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TWO-TWISTOR DESCRIPTION OF PARTICLES, STRINGS AND MEMBRANES

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1. Introduction

- elements of twistor theory
- from one-twistor to two-twistor geometry

2. Twistorial versus space-time formulations of relativistic point particle models

- one-twistor models ($m=0$)
- two-twistor models ($m \neq 0, s \neq 0, e \neq 0$)

3. Bosonic string from two twistors

- two Hamiltonian descriptions (P_μ^m or $P_{(uv)}$)
- ⇒ - purely two-twistor description
- remarks about quantization

4. Membrane from two twistors

- ⇒ - extension from strings
to membranes and p-branes

5. Outlook

- $D \neq 4$ and SUSY
- $D=11$: M-theory and BPS preons
- relation to "standard" (super) twistor string (Witten 2004, Berkovits 2009)

ONLY MINKOWSKI SPACE-TIME AND "MINKOWSKI" TWISTORS!
IN SECT. 1-4 ONLY $D=4$, NO SUPER

(01)

Our aim:

To find exact equivalence of
classical torsionel and space-time actions:

- Space-time formulation (phase space)
- intermediate spinor / space-time
- purely torsionel formulation

References:

Particles ($m \neq 0, S \neq 0$)

A. Bette,	hep-th/0405166 (PLB)
J. de Azcarraga	hep-th/0510266 (Int. J. Mod. Phys.)
A. Frydrych	hep-th/0510161 (PRD)
S. Fedork	
C. M. Espanya	
J.L.	

Strings :

S. Fedork	hep-th/060624 (PRD)
+ J.L.	

membrane:

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+ J.L.

arXiv:0706.2129

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1. INTRODUCTION

Elements of twistor theory:

Primary geometry described by conformal
 $SU(2,2)$ spinors \equiv twistors $Z_A \in T$ ($A=1,2,3,4$)

$$T: \quad Z_A = (\pi_\alpha, \omega^\alpha) \\ (\bar{Z}_A = (\bar{\pi}_\dot{\alpha}, \bar{\omega}^{\dot{\alpha}})) \quad \langle Z, \bar{Z} \rangle = \pi_\alpha \bar{\omega}^\alpha + \omega^\alpha \bar{\pi}_\alpha$$

\uparrow
 $SU(2,2)$ norm - special
choice

$$T^n = T \otimes \dots \otimes T \quad Z_{A;i} = (\pi_{\alpha;i}, \omega^{\alpha;i}) \quad (\pi_{\alpha;i})^* = \bar{\pi}_{\dot{\alpha};i} \\ (\omega^{\alpha;i})^* = \bar{\omega}^{\dot{\alpha};i}$$

Two basic formulae of twistor theory:

i) Incidence relation $T \leftrightarrow \mathbb{C}P^{3;1}$
twistors complex Minkowski space

(Penrose 1962) $\bar{\omega}^{\alpha;i} = i Z^{\alpha\beta} \bar{\pi}_{\beta;i}$ $Z^{\alpha\beta} \leftrightarrow z_\mu = x_\mu + iy_\mu$
 $\omega^{\dot{\alpha};i} = -i Z^{\dot{\alpha}\beta} \bar{\pi}_{\beta;i}$ $Z^{\dot{\alpha}\beta} \leftrightarrow \bar{z}_\mu = x_\mu - iy_\mu$

If $i=1,2$ ($n=2$):

$$Z^{\alpha\beta} = \frac{i}{f} \bar{\omega}^{\alpha;i} \bar{\pi}_{\beta;i} \quad f = \frac{1}{2} \bar{\pi}_{\alpha;i} \bar{\pi}^{\alpha;i}$$

(metric $a_{ij}b^j = a_i \epsilon^{ij} b_j$)

ii) Composite formulae for momenta:

$$n=1: \quad P_{\alpha\beta} = \pi_\alpha \bar{\pi}_\beta \quad (P_{\alpha\beta} P^{\alpha\beta} = 0 \Leftrightarrow P^2 = 0)$$

$$n > 1: \quad P_{\alpha\beta} = \pi_{\alpha;i} \bar{\pi}_{\beta;i} \quad (P^2 \geq 0)$$

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Canonical Liouville one-form (symplectic potential) on twistor space \mathbb{P}^n :

$$\Theta^{(1)} = \frac{i}{2} \sum_{i=1}^n (\bar{Z}^{\alpha;i} dZ_{\alpha;i} - d\bar{Z}^{\alpha;i} Z_{\alpha;i})$$

corresponds
to $\sum_{i=1}^n p_{\alpha;i} dx^{\alpha;i}$

$$= \frac{i}{2} \sum_{i=1}^n (\bar{\omega}^{\alpha;i} d\pi_{\alpha;i} + \bar{\pi}_{\alpha;i}^i d\omega^{\alpha;i} - H.C.)$$

Poisson brackets: (for $n=1$)

$$\{\pi_\alpha, \bar{\omega}^\beta\} = i \delta_\alpha^\beta \quad \{\bar{\pi}_\alpha, \omega^\beta\} = -i \delta_\alpha^\beta$$

↓ Quantization ↓

$$[\hat{\pi}_\alpha, \hat{\bar{\omega}}^\beta] = -i \delta_\alpha^\beta \quad [\hat{\bar{\pi}}_\alpha, \hat{\omega}^\beta] = i \delta_\alpha^\beta$$

Two polarizations \Rightarrow two different realizations
 \Rightarrow two twistor quantizations

a) conformal-covariant quantization with
complex holomorphic phase space (Penrose 1968)

coordinate space: $Z_A = (\pi_\alpha, \omega^\alpha)$

momentum space: $\bar{Z}_A = (\bar{\pi}_\alpha, \bar{\omega}^\alpha)$

b) quantization with real symplectic phase space

coordinate space: $(\pi_\alpha, \bar{\pi}_\alpha)$ (Woodhouse 1975)

momentum space: $(\omega^\alpha, \bar{\omega}^\alpha)$

a) \Rightarrow provides space-time fields

b) \Rightarrow provides fourmomentum space fields

(2a)

