Quantum Impulse Control

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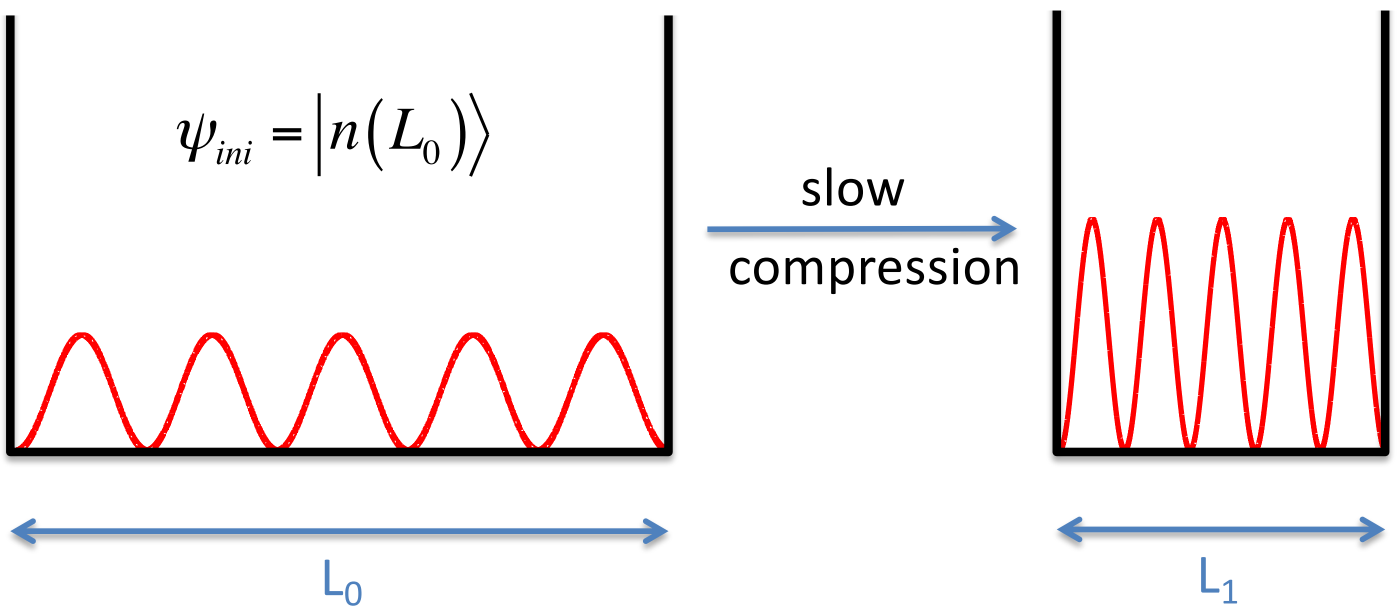
Quantum adiabatic theorem



$$\psi_{fin} = |n(L_1)\rangle$$

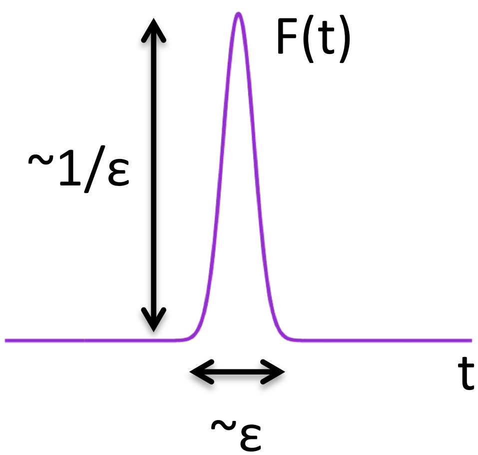
$$\hat{H}(\varepsilon t) \quad , \quad \varepsilon \to 0 \qquad \Delta t \propto \varepsilon^{-1}$$

What about the opposite limit : *fast* driving ? $\hat{H}\left(\frac{t}{\varepsilon}\right) \qquad \Delta t \propto \varepsilon$

… ***impulse control***

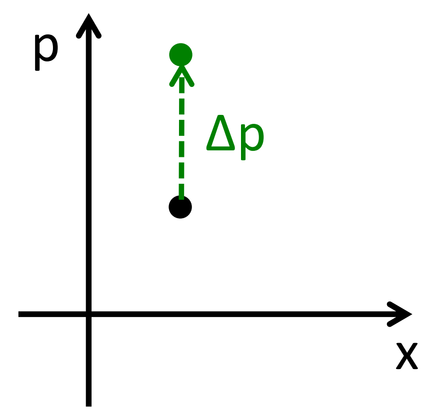
A simple example

<u>Classical</u> : time-dependent force $$-\frac{\varepsilon\tau_0}{2} \leq t \leq +\frac{\varepsilon\tau_0}{2}$$ duration = $\varepsilon\tau_0$



strength ~ 1/ε

$$\boxed{F(t) = \frac{1}{\varepsilon} f\left(\frac{t}{\varepsilon}\right)}$$

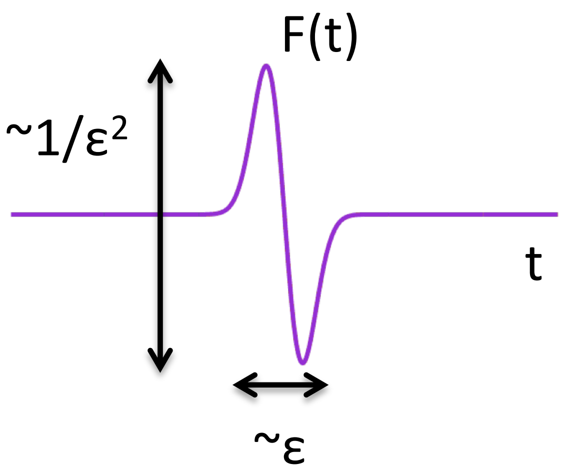


Newton's laws: $x \rightarrow x$

$\varepsilon \rightarrow 0$ $\quad p \rightarrow p + \Delta p$ $\quad \Delta p = \int f(\tau)\, d\tau$

