Refining MHV Diagrams

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hep-th/0612007 AB-Spence-Travaglini
arxiv:0704.0245 [hep-th] AB-Spence-Travaglini-Zoubos

Outline

- **Motivation**
- **MHV Diagrams**
  - Tree MHV Diagrams (Cachazo-Svrcek-Witten)
  - Loop MHV Diagrams (AB-Spence-Travaglini)
- **One-loop Amplitudes in Pure Yang-Mills from MHV diagram**
  - The all-minus helicity amplitudes
  - MHV diagrams from LC Lagrangian (Gorsky-Rosly, Mansfield, Morris-Ettle)
  - Completion of MHV diagram method
    - all-plus helicity amplitude from Lorentz violating counterterm
    - non-canonical, "holomophic" field transformation
- **Conclusions**

Goal

- The goal of this talk is to address two questions:

  I. Do MHV Diagrams (a la CSW) provide a new, complete, perturbative expansion of Yang-Mills?

  II. And if not can we find a suitable completion of the MHV diagram method?

Motivations

- Simplicity of scattering amplitudes unexplained by textbook Feynman methods: e.g. Park-Taylor formula for MHV tree amplitude $<--+++\ldots+>$
- New ideas from twistor string theory (initiated by Witten 2003) help to "explain" this simplicity
- Novel, practical tools to calculate physically relevant amplitudes ➡ LHC is coming
- New reorganisations of conventional perturbation theory

Scattering Amplitudes

Color Ordering:

$$\mathcal{A}^{tree}(\{p_i,\varepsilon_i\}) = \sum_\sigma Tr(T^{a_{\sigma_1}} \cdots T^{a_{\sigma_n}})\ \mathcal{A}(\sigma(p_1,\varepsilon_1),\ldots,\sigma(p_n,\varepsilon_n))$$

Colour-ordered partial amplitude

Spinor Helicity: $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$

$$\langle 12\rangle := \varepsilon_{ab}\lambda_1^a\lambda_2^b \quad , \qquad [12] := \varepsilon_{\dot{a}\dot{b}}\tilde{\lambda}_1^{\dot{a}}\tilde{\lambda}_2^{\dot{b}}$$

MHV Tree Amplitudes:

(Parke-Taylor)

$$\mathcal{A}(1^+,2^+,\ldots n^+) = 0\ ,$$

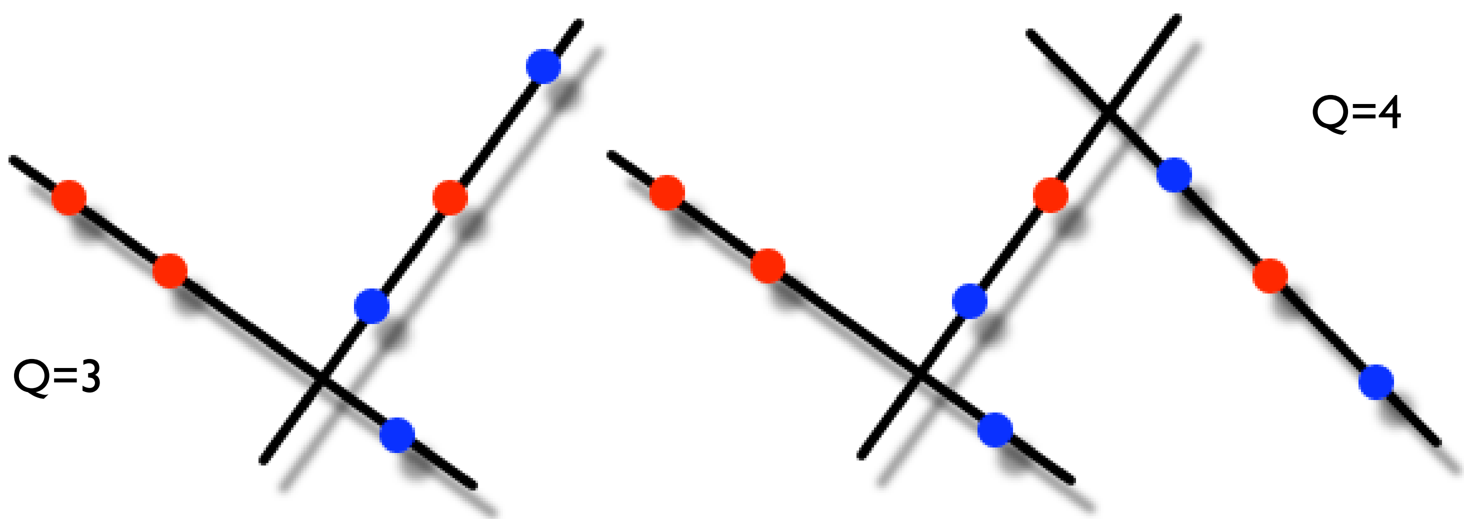
$$\mathcal{A}(1^-,2^+,\ldots n^+) = 0$$

Simple & Holomorphic!