GEOMETRICSPACE-TIME:

Point in complex space-time $C M^{3,1}$ \leftrightarrow 2-plane in T (point in $T \times T$)

Point in real space-time $M^{3,1}$ \leftrightarrow null 2-plane in T
 $\langle z_i, \bar{z}_j \rangle = t_{ij} = 0$
 $i, j = 1, 2$

Light rays in $C M^{3,1}$ \leftrightarrow point in T

Light rays in $M^{3,1}$ \leftrightarrow point in subspace $T^{(0)} \subset T$ of null twistors

light rays one can use for the description of massless objects \leftrightarrow one-twistor theory

Massive, nonconformal, tensionfull objects \leftrightarrow $n \geq 2$ twistors
(Hughston, 1978)

In general $\langle z_i, z_j \rangle = t_{ij} \neq 0$

(3)

Advantage of two-tensor space: one can map

$$(\Pi_{\alpha;i}, \omega^{\dot{\alpha}}_i, \bar{\Pi}_{\dot{\alpha};i}^{\dot{\beta}}, \bar{\omega}^{\alpha;i}) \rightarrow (x_\mu, \dots)$$

But surprise: x_μ defined by standard Penrose incidence relation do not commute:

$$\{x_\mu, x_\nu\} = -\frac{1}{M^2} \epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \quad \text{calculated from twistor PB}$$

$$P^\mu = (\delta^\mu)_{\alpha\beta} \Pi^{\alpha;i} \bar{\Pi}^{\dot{\beta};i} = P_\mu^{(1)} + P_\mu^{(2)} \quad - \text{2-twistor momentum}$$

$$W^\mu = (\delta^\mu)_{\alpha\beta} \Pi^{\alpha;i} \bar{\Pi}^{\dot{\beta};i} \delta(\zeta^r)^i_j t^r \quad - \text{2-twistor Pauli-Lubanski fourvector}$$

where

$$M^2 = P_\mu P^\mu = 2|f|^2 = \frac{1}{2} |\bar{\Pi}_{\dot{\alpha};i} \bar{\Pi}^{\dot{\alpha};i}|^2$$

One can calculate from twistor PB

$$\text{nonvanishing norms} \Rightarrow \{t^r, t^s\} = \epsilon^{rsu} t^u \quad \begin{matrix} r,s=1,2,3 \\ \text{SU}(2) \text{ PB algebra} \\ \text{(spin)} \end{matrix}$$

$$\text{Fourth generator } t^0 = t_1^i + t_2^i \quad t^r = \frac{1}{2} (\delta^r)^i_j t^i$$

$$\{t^0, t^r\} = 0 \quad \begin{matrix} \text{electric charge} \\ \text{extension of SU}(2) \\ \text{to U}(2) \end{matrix}$$

2. TWISTORIAL VERSUS SPACE-TIME FORMULATION OF RELATIVISTIC PARTICLE MODELS

a) One-twistor description of massless particles.

Three equivalent ways of introducing Liouville one-form $\Theta^{(1)}$: $P_{\alpha\beta}$ composite incidence relation $\sum \bar{z}^A dz_A$

$$\Theta^{(1)} = P_{\alpha\beta} dx^{\alpha\beta} \stackrel{i}{\simeq} \Pi_\alpha \bar{\Pi}_\beta dx^{\alpha\beta} \stackrel{i}{\simeq} \frac{i}{2} (\bar{w}^\alpha d\Pi_\alpha + \bar{\Pi}_\alpha d\bar{w}^\alpha)$$

We get subsequently three classically equivalent models of massless relativistic particles with helicity $h=0$:

- ① $S_1^{d=1} = \int d\tau (p_\mu \dot{x}^\mu - \lambda p_\mu p^\mu) \leftarrow \begin{array}{l} \text{relativistic phase} \\ \text{space description} \\ (x^\mu, p^\mu) \end{array}$
- ② $S_2^{d=1} = \int d\tau \Pi_\alpha \bar{\Pi}_\beta \dot{x}^{\alpha\beta} \leftarrow \begin{array}{l} \text{mixed space-time/spins} \\ \text{description} (\Pi_\alpha, \bar{\Pi}_\beta, x^\mu) \end{array}$
- ③ $S_3^{d=1} = \frac{i}{2} \int d\tau \{ (\bar{z}_A \dot{\bar{z}}^A - z_A \dot{\bar{z}}^A) + \lambda \langle z, \bar{z} \rangle \}$

free twistor particle model for null twistors

If helicity $h \neq 0$:

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SU(2,2) — introduce super twistors $Z_R = (z_A, \bar{z})$

$\langle z, \bar{z} \rangle \rightarrow \langle z, \bar{z} \rangle \langle z, \bar{z} \rangle + \langle z, \bar{z} \rangle \langle z, \bar{z} \rangle \leftarrow \text{fermionic}$

SU(2,3) — introduce "fermionic super twistors" $\langle z, \bar{z} \rangle + \langle z, \bar{z} \rangle$

$\{ \text{Bargmann} + \text{Ortal} \text{ arXiv 0704.0296} \} \text{ (S. Fedoruk + J.L. Gómez, hep-th/0506086)}$

(5)

b) Two-twistor description of massive charged particles with spin

Basic 2-twistorial Liouville one-form:

$$\Theta^{(1)} = \frac{i}{2} (z_{A;j} d\bar{z}^{A;j} - \bar{z}^{A;j} dz_{A;j}) \quad j=1,2$$

