s >> M_Planck^2)**
**Four-graviton Scattering at 2 loops**

**Eikonal phase** \delta in D=4
with one- and two-loop corrections
using the Regge-Gribov approach

confirmed by DiVecchia+'19

$$\delta = \frac{Gs}{\hbar}\left(\log\left(\frac{L_{IR}}{b}\right) + \frac{6\ell_s^2}{\pi b^2} + \frac{2G^2 s}{b^2}\left(1 + \frac{2i}{\pi}\log(\cdots)\right)\right)$$

**Modern techniques for amplitudes** (generalized unitarity; double copy; method of regions; IBPs; differential eqs; Bern, Dixon, Dunbar, Carrasco, Johansson, Cachazo et al., Bjerrum-Bohr et al., Cachazo-Guevara,…) can be used (Damour '17CheungRothsteinSolon'18) to improve the classical 2-body dynamics: need a quantum/classical dictionary.

# Classical scattering perturbation theory

$$\frac{dx_a^\mu}{d\sigma_a} = g^{\mu\nu}(x_a)p_{a\nu},$$

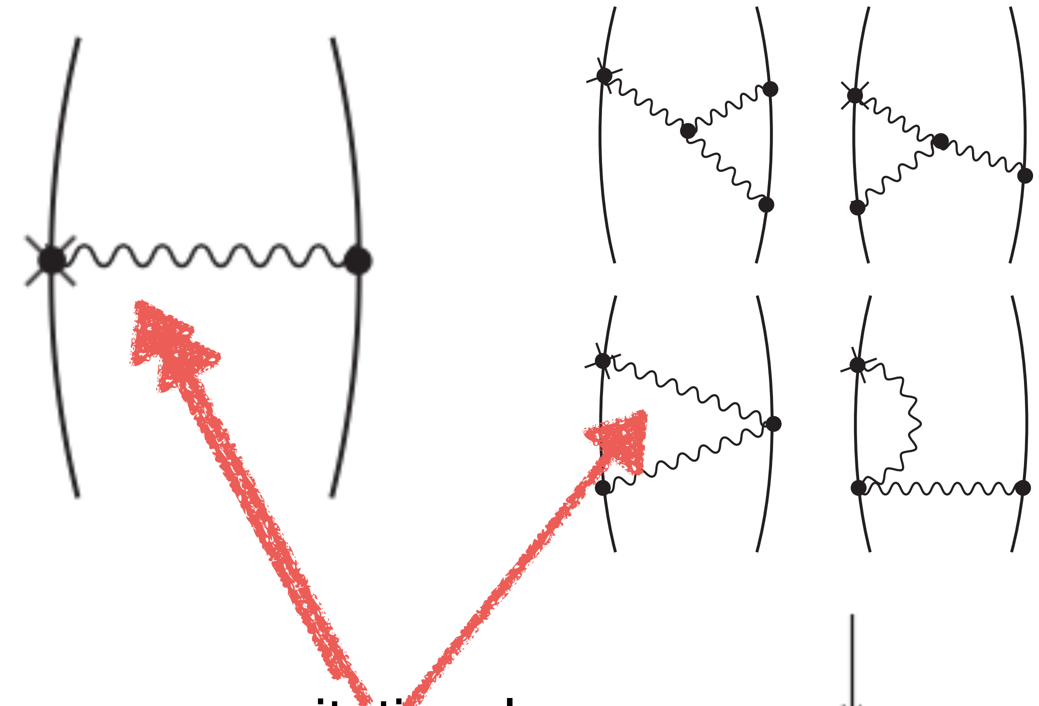
$$\frac{dp_{a\mu}}{d\sigma_a} = -\frac{1}{2}\partial_\mu g^{\alpha\beta}(x_a)p_{a\alpha}p_{a\beta}.$$

$$R^{\mu\nu} - \tfrac{1}{2}Rg^{\mu\nu} = 8\pi G T^{\mu\nu}$$

$$T^{\mu\nu}(x) = \sum \int d\sigma_a p_a^\mu p_a^\nu \frac{\delta^4(x - x_a(\sigma_a))}{\sqrt{g}}$$

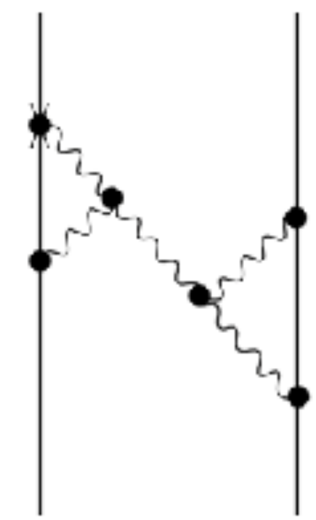
$$\Delta p_{a\mu} = \int_{-\infty}^{+\infty} d\sigma_a \frac{dp_{a\mu}}{d\sigma_a} = -\frac{1}{2}\int_{-\infty}^{+\infty} d\sigma_a \partial_\mu g^{\alpha\beta}(x_a)p_{a\alpha}p_{a\beta}.$$

$$\Delta p_{1\mu} = 2G\int d\sigma_1 d\sigma_2 p_{1\alpha}p_{1\beta} \times \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1) - x_2(\sigma_2))p_{2\alpha'}p_{2\beta'}$$



gravitational propagator

$$\mathcal{P}^{\alpha\beta;\alpha'\beta'}(x-y) = \left(\eta^{\alpha\alpha'}\eta^{\beta\beta'} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\alpha'\beta'}\right)\mathcal{G}(x-y)$$



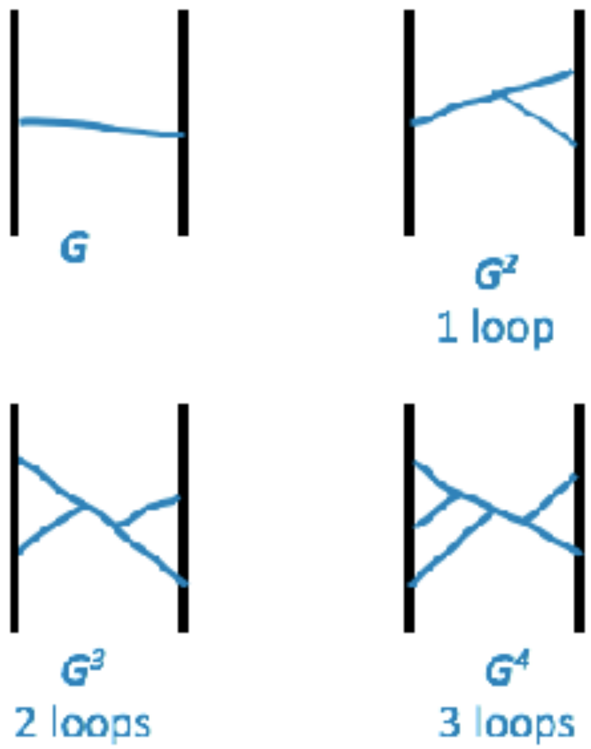
Approach initiated long ago: Rosenblum'78 Westpfahl'79,'85 Portilla'80 Bel et al.'81 limited by the technical difficulty of computing the integrals beyond G^2, ie at G^2=2-loop.

**Recently developed to compete with quantum-scattering approach: Kalin-Porto, Porto et al, Plefka et al, Dlapa-Kalin-Liu-Porto,…**

# Effective One-Body (EOB) approach: H + Rad-Reac Force

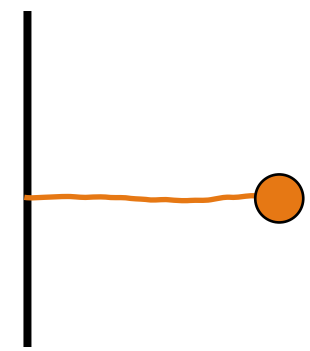
Historically rooted in QM: Brezin-Itzykson-ZinnJustin'70
eikonal scattering amplitude+ Wheeler's:`Think quantum mechanically'

Real 2-body system
(in the c.o.m. frame)




An effective particle of mass mu in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$



$$g_{\rm eff}^{\mu\nu}(X)$$

1:1 map

**mass-shell constraint**

$$0 = g_{\rm eff}^{\mu\nu}(X) P_\mu P_\nu + \mu^2 + Q(X, P)$$

Level correspondence
in the semi-classical limit:
**Bohr-Sommerfeld ->**
identification of
quantized action variables

$$J = \ell\hbar = \frac{1}{2\pi}\oint p_\varphi d\varphi$$

$$N = n\hbar = I_r + J$$

$$I_r = \frac{1}{2\pi}\oint p_r dr$$

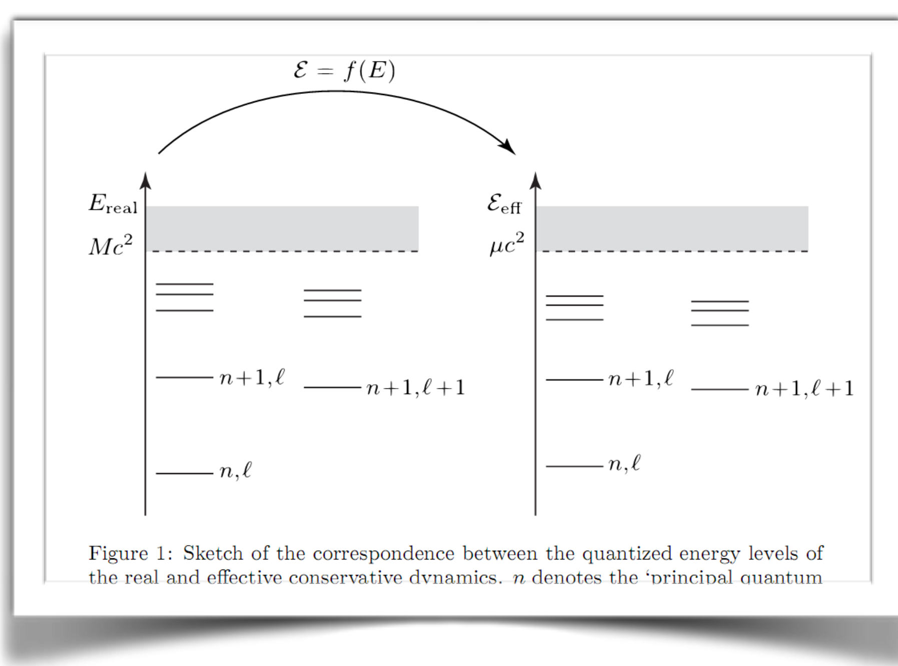


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. $n$ denotes the 'principal quantum