A simple example

<u>Classical</u> : time-dependent force $\quad -\dfrac{\varepsilon\tau_0}{2} \le t \le +\dfrac{\varepsilon\tau_0}{2}$ $\quad$ duration = $\varepsilon\tau_0$

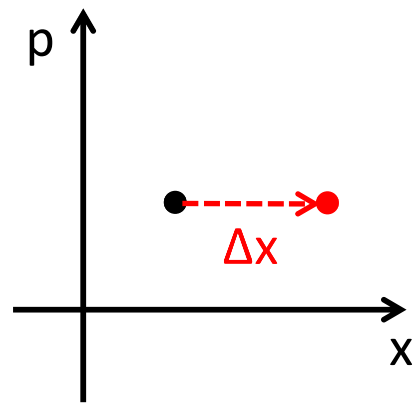


strength ~ $1/\varepsilon^2$

$$\boxed{F(t) = \frac{1}{\varepsilon^2} f\left(\frac{t}{\varepsilon}\right)}$$

$$\int f(\tau)\,d\tau = 0$$

balanced

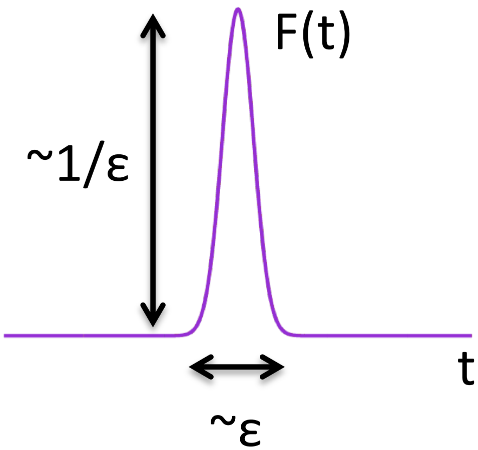


Newton's laws: $\quad x \to x + \Delta x \qquad \Delta x = -\dfrac{1}{m}\displaystyle\int \tau f(\tau)\,d\tau$

$\varepsilon \to 0 \qquad p \to p$

A simple example

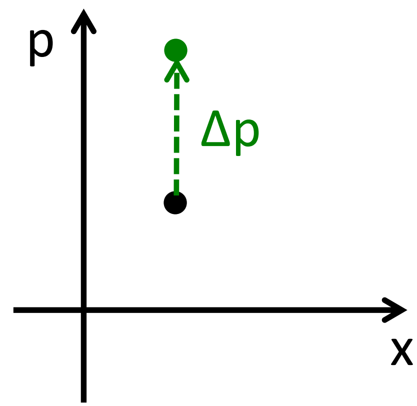
<u>Quantum</u>: time-dependent force $-\frac{\varepsilon\tau_0}{2} \le t \le +\frac{\varepsilon\tau_0}{2}$ duration = $\varepsilon\tau_0$



strength ~ 1/ε

$$\boxed{F(t) = \frac{1}{\varepsilon} f\left(\frac{t}{\varepsilon}\right)}$$

$$U(x,t) = -xF(t)$$



Schrödinger equation:

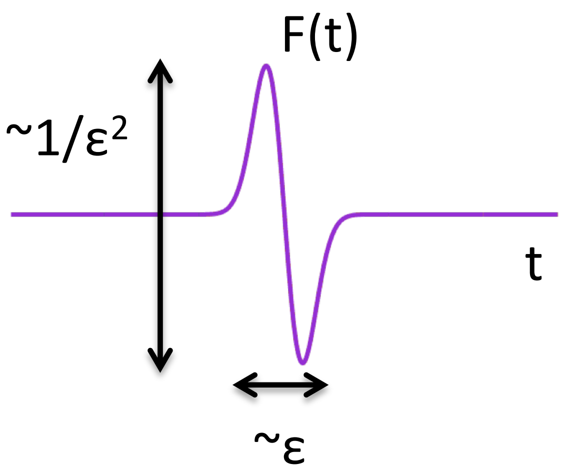
$\varepsilon \to 0$

$$\boxed{\psi_f(x) = e^{i\Delta p\, x/\hbar}\psi_i(x)}$$

$$\Delta p = \int f(\tau)\, d\tau$$

A simple example

<u>Quantum</u>: time-dependent force $-\frac{\varepsilon\tau_0}{2} \le t \le +\frac{\varepsilon\tau_0}{2}$ duration = $\varepsilon\tau_0$



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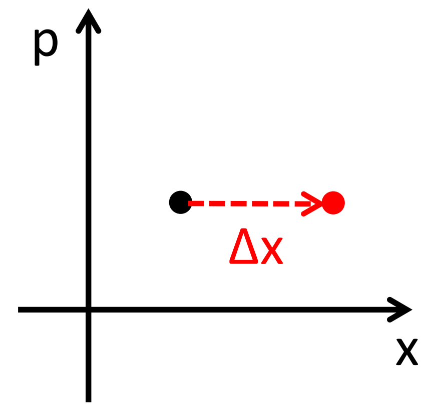