Introducing composite $z_\mu = x_\mu + i y_\mu$ one gets

$$\Theta^{(1)} = \Pi_{\alpha;j} \bar{\Pi}_{\beta;j}^i dx^{\alpha\beta} + i y^{\alpha\beta} (\Pi_{\alpha;j}^i d\bar{\Pi}_{\beta;j} - \bar{\Pi}_{\beta;j}^i d\Pi_{\alpha;j})$$

where

$$y^{\alpha\beta} = - \frac{1}{2|f|^2} t_i^j \Pi_{\alpha;j}^i \bar{\Pi}_{\beta;j}^i \leftarrow \begin{array}{l} \text{composite} \\ \text{imaginary} \\ \text{part of } C \end{array}$$

and one gets

$$(A) \Rightarrow \Theta^{(1)} = \Pi_{\alpha;j} \bar{\Pi}_{\beta;j}^i dx^{\alpha\beta} + \frac{i}{2} t_i^j \left(\frac{1}{f} \Pi_{\alpha;j}^i d\bar{\Pi}_{\beta;j} + \frac{1}{f} \bar{\Pi}_{\beta;j}^i d\Pi_{\alpha;j} \right)$$

This is basic Liouville one-form with noncommutative x

Two options:

- 1) To consider (A) as basis for the action of spinning massive particles (level 2)

$$S_2^{d=1} = \int d\sigma \left\{ \Pi_{\alpha;j} \bar{\Pi}_{\beta;j}^i \dot{x}^{\alpha\beta} + \frac{i}{2} t_i^j \cdot \left(\frac{1}{f} \Pi_{\alpha;j}^i \dot{\bar{\Pi}}_{\beta;j} + \frac{1}{f} \bar{\Pi}_{\beta;j}^i \dot{\Pi}_{\alpha;j} \right) + 2a R_a \right. \\ \left. a=1,2,3,4 \right\}$$

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Four physical constraints:

$$R_1 = 4\bar{f}f - m^2 = 0 \quad \leftarrow \text{mass constraints}$$

$$R_2 = \vec{t}^2 - s(s+1) = 0 \quad \leftarrow \text{spin}$$

$$R_3 = t_3 - m_3 = 0$$

$$R_4 = t_0 - q = 0 \quad \leftarrow \begin{matrix} \text{internal} \\ \text{Abelian charge} \end{matrix} \text{ (electric)}$$

This is a two-funder generalization of
Shupe Fuji model $S_{2-}^{d=1}$

one funder:

$$(x^\alpha, \Pi_\alpha, \bar{\Pi}_\beta) \Rightarrow (x^\alpha, \Pi_{\alpha;i}, \bar{\Pi}_{\beta; i}, t_i)$$

8 variables

two funders:

$$\uparrow \quad 16 \text{ variables}$$

noncommutative

Quantization:

Due to constraints the independent degrees of freedom which commute ("coordinates") are

$$8 \text{ degrees} \Rightarrow X^8 = (y_0 = \frac{1}{m^2} x_\mu p^\mu, \Pi_{\alpha;i}, \bar{\Pi}_{\alpha; i}) \quad \begin{matrix} \nearrow \text{contains} \\ \text{fourmomentum} \end{matrix} \quad \boxed{|\Pi_{\alpha;i} \Pi^{\alpha;i}|^2 = m^2}$$

The wave function $\Psi(X) = \Psi(y_0, \pi, \bar{\pi})$

Dependence on $y_0 \rightarrow$ solution of mass constraint

$$\frac{1}{i} \frac{\partial}{\partial y_0} \Psi(y_0, \pi, \bar{\pi}) = m^2 \Psi(y_0, \pi, \bar{\pi})$$

$$\Rightarrow \Psi(y_0, \pi, \bar{\pi}) = e^{-im^2 y_0} \chi(\pi, \bar{\pi}) = e^{-ipX} \chi(\pi, \bar{\pi})$$

Plane wave solutions with x_μ noncommutative

Remaining three equations:

$$\stackrel{\text{spin}}{\left\{ \begin{array}{l} (\hat{T}_r \hat{T}_r - s(s+1)) \chi(\pi, \bar{\pi}) = 0 \\ (\hat{T}_3 - m_s) \chi(\pi, \bar{\pi}) = 0 \end{array} \right.} \quad r=1,2,3$$

Abelian
charge $\Rightarrow (\hat{T}_0 - q) \chi(\pi, \bar{\pi}) = 0$

where \hat{T}_a is the differential realization of t_a :

$$\hat{T}_a = \frac{1}{2} (\delta_a)_{jk}^{\mu\nu} (\Pi_{\mu k} \frac{\partial}{\partial \Pi_{j;i}} - \bar{\Pi}_{i;j} \frac{\partial}{\partial \bar{\Pi}_{i;k}})$$

Doubly infinite spin/charge multiplets

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \quad q = \dots -2, -1, 0, 1, 2 \dots$$

for details \Rightarrow (Fedoruk, Fydrynski, J.L., Miguel-Espanya)
hep-th/0510266

ii) second option is to introduce commuting components $X_\mu = X_\mu + \Delta X_\mu$ by using modified Penrose incidence formula:

$$\tilde{\omega}^{\alpha;i} = i \left(\sum \tilde{\omega}^{\beta} \bar{\Pi}_{\beta;i} + \frac{t_1 - it_2}{f} \Pi^{\alpha;i} \right) \quad \begin{aligned} \sum \tilde{\omega}^\beta &= \\ &= \tilde{X}^{\alpha\beta} + i \tilde{Y}^{\alpha\beta} \end{aligned}$$

One gets (Bette, de Arcarraga, Miguel-Espanya, J.L.)
hep-th/0405166

$$\tilde{X}^{\alpha\beta} = X^{\alpha\beta} - \frac{1}{2f^2} \epsilon_{\alpha\beta\gamma\delta} \text{tr} \Pi^{\alpha;i} (\delta_s)_{i;\delta} \bar{\Pi}_{\beta;j}$$

commuting space-time coordinates break
"internal" SU(2) invariance to O(2)