**Crucial energy map**

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}$$

**as functions of I_r and I_phi=J**

# Gravitational Scattering and the GR 2-body problem

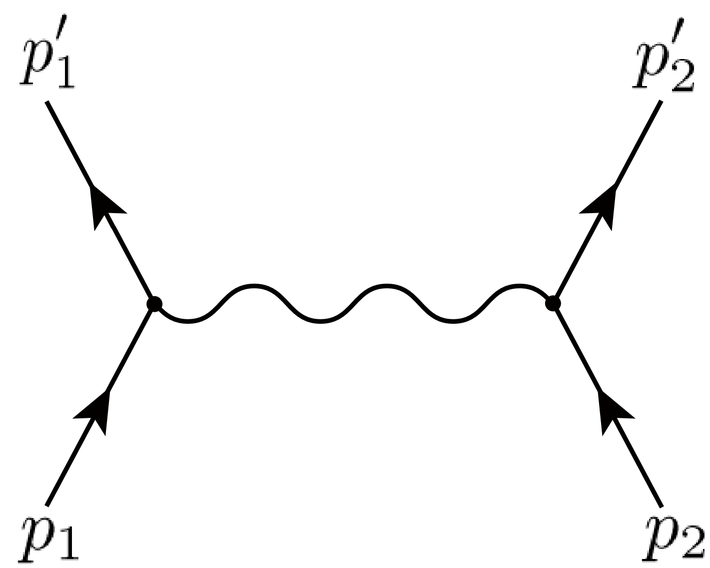
Beyond the PN approximation: (Possibly) High-energy Classical Scattering:
Post-Minkowskian (PM) approximation: expansion in G^n keeping all orders in v/c

Original EOB dictionary based on bound states
New (equivalent) dictionary for scattering states
applicable to the PM approximation (no restriction on v/c,
[Damour2016]

$$\chi_{\text{eff}}(\mathcal{E}_{\text{eff}}, J) = \chi_{\text{real}}(\mathcal{E}_{\text{real}}, J)$$

could recently exploit 'old' results by Bel-Martin '75-'81, Portilla '79,Westpfahl-Goller '79, Portilla '80, Bel-Damour-Deruelle-Ibanez-Martin'81,Westpfahl '85 to compute some pieces of the EOB dynamics to all orders in v/c.

**G^1**

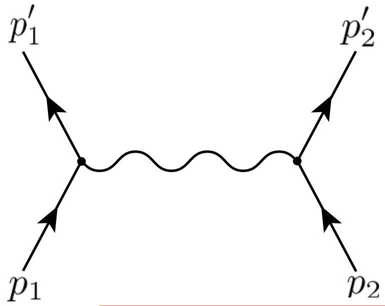



$$\frac{1}{2}\chi_{\text{class}}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\text{eff}},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\text{eff}},\nu) + O(G^3)$$

$$\frac{1}{2}\chi^{\text{real}}_{1PM} = 2\frac{G}{b\,p_{\text{c.m.}}}\frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}}{\mathcal{D}} = 2\frac{G}{J}\frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}}{\mathcal{D}}.$$

$$\mathcal{D}^2 = |p_1 \wedge p_2|^2 = -\frac{1}{2}(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu)(p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu}) = (p_1.p_2)^2 - p_1^2 p_2^2. \quad (52)$$

$$\frac{1}{2}\chi^{\text{real}}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2p_2^2}}.$$




# New results already at the 1PM order (linear in G)

Derivation of EOB energy map to all orders in v/c:

$$\frac{1}{2}\chi^{\rm real}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}.$$

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}.$$

$$ds^2_{\rm lin} = -(1 - 2\frac{GM}{r})dt^2 + (1 + 2\frac{GM}{r})dr^2 + r^2 d\Omega^2$$

to order $G^1$, the relativistic dynamics of a two-body system (of masses $m_1$ , $m_2$ ) is equivalent to the relativistic dynamics of an effective test particle of mass $\mu = m_1 m_2/(m_1 + m_2)$ moving in a Schwarzschild metric of mass $M = m_1 + m_2$, i.e. the rather complicated 1PM Hamiltonian of Ledvinka-Schaefer-Bicak2010: with

$$\begin{aligned}
H_{\rm lin} = {} & \sum_a \overline{m}_a + \frac{1}{4}G\sum_{a,b\neq a}\frac{1}{r_{ab}}\left(7\,\mathbf{p}_a\cdot\mathbf{p}_b + (\mathbf{p}_a\cdot\mathbf{n}_{ab})(\mathbf{p}_b\cdot\mathbf{n}_{ab})\right) - \frac{1}{2}G\sum_{a,b\neq a}\frac{\overline{m}_a\overline{m}_b}{r_{ab}} \\
& \times\left(1 + \frac{\mathbf{p}_a^2}{\overline{m}_a^2} + \frac{\mathbf{p}_b^2}{\overline{m}_b^2}\right) - \frac{1}{4}G\sum_{a,b\neq a}\frac{1}{r_{ab}}\frac{(\overline{m}_a\overline{m}_b)^{-1}}{(y_{ba}+1)^2 y_{ba}}\Big[2\Big(2(\mathbf{p}_a\cdot\mathbf{p}_b)^2(\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 \qquad (6) \\
& -2(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b)\mathbf{p}_b^2 + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2\mathbf{p}_b^4 - (\mathbf{p}_a\cdot\mathbf{p}_b)^2\mathbf{p}_b^2\Big)\frac{1}{\overline{m}_b^2} + 2\Big[-\mathbf{p}_a^2(\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 \\
& + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2(\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + 2(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b) + (\mathbf{p}_a\cdot\mathbf{p}_b)^2 - (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2\mathbf{p}_b^2\Big] \\
& + \Big[-3\mathbf{p}_a^2(\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2(\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + 8(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b) \\
& + \mathbf{p}_a^2\mathbf{p}_b^2 - 3(\mathbf{p}_a\cdot\mathbf{n}_{ba})^2\mathbf{p}_b^2\Big]y_{ba}\Big],
\end{aligned}$$

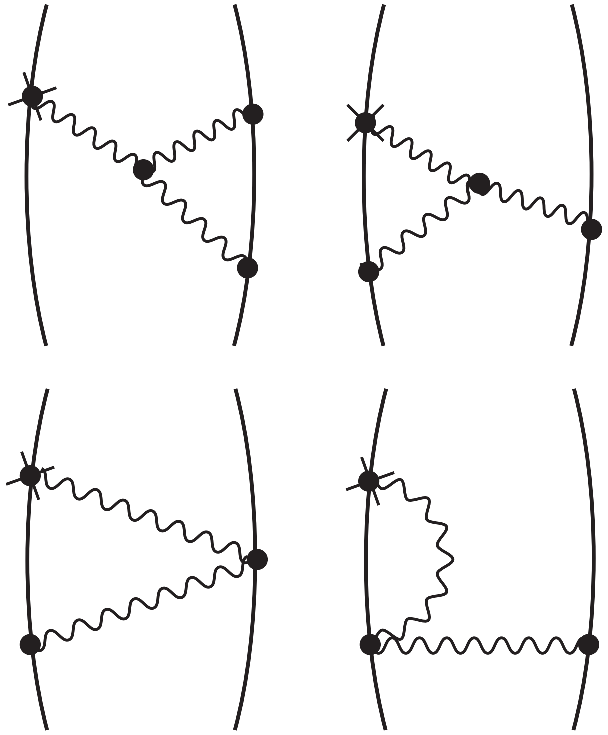
$$y_{ba} = \frac{1}{\overline{m}_b}\sqrt{m_b^2 + (\mathbf{n}_{ba}\cdot\mathbf{p}_b)^2}.$$

$$\overline{m}_a = (m_a^2 + \mathbf{p}_a^2)^{\frac{1}{2}}$$

# Classical Gravitational Scattering at the 2PM level (one-loop)

Damour'18, using Westpfahl-Goller '79, Bel-Damour-Deruelle-Ibanez-Martin'81,Westpfahl '85

**G^2**



$$\frac{1}{2}\chi_{\rm class}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\rm eff},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\rm eff},\nu) + O(G^3)$$

$$\chi_1(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{\hat{\mathcal{E}}_{\rm eff}^2 - 1}},$$

$$\chi_2(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{1+2\nu(\hat{\mathcal{E}}_{\rm eff}-1)}}.$$

$$\hat{\mathcal{E}}_{\rm eff} \equiv \frac{\mathcal{E}_{\rm eff}}{\mu} \equiv \frac{(E_{\rm real})^2 - m_1^2 - m_2^2}{2m_1m_2} = \frac{s - m_1^2 - m_2^2}{2m_1m_2}.$$

$$\frac{\mathcal{E}_{\rm eff}}{\mu} = \gamma = -\frac{p_1\cdot p_2}{m_1 m_2}$$

Effective EOB Hamiltonian transcription of chi2PM as a post-Schwarzschild Hamiltonian

$$0 = g^{\mu\nu}_{\rm Schwarz}P_\mu P_\nu + Q(\mathbf{R},\mathbf{P})$$ --> $$\frac{1}{2}(\chi(\mathcal{E}_{\rm eff},J) - \chi^{\rm Schw}(\mathcal{E}_{\rm eff},J)) = \frac{1}{4}\frac{\partial}{\partial J}\int d\sigma_{(0)}Q + O(G^4).$$

gauge-freedom $Q'(\mathbf{R},\mathbf{P}) = Q(\mathbf{R},\mathbf{P}) + \frac{d}{d\sigma_{(0)}}G(\mathbf{R},\mathbf{P})$ use an « energy gauge »

$$Q(\mathbf{R},\mathbf{P}) = \frac{G^2M^2}{R^2}q_2(\gamma,\nu) + O(G^3)$$

$$q_2(\gamma,\nu) = \frac{3}{2}(5\gamma^2-1)\left(1 - \frac{1}{h(\gamma,\nu)}\right)$$

$$h(\gamma,\nu) = \sqrt{1+2\nu(\gamma-1)}$$

# Simple Map: ConservativeScattering angle <-> EOB dynamics

scattering angle, and its expansion in:

$$\frac{1}{j} = \frac{Gm_1 m_2}{J}$$

$$\frac{1}{2}\chi = \Phi(E_{\rm real}, J; m_1, m_2, G)$$

TD'16-18
Bini-TD-Geralico'20

$$\frac{1}{2}\chi_{\rm class}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\rm eff},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\rm eff},\nu) + O(G^3)$$

$$0 = g_{\rm eff}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q$$

$$\chi_1(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{\hat{\mathcal{E}}_{\rm eff}^2 - 1}},$$

$$\chi_2(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\rm eff} - 1)}}$$

Westpfahl'85

$$g_{\rm eff}^{\mu\nu}$$

**Schwarzschild metric M=m1+m2**

$$Q = \left(\frac{GM}{R}\right)^2 q_2(E) + \left(\frac{GM}{R}\right)^3 q_3(E) + O(G^4)$$

$$q_2(\hat{\mathcal{E}}_{\rm eff},\nu) = -\frac{4}{\pi}[\chi_2(\hat{\mathcal{E}}_{\rm eff},\nu) - \chi_2^{\rm Schw}(\hat{\mathcal{E}}_{\rm eff})].$$

$$q_3(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{4}{\pi}\frac{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\hat{\mathcal{E}}_{\rm eff}^2 - 1}(\chi_2(\hat{\mathcal{E}}_{\rm eff},\nu) - \chi_2^{\rm Schw}(\hat{\mathcal{E}}_{\rm eff})) - \frac{\chi_3(\hat{\mathcal{E}}_{\rm eff},\nu) - \chi_3^{\rm Schw}(\hat{\mathcal{E}}_{\rm eff})}{\sqrt{\hat{\mathcal{E}}_{\rm eff}^2 - 1}}.$$

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}.$$

$$\frac{\mathcal{E}_{\rm eff}}{\mu} = \gamma = -\frac{p_1 \cdot p_2}{m_1 m_2}$$

$$q_2(\gamma,\nu) = \frac{3}{2}(5\gamma^2 - 1)\left(1 - \frac{1}{h(\gamma,\nu)}\right)$$

$$h(\gamma,\nu) = \sqrt{1 + 2\nu(\gamma - 1)}$$

Linear combinations of the scattering coefficients!

# Application to the ACV eikonal scattering phase (massless or ultra-relativistic scattering)

Amati-Ciafaloni-Veneziano'90+ Ciafaloni-Colferai'14+ Bern et al'20+ DiVecchia et al'20

$$\delta^{\rm eikonal} = \frac{1}{\hbar}(\delta^{\rm R} + i\delta^{\rm I}) + \text{quantum corr.}$$

$$\frac{1}{2}\chi^{\rm eikonal} = 2\frac{\gamma}{j} + \frac{16}{3}\frac{\gamma^3}{j^3} + \cdots$$

**valid in the HE limit gamma-> infty**

Using the chi—> Q dictionary this corresponds to the HE limits:

$$q_2^{\rm HE} = \frac{15}{2}\gamma^2$$

$$q_3^{\rm HE} = \gamma^2$$

i.e. an HE limit for the EOB mass-shell condition (TD'18)

$$0 = g_{\rm eff}^{\mu\nu}(X)P_\mu P_\nu + \mu^2 + Q(X,P)$$

$$0 = g_{\rm Schw}^{\mu\nu}P_\mu P_\nu + \left(\frac{15}{2}\left(\frac{GM}{R}\right)^2 + \left(\frac{GM}{R}\right)^3\right)P_0^2$$

# Translating quantum scattering amplitudes into classical dynamical information (1)

## The domain of validity of the Born-Feynman expansion

$$\mathcal{M}(s,t) = \mathcal{M}^{(\frac{G}{\hbar})}(s,t) + \mathcal{M}^{(\frac{G^2}{\hbar^2})}(s,t) + \cdots, \qquad \mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi \frac{G}{\hbar} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{-t}.$$

is

$$\frac{Gs}{\hbar v} \sim \frac{G E_1 E_2}{\hbar v} \ll 1$$

while the domain of validity of classical scattering is (Bohr 1948)

$$\frac{Gs}{\hbar v} \sim \frac{G E_1 E_2}{\hbar v} \gg 1$$

**Amati-Ciafaloni-Veneziano** faced this issue by assuming **eikonalization in b space**

$$\widetilde{\mathcal{A}}(s,b) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{\mathcal{A}(s,q^2)}{4pE} e^{-ib\cdot q}$$

$$1 + i\widetilde{\mathcal{A}}(s,b) = (1 + 2i\Delta(s,b))\, e^{2i\delta(s,b)}$$

classical phase

$$i\frac{\mathcal{A}(s,Q^2)}{4pE} = \int d^{D-2}b \left(e^{2i\delta(s,b)} - 1\right) e^{ib\cdot Q}$$

$$2\delta(s,b) = \frac{\Delta S_r(s,J)}{\hbar}$$

subtracted radial action of potential scattering

total classical momentum transfer:

$$Q^\mu = -\frac{\partial\, \mathrm{Re}\, 2\delta(s,b)}{\partial b^\mu}$$

# Translating quantum scattering amplitudes into classical dynamical information (2)

Damour'17: EOB potential Q(R,E) or W(R,E)
Cheung-Rothstein-Solon'18, Bern et al'19
different EFT potential V(R,P^2) and methods for
taking the classical limit at the integrand level,
and extracting the « classical part » of the scattering amplitude

EOB

$$Q^F(u,\mathcal{E}_{\rm eff}) = u^2 q_2(\mathcal{E}_{\rm eff}) + u^3 q_3(\mathcal{E}_{\rm eff}) + u^4 q_4^F(\mathcal{E}_{\rm eff}) + O(G^5)$$

$$w(r,p_\infty) = \frac{w_1(\gamma)}{r} + \frac{w_2(\gamma)}{r^2} + \frac{w_3(\gamma)}{r^3} + \frac{w_4(\gamma)}{r^4} + \cdots$$

EFT

$$H(\mathbf{P},\mathbf{X}) = \sqrt{m_1^2+\mathbf{P}^2} + \sqrt{m_2^2+\mathbf{P}^2} + V(R,\mathbf{P}^2)$$

$$V(R,\mathbf{P}^2) = G\frac{c_1(\mathbf{P}^2)}{R} + G^2\frac{c_2(\mathbf{P}^2)}{R^2} + G^3\frac{c_3(\mathbf{P}^2)}{R^3} + \cdots$$

**non-relativistic potential scattering !**

$$-\hat{\hbar}^2\Delta_{\mathbf{x}}\psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{r} + \frac{w_2}{r^2} + \frac{w_3}{r^3} + O\left(\frac{1}{r^4}\right)\right]\psi(\mathbf{x})$$

$$\mathcal{M}^{QFT}_{\rm classical} = \frac{8\pi G s}{\hbar} f^{EOB} = \mathcal{M}^{EFT}$$

# Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order

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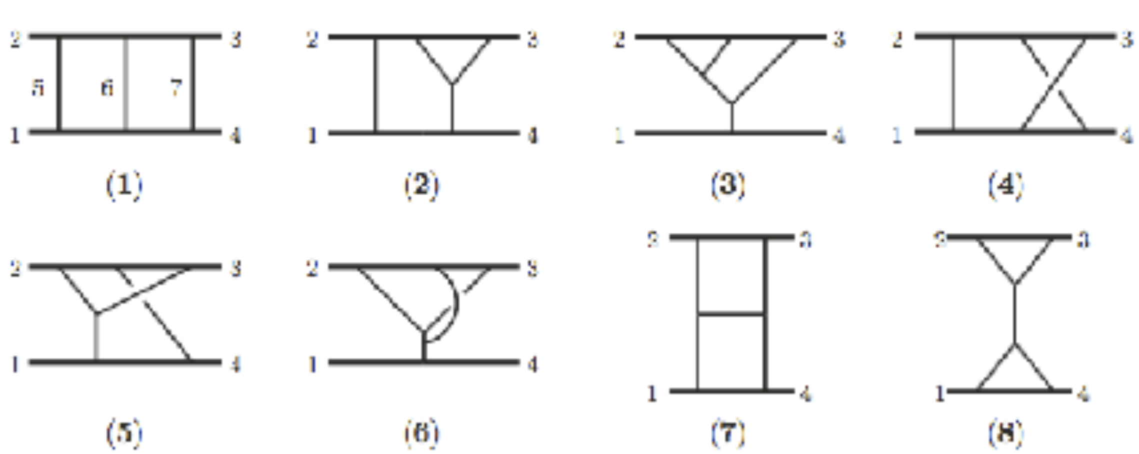
**two-loop level G^3**

(Received 28 January 2019; published 24 May 2019)

We present the amplitude for classical scattering of gravitationally interacting massive scalars at third post-Minkowskian order. Our approach harnesses powerful tools from the modern amplitudes program such as generalized unitarity and the double-copy construction, which relates gravity integrands to simpler gauge-theory expressions. Adapting methods for integration and matching from effective field theory, we extract the conservative Hamiltonian for compact spinless binaries at third post-Minkowskian order. The resulting Hamiltonian is in complete agreement with corresponding terms in state-of-the-art expressions at fourth post-Newtonian order as well as the probe limit at all orders in velocity. We also derive the scattering angle at third post-Minkowskian order and find agreement with known results.

the eight 2-loop diagrams contributing to the O(G^3/r^3) classical potential




**two-loop level**

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2 \xi}\Big[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu(3+12\sigma^2-4\sigma^4)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma(1-2\sigma^2)(1-5\sigma^2)}{(1+\gamma)(1+\sigma)}\Big] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}[3\gamma(1-2\sigma^2)(1-5\sigma^2)F_1 - 32m^2\nu^2(1-2\sigma^2)^3 F_2], \quad (8)$$

arcsinh

# 3PM computation (Bern-Cheung-Roiban-Shen-Solon-Zeng'19)

using a combination of techniques: generalized unitarity; BCJ double-copy; 2-loop amplitude of quasi-classical diagrams; **EFT transcription** (Cheung-Rothstein-Solon'18); **resummation of PN-expanded integrals for potential-gravitons**

$$\chi_3^{\rm cons} = \chi_3^{\rm Schw} - \frac{2\nu\sqrt{\gamma^2-1}}{h^2(\gamma,\nu)}\,\bar{C}^{\rm cons}(\gamma)$$

