Symmetry Resolution at High Energy

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Twenty-eighth Arnold Sommerfeld Lectures

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When CFT has global symmetry,
it is interesting to understand the **decomposition of its Hilbert space into irreducible representations** of the symmetry.

I would like to present **universal results** about the decomposition, applicable to any unitary CFT.

I will discuss the **high energy asymptotic behaviors** of the density of states and the three-point functions of conformal field theory and their **decompositions with respect to symmetries**.

Based on:

- with Harlow [2109.03838]
- with Kang, Lee [2206.14814]
- with Benjamin, Lee, Simmons-Duffin [2306.08031]
- with Pal, Sun, Zhang [in progress]

The dimensional analysis shows that the canonical partition function of any unitary quantum field theory in $\boldsymbol{d}$ **spacetime dimensions** behaves as,

$$\mathrm{Tr}\left[\, e^{-\beta H} \,\right] \approx \exp(\, f\, T^{d-1} + \cdots) \qquad (\, \beta = 1/T \to 0)\,.$$

For $d = 2$, the coefficient f is $\frac{\pi^2}{3} c$, the Cardy formula.

The purpose of this talk is to **refine this formula with respect to symmetry** and answer the question:

How is the high energy Hilbert space decomposed into irreducible representations of global and spacetime symmetries?

We also derive a behavior of **correlation functions** of these high energy states.

The dimensional analysis shows that the canonical partition function of any unitary quantum field theory in $\boldsymbol{d}$ **spacetime dimensions** behaves as,

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The purpose of this talk is to **refine this formula with respect to symmetry** and answer the question:

How is the high energy Hilbert space decomposed into irreducible representations of global and spacetime symmetries?

We also derive a behavior of **correlation functions** of these high energy states.

Asymptotic Density of States in High Energy **resolved with respect to Global Symmetry**

When the global symmetry G is a **finite group**, we will show

$$\frac{\mathrm{Tr}\left[\, U(g)\, e^{-\beta H}\,\right]}{\mathrm{Tr}[\, e^{-\beta H}\,]} \approx \delta(g,1),$$

in the limit of $\beta = 1/T \ \to 0$.

This means that the density of high energy states $\rho(\Delta, R)$ transforming in the irreducible representation R is,

$$\rho(\Delta, R) \approx \frac{(\dim R)^2}{|G|}\rho(\Delta), \quad \Delta \gg 1 \ .$$

Harlow + H.O.: 2111.04725

Asymptotic Density of States in High Energy **resolved with respect to Global Symmetry**

When the internal symmetry G is a **compact Lie group**, we show

$$\frac{\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]}{\mathrm{Tr}[\, e^{-\beta H}\,]} \approx \left(\frac{4\pi}{b\, T^{d-1}}\right)^{\dim G/2} \sum_R \dim R \cdot \chi_R(g) \exp\left(-\frac{c_2(R)}{b\, T^{d-1}}\right)$$

for $\beta = 1/T \;\to 0$.

This means that the density of high energy states $\rho(\Delta, R)$ transforming in the irreducible representation R is,

$$\rho(\Delta, R) \approx (\dim R)^2 \left(\frac{4\pi}{b'\Delta^{(d-1)/d}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{b'\Delta^{(d-1)/d}}\right] \rho(\Delta) \;.$$

Kang, Lee + H.O.: 2111.04725

Asymptotic Density of States in High Energy **resolved with respect to Global Symmetry**

The density of high energy states $\rho(\Delta, R)$ transforming in the irreducible representation R is,

- $\rho(\Delta, R) \approx \frac{(\dim R)^2}{|G|} \rho(\Delta)$, when G is a **finite group**.

Harlow + H.O.: 2111.04725

- $$\rho(\Delta, R) \approx (\dim R)^2 \left(\frac{4\pi}{b' \Delta^{(d-1)/d}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{b' \Delta^{(d-1)/d}}\right] \rho(\Delta) ,$$

when G is a **compact Lie group**.

Kang, Lee + H.O.: 2206.14814

Asymptotic Density of States in High Energy **resolved with respect to Global Symmetry**

$$\rho(\Delta, R) \approx \frac{(\dim R)^2}{|G|} \rho(\Delta)\,,$$ when G is a **finite group**.

Harlow + H.O.: 2111.04725

$$\rho(\Delta, R) \approx (\dim R)^2 \left(\frac{4\pi}{b'\Delta^{(d-1)/d}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{b'\Delta^{(d-1)/d}}\right] \rho(\Delta)\,,$$

when G is a **compact Lie group**.