Substituting \tilde{x}^{μ} in $\Theta^{(1)}$ one gets

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$$\Theta^{(1)} = p_{\mu} \tilde{dx}^{\mu} - i(\bar{\sigma}^{\dot{\alpha}};_1 d\bar{\pi}_{\dot{\alpha};2} - \sigma^{\alpha;1} d\pi_{\alpha;2}) + e d\phi$$

where the composite variables are

$$\begin{aligned}\sigma^{\alpha;i} &= \frac{1}{f} t_r(\sigma_r);_j \pi^{\alpha;j} & \bar{\sigma}^{\dot{\alpha};i} &= -\frac{1}{f} t_r(\bar{\sigma}_r);_j \bar{\pi}^{\dot{\alpha};j} \\ e &= t_0 + t_3 & \phi &= \frac{i}{2} \ln \frac{\tilde{t}}{f}\end{aligned}$$

We have 18 real variables ($\sigma_{\alpha} \equiv \sigma_{\alpha;1}$, $\tau_{\alpha} \equiv \tau_{\alpha;1}$)

$$x_{\mu}, p_{\mu}, \sigma_{\alpha}, \bar{\sigma}_{\dot{\alpha}}, \tau_{\alpha}, \bar{\tau}_{\dot{\alpha}}, e, \phi$$

$\tau_{\alpha}, \bar{\tau}_{\dot{\alpha}}$
primary!

which satisfy two identities

$$\begin{aligned}\tilde{R}_5 &= \tau_{\alpha} P^{\alpha\beta} \bar{\tau}_{\beta} - \frac{1}{2} P_{\alpha\beta} P^{\alpha\beta} = 0 \\ \tilde{R}_6 &= \tau^{\alpha} \sigma_{\alpha} - \bar{\tau}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}} = 0\end{aligned}\quad \left. \begin{array}{l} \text{encoding} \\ \text{Composite} \\ \text{structure} \\ 18-2=16 \\ \text{electric charge} \\ \downarrow \downarrow \end{array} \right\}$$

The action:

$$S_1^{d=1} = \int dt \{ p_{\mu} \dot{x}^{\mu} - i(\bar{\sigma}^{\dot{\alpha}} \dot{\bar{\pi}}_{\dot{\alpha}} - \sigma^{\alpha} \dot{\pi}_{\alpha}) + e \dot{\phi} + \sum_{A=1,2,\dots,6} \lambda_A \tilde{R}_A \}$$

\tilde{R}_5, \tilde{R}_6 - algebraic; $\tilde{R}_1, \dots, \tilde{R}_4$ - physical constraints:

$$\tilde{R}_1 = p^2 - m^2 = 0$$

$$\vec{t}^2 = t_1^2 + t_2^2 + t_3^2$$

$$\tilde{R}_2 = \vec{t}^2 - s(s+1) = 0$$

$$t_1 + it_2 = -\frac{1}{f} \bar{\sigma}_{\alpha} P^{\alpha\beta} \bar{\tau}_{\beta}$$

$$\tilde{R}_3 = t_3 - s_3 = 0$$

$$t_3 = \tau^{\alpha} \sigma_{\alpha}$$

$$\tilde{R}_4 = e - e_0 = 0$$

difference with R_4 : $t_0 \rightarrow e = t_0 + t_3$

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Quantization:

- \tilde{R}_5, \tilde{R}_6 are second class constraints, $\tilde{R}_1, \dots, \tilde{R}_4$ are first class
- one can introduce Dirac brackets, eliminating the constraints \tilde{R}_5, \tilde{R}_6 from the phase space
- differential realization of Dirac brackets in 8-dimensional "generalized momenta" space

$$P^8 = (p_\mu, \pi_\alpha, \bar{\pi}_{\dot{\alpha}}, \varphi; \tilde{R}_5 = 0) \leftarrow \text{Abelian}$$

One gets for $\sigma_\alpha, \bar{\sigma}_{\dot{\alpha}}, e$:

$$\hat{\sigma}^\alpha = \frac{\partial}{\partial \pi_\alpha} - p^\alpha \bar{\pi}_{\dot{\beta}} \frac{1}{p^2} \mathcal{D}$$

$$\hat{\bar{\sigma}}^{\dot{\alpha}} = - \frac{\partial}{\partial \bar{\pi}_{\dot{\alpha}}} + \pi_\alpha p^{\alpha \dot{\beta}} \frac{1}{p^2} \mathcal{D}$$

$$e = \frac{1}{i} \frac{\partial}{\partial \varphi}$$

$$\mathcal{D} = \pi_\alpha \frac{\partial}{\partial \pi_\alpha} + \bar{\pi}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\pi}_{\dot{\alpha}}}$$

The constraints $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_4 \rightarrow$ four wave equations with eigenvalues m, s, s_3, e_0

General solution:

$$\Psi(p_\mu, \pi_\alpha, \bar{\pi}_{\dot{\alpha}}, \varphi) = \sum_{n,m=0} \sum_{\substack{(\alpha_1 \dots \alpha_n) \\ (\beta_1 \dots \beta_m)}} \pi_{\alpha_1} \dots \pi_{\alpha_n} \bar{\pi}_{\dot{\beta}_1} \dots \bar{\pi}_{\dot{\beta}_m}$$

$$\cdot \Psi^{(\alpha_1 \dots \alpha_n)(\beta_1 \dots \beta_m)}(p_\mu, \varphi)$$

satisfies Bargmann-Wigner eq.
for free higher spin fields

phase dependence
 $\exp(i e_0 \varphi)$ determines
the charge e_0

for details → (Azcarraga, Fringuez, J.L., Miquel-Espanya,
hep-th/0510161)

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3. BOSONIC D=4 STRING MODELS FROM TWISTORS.

a) Two Hamiltonian descriptions of

a1) World-sheet momentum one-form (Negele, 1996) Nambu-Goto string

$$\Theta^{(1)} = P_\mu dx^\mu \Rightarrow \Theta^{(2)} = \underbrace{P_\mu^m d\xi_m}_m \wedge dx^\mu \quad m=0,1$$