**G^3 contrib. to H_EOB**

$$q_3^{\rm cons} = \frac{3}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{\gamma^2-1}\left(\frac{1}{h(\gamma,\nu)}-1\right) + \frac{2\nu}{h^2(\gamma,\nu)}\bar{C}^{\rm cons}(\gamma)$$

$$\overline{C}^{\rm cons}(\gamma) = \frac{2}{3}\gamma(14\gamma^2+25) + 2(4\gamma^4-12\gamma^2-3)\frac{\mathcal{A}(v)}{\sqrt{\gamma^2-1}}$$

$$h(\gamma,\nu) \equiv \frac{\sqrt{s}}{M} = \sqrt{1+2\nu(\gamma-1)}$$

$$\mathcal{A}(v) \equiv {\rm arctanh}(v) = \frac{1}{2}\ln\frac{1+v}{1-v} = 2\,{\rm arcsinh}\sqrt{\frac{\gamma-1}{2}}$$

**puzzling HE limits when compared to ACV and Akcay et al'12**

$$\frac{1}{2}\chi^{\rm cons} = 2\frac{\gamma}{j} + (12 - 8\ln(2\gamma))\frac{\gamma^3}{j^3} + O(G^4)$$

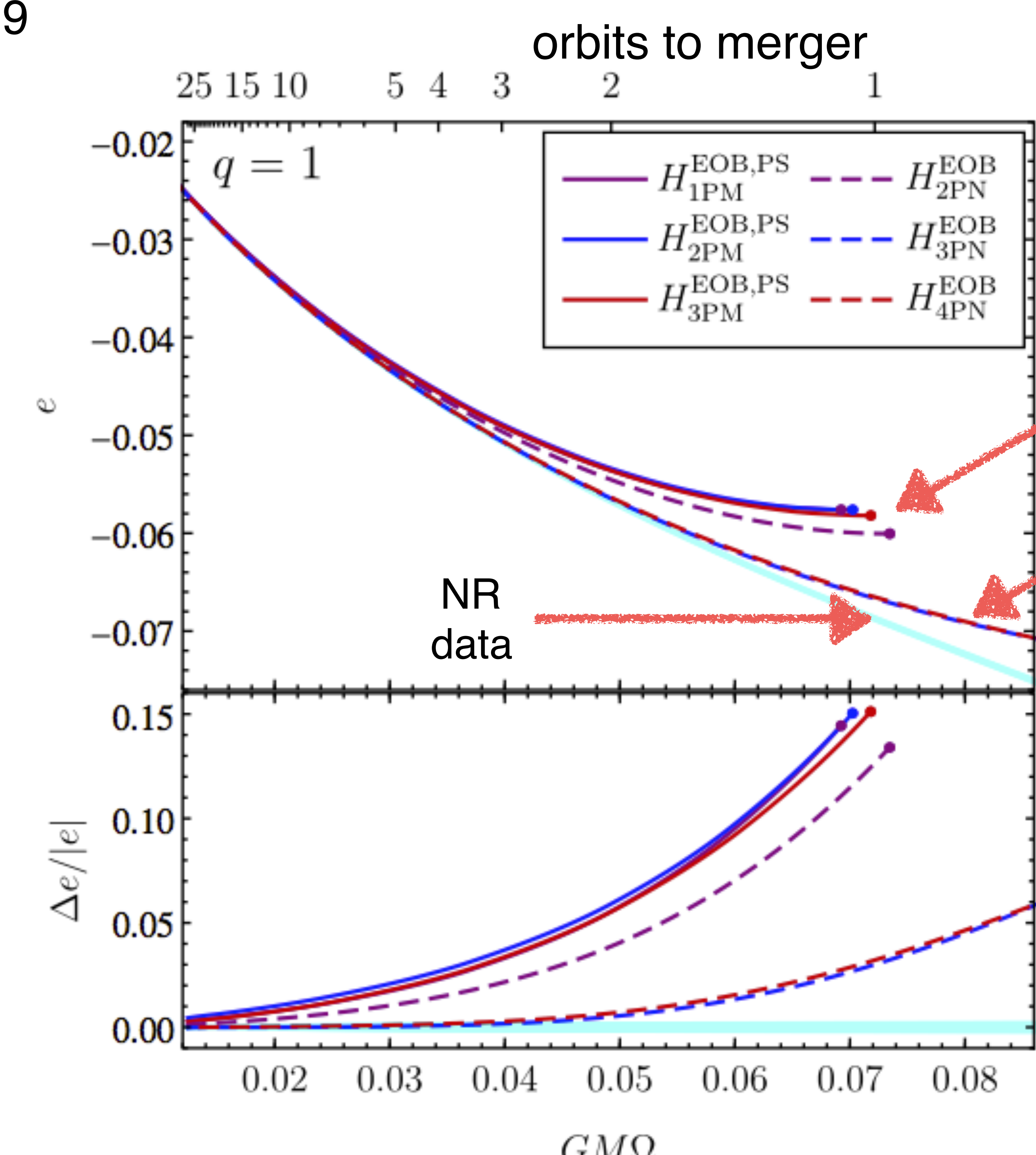
$q_3^{\rm cons} \approx +8\ln(2\gamma)\gamma^2$ instead of $q_3^{\rm ACV} \approx +1\gamma^2$

**confirmations:** 5PN (Bini-TD-Geralico'19); 6PN (Blümlein-Maier-Marquard-Schäfer'20, Bini-TD-Geralico'20); 3PM (Cheung-Solon'20, Kälin-Porto'20)

# Comparison of 3PM Hamiltonian to NR energetics

Antonelli et al. '19

fractional binding energy versus angular frequency




PM Hamiltonians up to G^3

PN Hamiltonians used in LIGO-Virgo data-analysis

includes terms O(G^5) and resummation

The main current interest of gravitational scattering results is **conceptual**, rather than directly practical

# Conservative and Radiative Aspects of the Dynamics

selected by Bern's integration method of regions

1PM

2PM

$$\dot{\mathbf{p}}_a \sim G\left(1+\frac{1}{c^2}+\frac{1}{c^4}+\frac{1}{c^6}+\frac{1}{c^8}+\frac{1}{c^{10}}\right)+$$
$$+G^2\left(\frac{1}{c^2}+\frac{1}{c^4}+\frac{1}{c^5}+\frac{1}{c^6}+\frac{1}{c^7}+\frac{1}{c^8}+\frac{1}{c^9}+\frac{1}{c^{10}}\right)+$$
$$+G^3\left(\frac{1}{c^4}+\frac{1}{c^5}+\frac{1}{c^6}+\frac{1}{c^7}+\frac{1}{c^8}+\frac{1}{c^9}+\frac{1}{c^{10}}\right)+$$
$$+G^4\left(\frac{1}{c^6}+\frac{1}{c^7}+\frac{1}{c^8}+\frac{1}{c^9}+\frac{1}{c^{10}}\right)+$$
$$+G^5\left(\frac{1}{c^8}+\frac{1}{c^9}+\frac{1}{c^{10}}\right)+$$
$$+G^6\frac{1}{c^{10}}$$

1PN

2PN

2.5PN LO rad reac

4PN (G^4,G^5) separation well-defined

5PN same order as F_radreac^2!

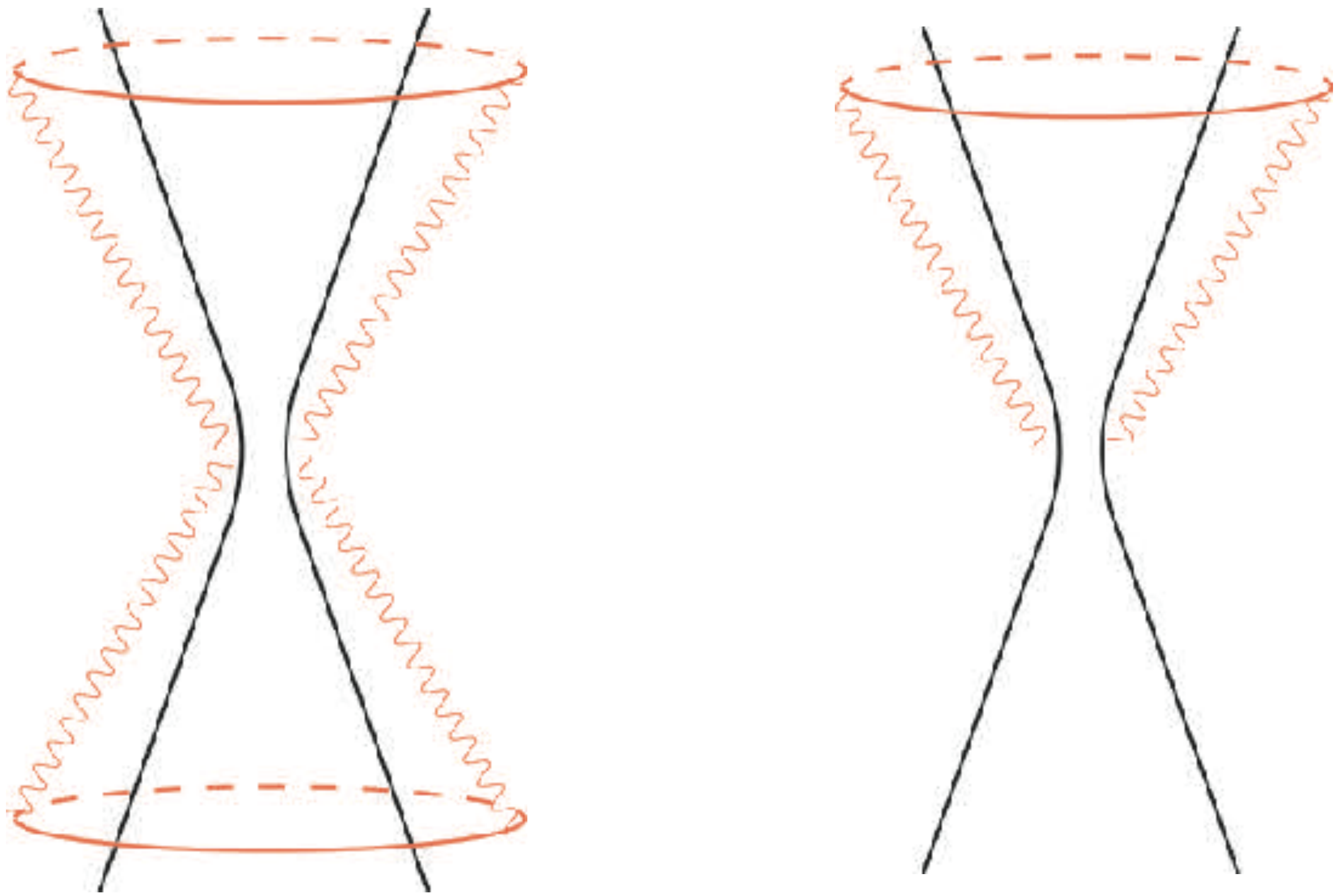
**Black: time-even and conservative**

**Red: time-odd and dissipative**

**Blue: nonlocal-in-time but decomposable in conservative and dissipative**

**Purple: ambiguous in various ways ??**

# Conservative vs Radiation-reacted Classical Gravitational Scattering



Radiation-reaction effects enter scattering at G^3/c^5 (Bini-TD'12)

$$\frac{1}{2}\chi^{\rm rad} = +\frac{8G^3}{5c^5}\frac{m_1^3 m_2^3}{J^3}\nu v^2 + \cdots$$