Kang, Lee + H.O.: 2206.14814

By the AdS/CFT correspondence, these formulae also teach us about charged black holes. For example, **if the theory contains fermions, a half of large black hole microstates are fermionic.**

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

$$\mathrm{Tr}\left[\, e^{-\beta(H+i\vec{\Omega}\cdot\vec{L})} \,\right] = \int_0^\infty \sum_{\ell=0}^{\infty} \rho_d(\Delta,\ell)\, e^{-\beta(\Delta+\varepsilon_d)+i\beta\vec{\Omega}\cdot\vec{\ell}}\, d\Delta,$$

where H and $\vec{L}$ are the Hamiltonian and the angular momentum on S^{d-1},
$\rho_d(\Delta,\ell)$ is the density of local operators with scaling dimension Δ and spin $\vec{\ell}$,

and ε_d is the Casimir energy on S^{d-1} given by,

$$\varepsilon_d = \frac{(d-1)!!}{(-2)^{d/2}}\, a_d \quad \text{for } d\text{: even,}$$

and $\left\langle T^\mu_\mu \right\rangle = \frac{a_d}{(-4\pi)^{d/2}} E_d + \cdots$ (Euler density: E_d)

$$\varepsilon_d = 0 \quad \text{for } d\text{: odd.}$$

For example, $\varepsilon_{d=2} = -\frac{c}{12}$

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

For $\Delta, \ell \gg 1$ and $|\Delta - \ell| \sim \sqrt{f\Delta}$,

$$\rho_d(\Delta,\ell) \approx \exp\left[\frac{d}{d-1}(\Delta+\varepsilon_d)^{\frac{d-1}{d}}\left(\frac{f(d-1)\mathrm{vol}\, S^{d-1}}{2}\left(1+\sqrt{1+\frac{(d-3)(d-1)\,\ell^2}{(\Delta+\varepsilon_d)^2}}\right)\right)^{1/d} \times\left(\frac{d-2}{d-3}-\frac{1}{d-3}\sqrt{1+\frac{(d-3)(d-1)\ell^2}{(\Delta+\varepsilon_d)^2}}\right)^{1-2/d}\right].$$

Shaghoulian: 1512.06855
Benjamin, Lee, Simmons-Duffin + H.O.: 2306.08031

When $d=2$, this becomes,

$$\rho_{d=2}(\Delta,\ell) \approx \exp\left[\sqrt{\frac{2c}{3}}\pi\left(\sqrt{\frac{\Delta+\ell-c/12}{2}}+\sqrt{\frac{\Delta-\ell-c/12}{2}}\right)\right],$$

reproducing the Cardy formula.

Symmetry-Resolved Density of States

- $\rho(\Delta, R) \approx \frac{(\dim R)^2}{|G|} \rho(\Delta)$, when G is a **finite group**.

Harlow + H.O.: 2111.04725

- $\rho(\Delta, R) \approx (\dim R)^2 \left(\frac{4\pi}{b' \Delta^{(d-1)/d}} \right)^{\dim G/2} \exp\left[- \frac{c_2(R)}{b' \Delta^{(d-1)/d}} \right] \rho(\Delta)$,

when G is a **compact Lie group**. Kang, Lee + H.O.: 2111.04725

- $$\rho_d(\Delta, \ell) \approx \exp\left[\frac{d}{d-1} (\Delta + \varepsilon_d)^{\frac{d-1}{d}} \left(\frac{f(d-1) \mathrm{vol}\, S^{d-1}}{2} \left(1 + \sqrt{1 + \frac{(d-3)(d-1)\,\ell^2}{(\Delta + \varepsilon_d)^2}} \right) \right)^{1/d} \times \left(\frac{d-2}{d-3} - \frac{1}{d-3} \sqrt{1 + \frac{(d-3)(d-1)\ell^2}{(\Delta + \varepsilon_d)^2}} \right)^{1-2/d} \right].$$

for **spacetime spin** ℓ. Shaghoulian: 1512.06855
Benjamin, Lee, Simmons-Duffin + H.O.: 2306.08031

Three-Point Function at Large Conformal Dimensions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \sum c_{123}^{s}\; V^{s}(x_1,x_2,x_3),$$

$d = 2$

$$c_{123}\sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{13/2} f\, e^{\frac{9}{4}\left(\pi^2 f^2 \Delta\right)^{1/3}}}{3^{9/2}\pi^{1/2}\Delta^{5/2}}$$

$d = 3$

$$c_{123}^{s}\sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{49/8} f^{9/8} e^{\frac{3}{2}(2\pi f \Delta)^{1/2}}}{3^{19/4}\pi^{1/4}\Delta^{31/8}}$$

$$\times \prod_{i=1}^{3}(2\ell_i + 1)\left(\frac{2\ell_i}{\ell_i + q_I}\right)$$

$s = [q_1, q_2, q_3]$: conformal block parameters

Benjamin, Lee, Simmons-Duffin + H.O.: 2306.08031

The single parameter f controls all of these.

