# The twistor programme and twistor strings

## From twistor strings to quantum gravity?

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Quantum gravity requires a pregeometry for space-time.

**Penrose's Proposal:**
Twistor space is the fundamental arena for physics.

Flat correspondence:

- Complex space-time $\mathbb{M} = \mathbb{C}^4$, coords $x^{AA'}$, ${\scriptstyle A=0,1, A'=0',1'}$ flat metric $\mathrm{d}s^2 = \mathrm{d}x^{AA'}\mathrm{d}x^{BB'}\varepsilon_{AB}\varepsilon_{A'B'}$, $\varepsilon_{AB} = \varepsilon_{[AB]}$ etc..
- Twistor space $\mathbb{T} = \mathbb{C}^4$, coords $Z^\alpha = (\omega^A, \pi_{A'})$, ${\scriptstyle \alpha=(A,A')}$. Projective twistor space $\mathbb{PT} = \{\mathbb{T} - \{0\}\}/\{Z \sim \lambda Z, \lambda \in \mathbb{C}^*\} = \mathbb{CP}^3$.
- Incidence relation

$$\omega^A = ix^{AA'}\pi_{A'}.$$

$$\{\text{Point } x \in \mathbb{M}\} \longleftrightarrow \{L_x = \mathbb{CP}^1 \subset \mathbb{PT}\}, \text{ hgs coords } \pi_{A'}.$$

# Penrose transform

Massless fields $\longleftrightarrow$ cohomology on $\mathbb{PT}' = \mathbb{PT} - L_\infty = \{\pi_{A'} \neq 0\}$

$$\phi(x)_{A'_1 A'_2 \ldots A'_n} = \oint_{L_x} \pi_{A'_1} \pi_{A'_2} \ldots \pi_{A'_n}\, f\, \pi_{B'} \mathrm{d}\pi^{B'}\,, \quad f \in H^1(\mathbb{PT}', \mathcal{O}(-n-2))$$

$$\phi(x)_{A_1 A_2 \ldots A_n} = \oint_{L_x} \frac{\partial^n f}{\partial\omega^{A_1} \partial\omega^{A_2} \ldots \partial\omega^{A_n}} \pi_{B'} \mathrm{d}\pi^{B'}\,, \quad f \in H^1(\mathbb{PT}', \mathcal{O}(n-2))$$

- homogeneous fns $\mathcal{O}(n)$ are $f(Z)$, $f(\lambda Z) = \lambda^n f(Z)$, $\lambda \in \mathbb{C}^*$.
- Čech cohomology for open cover $\{U_i\}$

  $H^1(\mathcal{O}) = \{\text{hol. fns } f_{ij} \text{ on } U_i \cap U_j, f_{ij} + f_{jk} + f_{ki} = 0\} / \{f_{ij} = g_i - g_j, \}$

- Dolbeault cohomology

$$H^1(\mathcal{O}(n)) = \{f \in \Omega^{(0,1)}(n) |\, \bar\partial f = 0\} / \{f = \bar\partial g\}$$

- $n = 0, -4$ for ASD, SD Maxwell
- $n = 2, -6$ for linearized gravity (note parity asymmetry).

$H^1(\mathcal{O})$: ASD Maxwell fields $\leftrightarrow$ holomorphic line bundles.

For vector bundles:

**Theorem (Ward)**

*ASD Yang-Mills fields, $D = d + A$ with $D^2 = F$ with $F^+ = 0$, on $\tilde{E} \to \mathbb{M}$ are in $1:1$ correspondence with holomorphic Bundles $E \to \mathbb{PT}'$ trivial on each $L_x$.*

> Easiest proof is in Euclidean signature where $p : \mathbb{PT}' \to \mathbb{M}$;
> Can pull back $(\tilde{E}, D) \to \mathbb{M}$ to give $(E, \bar{\partial}) \to \mathbb{PT}'$.
> For such bundles $\bar{\partial}_E^2 = F^+$, so
>
> $$\text{bundle is holomorphic} \;\Leftrightarrow F^+ = 0.$$

[Reverse direction requires some complex analysis.]