Schrödinger equation:
$\varepsilon \to 0$

$$\boxed{\psi_f(x) = \psi_i(x - \Delta x)}$$

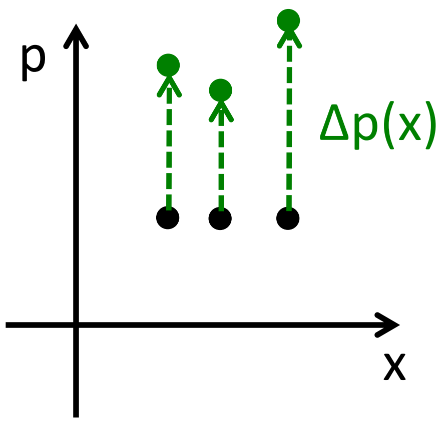
$$\Delta x = -\frac{1}{m}\int \tau f(\tau)\, d\tau$$

The general case - classical

$$H\left(x,p,t\right)=\frac{p^2}{2m}+U_0\left(x\right)+\frac{1}{\varepsilon^k}U_I\left(x,\frac{t}{\varepsilon}\right) \qquad \tau=\frac{t}{\varepsilon} \quad , \quad \tau_{i,f}=\pm\frac{\tau_0}{2}$$

$k=1$

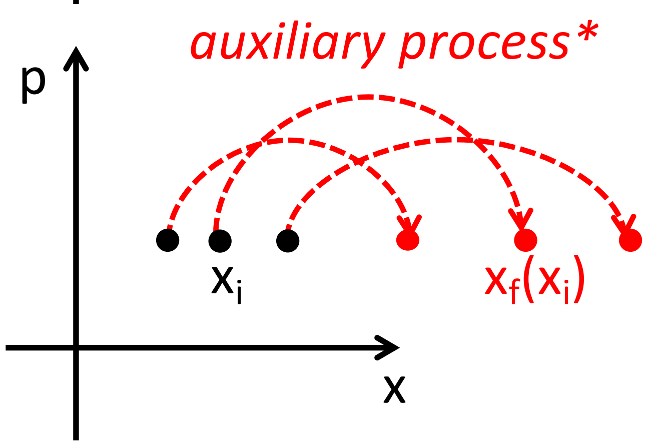
$$x \rightarrow x$$
$$p \rightarrow p+\Delta p\left(x\right)$$



$k=2$

*balanced**

$$x_i \rightarrow x_f\left(x_i\right)$$
$$p_i \rightarrow p_i$$



*to be defined

The general case - quantum

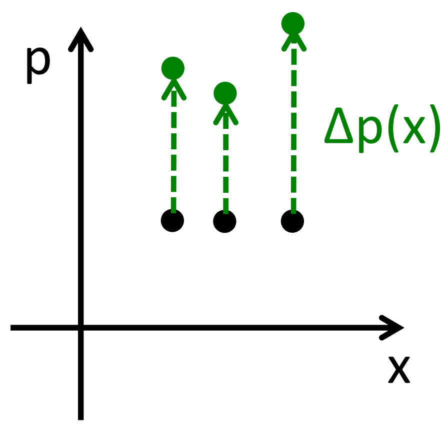
$$\hat{H}(t) = -\frac{\hbar^2}{2m}\nabla^2 + U_0(x) + \frac{1}{\varepsilon^k} U_I\left(x, \frac{t}{\varepsilon}\right) \qquad \tau = \frac{t}{\varepsilon} \quad , \quad \tau_{i,f} = \pm\frac{\tau_0}{2}$$

Given $\psi_i(x) \equiv \psi(x, \tau_i)$, solve for $\psi_f(x) \equiv \psi(x, \tau_f)$ as $\varepsilon \to 0$

$k = 1$

$$\boxed{\psi_f(x) = \psi_i(x) \exp\left[\frac{i}{\hbar}\Delta S(x)\right]}$$

$$\Delta p(x) = \frac{\partial}{\partial x}\Delta S(x)$$



The general case - quantum

$$\hat{H}(t) = -\frac{\hbar^2}{2m}\nabla^2 + U_0(x) + \frac{1}{\varepsilon^k}U_I\left(x, \frac{t}{\varepsilon}\right) \qquad \tau = \frac{t}{\varepsilon} \quad , \quad \tau_{i,f} = \pm\frac{\tau_0}{2}$$

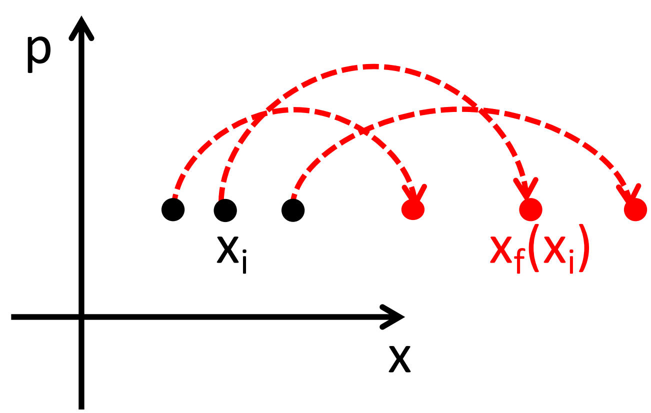
Given $\psi_i(x) \equiv \psi(x, \tau_i)$, solve for $\psi_f(x) \equiv \psi(x, \tau_f)$ as $\varepsilon \to 0$