Density of states for any d

$$\rho_d(\Delta,\ell) \approx \exp\left[\frac{d}{d-1}(\Delta+\varepsilon_d)^{\frac{d-1}{d}}\left(\frac{f(d-1)\mathrm{vol}\,S^{d-1}}{2}\left(1+\sqrt{1+\frac{(d-3)(d-1)\,\ell^2}{(\Delta+\varepsilon_d)^2}}\right)\right)^{1/d} \times\left(\frac{d-2}{d-3}-\frac{1}{d-3}\sqrt{1+\frac{(d-3)(d-1)\ell^2}{(\Delta+\varepsilon_d)^2}}\right)^{1-2/d}\right].$$

Three-point function at $d=2$

$$c_{123}\sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta}\frac{2^{13/2}f\,e^{\frac{9}{4}(\pi^2 f^2\Delta)^{1/3}}}{3^{9/2}\pi^{1/2}\Delta^{5/2}}$$

Three-point function at $d=3$

$$c^S_{123}\sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta}\frac{2^{49/8}f^{9/8}e^{\frac{3}{2}(2\pi f\Delta)^{1/2}}}{3^{19/4}\pi^{1/4}\Delta^{31/8}}\prod_{i=1}^{3}(2\ell_i+1)\left(\frac{2\ell_i}{\ell_i+q_I}\right)$$

Benjamin, Lee, Simmons-Duffin + H.O., 2306.08031

Proof Method

- **Generalized Noether Theorem**
- **Thermal Effective Field Theory**

ANNALS OF PHYSICS **170**, 1–17 (1986)

On Noether's Theorem in Quantum Field Theory

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Received April 29, 1985

Extending the construction of local generators of symmetries in (S. Doplicher, *Commun. Math. Phys.* **85** (1982), 73; S. Doplicher and R. Longo, *Commun. Math. Phys.* **88** (1983), 399) to space-time and supersymmetries, we establish a weak form of Noether's theorem in quantum field theory. We also comment on the physical significance of the "split property," underlying our analysis, and discuss some local aspects of superselection rules following from our results.

Let us now turn to our main objective: given any global (internal or space-time) symmetry transformation $U(g)$ and any bounded region $\mathcal{O}$, we want to exhibit local unitary operators[2] $U_\Lambda(g)$ which induce the same action on $\mathfrak{F}(\mathcal{O})$ as $U(h)$, i.e.,

$$U_\Lambda(g)\,FU_\Lambda(g)^{-1} = U(g)\,FU(g)^{-1} \qquad \text{for} \quad F \in \mathfrak{F}(\mathcal{O}). \tag{1.4}$$

~~Generalized~~ Noether Theorem:

If the symmetry G is a compact Lie group, for every element $g \in G$, there is a one-form operator J on the Hilbert space which is conserved, $d^{*}J = 0$.

For any region R of a Cauchy surface, we can define a unitary operator $U(g,R)$ to implement the symmetry on the Hilbert space by,

$$U(g,R) = \exp\left[\int_R {}^{*}J\right].$$

If R is a union of disjoint subregions, $R = \cup_i R_i$

$$U(g,R) = \otimes_i U(g,R_i)$$

Generalized Noether Theorem:

For a region R of a Cauchy surface, we can define a unitary operator $U(g, R)$ for every element g of the symmetry group G.

When R is a union of disjoint subregions, $R = \cup_i R_i$,
this unitary operator can be expressed as a product of operators associated to each region:

$$U(g, R) = \otimes_i U(g, R_i) \quad .$$

- It is obvious for continuous symmetry with a Noether current.
- The theorem also holds for discrete symmetry.

Generalized Noether Theorem:

For a region R of a Cauchy surface, we can define a unitary operator $U(g, R)$ for every element g of the symmetry group G.

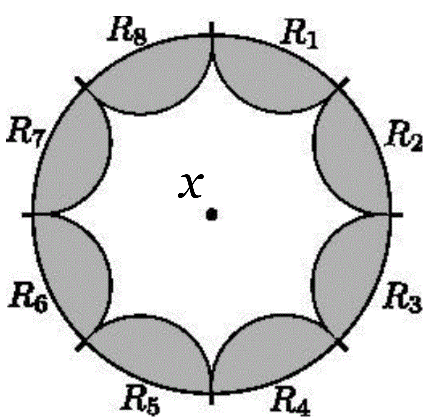
When R is a union of disjoint subregions, $R = \cup_i R_i$,
this unitary operator can be expressed as a product of operators associated to each region:

$$U(g, R) = \otimes_i U(g, R_i) \quad .$$

In my colloquium on Wednesday,
I discussed its application to quantum gravity.