## Theorem (Penrose)

$$\left\{\begin{matrix}\textit{Deformations of complex} \\ \textit{structure, } \mathbb{PT}' \rightsquigarrow \mathscr{PT}\end{matrix}\right\} \stackrel{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{asd deformations of con-} \\ \textit{formal structure } (\mathbb{M},\eta) \rightsquigarrow \\ (M,[g]).\end{matrix}\right\}$$

**For Ricci flat** $g \in [g]$, $\mathscr{PT}$ *must have*

- *A holomorphic fibration* $p : \mathscr{PT} \to \mathbb{CP}^1$
- *A Poisson structure* $\{,\}_I$*, on fibres of* $p$*, valued in* $p^*\mathcal{O}(-2)$*.*

**Main ideas**: We deform $\mathscr{PT}$ by plate tectonics or changing $\bar{\partial}$.

Ricci-flat linearised deformations $H^1(\mathcal{O}(2)) \stackrel{\{,\}_I}{\hookrightarrow} H^1(T^{1,0}\mathbb{PT})$.
The $\mathbb{CP}^1$s in $\mathscr{PT}$ survive deformation. Define

$$M = \{\text{moduli space of degree-1 } \mathbb{CP}^1\text{s} \subset \mathscr{PT}\}.$$

$x, y \in M$ connected by a light ray $\Leftrightarrow$ Incidence $\mathbb{CP}^1_x \cap \mathbb{CP}^1_y \neq \emptyset$.
$\rightsquigarrow$ ASD conformal structure, $[g]$, Weyl$^+ = 0$ on $M$.

# Supersymmetric extensions

**Super-twistor space [Ferber]:** $\mathbb{PT}_{[N]} = \mathbb{CP}^{3|N}$,
homogeneous coords $(Z^\alpha, \eta^i)$, $i = 1, \ldots, N$, $\eta^i$ anticommuting.

**Correspondence:** $\mathbb{PT}_{[N]} \supset \mathbb{CP}^1 \longleftrightarrow (x^{AA'}, \theta^{iA'}) \in \mathbb{M}^+_{[N]}$,
$\mathbb{M}^+_{[N]}$ =chiral super-Minkowski space. Incidence relation

$$\omega^A = x^{AA'}\pi_{A'}\,, \qquad \eta^i = \theta^{iA'}\pi_{A'}$$

Penrose-Ward transforms extends, e.g. $a_s \in H^1(\mathbb{PT}'_{[4]}, \mathcal{O})$
expands:

$$a_s = a + \eta^i\psi_i + \eta^i\eta^j\,\phi_{ij} + \eta^i\eta^j\eta^k\epsilon_{ijkl}\,\tilde{\psi}^l + \eta^1\eta^2\eta^3\eta^4\,b$$

$\leftrightarrow$ susy multiplet $A_s := (A, \Psi_i, \Phi_{ij}, \tilde{\Psi}^i, B)$ helicity $-1$ to 1 on $\mathbb{M}^+_{[4]}$.

# Supersymmetric Ward transform

**SUSY Ward transform:**
For $a_s \in \Omega^{0,1}(ad_{G_{\mathbb{C}}})$, take d-bar operator $\bar{\partial}_{a_s} = \bar{\partial}_0 + a_s$

$\{$Hol vector bundle $E \to \mathbb{PT}_{[4]}\} \longleftrightarrow \{N = 4$ SYM multiplet on $\mathbb{M}^+_{[4]}.\}$

**But,** the interactions are an ASD truncation:
cf Chalmers-Siegel action for $A \in \Omega^1(ad_G)\,, \quad B \in \Omega^{2+}(ad_G)$

$$S_{\rm asd}[A, B] = \int_{\mathbb{M}_{\mathbb{R}}} B \wedge F^+\,, \qquad \rightsquigarrow \quad F^+ = 0\,, \quad [\mathrm{d} + A, B] = 0\,.$$

To extend to full YM:

$$S[A, B] = \int_{\mathbb{M}_{\mathbb{R}}} B \wedge F^+ - \frac{\epsilon}{2} B \wedge B\,, \quad \rightsquigarrow \quad F^+ = \epsilon B\,, \quad [d + A, F^+] = 0.$$