$$\mathcal{A}_{MHV}(1^+\ldots i^-\ldots j^-\ldots n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

Amplitudes in Twistor Space

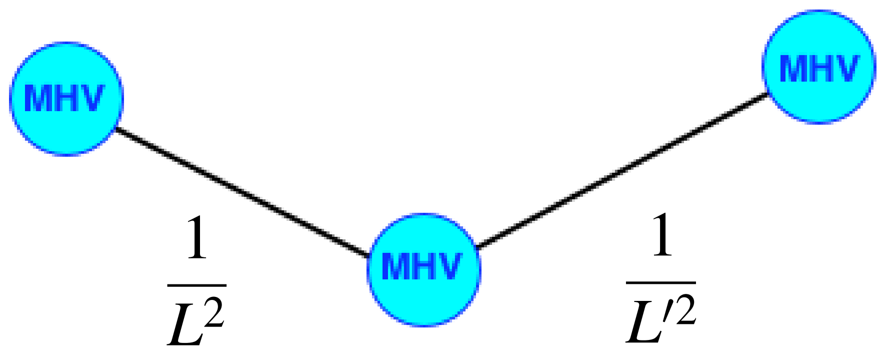
- Witten (2003): Simplicity can be "explained"
  - MHV Amplitudes holomorphic ➔ line in twistor space
- moreover: L-loop amplitudes with Q negative helicity gluons localise in twistor space on curves of degree=Q-1+L and genus<=L



MHV Diagrams

- MHV amplitude (holomorphic) = line in twistor space = pointlike interaction in Minkowski space (Witten, Cachazo-Svrcek-Witten)
- CSW Rules (Cachazo-Svrcek-Witten)
  - MHV amplitudes promoted to local vertices
    - off-shell continuation $L_\mu = l_\mu + z\,\eta_\mu\ ,\ \lambda_a \sim L_{a\dot{a}}\tilde{\eta}^{\dot{a}}$
  - Connect MHV vertices with scalar propagators
  - Note: η is related to LC gauge fixing

MHV Diagrams cont'd



typical MHV diagram
contributing to
an NNMHV amplitude

-

- Covariance (= η−independence) is achieved after summing all MHV diagrams (CSW)

- Equivalence with results of Feynman diagram calculations

  - On-Shell Recursion Relations (Britto-Cachazo-Feng-Witten)

  - MHV diagrams = special recursion relation (Risager) (shifts invisible since MHV amplitudes are holom.)

From Trees to Loops (AB-Spence-Travaglini)

- Original prognosis from twistor string theory was negative (Berkovits-Witten), Conformal SUGRA modes spoil duality

- Try anyway:

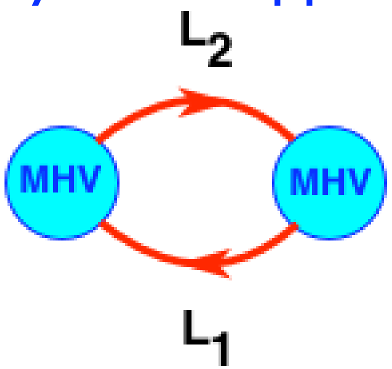
  - Connect MHV vertices, using the same off-shell continuation as for trees

  - sum over all MHV diagrams and internal particle species

  - Find measure, perform loop integration

  - different from unitarity-based approach

$$\int d\mathcal{M} \sum_{m_1, m_2, h}$$



The loop measure

$$d\mathcal{M} = \frac{d^4 L_1}{L_1^2 + i\varepsilon} \frac{d^4 L_2}{L_2^2 + i\varepsilon} \delta^{(4)}(L_2 - L_1 + P_L)$$

- Use $L = l + z\eta$ and $L \to (l, z)$

$$\frac{d^4 L}{L^2 + i\varepsilon} = \frac{dz}{z} \quad \times \quad d^4 l \, \delta^{(+)}(l^2)$$

dipersive measure **x** phase space measure

- Loop integral becomes (use dimensional regularisation) (Dispersion integral) x (2-particle LIPS integral)
- Sum of all MHV diagrams is covariant
- MHV 1-loop amplitudes in N=4/N=1 SYM (agree with Bern-Dixon-Dunbar-Kosower)