No Global Symmetry in AdS Quamtum Gravity

If a gravitational theory has global symmetry,
there must be a bulk local operator that transforms faithfully into another local operator.



Symmetry generator,

$$U(g,R) = \otimes_i \, U(g,R_i)$$

commute with the local operator at x in the bulk.

Harlow + H.O.:1810.05337
1810.05338

Contradiction

Thermal Effective Field Theory

Thermal Effective Field Theory

Consider a background gauge field coupled to G on $S^1_\beta \times \Sigma_{d-1}$.
At high temperature, we can write down a low energy effective action by dimensionally reducing on S^1_β,

$$S_{\text{eff}} = \int_{\Sigma_{d-1}} \sqrt{G} dx^{d-1} \left(T^{d-1} V(g) + \cdots \right) \quad ,$$

where $g \in G$ is the holonomy around S^1_β.
The kinetic terms such as $(\partial\phi)^2$ and F^2 are suppressed by $1/T^2$.

By diffeomorphism invariance, the potential $V(g)$ at high temperature is independent of the geometry of Σ_{d-1}.

Thermal Effective Field Theory

$$S_{\text{eff}} = \int_{\Sigma_{d-1}} \sqrt{G} dx^{d-1} \left(T^{d-1} V(g) + \cdots \right)$$

By setting g to be constant,

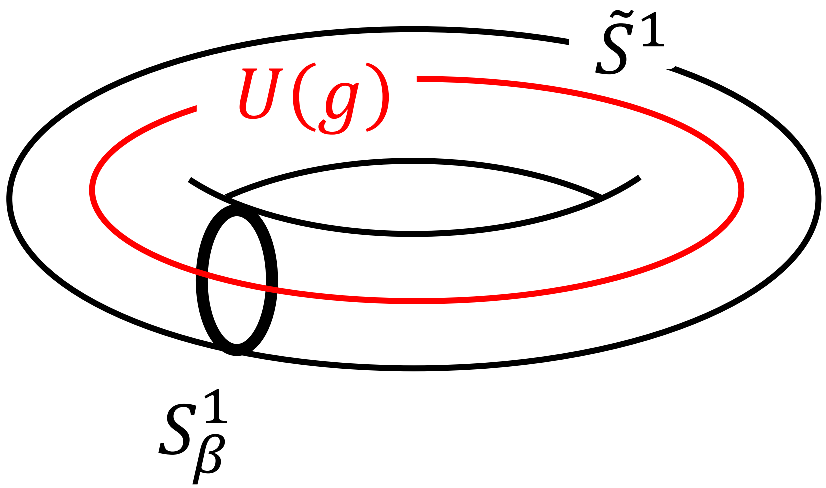
$$\text{Tr}\left[\, U(g)\, e^{-\beta H} \,\right] \approx\; e^{-T^{d-1}\, V(g)\, \text{vol}(\Sigma_{d-1})} \times \text{Tr}\left[\, e^{-\beta H} \,\right]$$

By choosing $\Sigma_{d-1} = \tilde{S}^1 \times \Sigma_{d-2}$, exchanging S^1_β and $\tilde{S}^1$, and rescaling the whole spacetime $S^1_\beta \times \tilde{S}^1 \times \Sigma_{d-2}$ by T,
we can identify:

$V(g)$ = **tension of the domain wall**
implementing the twist by g.

Thermal Effective Field Theory

$$\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right] \approx \; e^{-T^{d-1}\, V(g)\, \mathrm{vol}(\Sigma_{d-1})} \times \mathrm{Tr}\left[\, e^{-\beta H} \,\right]$$



$\times \;\; \Sigma_{d-2}$

By exchanging S^1_β and $\tilde{S}^1$, and rescaling the whole space by T, we can interpret:

$$T \; \times \left(T^{d-2}\, \mathrm{vol}(\Sigma_{d-2}) \right) \times \; V(g)$$

- T: Circumference of rescaled $\tilde{S}^1$
- $T^{d-2}\, \mathrm{vol}(\Sigma_{d-2})$: Volume of rescaled Σ_{d-2}
- $V(g)$: Domain wall tension

Thermal Effective Field Theory

$$\mathrm{Tr}\left[\,U(g)\,e^{-\beta H}\,\right] \approx\; e^{-T^{d-1}\,\boldsymbol{V(g)}\,\mathrm{vol}(\Sigma_{d-1})} \times \mathrm{Tr}\left[\,e^{-\beta H}\,\right]$$

Since $U(g)$ is unitary, $\mathrm{Tr}\left[\,U(g)\,e^{-\beta H}\,\right] \leq \mathrm{Tr}\left[\,e^{-\beta H}\,\right]$.

