

Non-commutative Field Theory with Twistor Coordinates

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Outline

- I. *Non-commuting coordinates*
- II. *Non-commutative Field Theory on Moyal Plane*
- III. *Twistor Theory*
- IV. *Quantum Fields in Twistor Space*

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Non-commuting Coordinates

Motivation

- Uncover the fabric of space-time
- UV completion of the standard model
- Quantization and unification of gravity (hopefully, if necessary)

Simplest and tractable example is the Groenewold-Moyal plane \mathcal{R} with

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}$$

A radical step – the algebra of functions (fields) on \mathcal{R} is modified
– the product is deformed to a star (Moyal) product:

$$(\phi_1 \star \phi_2)(x) = e^{\frac{i}{2}\Theta^{\mu\nu}\partial_\mu^y\partial_\nu^z} \phi_1(y)\phi_2(z)|_{y=z=x}$$

Interesting mathematics. Appears in open string theory in the presence of a constant B-field $B=\Theta$ (Seiberg-Witten, '99). But is it physically sensible?

Non-commutative Field Theory

Is it physically sensible?

Non-commutative Lagrangians involve non-local interactions with star products

$$\int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda \phi^4 \right]$$

$$\rightarrow \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda \phi \star \phi \star \phi \star \phi \star \phi \right]$$

Free Feynman propagators are not affected, but the perturbative interaction vertices are modified by the factors

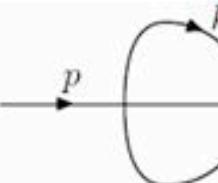
$$e^{i \sum_{i < j} C_{ij} k_{i\mu} k_{j\nu} \Theta^{\mu\nu}} \quad (\text{Minwalla et al, '99})$$

They affect UV and IR behavior of Feynman diagrams



Non-commutative Field Theory

Is it physically sensible?


$$= \frac{\lambda}{12} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ip_\mu k_\nu \Theta^{\mu\nu}}}{k^2 + m^2}$$

from Rivasseau, '07

$$= \frac{\lambda}{48\pi^2} \sqrt{\frac{m^2}{(\Theta p)^2}} K_1(\sqrt{m^2(\Theta p)^2}) \underset{p \rightarrow 0}{\simeq} p^{-2}.$$

UV-IR mixing: $\Lambda \rightarrow \infty$, $p \rightarrow 0$ limits' order matters

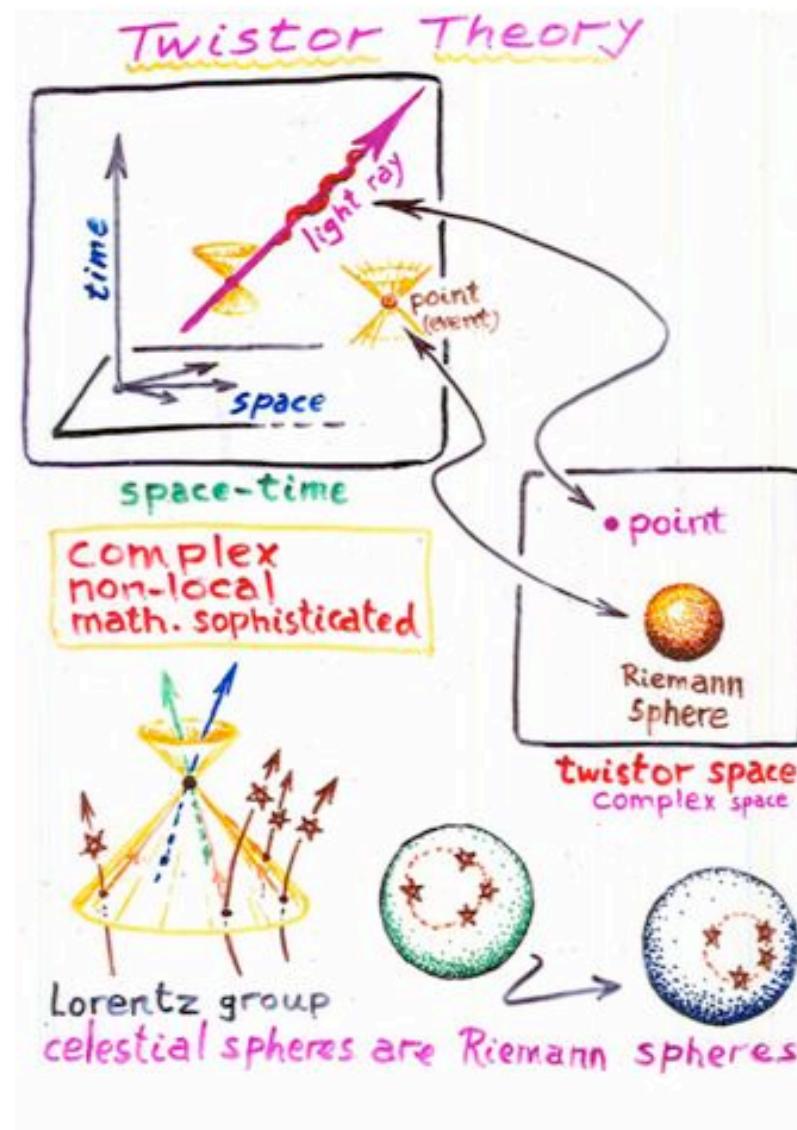
**Renormalizable Φ^4 , without UV/IR mixing, can be
constructed by modifying quadratic terms**