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**On-shell generating function** for amplitudes with full Yang-Mills interactions is

$$\mathcal{A}[A_s] = \sum_{d=1}^{\infty} \int_{\mathscr{M}^d_{[4]}} \mathrm{Det}(\bar{\partial}_{a_s}|_C)\, \mathrm{d}\mu$$

$\mathscr{M}^d_{[4]}$ = contour in moduli space of *connected* algebraic curves $C$ degree $d$ in $\mathbb{PT}_{[4]}$, $C \in \mathscr{M}^d_{[4]}$, $\mathrm{d}\mu$ some measure.

**Selection rules:**

- $\mathscr{M}^d_{[4]}$ contribution $\leftrightarrow$ processes with $d + 1 - l$ SD gluons.
- $l :=$Number of loops, $g :=$ genus of $C$, then $g \leq l$.

$\rightsquigarrow$ concrete algebraic formulae for all tree amplitudes [RSV].

If $C = d$ lines then $\mathscr{M}^d_{[4]} = (\mathbb{M}^+_{[4]})^d/\mathrm{Sym_d}$, *but* must include propagators from holomorphic Chern-Simons action

$$S_{\mathrm{asd}}[a_s] = \int_{\mathbb{PT}_s} \mathrm{CS}(a_s) \wedge \Omega_{[4]}\,, \qquad \Omega_{[4]} = \epsilon_{\alpha\beta\gamma\delta} Z^\alpha \mathrm{d}Z^\beta \mathrm{d}Z^\gamma \mathrm{d}Z^\delta \prod_{i=1}^{4} \mathrm{d}\eta^i$$

$\rightsquigarrow$ diagrammatic formalism with MHV diagrams glued together with Chern-Simons propagators.Gluing is done on-shell.

Gukov, Motl & Nietzke argue $\Leftrightarrow$ connected version.

**Path integral formulation:** The above can be expressed as

$$\mathcal{A}[A_s] = \int Da'_s \, \mathrm{e}^{\left(\int_{\mathbb{PT}_{[4]}} \mathrm{CS}(a'_s)\wedge\Omega_{[4]}\right)} \sum_{d=1}^{\infty} \int_{\mathscr{M}^d_{[4]}} \mathrm{Det}(\bar\partial_{a'_s}|_C)\, \mathrm{d}\mu$$

Here $a'_s$ are off-shell, but $\rightarrow a_s$, on-shell, at $\infty$.
$\rightsquigarrow$ twistor action, see Boels talk.

**String models:** $\mathbb{PT}_{[4]}$ is Calabi-Yau

- **Witten:** B-model in twistor space coupled to D1-instantons (the holomorphic curves $C$ of $\mathscr{M}^d_{[4]}$).
- **Berkovits:** Open string action on Riem-surface $C$ with bdy, $Z^I : C \to \mathbb{T}_{[4]}$, $Y_I : C \to \Omega^{1,0}(C) \otimes \mathbb{T}^*_{[4]}$, $A \in \Omega^{0,1}(C)$

$$S[Z^I, Y_I, A] = \int_C Y_I \bar{\partial} Z^I + A Y_I Z^I + \text{Complex conjugate}$$

fields are real on boundary ($\leftrightarrow$ split signature for $\mathbb{M}$).
($C \in \mathscr{M}^d_{[4]}$ are worldsheet instantons.)

**But:** [Berkovits, Witten] Amplitudes contain $N = 4$ conformal supergravity; wont decouple from loops.

Theories depend on $\mathbb{C}$-structure $J$ of $\mathbb{PT}_{[4]}$ and:
**Witten:** B-field $b \in \Omega^{1,1}(\mathbb{PT}_{[4]})$ coupled in via $\int_C b$
**Berkovits:** 1-form $g_I(Z)\partial Z^I$ coupled in by $\int_{\partial C} g_I \partial Z^I$.
$(J, g_I)$ or $(J, b) \longleftrightarrow$ spectrum of $N = 4$ conf sugra.

This is a problem for gauge theory applications, but an opportunity for quantum gravity (although wrong theory....).