Therefore, $V(g) \geq 0$.

However, we need to show the stronger inequality $V(g) > 0$ when $g \neq 1$.

Equivalently, we need to show that $V(g)$ has the global minimum at $g = 1$, where $V(1) = 0$.

The domain wall tension $V(g)$ has the **global minimum at $g = 1$**.

Pal, Sun, Zhang + H.O., in preparation.

To prove this, we use:

1. Generalized Noether Theorem:

On Noether's Theorem in Quantum Field Theory

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For any bounded region *R* of the Minkowski space, there is a unitary operator $U(g, R)$ which implements the g-action across *R*.

- For **continuous symmetry**, $U(g, R)$ can be expressed in terms of the Noether currents.
- The theorem also holds for **discrete symmetry** assuming the split property of quantum field theory.

2. Uniqueness of the Ground State in the Untwisted Sector

The domain wall tension $V(g)$ has the **global minimum at $g = 1$**.

Pal, Sun, Zhang + H.O., in progress

- $V(g)$ is related to $\langle 0|U(g,R)|0\rangle$.
- $U(g,R)|0\rangle$ for $g \neq 1$ is distinct from the vacuum $|0\rangle$ since a charged local operator can detect the location of R.
- Since $U(g,R)$ is unitary, one can derive inequalities such as $\langle 0|0\rangle > |\langle 0|U(g,R)|0\rangle|$ for $g \neq 1$.

$V(g) > 0$ for $g \neq 1$ follows from these observations.

$\boldsymbol{G}$: Finite Group

Since $V(g)$ has the global minimum at $g = 1$,

$$\exp\left(-\,T^{d-1}\,V(g)\,\mathrm{vol}(\Sigma_{d-1})\right) \propto \delta(g,1) \text{ at high temperature.}$$

Therefore,

$$\mathrm{Tr}\left[\,U(g)\,e^{-\beta H}\,\right] \approx \delta(g,1) \times \mathrm{Tr}\left[\,e^{-\beta H}\,\right].$$

For a finite group, $\delta(g,1) = \sum_R \frac{\mathbf{dim}\,\boldsymbol{R}}{|\boldsymbol{G}|}\chi_R(g)$.

The density of high energy states $\rho(\Delta, R)$ transforming in R is then,

$$\rho(E,R) \approx \frac{(\mathbf{dim}\,\boldsymbol{R})^{\mathbf{2}}}{|\boldsymbol{G}|}\,\rho(E)\ .$$

G : Compact Lie Group

If G is a compact Lie group,

$$V\left(e^{i\phi}\right) \mathrm{vol}(\partial R) = f - \frac{b}{4}\langle \phi, \phi \rangle + \cdots .$$

$$\mathrm{Tr}\left[\, U\left(g = e^{i\phi}\right) e^{-\beta H} \,\right] \approx \exp\left(f\, T^{d-1} - \frac{b}{4} T^{d-1}\langle \phi, \phi \rangle + \cdots\right).$$

Since $T^{(\mathrm{d}-1)\dim G/2} e^{-\frac{b}{4}T^{d-1}\langle \phi,\phi \rangle}$ is a solution to the heat equation on the group manifold G with $\tau = 1/T^{d-1}$ as the time variable,

$$\frac{\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]}{\mathrm{Tr}[\, e^{-\beta H} \,]} \approx \left(\frac{4\pi}{b\, T^{d-1}}\right)^{\dim G/2} \sum_R \dim R \cdot \chi_R(g) \exp\left(-\frac{c_2(R)}{b\, T^{d-1}}\right)$$

G : Compact Lie Group

$$\frac{\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]}{\mathrm{Tr}[\, e^{-\beta H} \,]} \approx \left(\frac{4\pi}{b\, T^{d-1}}\right)^{\dim G/2} \sum_R \dim R \cdot \chi_R(g) \exp\left(-\frac{c_2(R)}{b\, T^{d-1}}\right)$$

The density of high energy states $\rho(\Delta, R)$ transforming in the irreducible representation R is,

$$\rho(\Delta, R) \approx (\dim R)^2 \left(\frac{4\pi}{b' \Delta^{(d-1)/d}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{b' \Delta^{(d-1)/d}}\right] \rho(\Delta) .$$

Examples

We have verified

$$\mathrm{Tr}\left[\, U\left(g = e^{i\phi}\right) e^{-\beta H} \,\right] \approx \exp\left(f\, T^{d-1} - \frac{b}{4} T^{d-1} \langle \phi, \phi \rangle + \cdots \right).$$

for free field theories and holographic CFTs and computed the coefficients a and b.