(Grosse, Wulkenhaar '04)

**In general, no significant improvement in UV – Feynman diagrams
still have the same degree of divergences as commutative QFT**

Moyal Plane non-commutativity isn't too useful..

from Penrose, 2005 Oxford LMS Workshop



Twistor Theory (Penrose '67)

Twistors: Spinors representing null geodesics (light rays, world lines) in M^4

Intersections \rightarrow Points

Notation

(Penrose, Rindler '86)

$$\begin{aligned} V^{AA'} &= \begin{bmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{bmatrix} \end{aligned}$$

momentum $p_{AA'} = \bar{\pi}_A \pi_{A'}$

$$L^{AA'BB'} = i\omega^{(A}\bar{\pi}^{B)}\epsilon^{A'B'} - i\epsilon^{AB}\bar{\omega}^{(A'}\pi^{B')}$$

↑ angular momentum

Twistor Theory

Twistors

$$Z^\alpha = (\omega^A, \pi_{A'}) \quad \alpha = 1, 2, 3, 4$$

$$\omega^A = \overset{\circ}{\omega}{}^A - ix^{AA'} \overset{\circ}{\pi}_{A'} \quad , \quad \pi_{A'} = \overset{\circ}{\pi}_{A'}$$

↑ reference point (\rightarrow line) of angular momentum L

Dual Twistors

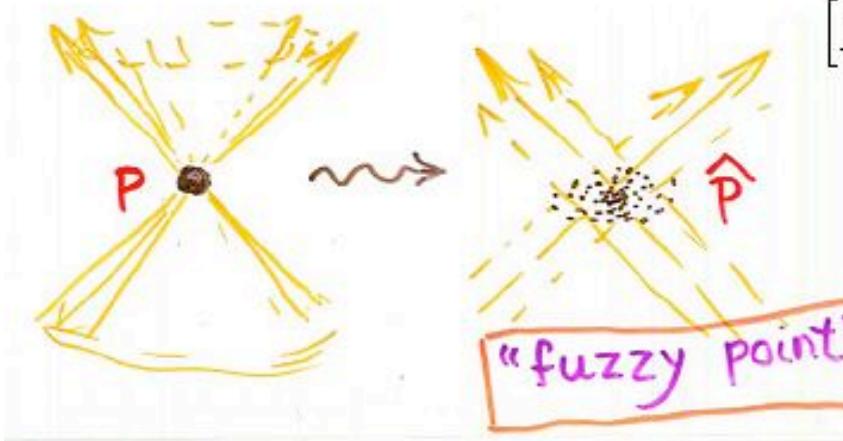
$$\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})$$