Optimal formulation should be as a twisted $(0, 2)$-model.

- Model depends only on $\mathbb{C}$-str, and $b$-field, on $\mathbb{PT}_{[4]}$ and bundle $E \to \mathbb{PT}_{[4]}$.
- Gives a Dolbeault formulation that allows off-shell fields.
- Allows $C_2(E) \neq 0$ to incorporate instantons
- Gives interpretation of $b$ as twisting of Courant bracket in the context of Hitchin's generalised complex structures.
- Witten, hep-th/0504078 gives correspondence with Berkovits type models as sheaves of chiral algebras.
- Directly gives generating functions of amplitudes as integrals of det $\bar{\partial}|_C$ over instanton moduli spaces.

We 'prove' Berkovits-Witten's conjecture for conformal Sugra:

Off-shell theory depends on

- $\mathscr{P}\mathscr{T}_{[4]}$, with Calabi-Yau *almost* complex structure $J$
- $b \in \Omega_J^{(1,1)}(\mathscr{P}\mathscr{T}_{[4]})$

Best guess (Berkovits-Witten) for string field theory is action

$$S[J,b] = \int_{\mathscr{P}\mathscr{T}_{[4]}} (N(J) \wedge \lrcorner\, b) \wedge \Omega_{[4]} \tag{1}$$

where $N(J) = \bar{\partial}_J^2 \in T^{(1,0)} \otimes \Omega^{(0,2)}$, is Nijenhuis tensor of $J$.

Field equations: $N(J) = 0 = \bar{\partial} b$ ; so $J$ is integrable.
Gauge freedom: $b \to b + \bar{\partial}\chi + \partial\xi$.
$\leadsto \mathscr{P}\mathscr{T}_{[4]}$ is a complex manifold, $\partial b \in H^1_{\bar{\partial}}(\mathscr{P}\mathscr{T}_{[4]}, \mathrm{d}\Omega^1)$.

**Gives:** Spectrum of $N = 4$ conf sugra with ASD couplings.

Instantons are pseudo-holomorphic maps $Z : C \to \mathscr{PT}_{[4]}$ and contribute

$$\sum_d \int_{\mathscr{M}^d_{[4]}} \mathrm{d}\mu \, \mathrm{e}^{\left(\int_C b\right)} \tag{2}$$

to string field theory action, where $C$ is an instanton and $\mathscr{M}^d_{[4]}$ a contour in the moduli space of instantons of degree-$d$.

**Disconnected prescription:** $C = \cup_{i=1}^d L_{x_i}$, $L_{x_i}$ degree-1, $x_i \in M_{[4]} =$ real space of degree-1 instantons so $\mathscr{M}^d_{[4]} = (M_{[4]})^d / \mathrm{Sym}_d$ . Thus instanton contribution is

$$\sum_d \int_{\mathscr{M}^d_{[4]}} \mathrm{d}\mu \, \mathrm{e}^{\left(\int_{\cup_i L_{x_i}} b\right)} = \sum_d \int_{(M_{[4]})^d / \mathrm{Sym}_d} \prod_{i=1}^d \mathrm{e}^{\left(\int_{L_{x_i}} b\right)} \mathrm{d}^{4|8} x_i$$

$$= \sum_d \frac{1}{d!} \left( \int_{M_{[4]}} \mathrm{e}^{\int_{L_x} b} \right)^d = \exp \left( \int_{M_{[4]}} \mathrm{e}^{\int_{L_x} b} \right)$$

Thus path integral becomes

$$\int DJ\, Db \exp\left(\int_{\mathscr{P}\mathscr{T}_{[4]}} (N \wedge \lrcorner\, b) \wedge \Omega_{[4]} + \int_{M_{[4]}} \mathrm{e}^{\int_{L_x} b} \mathrm{d}^{4|8}x\right)$$

giving a string-field theory action (incorporating instantons)

$$S[J, b] = \int_{\mathscr{P}\mathscr{T}_{[4]}} (N \wedge \lrcorner\, b) \wedge \Omega_{[4]} + \int_{M_{[4]}} \mathrm{e}^{\int_{L_x} b} \mathrm{d}^{4|8}x \qquad (3)$$

## Theorem (M, hep-th/0507269)

*Let the real slice of $M_{[4]}$ arise from Euclidean signature reality conditions, and assume that only spin-2 parts of $N = 4$ csugra spectrum are present*

**Then:** *(3) is equivalent to conformal supergravity action $\int_{M_{[4]}} |C|^2$ on (Euclidean signature) space-time.*

[See Rutger Boels talk for gauge theory twistor actions.]