- For free massless scalars with $G = U(1)$,
  - $f = 2\zeta(d)$ and $b = 4\zeta(d-2)$ for $d \geq 4$.

- For free Weyl spinors with $G = U(1)$,
  - $f = 3\zeta(3), b = 16 \log 2$ for $d = 3$
  - $f = \zeta(2) = \pi^2/6, b = 1$ for $d = 2$

Examples

In holographic CFTs with non-abelian G,
there are **two types of bulk geometries** relevant to
$\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]$ above the Hawking-Page transition:

1. The Reissner–Nordström solution with a commutative gauge field in the non-abelian theory by $U(1)^{\mathrm{rank}\, G} \in G$
2. The genuinely non-abelian solution (*i.e.*, with non-abelian hair)

Their relative stability has been an outstanding question.

Reviews: Volkov, Galt'sov: 9810070
Winstanley: 0801.0527

Examples

In holographic CFTs with non-abelian G,
there are **two types of bulk geometries** relevant to
$\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]$ above the Hawking-Page transition:

- Our task is simplified since the two solutions converge at high temperature.
- We found that the **solution with non-abelian hair is more stable** if we take $1/T$ effects into account.

$$f = \left(\frac{4\pi}{d}\right)^{d-1} \frac{w_{d-1}\ell^{d-1}}{4dG_N}\,, \qquad b = \left(\frac{4\pi}{d}\right)^{d-2} \frac{4(d-2)w_{d-1}\ell^{d-1}}{e^2}$$

G_N: Newton constant, e: gauge coupling, ℓ: AdS radius, w_{d-1}: area of the unit sphere

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

Universal behavior with respect to the angular momentum $\vec{M}$

$$\mathrm{Tr}\left[\, e^{-\beta(H+i\vec{\Omega}\cdot\vec{L})} \,\right] \approx \exp\left(\frac{G(T,\vec{\Omega})}{\prod_i (1+\Omega_i^2)} + \cdots\right)$$

When $\Delta, \ell \gg 1$ and $|\,\Delta - \ell\,| \sim \sqrt{f\Delta}$, we can use the saddle-point approximation to invert the Laplace transform,

$$\int_0^\infty d\Delta \sum_{J=0}^\infty \rho_d\left(\Delta, \vec{\ell}\right) e^{-\beta(\Delta+\varepsilon_d)+i\beta\vec{\Omega}\cdot\vec{\ell}} = \mathrm{Tr}\left[\, e^{-\beta(H+i\vec{\Omega}\cdot\vec{L})} \,\right],$$

to calculate $\rho_d\left(\Delta, \vec{\ell}\right)$.

- Bhattacharya, Lahiri, Longanayagam, Minwalla: 0708.1770
- Shaghoulian: 1512.06855
- Benjamin, Lee, Simmons-Duffin + H.O.: 2306.08031

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

For $\Delta, \ell \gg 1$ and $|\,\Delta - \ell\,| \sim \sqrt{f\Delta}$,

$$\rho_d(\Delta,\ell) \approx \exp\left[\frac{d}{d-1}(\Delta+\varepsilon_d)^{\frac{d-1}{d}}\left(\frac{f(d-1)\mathrm{vol}\,S^{d-1}}{2}\left(1+\sqrt{1+\frac{(d-3)(d-1)\,\ell^2}{(\Delta+\varepsilon_d)^2}}\right)\right)^{1/d}\right.$$
$$\left.\times\left(\frac{d-2}{d-3}-\frac{1}{d-3}\sqrt{1+\frac{(d-3)(d-1)\ell^2}{(\Delta+\varepsilon_d)^2}}\right)^{1-2/d}\right].$$

When $d=2$, this becomes,

$$\rho_{d=2}(\Delta,\ell) \approx \exp\left[\sqrt{\frac{2c}{3}}\pi\left(\sqrt{\frac{\Delta+\ell-c/12}{2}}+\sqrt{\frac{\Delta-\ell-c/12}{2}}\right)\right],$$

reproducing the Cardy formula

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

Benjamin, Lee, Simmons-Duffin + H.O., to appear.

Higher-Dimensional Generalization of the Cardy Formula

$$\rho_d(\Delta,\ell) \approx \exp\left[\frac{d}{d-1}(\Delta+\varepsilon_d)^{\frac{d-1}{d}}\left(\frac{f(d-1)\mathrm{vol}\,S^{d-1}}{2}\left(1+\sqrt{1+\frac{(d-3)(d-1)\,\ell^2}{(\Delta+\varepsilon_d)^2}}\right)\right)^{1/d}\right.$$
$$\left.\times\left(\frac{d-2}{d-3}-\frac{1}{d-3}\sqrt{1+\frac{(d-3)(d-1)\ell^2}{(\Delta+\varepsilon_d)^2}}\right)^{1-2/d}\right].$$

for $\Delta,\ell \gg 1$ and $|\,\Delta-\ell\,| \sim \sqrt{f\Delta}$,

- The formula agrees with the **Kerr black hole entropy** and **free field calculations**.
- We can also calculate **sub-leading terms** systematically.