Radiation-reaction effects in scattering play a crucial role at **high-energy**

(DiVecchia-Heissenberg-Russo-Veneziano'20, TD'21, Hermann-Parra-Martinez-Ruf-Zeng'21,….)

they resolve the puzzle of the discrepancy between the HE limit of

Amati-Ciafaloni-Veneziano'90(+ Ciafaloni-Colferai'14), and the G^3 result of Bern et al'19,20

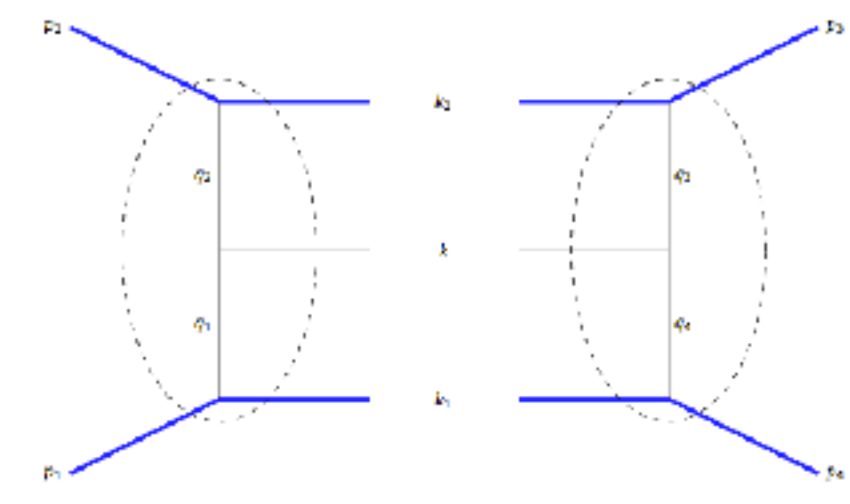
# Universality of ultra-relativistic gravitational scattering from analyticity/crossing (DiVecchia-Heissenberg-Russo-Veneziano'20)

**ultra-relativistic** eikonal phase: $\delta(s,b) = \delta_0(s,b) + \delta_2(s,b)$

$$\mathrm{Re}(2\delta_2) = \frac{\pi}{2\log s}\,\mathrm{Im}(2\delta_2) - \frac{\delta_0}{s}(\nabla 2\delta_0)^2 + \mathcal{O}\left(\frac{1}{\log s}\right)$$

IR finite

IR divergent



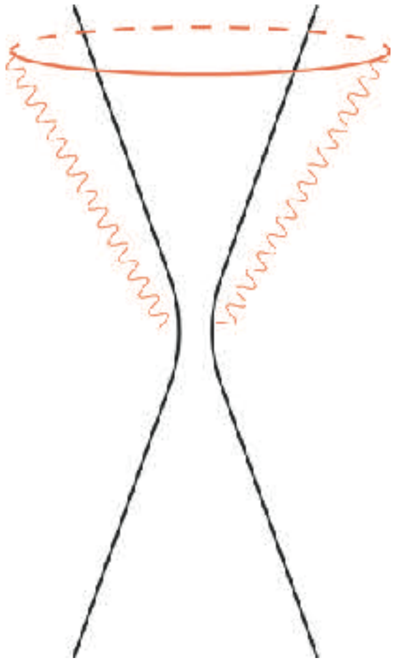
$$\mathrm{Im}\,\widetilde{A_2^{(3p)}}(s,b) \simeq \frac{1}{2s}\frac{(8G_N s)^3 \log s\,\Gamma^3(1-\epsilon)}{16(\pi b^2)^{1-3\epsilon}}\left[-\frac{1}{4\epsilon}+\frac{1}{2}+\mathcal{O}(\epsilon)\right]$$

$$\mathrm{Re}(2\delta_2) \simeq \frac{4G_N^3 s^2}{\hbar b^2}$$

universality of HE result = ACV, thanks to radiative effects

**+** DiVecchia-Heissenberg-Russo-Veneziano'21: **Radiation Reaction from Soft Theorems**

# Radiation-Reaction Contribution to the (transverse) Classical Scattering Angle at G^3 (TD 2010.01641)



$$\chi^{\mathrm{tot}} = \chi^{\mathrm{cons}} + \chi^{\mathrm{rad}}$$

where, to first order in Rad-Reac, one has (Bini-TD'12)

$$\chi^{\mathrm{rad}}(E,J) = -\frac{1}{2}\frac{\partial \chi^{\mathrm{cons}}}{\partial E} E^{\mathrm{rad}} - \frac{1}{2}\frac{\partial \chi^{\mathrm{cons}}}{\partial J} J^{\mathrm{rad}}$$

O(G^2)

[TD-Deruelle'81]

chi^cons=O(G^1)

O(G^3)

O(G^4)

O(G^3)

$$h_{ij}^{TT} = \frac{f_{ij}(t-r,\theta,\phi)}{r} + O\left(\frac{1}{r^2}\right)$$

$$J_k^{\mathrm{rad}} = \frac{\epsilon_{kij}}{16\pi G}\int du\, d\Omega \left[ f_{ia}\partial_u f_{ja} - \frac{1}{2} x^i \partial_j f_{ab} \partial_u f_{ab} \right]$$

DeWitt'71, Thorne'80
Kovacs-Thorne'77, Bel et al'81,
Westpfahl'85

$$\mathcal{I}(v) = -\frac{16}{3} + \frac{2}{v^2} + \frac{2(3v^2-1)}{v^3}\mathcal{A}(v)$$

$$\mathcal{A}(v) \equiv \mathrm{arctanh}(v) = \frac{1}{2}\ln\frac{1+v}{1-v}$$

$$\frac{1}{2}\chi^{\mathrm{rad}}(\gamma, j, \nu) = +\frac{\nu}{h^2(\gamma,\nu) j^3}(2\gamma^2-1)^2 \mathcal{I}(v) + O(G^4)$$

$$\frac{1}{2}(\chi^{\mathrm{cons}} + \chi^{\mathrm{rad}}) = 2\frac{\gamma}{j} + \frac{16}{3}\frac{\gamma^3}{j^3} = \chi^{\mathrm{ACV}}$$

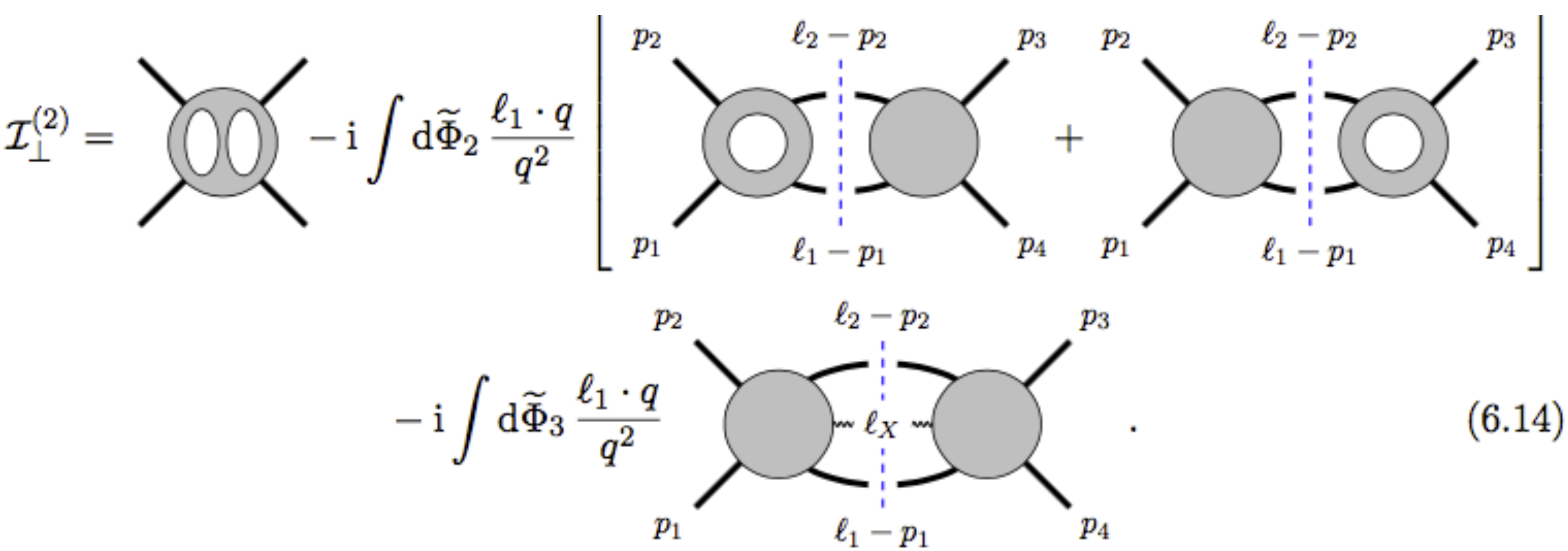
# Translating quantum scattering amplitudes into classical dynamical information (3)