$k = 2$

balanced

$$\boxed{\psi_f(x_f) = \psi_i(x_i)\left|\frac{\partial x_f}{\partial x_i}\right|^{-1/2}}$$

$$x_f = x_f(x_i)$$



The general case - quantum

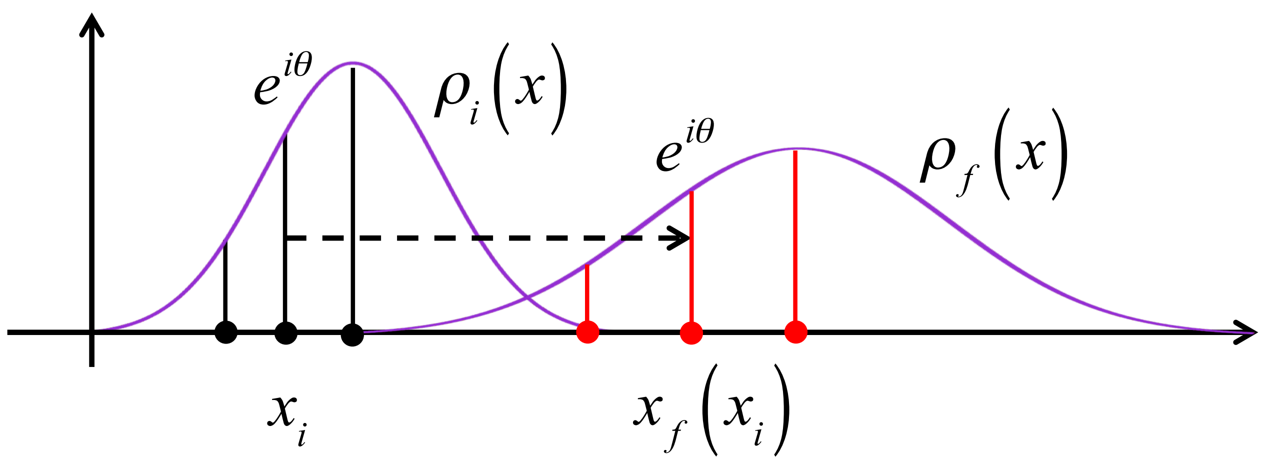
$$\hat{H}(t) = -\frac{\hbar^2}{2m}\nabla^2 + U_0(x) + \frac{1}{\varepsilon^k} U_I\left(x, \frac{t}{\varepsilon}\right) \qquad \tau = \frac{t}{\varepsilon} \quad , \quad \tau_{i,f} = \pm\frac{\tau_0}{2}$$

$k = 2$

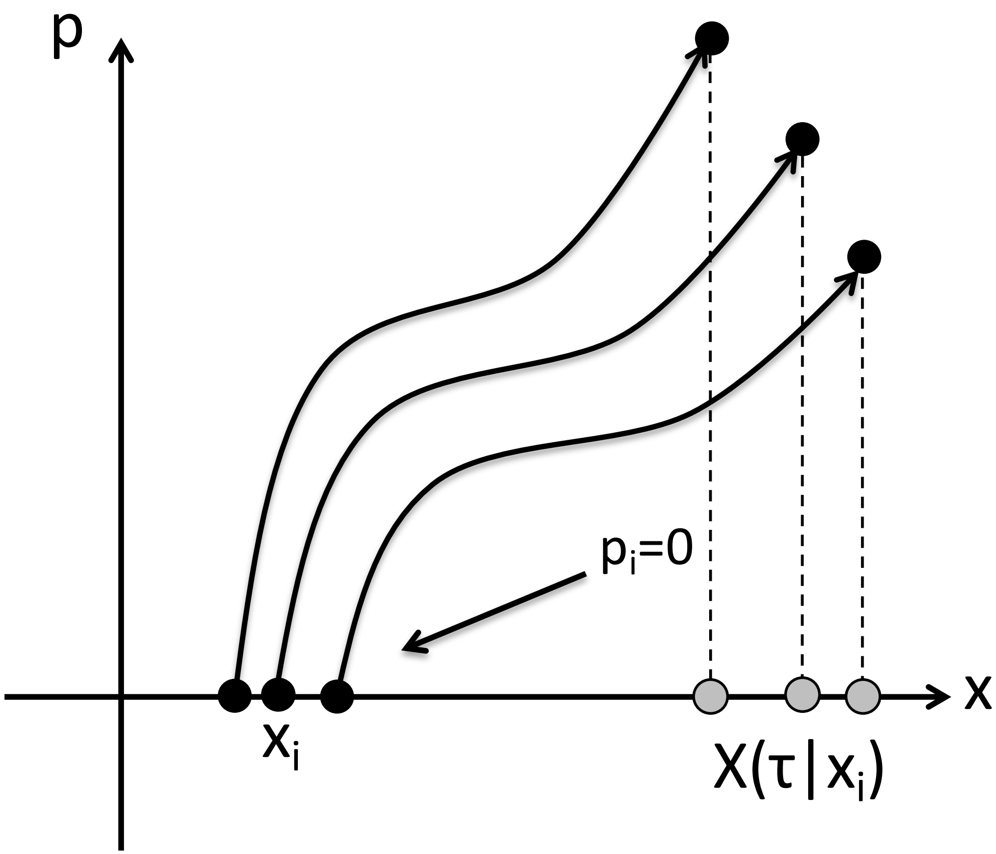
balanced

$$\boxed{\psi_f(x_f) = \psi_i(x_i)\left|\frac{\partial x_f}{\partial x_i}\right|^{-1/2}}$$