Symmetry-Resolved Density of States

- $\rho(\Delta, R) \approx \dfrac{(\dim R)^2}{|G|}\rho(\Delta)$, when G is a **finite group**.

Harlow + H.O.: 2111.04725

- $$\rho(\Delta, R) \approx (\dim R)^2 \left(\frac{4\pi}{b'\Delta^{(d-1)/d}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{b'\Delta^{(d-1)/d}}\right] \rho(\Delta) \quad ,$$

when G is a **compact Lie group**. Kang, Lee + H.O.: 2206.14814

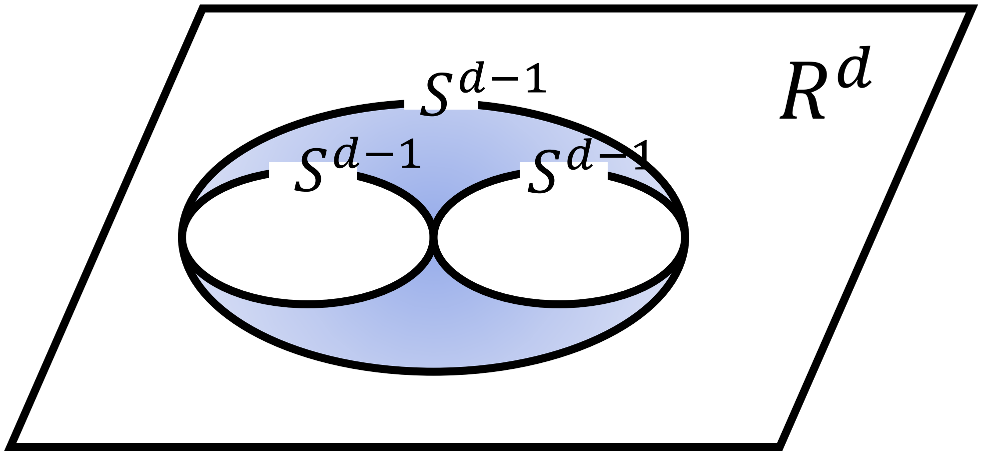
- $$\rho_d(\Delta, \ell) \approx \exp\left[\frac{d}{d-1}(\Delta+\varepsilon_d)^{\frac{d-1}{d}}\left(\frac{f(d-1)\mathrm{vol}\, S^{d-1}}{2}\left(1+\sqrt{1+\frac{(d-3)(d-1)\,\ell^2}{(\Delta+\varepsilon_d)^2}}\right)\right)^{1/d} \times\left(\frac{d-2}{d-3}-\frac{1}{d-3}\sqrt{1+\frac{(d-3)(d-1)\ell^2}{(\Delta+\varepsilon_d)^2}}\right)^{1-2/d}\right].$$

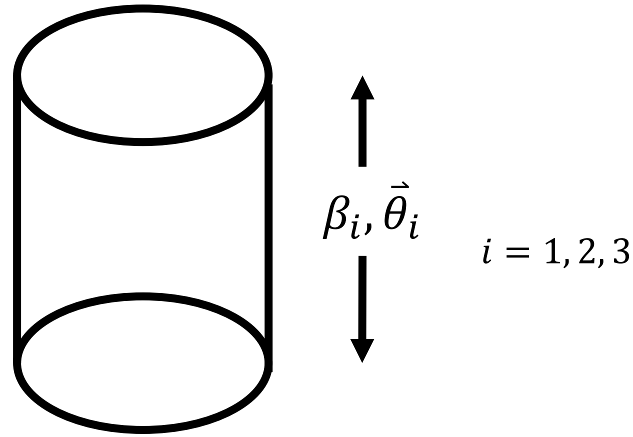
for **spacetime spin** ℓ.