Kosower-Maybee-O'Connell'19 formalism for any observable O

$\Delta O = \langle \text{out}|\mathbb{O}|\text{out}\rangle - \langle \text{in}|\mathbb{O}|\text{in}\rangle$ with |out> = S |in> and S=1+i T

$$\Delta O = \langle \text{in}|i[O,T]|\text{in}\rangle + \langle \text{in}|T^\dagger[O,T]|\text{in}\rangle$$

Hermann-Parra-Martinez-Ruf-Zeng'21 making use of: generalized unitarity, reverse unitarity ( for phase-space integrals), method of regions, integration by parts canonical differential eqs applied KMOC to O= p_1^mu and p_rad^mu



## momentum transfer (impulse)

(Hermann-Parra-Martinez-Ruf-Zeng’21)

$$\Delta p^{\mu,(2)}_{1,\perp,\mathrm{cons}} = \frac{G^3M^4\nu}{|b|^3}\frac{2}{\sqrt{\sigma^2-1}}\frac{b^\mu}{|b|}\left[h^2(\sigma,\nu)\left(16\sigma^2-\frac{1}{(\sigma^2-1)^2}\right)\right. \qquad (7$$

$$\left. -\frac{4}{3}\nu\,\sigma\left(14\sigma^2+25\right)-8\nu\left(4\sigma^4-12\sigma^2-3\right)\frac{\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right]$$

$$\Delta p^{\mu,(2)}_{1,u,\mathrm{cons}} = \frac{G^3M^5\nu^2}{|b|^3}\frac{3\pi\left(2\sigma^2-1\right)\left(5\sigma^2-1\right)}{2\left(\sigma^2-1\right)}\left[\frac{1}{m_1}\check{u}_1^\mu-\frac{1}{m_2}\check{u}_2^\mu\right]$$

$$\Delta p^{\mu,(2)}_{1,\mathrm{rad}} = \frac{G^3M^4\nu^2}{|b|^3}\left\{\frac{4}{\sqrt{\sigma^2-1}}\frac{b^\mu}{|b|}\left[f_1^{\mathrm{LS}}(\sigma)+f_3^{\mathrm{LS}}(\sigma)\frac{\sigma\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right]\right.$$

$$\left.+\pi\,\check{u}_2^\mu\left[f_1(\sigma)+f_2(\sigma)\log\left(\frac{\sigma+1}{2}\right)+f_3(\sigma)\frac{\sigma\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right]\right\}.$$

$$f_1^{\mathrm{LS}}(\sigma) = -\frac{(2\sigma^2-1)^2(5\sigma^2-8)}{3(\sigma^2-1)^{3/2}},$$

$$f_3^{\mathrm{LS}}(\sigma) = \frac{2(2\sigma^2-1)^2(2\sigma^2-3)}{(\sigma^2-1)^{3/2}},$$

$$f_1(\sigma) = \frac{210\sigma^6-552\sigma^5+339\sigma^4-912\sigma^3+3148\sigma^2-3336\sigma+1151}{48\left(\sigma^2-1\right)^{3/2}},$$

$$f_2(\sigma) = -\frac{35\sigma^4+60\sigma^3-150\sigma^2+76\sigma-5}{8\sqrt{\sigma^2-1}},$$

$$f_3(\sigma) = \frac{\left(2\sigma^2-3\right)\left(35\sigma^4-30\sigma^2+11\right)}{8(\sigma^2-1)^{3/2}}.$$

## radiated momentum

$$\Delta R^\mu = \frac{G^3m_1^2m_2^2}{|b|^3}\frac{u_1^\mu+u_2^\mu}{\sigma+1}\mathcal{E}(\sigma)+\mathcal{O}(G^4)$$

$$\frac{\mathcal{E}(\sigma)}{\pi} = f_1(\sigma)+f_2(\sigma)\log\left(\frac{\sigma+1}{2}\right)+f_3(\sigma)\frac{\sigma\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}$$

see also: DiVecchia et al, Riva-Vernizzi’21,Bjerrum-Bohr-Plante-Vanhove-Damgaard, useful results concerning the **waveform** (using QFT integration methods)…

potential gravitons only

# Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$

Zvi Bern,[1] Julio Parra-Martinez,[2] Radu Roiban,[3] Michael S. Ruf,[4]
Chia-Hsien Shen,[5] Mikhail P. Solon,[1] and Mao Zeng[6]
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three-loop
level
G^4

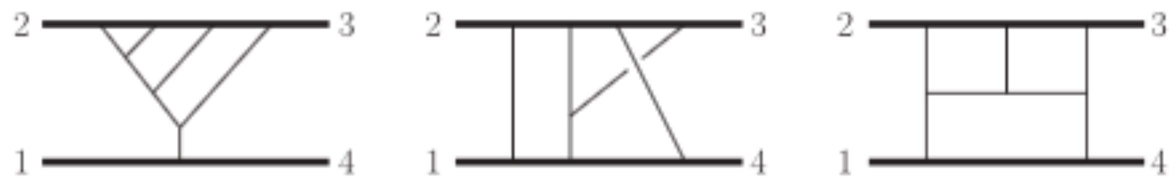



FIG. 2. Sample diagrams at $\mathcal{O}(G^4)$. From left to right: a contribution in the probe limit, a nonplanar diagram that contains iteration terms, and a diagram that contains contributions related to the tail effect.

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{1/3}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^p + \nu\left(\frac{\mathcal{M}_4^t}{\epsilon} + \mathcal{M}_4^f\right)\right] + \int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1},$$

$$\mathcal{M}_4^p = -\frac{35(1-18\sigma^2+33\sigma^4)}{8(\sigma^2-1)}, \quad \mathcal{M}_4^t = h_1 + h_2 \log\left(\frac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}},$$

$$\begin{aligned}\mathcal{M}_4^f = {} & h_4 + h_5 \log\left(\frac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9 \left[\operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\frac{\sigma+1}{2}\right)\right] \\ & + h_{10}\left[\operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\operatorname{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) - \operatorname{Li}_2\left(\frac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right)\right] \\ & + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\operatorname{Li}_2(1-\sigma-\sqrt{\sigma^2-1}) - \operatorname{Li}_2(1-\sigma+\sqrt{\sigma^2-1}) + 5\operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - 5\operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right)\right. \\ & \left. + 2\log\left(\frac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] + h_{12} K^2\left(\frac{\sigma-1}{\sigma+1}\right) + h_{13} K\left(\frac{\sigma-1}{\sigma+1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + h_{14} E^2\left(\frac{\sigma-1}{\sigma+1}\right), \qquad (6)\end{aligned}$$

# Classical scattering perturbation theory enhanced by using QFT integration methods

$$\frac{dx_a^\mu}{d\sigma_a} = g^{\mu\nu}(x_a)p_{a\nu},$$

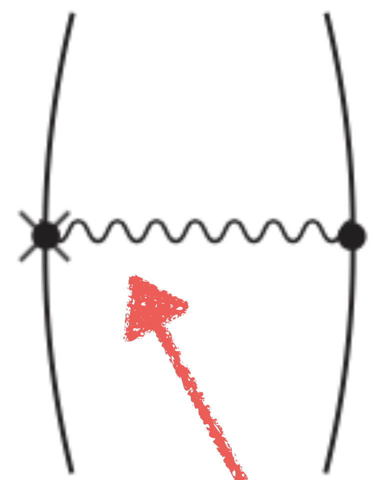
$$\frac{dp_{a\mu}}{d\sigma_a} = -\frac{1}{2}\partial_\mu g^{\alpha\beta}(x_a)p_{a\alpha}p_{a\beta}.$$

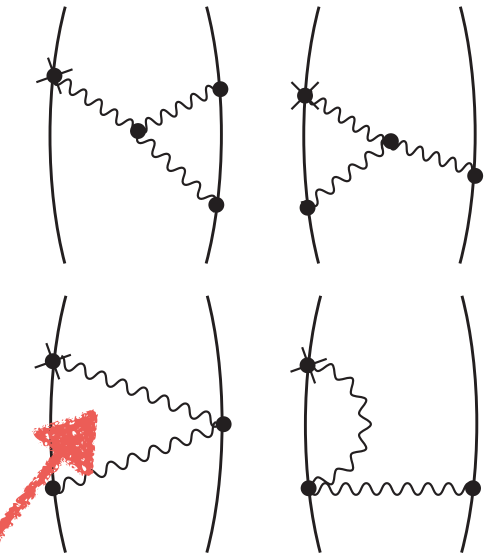
$$R^{\mu\nu} - \tfrac{1}{2}Rg^{\mu\nu} = 8\pi G T^{\mu\nu}$$

$$T^{\mu\nu}(x) = \sum \int d\sigma_a p_a^\mu p_a^\nu \frac{\delta^4(x - x_a(\sigma_a))}{\sqrt{g}}$$

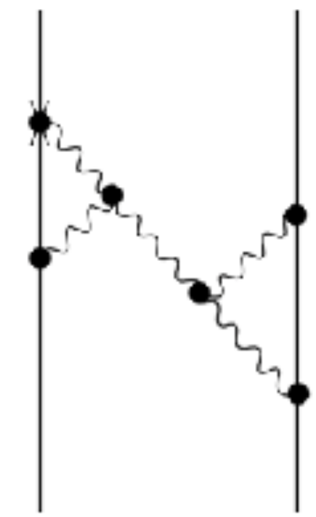
$$\Delta p_{a\mu} = \int_{-\infty}^{+\infty} d\sigma_a \frac{dp_{a\mu}}{d\sigma_a} = -\frac{1}{2}\int_{-\infty}^{+\infty} d\sigma_a \partial_\mu g^{\alpha\beta}(x_a)p_{a\alpha}p_{a\beta}.$$

$$\Delta p_{1\mu} = 2G\int d\sigma_1 d\sigma_2 p_{1\alpha}p_{1\beta} \times \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1) - x_2(\sigma_2))p_{2\alpha'}p_{2\beta'}$$





with retarded propagator



$$\mathcal{P}^{\alpha\beta;\alpha'\beta'}(x-y) = \left(\eta^{\alpha\alpha'}\eta^{\beta\beta'} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\alpha'\beta'}\right)\mathcal{G}(x-y)$$

with retarded propagator

Approach initiated long ago: Rosenblum’78 Westpfahl’79,’85 Portilla’80 Bel et al.’81 limited by the technical difficulty of computing the integrals beyond G^2, ie at G^2=2-loop.