$$p^0 = \frac{1}{\sqrt{2}}(Z^3 \bar{Z}_1 + Z^2 \bar{Z}_0), \dots$$

$$L^{01} = -L^{10} = \frac{i}{2}(Z^0 \bar{Z}_0 - Z^1 \bar{Z}_1 - Z^2 \bar{Z}_2 + Z^3 \bar{Z}_3), \dots$$

from Penrose, 2005 Oxford LMS Workshop

Twistor Quantization



$$[L_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu$$

↓
canonical quantization
of Poincare algebra

$$[Z^\alpha, \bar{Z}_\beta] = \hbar \delta_\beta^\alpha$$

$$[\omega^A, \bar{\pi}_B] = \hbar \delta_B^A$$

$$[\pi_{A'}, \bar{\omega}^{B'}] = \hbar \delta_{A'}^{B'}$$

n.b. units: $\omega = x\sqrt{p}$, $\pi = \sqrt{p}$

Fuzzy space-time needs no quantum gravity? What are the consequences of such non-commutativity?



Quantum Field Theory in Twistor Space

Who can resist the power of holomorphy?



Passage devant le rocher des sirènes. Liebig 1914. Photo ©Maurer Förde-GML

Z^α and \bar{Z}_β are conjugate variables

\Rightarrow holomorphic fields $f(Z^\alpha) = f(\omega, \pi)$ are justified

Penrose transform: $f(Z^\alpha) \rightarrow \phi(x), \quad \square\phi = 0$

Beautiful, but ...

- No connection with second quantization (Lagrangian formalism)
- Fuzzy space-time not visible: it's hidden by holomorphy

Quantum Fields in Twistor Space

Non-holomorphic	Holomorphic
$\phi(\omega, \bar{\omega})$	$f(\omega, \pi)$
<i>Direct map to M^4</i>	<i>Penrose Transform</i>

Need 2 twistors:

$$Z_a^\alpha = (\omega_a^A, \pi_{aA'}) \quad a = 1, 2$$

c.f. ambitwistors, two-twistors...

$$x^{AA'} = \frac{i}{\pi_{1B'} \pi_{2B'}^*} [(\omega_1^A - \dot{\omega}_1^A) \pi_2^{A'} - (\omega_2^A - \dot{\omega}_2^A) \pi_1^{A'}]$$

↑ points ≡ intersections

(pseudo)Reality constraint (symbolically): $\bar{\omega} = -\pi^{-1} \omega \bar{\pi}$

Quantum Fields in Twistor Space

QUANTIZATION OF
TWISTOR COORDINATES

$$[Z_a^\alpha, \bar{Z}_{b\beta}] = \hbar \delta_\beta^\alpha \delta_{ab}$$



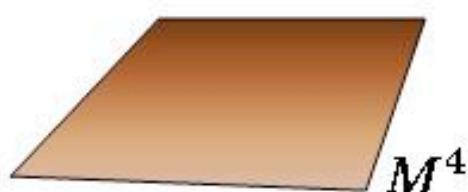
$$[\omega, \bar{\omega}] = 0$$



FIELDS WITH
TWISTOR COORDINATES

$$\phi = \phi(\omega(x), \bar{\omega}(x))$$

FIX $\pi \longrightarrow l_{AA'} = \bar{\pi}_{1A}\pi_{1A'} + \bar{\pi}_{2A}\pi_{2A'} \quad l^2 = \mu^2 > 0$



LORENTZ SYMMETRY BROKEN BY TIME-LIKE l
 $SO(1,3) \rightarrow SO(3)$

Non-commuting Twistor Coordinates

$$[\omega_a^A(x_1), \bar{\omega}_b^{A'}(x_2)] = i\hbar(x_2^{AA'} - x_1^{AA'})\delta_{ab}$$

LOCALLY COMMUTING BUT

**NON-LOCALLY NON-COMMUTING
UNCERTAINTY GROWS WITH SEPARATION (LIKE IN A FOG...)**

BROKEN LORENTZ SYMMETRY $\vec{l} = (\mu, \vec{0})$

$SO(1,3) \rightarrow SO(3)$

SO IT'S A...