$$\psi(x) = \sqrt{\rho(x)}\, e^{i\theta(x)}$$



Auxiliary process



trajectories =
evolution under

$$H = \frac{p^2}{2m} + U_I(x,\tau)$$

$$\tau_i \le \tau \le \tau_f$$

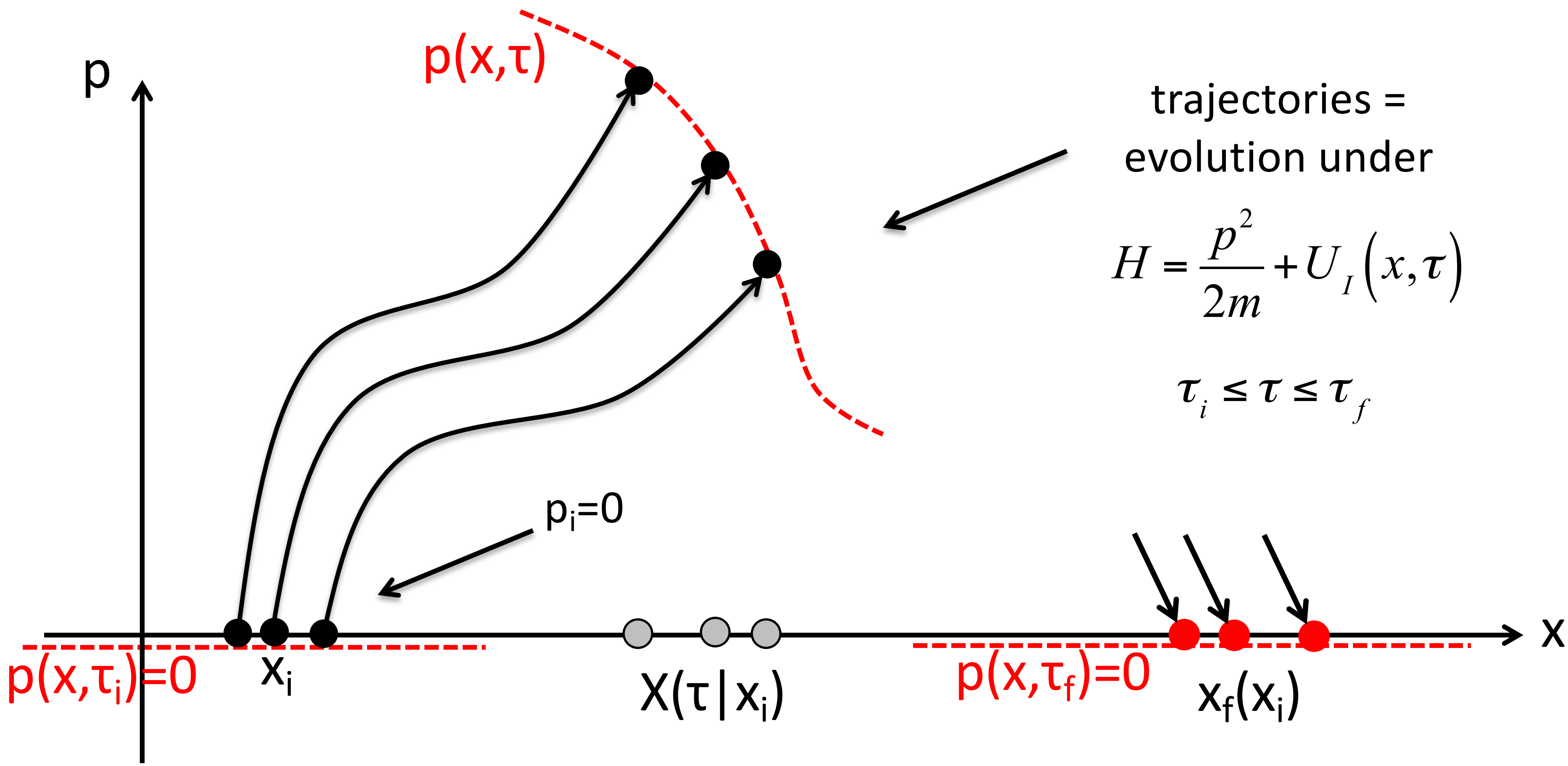
balance:

$$\int_{\tau_i}^{\tau_f} F(\tau|x_i)\, d\tau = 0 \quad \forall\, x_i$$

(Δp=0)