P_μ^0, P_μ^1 - generalized string momenta $\xrightarrow{\text{one-form}} [P_\mu^m] = M$

The action:

$$[T] = M^2$$

$$S_1^{d=2} = \int d^2\xi \left[\tilde{P}_\mu^m \partial_m X^\mu + \frac{1}{2T} (-g)^{-\frac{1}{2}} g_{mn} P_\mu^m P_\nu^n \right]$$

Equations of motions:

$$(1) \quad \partial_m P_\mu^m = 0 \quad \leftarrow \delta X$$

$$(2) \quad P_\mu^m = -T(-g)^{\frac{1}{2}} g^{mn} \partial_n X_\mu \quad \leftarrow \delta P$$

$$(3) \quad P_\mu^m P_\nu^n = \frac{1}{2} g^{mn} g_{KL} P_\mu^K P_\nu^L \quad \leftarrow \delta g$$

Inserting P_μ^1 one gets $(P_\mu^0 \equiv P_\mu)$

$$S_1^{d=2} = \int d^2\xi \left[P_\mu \dot{X}^\mu - \frac{\lambda}{2} \left(T^{-1} P_\mu^2 + T X_\mu^{12} \right) - g P_\mu X^\mu \right]$$

This is Hamiltonian formulation of Nambu-Goto string (with two Virasoro constraints):

$$S^{d=2} = -T \int d^2\xi \sqrt{-\det G^{(2)}} \quad G_{mn}^{(2)} = \partial_m X^\mu \partial_n X_\mu$$

(11)

a2) Tensorial momenta formulation (Gursey 1975)

$$\theta^{(1)} = P_u dx^u \rightarrow \theta^{(2)} = \underline{P_{uv} dx^u \wedge dx^v}$$

$$dS^{uv} = dx^u \wedge dx^v \Leftarrow \underline{\text{dynamical world sheet}} \\ \underline{\text{surface element}}$$

The action:

$$\tilde{S}_1^{d=2} = \sqrt{2} \int d^2\zeta \left[P_{uv} \partial_m X^u \partial_n X^v \epsilon^{mn} - \Lambda \left(P_{uv} P^{uv} + \frac{T^2}{4} \right) \right]$$

Recalls action for massive relativistic particle with
mass m \leftrightarrow tension T

Equations of motion - the algebraic one:

$$(2') P^{uv} = \frac{1}{2\Lambda} T^{uv} \quad \gamma^{uv} = \partial_m X^u \partial_n X^v \epsilon^{mn} \Leftarrow 8P_{uv}$$

and one gets using $\det G^{(2)} = \frac{1}{2} T^{uv} T_{uv}$

$$\tilde{S}_1^{d=2} = \frac{1}{\sqrt{2}} \int d^2\zeta \left[\Lambda^{-1} \det G^{(2)} - \Lambda T^2 \right]$$

Eliminating Λ one gets the Nambu-Goto
(equivalent to Polyakov action) action.

Remark: P_{uv} (see (2')) satisfies the algebraic
relations:

$$P_{uv} P^{uv} + \frac{T^2}{4} = 0 \quad \underline{T=0: \text{Schild action}}$$

$$P_{uv} \tilde{P}^{uv} = 0 \quad \tilde{P}^{uv} = \frac{1}{2} \epsilon^{uvgc} P_{gc}$$

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b) From space-time to torsorial formulation

b1) Siegel formulation

String generalization of composite momentum

$$P_{\alpha\beta}^m = \sqrt{-g} e_a^m \bar{\lambda}_{\dot{\beta}}^i (\eta^a)_{ij} \lambda_{\dot{a}}^j \quad [\lambda_{\dot{a}}^i] = M$$

zweibein $D=2$ Dirac $D=2$ fields
 matrices ($a=0,1$) $i,j=1,2$

One gets the action (Soroka, Sorokin, Tkach, Volkov 1989)

$$\begin{aligned} S_2^{d=2} = & \int d^2 \xi \sqrt{-g} \left[\bar{\lambda}_{\dot{\beta}}^i (\eta^a)_{ij} \lambda_{\dot{a}}^j \partial_a X^{\dot{a}\dot{\beta}} + \right. \\ & \left. + \frac{1}{2T} (\lambda^{\dot{a}i} \lambda_{\dot{a}i}) (\bar{\lambda}_{\dot{a}}^j \bar{\lambda}_{\dot{a}j}) \right] \end{aligned}$$

intermediate
 spinor-
 space-time
 (1) level

Phase space description ($\lambda_{\dot{a}i}^* = \bar{\lambda}_{\dot{a}}^i$ etc.):

$$(X_{\alpha\beta}, \lambda_{\dot{a}i}, \bar{\lambda}_{\dot{a}}^i, e_m^a) + (P_{\alpha\beta}, \Pi^{\dot{a}i}, \bar{\Pi}^{\dot{a}i}, p^{(e)a})$$

"Coordinates" "momenta"

One can derive two first class constraints

$$F = \lambda_{\dot{a}i} \Pi^{\dot{a}i} + \bar{\lambda}_{\dot{a}}^i \bar{\Pi}^{\dot{a}i} - 2 e_m^a p^{(e)a} = 0$$

$$G = i(\lambda_{\dot{a}i} \Pi^{\dot{a}i} - \bar{\lambda}_{\dot{a}}^i \bar{\Pi}^{\dot{a}i})$$

generating two local gauge invariances of SSTV model:

$$\lambda'_{\dot{a}i} = e^{i(b+ic)} \lambda_{\dot{a}i} \quad \bar{\lambda}'_{\dot{a}}^i = e^{-i(b-ic)} \bar{\lambda}_{\dot{a}}^i \quad e'_m = e^c e_m^a$$