**Recently developed to compete with quantum-scattering approach: Kalin-Porto, Porto et al, Plefka et al, Dlapa-Kalin-Liu-Porto,…**

# Tutti-Frutti method

(**Bini-TD-Geralico** '19,'10'21)

SIXTH POST-NEWTONIAN LOCAL-IN-TIME DYNAMIC

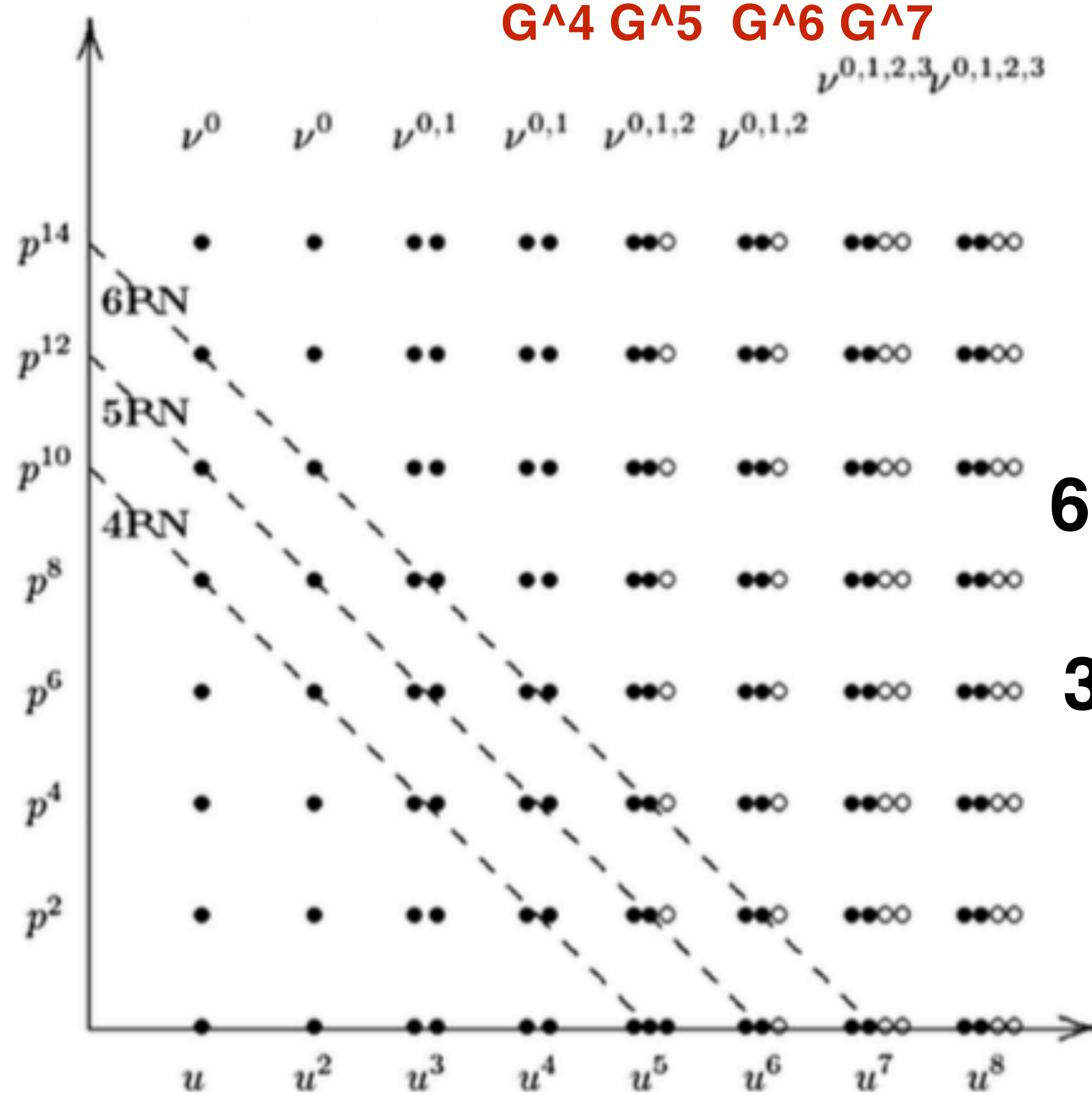
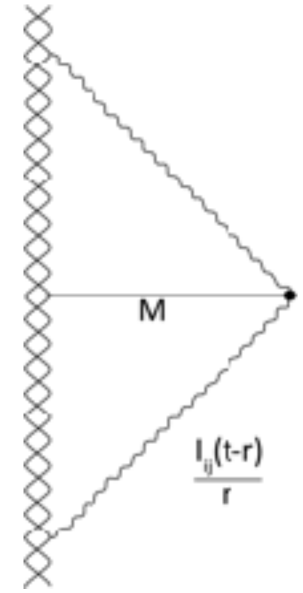


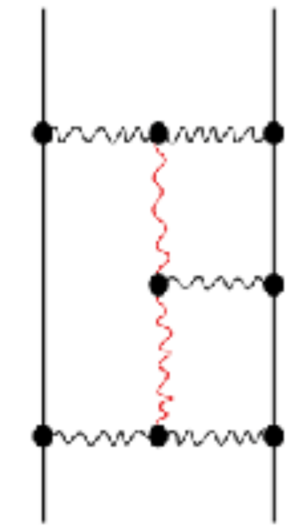

FIG. 1. Schematic representation of the irreducible information contained, at each post-Minkowskian level (keyed by a power of $u = GM/r$), in the local dynamics. Each vertical column of dots describes the post-Newtonian expansion (keyed by powers of $p^2$) of an energy-dependent function parametrizing the scattering angle. The various columns at a given post-Minkowskian level correspond to increasing powers of the symmetric mass-ratio $\nu$. See text for details.

**6PN dynamics complete at 3PM and 4PM**

# Inclusion of conservative nonlocal radiation-graviton effects

Use **Delaunay-averaging expansions in e or p_r**
(TD-Jaranowski-Schaefer'15, Bini-TD-Geralico…)



**Starting at G^4/c^8, dynamics contains a nonlocal action**

$$S^{4+5\mathrm{PN}}_{\mathrm{nonloc}}[x_1(s_1),x_2(s_2)] = \frac{G^2\mathcal{M}}{c^3}\int dt\,\mathrm{PF}_{2r^h_{12}(t)/c} \times \int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{1\mathrm{PN}}(t,t').$$

$$\mathcal{F}^{\mathrm{split}}_{1\mathrm{PN}}(t,t') = \frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t') + \frac{1}{189c^2}I^{(4)}_{abc}(t)I^{(4)}_{abc}(t') + \frac{16}{45c^2}J^{(3)}_{ab}(t)J^{(3)}_{ab}(t')\right).$$

For elliptic motions, the 4PN nonlocal EOB Hamiltonian reads

$$A(u) = 1-2u+2\nu u^3+\left(\frac{94}{3}-\frac{41\pi^2}{32}\right)\nu u^4+\left(\left(\frac{2275\pi^2}{512}-\frac{4237}{60}+\frac{128}{5}\gamma_E+\frac{256}{5}\ln 2\right)\nu+\left(\frac{41\pi^2}{32}-\frac{221}{6}\right)\nu^2+\frac{64}{5}\nu\ln u\right)u^5, \tag{8.1a}$$

$$\bar{D}(u) = 1+6\nu u^2+(52\nu-6\nu^2)u^3+\left(\left(-\frac{533}{45}-\frac{23761\pi^2}{1536}+\frac{1184}{15}\gamma_E-\frac{6496}{15}\ln 2+\frac{2916}{5}\ln 3\right)\nu + \left(\frac{123\pi^2}{16}-260\right)\nu^2+\frac{592}{15}\nu\ln u\right)u^4, \tag{8.1b}$$

$$\hat{Q}(\mathbf{r}',\mathbf{p}') = \left(2(4-3\nu)\nu u^2+\left(\left(-\frac{5308}{15}+\frac{496256}{45}\ln 2-\frac{33048}{5}\ln 3\right)\nu-83\nu^2+10\nu^3\right)u^3\right)(\mathbf{n}'\cdot\mathbf{p}')^4 + \left(\left(-\frac{827}{3}-\frac{2358912}{25}\ln 2+\frac{1399437}{50}\ln 3+\frac{390625}{18}\ln 5\right)\nu-\frac{27}{5}\nu^2+6\nu^3\right)u^2(\mathbf{n}'\cdot\mathbf{p}')^6+\mathcal{O}[\nu u(\mathbf{n}'\cdot\mathbf{p}')^8].$$

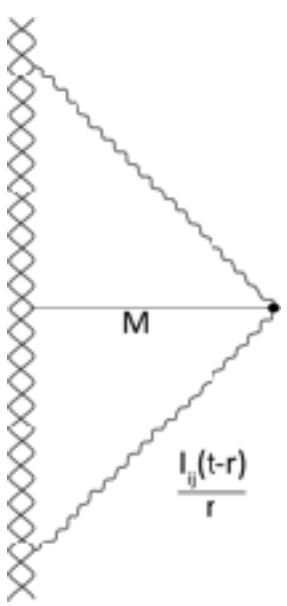
# Conservative Radiative Contributions to the Classical Scattering at 6PN and G^4

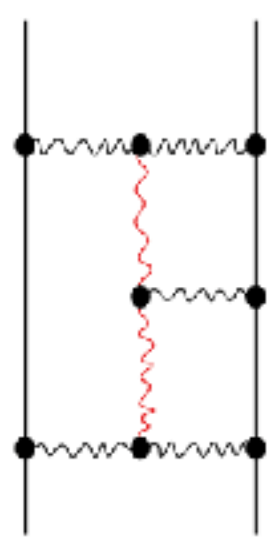
Recent amplitude computation of **potential-graviton** contribution to **conservative** 4PM (G^4) dynamics (Bern et al '21);
additional **radiation-graviton conservative** contrib. (Bini-TD-Geralico'21)

$$\tilde{\chi}_4 - \chi_4^{\rm Schw} = \pi\nu\hat{A}(p_\infty) - \hat{\mathcal{E}}(p_\infty)\ln\left(\frac{p_\infty}{2}\right)$$

$$\hat{A}(p_\infty) = -\frac{15}{4} + \left(\frac{123}{256}\pi^2 - \frac{557}{16}\right)p_\infty^2 + \left(\frac{33601}{16384}\pi^2 - \frac{6113}{96}\right)p_\infty^4 + \left(\frac{93031}{32768}\pi^2 - \frac{615581}{19200}\right)p_\infty^6 + \left(\frac{29201523}{33554432}\pi^2 - \frac{5824797}{627200}\right)p_\infty^8.$$

$$\mathcal{M}_4^{\rm radgrav} = \mathcal{M}_4^{\rm radgrav,finite} + 4\mathcal{M}_4^t\ln\left(\frac{p_\infty}{2}\right) = \mathcal{M}_4^{\rm radgrav,finite} + 16p_\infty\hat{\mathcal{E}}\ln\left(\frac{p_\infty}{2}\right)$$