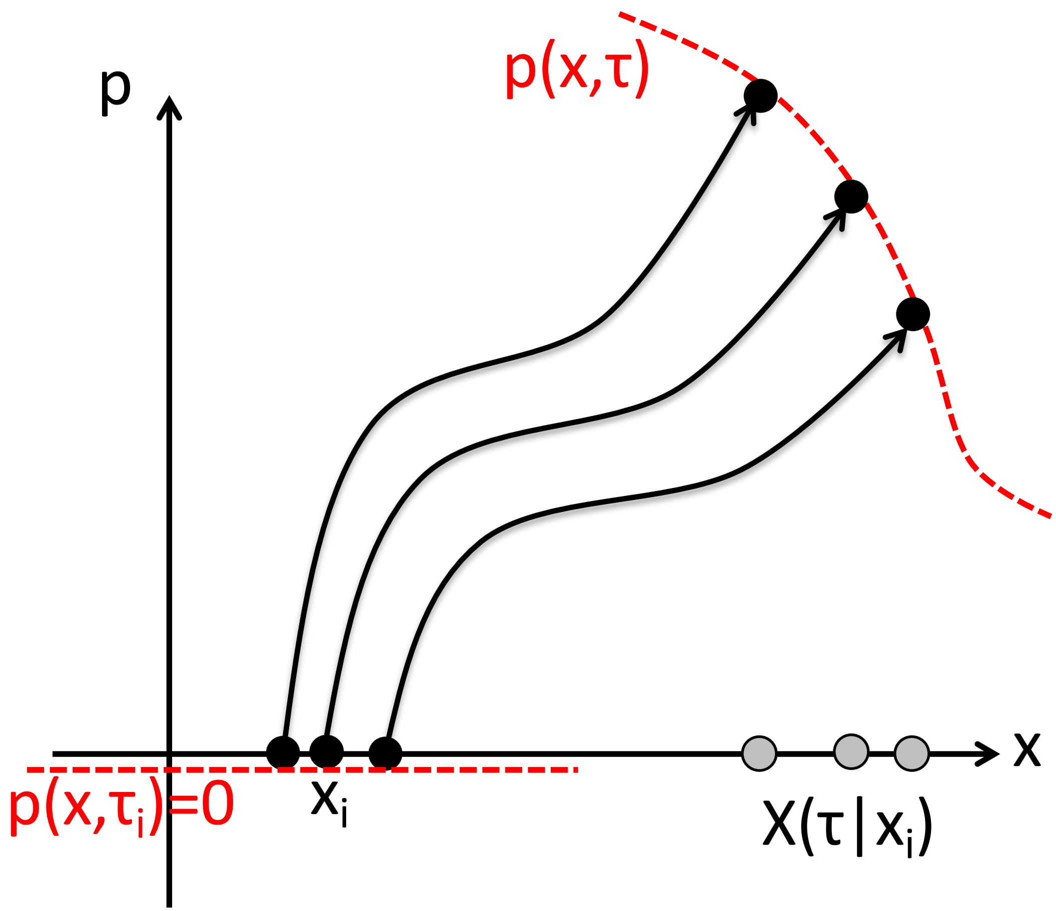
$$F(\tau|x_i) = -\frac{\partial U_I}{\partial x}\left(X(\tau|x_i)\right)$$ = force along trajectory evolving from x_i

Auxiliary process



$x_f(x_i) = X(\tau_f|x_i)$ = final point reached by trajectory evolving from x_i

Generating function S(x,τ)



$$p(x,\tau) = \nabla S(x,\tau)$$

$$S(X,\tau) = \underbrace{S(x_i,\tau_i)}_{0} + \int_{\tau_i}^{\tau} L\,d\tau'$$

$$\frac{\partial S}{\partial \tau} + \frac{(\nabla S)^2}{2m} + U_I(x,\tau) = 0$$

Hamilton-Jacobi equation

$$v(x,\tau) \equiv \frac{1}{m}\nabla S(x,\tau) \qquad \frac{dX}{d\tau} = v(X,\tau)$$

Derivation – sketch

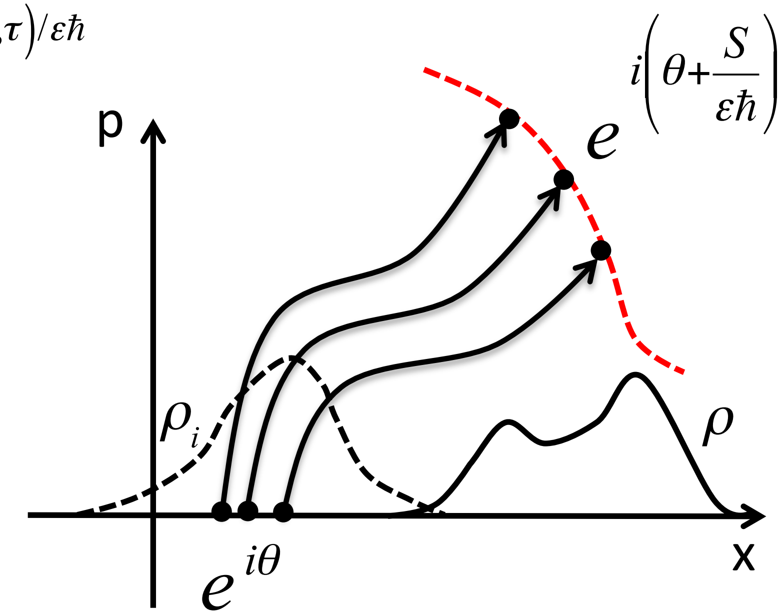
$$\hat{H}(t) = -\frac{\hbar^2}{2m}\nabla^2 + U_0(x) + \frac{1}{\varepsilon^2}U_I\left(x, \frac{t}{\varepsilon}\right) \qquad \tau = \frac{t}{\varepsilon} \quad , \quad \tau_i \le \tau \le \tau_f$$

- *Ansatz*: $\psi(x,t) = \sqrt{\rho(x,\tau)} \cdot e^{i\theta(x,\tau)} \cdot e^{iS(x,\tau)/\varepsilon\hbar}$

- Schrödinger eqn: $\varepsilon \to 0$

$$\boxed{\begin{aligned} \dot{\rho} + \nabla\cdot(v\rho) &= 0 \\ \dot{\theta} + v\cdot\nabla\theta &= 0 \end{aligned}}$$

$$v(x,\tau) \equiv \frac{\nabla S}{m}$$



Derivation – sketch

$$\psi(x,t) = \sqrt{\rho(x,\tau)} \cdot e^{i\theta(x,\tau)} \cdot e^{iS(x,\tau)/\varepsilon\hbar}$$