Returning to the twistor programme, this resolves old questions:

- Quantize metrics on $\mathbb{M} \rightsquigarrow$ well defined events but fuzzy lightcones—bad (fields dont know how to propagate),
- Quantize on twistor space $\rightsquigarrow$ fuzzy events but well defined light rays—good.

**Note:** we only have ASD (leg break) data.
For physics, we need the SD (googly) part also.

Lets quantize anyway!
But, do we quantize the $\mathbb{CP}^1$s or the complex structures?

**Answer:** In string theory, quantization of the $\mathbb{CP}^1$s leads to quantization of the complex structures.

**But:** Above shows that this leads to conf sugra. How do we get Einstein (super-) gravity?

**Basic idea:** Conformal gravity contains Einstein gravity; eliminate superfluous fields by imposing symmetries.

**Berkovits string on $\mathbb{T}_{[4]}$:**

Consider strings on $\mathbb{T}_{[4]}$, action $S = \int Y_I \bar\partial Z^I$;
reduce to $\mathbb{PT}_{[4]}$ by gauging symmetry along $\Upsilon = Z^I \partial/\partial Z^I$ by including field $A \in \Omega^{0,1}(C)$ in the action

$$S = \int_C Y_I \bar\partial Z^I + A \wedge J_\Upsilon , \qquad J_\Upsilon = Y_I Z^I \, ;$$

gauge freedom $A \to A + \bar\partial\lambda$ gives $(Y_I, Z^I) \to (\mathrm{e}^{-\lambda} Y_I, \mathrm{e}^{\lambda} Z^I)$.

**Reduction to Einstein:**

We gauge symmetries $\leftrightarrow$ 1-forms pulled back from $\mathbb{CP}^1$ or $\mathbb{CP}^{1|N}$ for projections $\mathbb{PT}_{[N]} \to \mathbb{CP}^{1|N} \to \mathbb{CP}^1$.

- For a general choice of such symmetries there is an anomaly.
- For $\mathbb{CP}^{R|N} \to \mathbb{CP}^{1|N}$, we require $R = 3$, no restriction on $N$!.
- In particular we obtain $N = 4$ and $h$, $\tilde{h}$ above give spectrum of $N = 4$ supergravity.
- For $\mathbb{PT}'_{[4]} \to \mathbb{CP}^1$ we obtain a theory with spectrum of $N = 8$ supergravity.
- We obtain correct interactions for ASD couplings from string perturbation theory.
- It is an open question as to whether the instanton contributions give couplings of the full theory as they do for conformal supergravity.
  Perhaps just ASD truncation, so back to square one???

# Twistor action, self-dual case

hep-th/0706.??; joint with Martin Wolf

linear gravitational field $\leftrightarrow h \in H^1(\mathbb{PT}', \mathcal{O}(2))$

For $N = 8$, i.e., on $\mathbb{CP}^{3|8}$, this gives full multiplet.

$\rightsquigarrow$ inf. deformation $\{h, \cdot\}_I =$ Hamiltonian vector field using $\{,\}_I$.

In the full nonlinear ASD case, *h* can be regarded as a finite deformation of a 'contact form' and determines the $\mathbb{C}$-structure.

Full non-linear ASD field equations become

$$\bar{\partial}_0 h + \{h, h\} = 0$$

with Chern-Simons-like action

$$S[h] = \int_{\mathbb{PT}} (h \wedge \bar{\partial} h + \frac{2}{3}\{h, h\}h) \wedge \Omega$$

$\Omega =$ natural wt -4 hol. volume form.

It remains to relate this to (part of) a twistor-string theory.