One can fix the gauges (b, c) as follows:

$$A = \lambda_{\alpha i} \lambda^{\alpha i} - T = 0 \quad \bar{A} = \bar{\lambda}_{\dot{\alpha} i} \bar{\lambda}^{\dot{\alpha} i} - T = 0 \quad T \text{ real}$$

i.e. the last term in SSTV action is like a $D=2$ cosmological term

$$\frac{T^2}{2T} \sqrt{-g} \quad \sqrt{-g} = \det e \equiv e$$

We use world-sheet Penrose incidence relations

$$M_i^\alpha = \lambda_{\alpha i} X^{\alpha i} \quad \bar{M}_i^\alpha = X^{\alpha i} \bar{\lambda}_{\alpha i} \quad [u_i^\alpha] = 1$$

and observe that the reality of X_μ requires

$$t^i_j = \bar{\lambda}_{\dot{\alpha} i} M_j^\alpha - \bar{M}^{\dot{\alpha} i} \lambda_{\dot{\alpha} j} = 0 \Leftarrow \begin{array}{l} \text{two} \\ \text{null} \\ \text{factors!} \end{array}$$

We get after eliminating $X^{\alpha i}$

$$S_3^{d=2} = \int d^2 \xi \left\{ \frac{1}{2} e e_m^a \left[\bar{\lambda}_{\dot{\alpha} i} (\bar{g}^a)_{ij} \partial_m \lambda^{\dot{\alpha} j} - u^{\dot{\alpha} i} (\bar{g}^a)_{ij} \partial_m \lambda^{\dot{\alpha} j} + \text{c.c.} \right] + \frac{T}{2} e + \Lambda_i^j t^i_j + \Lambda A + \bar{\Lambda} \bar{A} \right\}$$

Further step: we solve algebraic eq. for zweibein Lagrange multiplier

$$e_m^a = -\frac{1}{T} \left[\partial_m \bar{Z}^{\dot{\alpha} i} (\bar{g}^a)_{ij} Z_A^j - \bar{Z}^{\dot{\alpha} i} (\bar{g}^a)_{ij} \partial_m Z_A^j \right]$$

where $Z_A^i = (\lambda_{\alpha i}^i, \mu^{\dot{\alpha} i})$, $\bar{Z}^{\dot{\alpha} i} = (\bar{\mu}^{\dot{\alpha} i}, -\bar{\lambda}^{\dot{\alpha} i})$

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Further we recall that

$$t_i \delta = Z_{Ai} \bar{Z}^{Aj}$$

After eliminating zweibein e^m_a one gets

$$S_3^{d=2} = \int d^2\zeta \left\{ \frac{1}{4T} \epsilon_{ab} \left[\partial_m \bar{Z}^A g^{ab} Z_A - \bar{Z}^A g^{ab} \partial_m Z_A \right] + \text{twistorial!} \cdot \left[\partial_m \bar{Z}^A g^b Z_A - \bar{Z}^A g^b \partial_m Z_A \right] + \Lambda_f^i (Z_{Ai} \bar{Z}^{Aj}) + \Lambda_A + \bar{\Lambda} \bar{A} \right\}$$

We get basic fourlinear twistor string action

The action can be derived from the following Liouville two-form:

$$\theta^{(2)} = \underbrace{\theta_1^{(1)} \wedge \theta_2^{(1)}}_{\theta^{(2)}}$$

by introducing the world sheet embedding

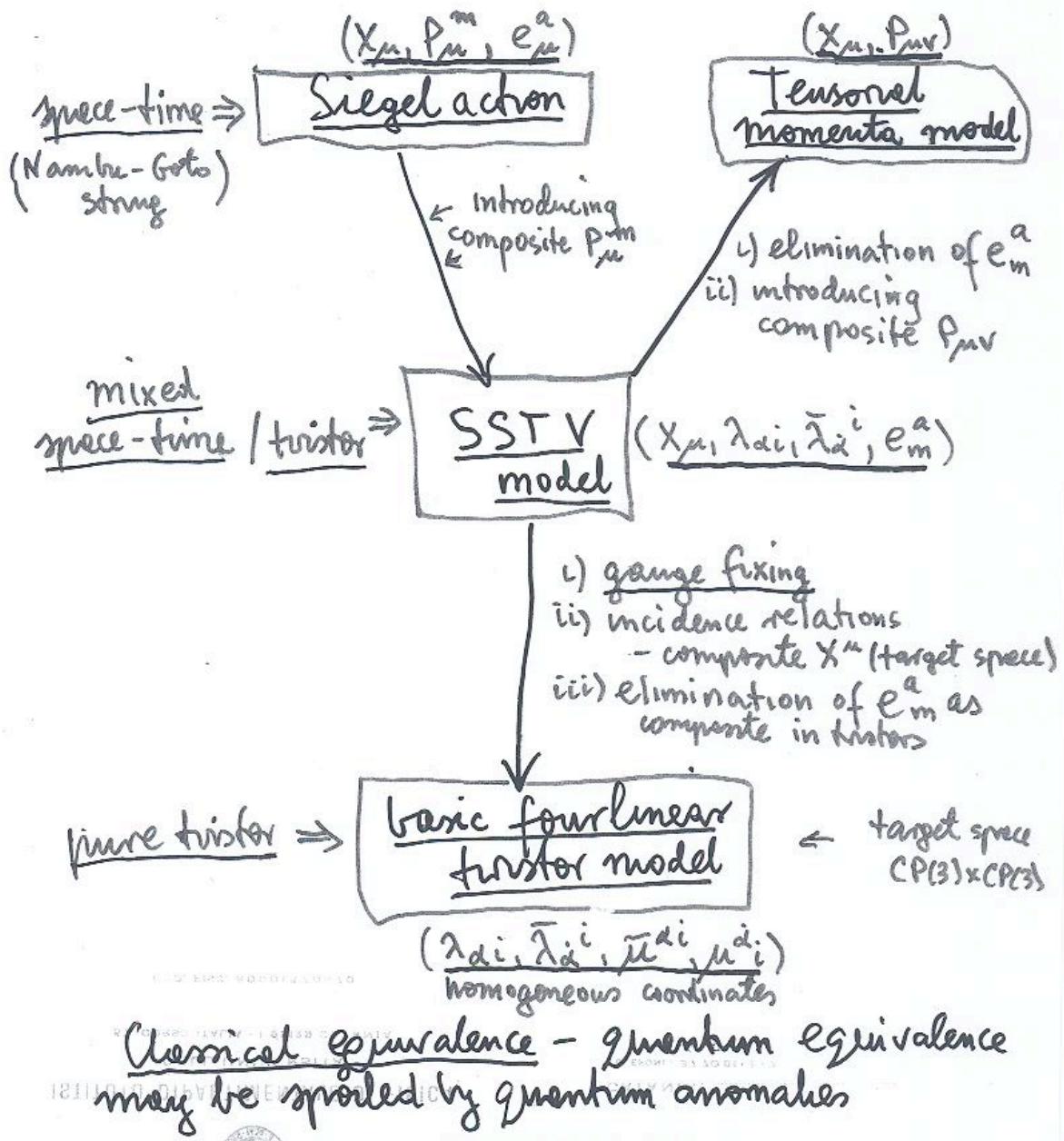
$$(\sigma, \tau) \rightarrow Z_A^i(\sigma, \tau)$$