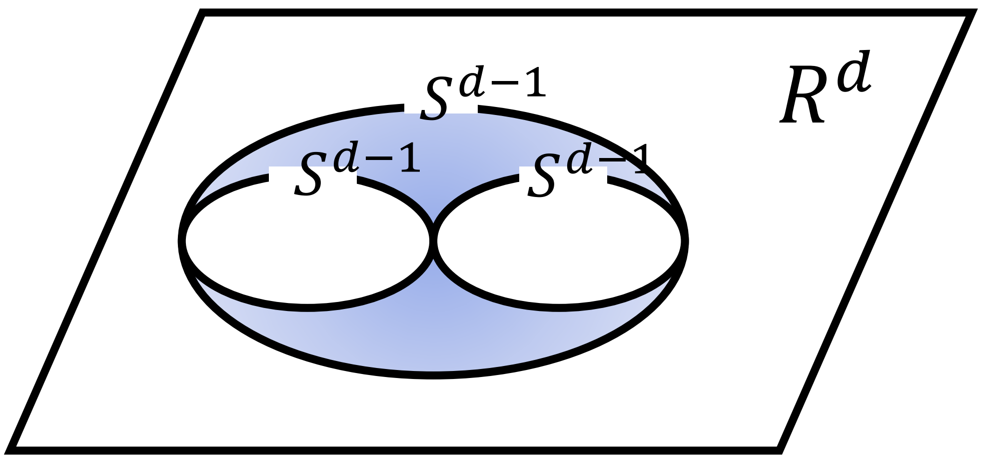
Shaghoulian: 1512.06855
Benjamin, Lee, Simmons-Duffin + H.O.: 2306.08031

Three-Point Function at Large Conformal Dimensions

Benjamin, Lee, Simmons-Duffin + H.O., to appear.







Glue three S^{d-1}'s by three cylinders.

Three-Point Function at Large Conformal Dimensions

In $d > 2$, three-point functions have several conformal blocks.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \sum_s c_{123}^s \; V^s(x_1, x_2, x_3).$$

We have a formula for general d.

For simplicity, let me show you the ones for $d = 2$ and 3 when $\Delta_1 = \Delta_2 = \Delta_3 = \Delta$.

Three-Point Function at Large Conformal Dimensions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \sum c^s_{123}\ V^s(x_1,x_2,x_3),$$

$d = 2$

$$c_{123}\ \sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{13/2} f e^{\frac{9}{4}\left(\pi^2 f^2 \Delta\right)^{1/3}}}{3^{9/2}\pi^{1/2}\Delta^{5/2}}$$

$d = 3$

$$c^s_{123}\ \sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{49/8} f^{9/8} e^{\frac{3}{2}(2\pi f \Delta)^{1/2}}}{3^{19/4}\pi^{1/4}\Delta^{31/8}} \times \prod_{i=1}^{3}(2\ell_i+1)\left(\frac{2\ell_i}{\ell_i+q_I}\right)$$

$s = [q_1, q_2, q_3]$: conformal block parameters

Benjamin, Lee, Simmons-Duffin + H.O.: 2306.08031

Symmetry-Resolved Density of States

- $\rho(\Delta, R) \approx \frac{(\dim R)^2}{|G|}\rho(\Delta)$, when G is a **finite group**.

Harlow + H.O.: 2111.04725

- $\rho(\Delta, R) \approx (\dim R)^2 \left(\frac{4\pi}{b'\Delta^{(d-1)/d}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{b'\Delta^{(d-1)/d}}\right] \rho(\Delta)$,

when G is a **compact Lie group.** Kang, Lee + H.O.: 2206.14814

- $$\rho_d(\Delta, \ell) \approx \exp\left[\frac{d}{d-1}(\Delta+\varepsilon_d)^{\frac{d-1}{d}}\left(\frac{f(d-1)\mathrm{vol}\, S^{d-1}}{2}\left(1+\sqrt{1+\frac{(d-3)(d-1)\,\ell^2}{(\Delta+\varepsilon_d)^2}}\right)\right)^{1/d} \times\left(\frac{d-2}{d-3}-\frac{1}{d-3}\sqrt{1+\frac{(d-3)(d-1)\ell^2}{(\Delta+\varepsilon_d)^2}}\right)^{1-2/d}\right].$$

for **spacetime spin** ℓ.

Shaghoulian: 1512.06855
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Three-Point Function at Large Conformal Dimensions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \sum c^s_{123}\; V^s(x_1,x_2,x_3),$$

$d = 2$

$$c_{123}\sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{13/2} f e^{\frac{9}{4}(\pi^2 f^2 \Delta)^{1/3}}}{3^{9/2}\pi^{1/2}\Delta^{5/2}}$$

$d = 3$

$$c^s_{123}\sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{49/8} f^{9/8} e^{\frac{3}{2}(2\pi f\Delta)^{1/2}}}{3^{19/4}\pi^{1/4}\Delta^{31/8}} \times \prod_{i=1}^{3}(2\ell_i+1)\left(\frac{2\ell_i}{\ell_i+q_I}\right)$$

$s = [q_1, q_2, q_3]$: conformal block parameters

Benjamin, Lee, Simmons-Duffin + H.O., 2306.08031

Thank you for your attention.