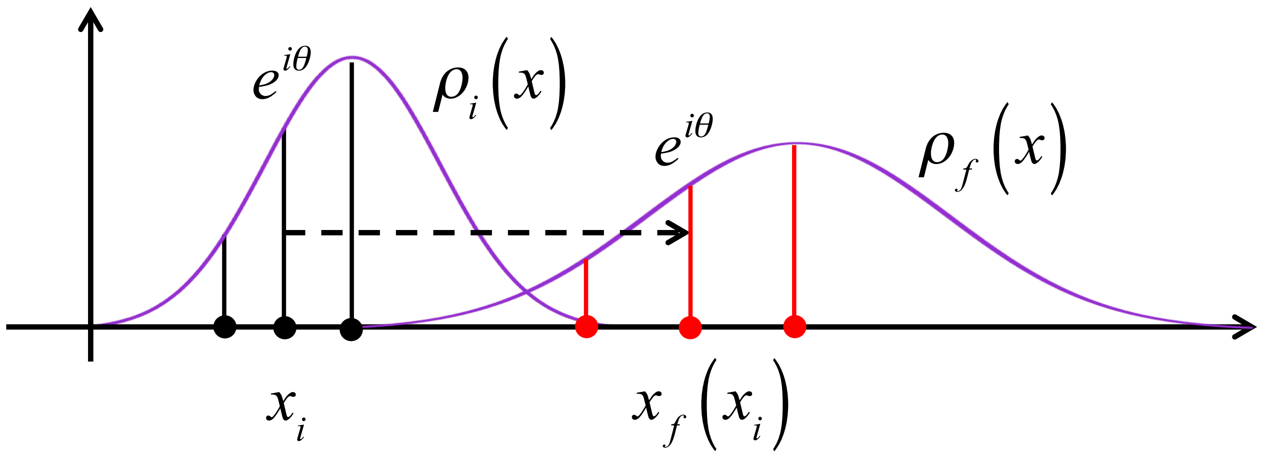
balance (p_f=0) $\longrightarrow$ $S(x,\tau_f) = S_f = const.$ … ignore!

$$\dot{\rho} + \nabla \cdot (v\rho) = 0 \qquad \rho_f(x_f) = \rho_i(x_i)\left|\frac{\partial x_f}{\partial x_i}\right|^{-1}$$

$$\dot{\theta} + v \cdot \nabla\theta = 0 \qquad \theta_f(x_f) = \theta_i(x_i)$$

$$\boxed{\psi_f(x_f) = \psi_i(x_i)\left|\frac{\partial x_f}{\partial x_i}\right|^{-1/2}}$$

Why it works – intuition



$$\psi_f(x_f) = \psi_i(x_i)\left|\frac{\partial x_f}{\partial x_i}\right|^{-1/2}$$

effective propagator: $$K(x|x_i) = \delta(x - x_f)\left|\frac{\partial x_f}{\partial x_i}\right|^{1/2}$$

only a single path contributes ??

during the impulse:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U_0\psi + \frac{1}{\varepsilon^2}U_I\psi$$

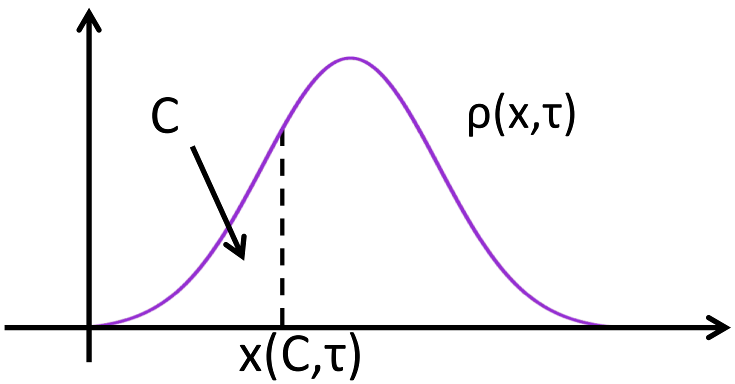
$$i(\varepsilon\hbar)\frac{\partial\psi}{\partial\tau} = -\frac{(\varepsilon\hbar)^2}{2m}\nabla^2\psi \cancel{+ \varepsilon^2 U_0\psi} + U_I\psi \qquad \tau = \frac{t}{\varepsilon}$$

The limit ε -> 0 is effectively the semiclassical limit!

Controlling wavefunctions with impulses

$$\psi_i(x) \rightarrow \psi_f(x)$$

given desired



$$\rho_{i,f}(x) = \left|\psi_{i,f}(x)\right|^2$$

construct ρ(x,τ) to interpolate from $\rho_i(x)$ to $\rho_f(x)$

$$C(x,\tau) \equiv \int_{-\infty}^{x} dx' \rho(x',\tau)$$

cumulative probability distribution

$$v(x,\tau) = \left.\frac{dx}{d\tau}\right|_C = -\frac{\partial C / \partial \tau}{\partial C / \partial x}$$

$$a(x,\tau) \equiv v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial \tau}$$