Extension to bosonic twistor p-brane as composite of $p+1$ twistor fields $Z_A^i(\sigma_1 \dots \sigma_{p+1}, \tau)$

$$\theta^{(p+1)} = \underbrace{\theta_1^{(1)} \wedge \theta_2^{(1)} \wedge \dots \wedge \theta_{p+1}^{(1)}}_{\theta^{(p+1)}} \quad \begin{matrix} m \\ (p+1)\text{-dim.} \end{matrix} \text{fields}$$

$$p=2 \text{-membrane} \Rightarrow \text{Sect. 4}$$

LINKS BETWEEN CLASSICAL BOSONIC STRING ACTIONS (Fedoruk + J.L. hep-th/0606245)



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Quantization:

The four linear twistor string action is
linear in time derivative:

$$\frac{1}{T} \epsilon^{mn} (\bar{Z}^{A1} \partial_m Z_{A1} - \partial_m \bar{Z}^{A1} Z_{A1}) \cdot \begin{array}{l} m,n=0,1 \\ A,B=1..4 \end{array}$$

$$\cdot (\bar{Z}^{B2} \partial_n Z_{B2} - \partial_n \bar{Z}^{B2} Z_{B2}) =$$

$$= Q_2 (\bar{Z}^{A1} \dot{Z}_{A1} - \dot{\bar{Z}}^{A1} Z_{A1}) -$$

$$- Q_1 (\bar{Z}^{A2} \dot{Z}_{A2} - \dot{\bar{Z}}^{A2} Z_{A2})$$

where

$$Q_i = \frac{1}{T} (\bar{Z}^{Bi} Z'_{Bi} - \bar{Z}'^{Bi} Z_{Bi}) \quad \left(\begin{array}{l} \dot{Z} = \frac{\partial Z}{\partial \tau} \\ Z' = \frac{\partial Z}{\partial \sigma} \end{array} \right)$$

Constraints:

a) $D^{Ai} = P^{Ai} - \epsilon^{ij} Q_j \bar{Z}^{Ai} \approx 0$

b) $\bar{D}_{Ai} = \bar{P}_{Ai} + \epsilon_{ij} Q_j Z_{Ai} \approx 0 \quad Q_j = Q_j(Z)$

c) $V_{ij} = Z_{Ai} \bar{Z}^{Aj} \approx 0 \quad \text{null string pistons}$

-AB-
ejectors

d) $I_{AB} Z^{A1} Z^{B2} = \frac{T}{2} \quad (\leftarrow \lambda_{Ai} \lambda^{Ai} = T)$

e) $I_{AB} \bar{Z}^{A1} \bar{Z}^{B2} = \frac{T}{2} \quad (\leftarrow \lambda_{Ai} \lambda^{Ai} = T)$

DIRAC QUANTIZATION of a)-e) \Rightarrow under consideration

(17)

4. TWO-TWISTOR DESCRIPTION OF MEMBRANE

The presented scheme for strings can be generalized in two-twistor space to membranes.

For p -branes - one needs $2^{\left[\frac{p+1}{2}\right]}$ twistors

a) Extension of Siegel momentum formulation to arbitrary p

$$\theta^{(1)} = p_\mu dx^\mu \rightarrow \theta^{(p+1)} = \overset{P\text{-form}}{P_\mu^{(p)}} \wedge dx^\mu$$

Siegel action

$$S = \int d^{p+1} \xi (P_\mu^m \partial_m X^\mu + \frac{1}{2T} (-g)^{-\frac{1}{2}} g_{mn} P_\mu^m P^\mu_n + \frac{T}{2} \sum_{m,n} (p-1) (-g)^{\frac{1}{2}})$$

Polyakov-type action

$$\Downarrow P_\mu^m = -T(-g)^{\frac{1}{2}} g^{mn} \partial_n X_\mu$$

$$\Downarrow S = -\frac{T}{2} \int d^{p+1} \xi (-g)^{\frac{1}{2}} [g^{mn} \partial_m X^\mu \partial_n X_\mu - (p-1)]$$

Hamiltonian

Putting $\dot{P}_\mu = P_\mu$, eliminating P_μ^m

$$S = P_\mu \dot{X}^\mu - \frac{\sqrt{-g}}{2} \det(h^{mn}) \left[\frac{1}{2} P_\mu P^\mu + T \det g_{mn} \right]_{m=1,2,\dots,p} - h_{\mu n} h^{nm} (P_\mu \partial_m X^\mu)$$

($p+1$) Virasoro conditions

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b) Intermediate spinor-space-time formulation