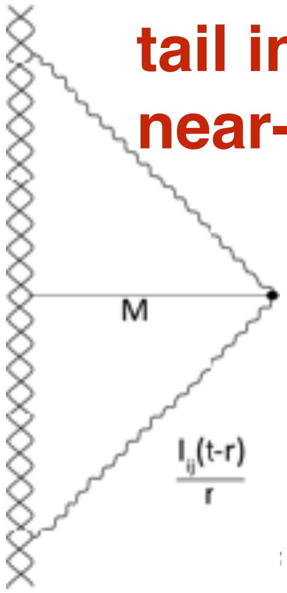


$$\mathcal{M}_4^{\rm radgrav,finite} = \frac{12044}{75}p_\infty^2 + \frac{212077}{3675}p_\infty^4 + \frac{115917979}{793800}p_\infty^6.$$

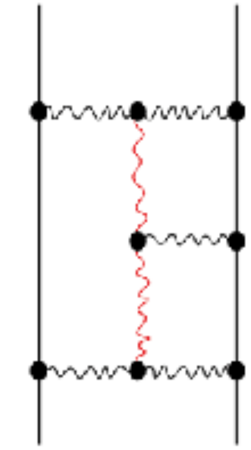
**soft (radiationlike) gravitons**

# Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at $\mathcal{O}(G^4)$




**tail in near-zone**

Zvi Bern,[1] Julio Parra-Martinez,[2] Radu Roiban,[3] Michael S. Ruf,[1]
Chia-Hsien Shen,[4] Mikhail P. Solon,[1] and Mao Zeng[5]



## Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-eccentricity Expansion

Christoph Dlapa,[1] Gregor Kälin,[1] Zhengwen Liu,[1] and Rafael A. Porto[1]

$$G^4 M^7 \nu^2 |\boldsymbol{q}| \pi^2 \left[\mathcal{M}_4^{\rm p} + \nu\left(4\mathcal{M}_4^{\rm t}\log\left(\tfrac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem}\right)\right] + \int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1},$$

$$\begin{aligned}
\mathcal{M}_4^{\rm p} &= -\frac{35\left(1-18\sigma^2+33\sigma^4\right)}{8\left(\sigma^2-1\right)}\,,\\
\mathcal{M}_4^{\rm t} &= r_1 + r_2\log\left(\tfrac{\sigma+1}{2}\right) + r_3\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\,,\\
\mathcal{M}_4^{\pi^2} &= r_4\pi^2 + r_5\,\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_6\,\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_7\,\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right),\\
\mathcal{M}_4^{\rm rem} &= r_8 + r_9\log\left(\tfrac{\sigma+1}{2}\right) + r_{10}\tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11}\log(\sigma) + r_{12}\log^2\left(\tfrac{\sigma+1}{2}\right) + r_{13}\tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\log\left(\tfrac{\sigma+1}{2}\right) + r_{14}\tfrac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1}\\
&\quad + r_{15}\,\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + r_{16}\,\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) + r_{17}\tfrac{1}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right].
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mathcal{M}_4^{\rm radgrav,f} = {} & \frac{12044}{75}p_\infty^2 + \frac{212077}{3675}p_\infty^4 + \frac{115917979}{793800}p_\infty^6\\
& - \frac{9823091209}{76839840}p_\infty^8 + \frac{115240251793703}{1038874636800}p_\infty^{10}\\
& - \frac{411188753665637}{4155498547200}p_\infty^{12} + \cdots,
\end{aligned} \tag{6}$$

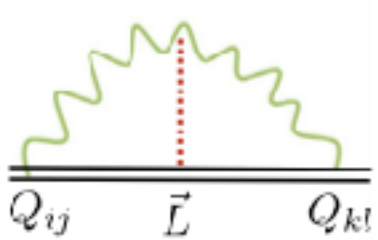
whose first three terms match the sixth PN order result in Eq. (6.20) of Ref. [42].

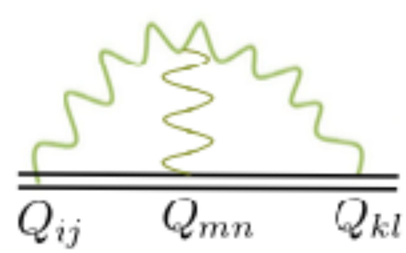
# Puzzles Concerning Radiative Contributions

high-energy limits?
G^3 energy loss too large
Conservative G^4 scattering diverges
cf ACV motivation: BH formation
in HE scattering

low-energy discrepancy at 5PN between

Foffa-Sturani'19,21 Bluemlein et al'21 and Bini-TD-Geralico




$$\begin{aligned}
S_{QQL} &= C_{QQL} G^2 \int dt I^{(4)}_{is} I^{(3)}_{js} \varepsilon_{ijk} L_k \\
S_{QQQ_1} &= C_{QQQ_1} G^2 \int dt I^{(4)}_{is} I^{(4)}_{js} I_{ij} \,, \\
S_{QQQ_2} &= C_{QQQ_2} G^2 \int dt I^{(3)}_{is} I^{(3)}_{js} I^{(2)}_{ij} \,.
\end{aligned}$$

**TF-constraint on 5PN O(nu^2) EFT radiative terms**

$$0 = \frac{2973}{350} - \frac{69}{2} C_{QQL} + \frac{253}{18} C_{QQQ_1} + \frac{85}{9} C_{QQQ_2}$$

# A tale of many Green's functions

$$G_{\rm ret}(x) = \frac{\delta(t - r/c)}{r} \qquad G_{\rm ret} = \mathrm{P}\frac{1}{k^2} + i\pi\,\mathrm{sign}(k^0)\delta(k^2)$$

$$G_{\rm sym}(x) = \frac{\delta(t - r/c) + \delta(t + r/c)}{2r} \qquad G_{\rm sym} = \mathrm{P}\frac{1}{k^2}$$

$$G^{\rm PN}_{\rm sym}(x) = \frac{\delta(t)}{r} + \frac{r}{2c^2}\ddot{\delta}(t) + \cdots \qquad G^{\rm PN}_{\rm sym} = \frac{1}{\mathbf{k^2}} + \frac{\omega^2}{c^2\mathbf{k^4}} + \cdots$$

$$G_{\rm F}(x) = \frac{i}{\pi(t^2 - r^2 + i0)} \qquad G_{\rm F} = \mathrm{P}\frac{1}{k^2} + i\pi\delta(k^2)$$

+ issues of: <in,out>; <in,in>, FWF, Schwinger-Keldysh,…

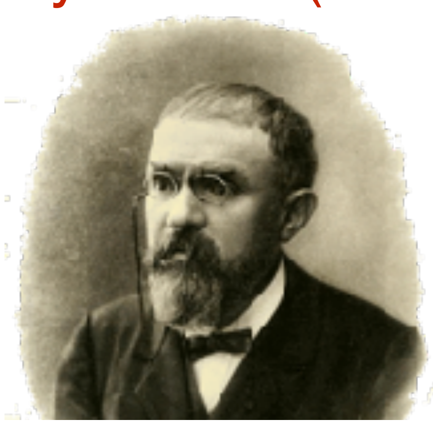
Within the <in,in> approach (which involves causal exchanges) Foffa-Sturani advocate to extract the conservative part by taking the time symmetric part of the result

+ **issue of scattering effects quadratic in radiation-reaction**
+ issue of separating/combining conservative and radiative effects

# Conclusions

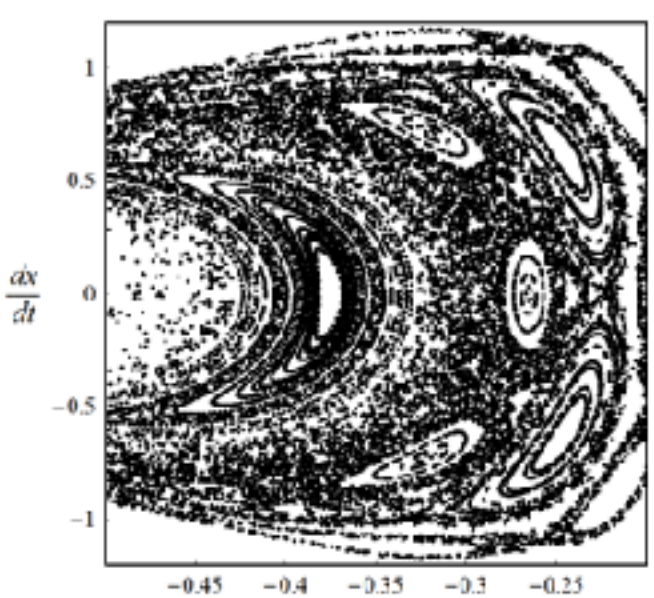
- Analytical approaches to GW signals play a crucial role (in conjunction with Numerical Relativity simulations) for the detection, interpretation and parameter estimation of coalescing binary systems (BBH and BNS). It is important to further improve our analytical knowledge for future GW detectors: second generation ground-based detectors, space detectors, second generation ground-based detectors.

- Quantum (and classical) scattering approaches have given new results of great conceptual interest, and also of potential interests for GW detection. The fruitful dialogue between QFT, EFT, PN, PM, EOB, Tutti-Frutti methods must be vigorously pursued. Discrepancies must be resolved to complete the determination of the 5PN dynamics (of direct utility for LIGO-Virgo). Radiative effects are still puzzling.



## Henri Poincaré

**«Il n’y a pas de problèmes résolus, il y a seulement des problèmes plus ou moins résolus »**




**«There are no (definitely) solved problems, there are only more or less solved problems »**