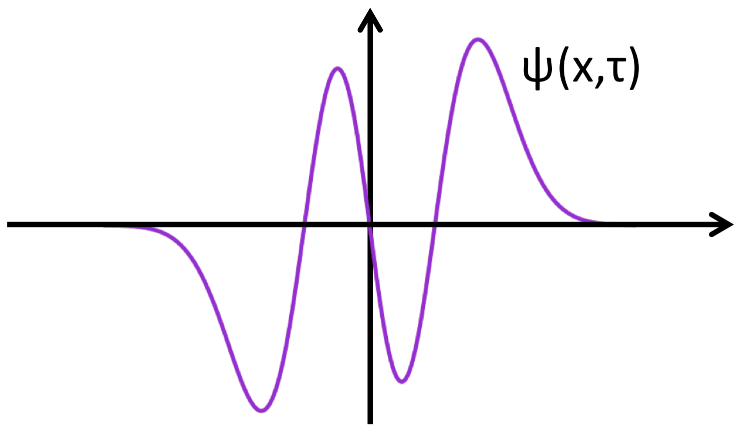
$$\boxed{ma(x,\tau) = -\frac{\partial U_I}{\partial x}(x,\tau)}$$

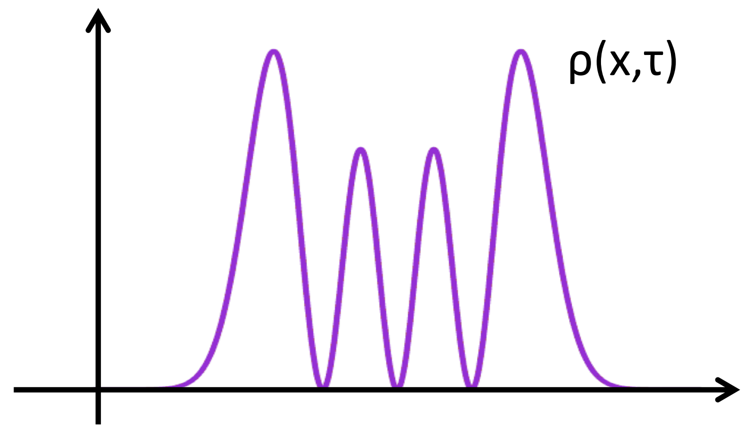
- 2nd order impulse: $\rho_i(x) \rightarrow \rho_f(x)$
- 1st order impulse: $\theta_i(x) \rightarrow \theta_f(x)$

A fly in the ointment

$$v\left(x,\tau\right) = -\frac{\partial C / \partial \tau}{\partial C / \partial x} = -\frac{\partial C / \partial \tau}{\rho}$$

might diverge where $\rho(x,\tau)=0$!





$\longrightarrow$ “no flux” criterion:

the probability between adjacent nodes must remain constant with τ

Summary and perspectives

$$\hat{H}(t) = -\frac{\hbar^2}{2m}\nabla^2 + U_0(x) + \frac{1}{\varepsilon^k} U_I\left(x, \frac{t}{\varepsilon}\right)$$

k = 1, 2

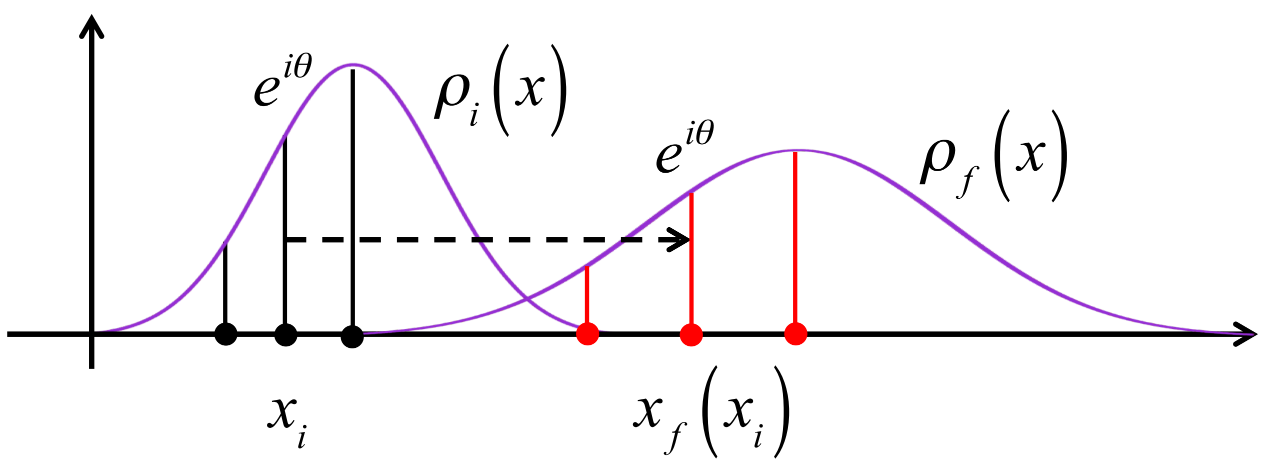
first-order impulse (k=1)

$$\boxed{\psi_f(x) = \psi_i(x)\exp\left[\frac{i}{\hbar}\Delta S(x)\right]}$$

second-order impulse (k=2)

$$\boxed{\psi_f(x_f) = \psi_i(x_i)\left|\frac{\partial x_f}{\partial x_i}\right|^{-1/2}}$$

$$x_f = x_f(x_i)$$



+ no-flux criterion

Summary and perspectives

$$\hat{H}(t) = -\frac{\hbar^2}{2m}\nabla^2 + U_0(x) + \frac{1}{\varepsilon^k} U_I\left(x, \frac{t}{\varepsilon}\right) \qquad k = 1, 2$$

- extend to multiple degrees of freedom (~ done)

- extend to many-particle systems
  mean field approach? e.g. Gross-Pitaevskii eqn.

- experimental realizations
  clarify difference between first & second order impulses

- work around the no-flux criterion (those pesky nodes)