In case of arbitrary Cartan-Penrose relation

$$P_\mu^m = \sqrt{-g} e_a^m \bar{\lambda}^{ai} (\varrho^a)_i{}^j \lambda_{\beta j} (\gamma^\mu)_a{}^{\hat{\beta}}$$

determine $(p+1)$ -bein world-volume
 Dirac matrices $i = 1 \dots 2^{\lfloor \frac{p+1}{2} \rfloor}$ $\hat{\beta} = 1 \dots 2^{\lfloor \frac{D}{2} \rfloor}$
 \uparrow \uparrow \uparrow
 D-dimensional
 Dirac matrices

For $D=4$ membrane ($i, j = 1, 2$, $a, m = 0, 1, 2$)

$$P_{\alpha\beta}^m = \sqrt{-g} \bar{\lambda}_{\beta}{}^i (\varrho^a)_i{}^j \lambda_{\alpha j}$$

One gets

$$S = \int d^3\xi \sqrt{-g} (\bar{\lambda}_\beta \varrho^m \lambda_\alpha \partial_m \chi^{\beta\alpha} + 2T + \lambda A)$$

Lagrange
 multiplier

The constraint: $A = (\lambda_\alpha \lambda^{ai})(\bar{\lambda}_\beta \bar{\lambda}^{aj}) - 2T^2 = 0$

This "membrane condition" can be derived in Siegel formulation as expressing the following constraint:

$$h_{mn} P_\mu^m P^{\mu n} = -\sqrt{-g} T^2$$

Difference with string case: There $A=0$ can be only achieved as local gauge fixing
 - for membrane $A=0$ follows from field eq.

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c) Purely twistorial membrane action ($D=4$)

$$\mu_i^\alpha = X^\alpha \beta^\lambda \lambda_{\beta i} \quad \bar{\mu}^{\dot{\alpha} i} = \bar{\lambda}_{\dot{\alpha}}^c X^{\dot{\alpha} c}$$

$X^{\alpha \beta}$ $\Leftrightarrow V_i{}^j = \lambda_{\alpha i} \bar{\lambda}^{\dot{\alpha} j} - u_i{}^{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha} j} = 0$
Hermitian null twistors

If

$$Z_{Ai} = (\lambda_{Ai}, \mu_i^\alpha), \quad \tilde{Z}^{Ai} = (\bar{\mu}^{\dot{\alpha} i}, -\bar{\lambda}^{\dot{\alpha} i})$$

one gets $\overset{\text{date}}{\rightarrow}$

$$S = \int d^3\xi \left[\frac{1}{2} \sqrt{-g} e_a^m (\partial_m \tilde{Z}^A g^{ab} Z_A - \tilde{Z}^A g^{ab} \partial_m Z_A) + 2 \sqrt{-g} T + \Lambda A + \Lambda_i{}^j V_j{}^i \right]$$

Eliminating e_a^m (dreibein) one gets
purely twistorial action:

$$S = -\frac{1}{48T^2} \int d^2\xi (\epsilon_{abc} \epsilon^{mnk} \theta_{(1)m}^a \theta_{(1)n}^b \theta_{(1)k}^c + \Lambda A + \Lambda_i{}^j V_j{}^i)$$

where

$$\theta_{(1)m}^a = \frac{\partial \tilde{Z}^A}{\partial \xi_m} (\eta^a)_i{}^j Z_{Aj} - \tilde{Z}^A (\eta^a)_i{}^j \frac{\partial Z_{Aj}}{\partial \xi_m}$$

Action induced on world-volume by 3-form:

$$\theta_{(3)} = \epsilon_{abc} \theta_{(1)}^a \wedge \theta_{(1)}^b \wedge \theta_{(1)}^c$$

(2a)

5. OUTLOOK

i) Arbitrary D

One can look on twists in two ways:

- as describing the space of all complex structures on \mathbb{R}^{2k} \Rightarrow pure spinors approach (Hugston, Shaw, Mason, Bedcovits)
It has advantages if we pass to curved spaces.

- as fundamental representation of (generalized) conformal group (Cederwall, Solitarev, Kharlov group)

In superstring theory:

interesting $D = 3, 4, 6, 10$

In M-theory (supermembrane): $D = 11$

In $D = 3, 4, 6, 10$ one can introduce symmetries corresponding to (super) groups on fields R, C, H, O

(Lorentz, conformal and superconformal symmetries)

One gets

$$D=3 \xleftarrow{C \rightarrow R} \text{Penrose formalism} \xrightarrow{C \rightarrow H} D=6$$

Moufang plane \Rightarrow For octonions does not work (only $SL(2; \mathbb{O}) \cong O(9, 1)$ can be correctly formulated)

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In $D=10$:Lorentz spinors:16-component real
Majorana-Weyl spinorstwisters:

32-component, real

If unconstrained \Rightarrow generalized conformal group $Sp(32)$ generalized superconf. group $OSp(1|32)$
 $Sp(32) \rightarrow O(10,2) \leftrightarrow$ constraints (Vasiliev)ii) In $D=11$: M-theory, no pure spinorsMakes more sense unconstrained

- spinors - 32 real components

- twisters - 64 real components

because of extension to M-algebra:

$$\{Q_A, Q_B\} = P_{AB} \quad A, B = 1 \dots 32$$

$$P_{AB} = (\gamma^{\mu} C)_{AB} P_{\mu} + (\gamma^{[uv]} C)_{AB} Z_{[uv]} +$$

$$+ (\gamma^{[u_1 \dots u_5]} C)_{AB} Z_{[u_1 \dots u_5]} \quad \begin{matrix} 528 \\ \text{Abelian} \\ \text{charges} \end{matrix}$$

M-algebraic extension of Penrose-Carter

$$\{Q_A, Q_B\} = P_{AB} = \sum_{i=1}^n \lambda_A^{(i)} \lambda_B^{(i)} \quad \text{formulae:} \quad 0 \leq n \leq 32$$

 $\lambda_A^{(i)}$ are called BPS preons (Bando, Azucarape, Izquierdo, J.L., 2001)