FØGGÆTHER®



Free Fields propagating in *free field theory*

$$\phi(\omega, \bar{\omega}) = \int \frac{d^3 \vec{p}}{\sqrt{(2\pi)^3 2|\vec{p}|}} \left(a_{\vec{p}} e^{\omega_a^A \tilde{\omega}_A^a / \hbar} + a_{\vec{p}}^\dagger e^{\bar{\omega}_{A'}^a \bar{\omega}_a^{A'}/ \hbar} \right)$$

SECOND QUANTIZATION:

$$p_{AA'} = \tilde{\omega}_A^a \pi_{aA'} \quad [a_{\vec{p}}, a_{\vec{p}'}^\dagger] = \delta^3(\vec{p} - \vec{p}')$$

FEYNMAN PROPAGATOR:

$$iD(x' - x) = \langle 0 | T(\phi(\omega', \bar{\omega}') \phi(\omega, \bar{\omega})) | 0 \rangle$$


NON-COMMUTATIVE (BAKER-HAUSDORFF)
 $e^A e^B = e^{(A+B+\frac{1}{2}[A,B])}$

$$e^{\omega_a^A(x') \tilde{\omega}_A^a(p')/\hbar} e^{\bar{\omega}_{A'}^b(p) \bar{\omega}_b^{A'}/\hbar}$$

$$= e^{[\omega_a^A(x') \tilde{\omega}_A^a(p') + \bar{\omega}_{A'}^a(p) \bar{\omega}_a^{A'}/\hbar + \frac{i}{2}(x-x')^{AA'} \tilde{\omega}_A^a(p') \bar{\omega}_{A'}^a(p)]/\hbar}$$

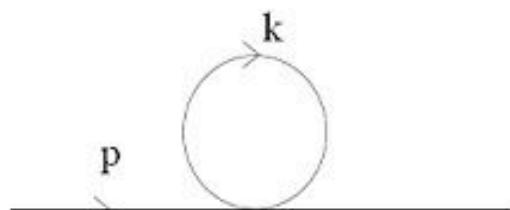
Feynman Propagator

$$\begin{aligned} iD(x' - x) &= \int \frac{d^3 \vec{p}}{(2\pi)^3 2|\vec{p}|} [e^{-i\vec{p} \cdot (x' - x)(p+2l)^2/(4\mu^2 \hbar)} \theta(t' - t) \\ &\quad + e^{i\vec{p} \cdot (x' - x)(p+2l)^2/(4\mu^2 \hbar)} \theta(t - t')] \\ &= i \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x' - x)} \tilde{D}(k) \\ \tilde{D}(k) &= \frac{1}{k^2} \times \frac{2}{\sqrt{|\vec{k}|/\mu+1}(\sqrt{|\vec{k}|/\mu+1}+1)} \\ UV : |\vec{k}| \gg \mu \quad \tilde{D}(k) &\sim \frac{1}{k^3} \quad \Bigg| \quad IR : |\vec{k}| \ll \mu \quad \tilde{D}(k) \sim \frac{1}{k^2} \\ \text{ABOVE } \boldsymbol{\mu} \text{ NON-COMMUTATIVITY SCALE} &\qquad \qquad \qquad \text{STANDARD PROPAGATOR} \end{aligned}$$

Interacting Fields (very preliminary)

Coordinates are locally commuting → local interactions unchanged

UV behavior of Feynman diagrams determined by propagators


$$\sim \int \frac{d^4 k}{k^3}$$

Linear (instead of quadratic)
UV divergence
No UV/IR mixing

gauge theories in ~~good sector~~: perturbatively finite ?

Conclusions

- Non-local (foggy) non-commutativity in twistor space
- Determined by 2 constants: \hbar, μ
- Lorentz symmetry broken: preferred time direction, fundamental time unit $\tau = \hbar/(\mu c^2)$ and rotational invariance in the \mathcal{A} Ether frame
- Assuming \mathcal{A} Ether = CMB $\rightarrow \mu > 10 \text{ TeV}$
- Foggy spacetime tames UV divergences of QFT, no UV/IR mixing
- Many open questions: interacting QFT formalism, divergences, gravity, ...



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