Evidence for MHV method

- Tree Level Amplitudes → complete proof (CSW, Britto-Cachazo-Feng-Witten, Risager)
- One-Loop Amplitudes in Supersymmetric Yang-Mills
  - MHV method at one-loop → Explicit calculation of complete MHV one-loop Amplitudes in N=4 SYM (AB-Spence-Travaglini) & N=1 SYM (Bedford-AB-Spence-Travaglini, Quigley-Rozali)
  - "Proof" at one-loop for generic amplitudes in SYM
    - Feynman Tree-Theorem → covariance
    - unitarity cuts
    - soft & collinear limits

Without Supersymmetry

- Cut-constructible part of MHV 1-loop amplitudes in pure Yang-Mills from MHV diagrams (Bedford-AB-Spence-Travaglini)
  - First result for QCD from MHV diagrams
- Amplitudes in pure Yang-Mills not 4D cut-constructible
  - rational terms are missed by MHV diagrams
- Look for alternative techniques, like on-shell recursions... (Bern, Dixon, Kosower, Berger, Forde...)
- ... or try to refine the MHV diagram method

Related Problems

$$A_n^{1-\text{loop}}(1^+,\ldots,n^+) = \frac{-i}{48\pi^2} \sum_{1\le l_1<l_2<l_3<l_4\le n} \frac{\mathrm{Tr}\left(\frac{1-\gamma^5}{2}\hat{l}_1\,\hat{l}_2\,\hat{l}_3\,\hat{l}_4\right)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

(Bern-Chalmer-Dixon-Kosower, Mahlon)

- All-plus helicity one-loop amplitude in pure YM is missing
  - finite
  - vanishes in supersymmetric theories
  - can be computed in Self-Dual-Yang-Mills
- we claimed that MHV diagrams miss rational terms
  with one exception...

The all-minus amplitude

(AB, Spence, Travaglini)

- Can be drawn with MHV vertices (unlike the all-plus) ➡ MHV diagrams should be able to produce it!
- For n-point <––– ....–> amplitude we need n 3-point MHV vertices:
  - Recall: D=Q-1+L
- Crucial: 3-point MHV vertices = LC vertices ➡ result a priori correct

$$v^{(3)}_{\bar{\phi}A_z\phi}(L_1,k,L_2) = \frac{\langle \eta | L_1 | k]}{\langle \eta k \rangle}$$

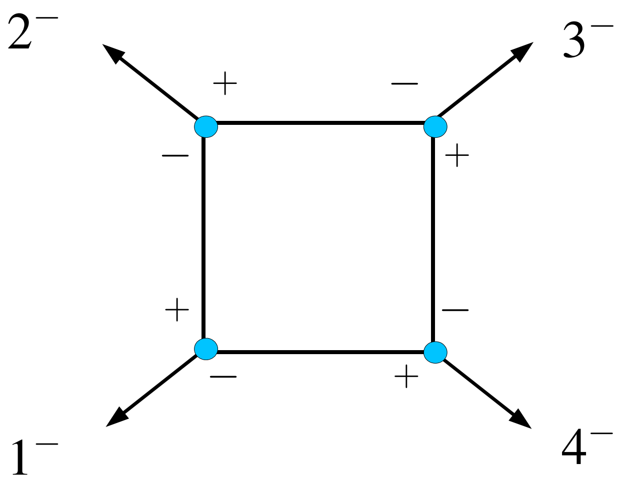
Explicit Calculation

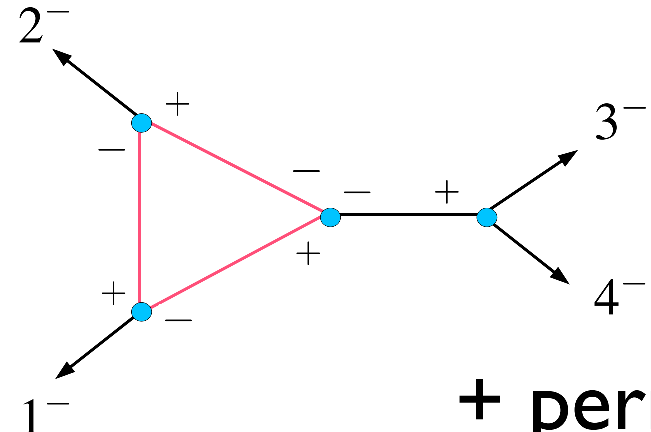
Supersymmetric decomposition of 1-loop amplitudes

$$\mathcal{A}_g = (\mathcal{A}_g + 4\mathcal{A}_f + 3\mathcal{A}_s) - 4(\mathcal{A}_f + \mathcal{A}_s) + \mathcal{A}_s$$

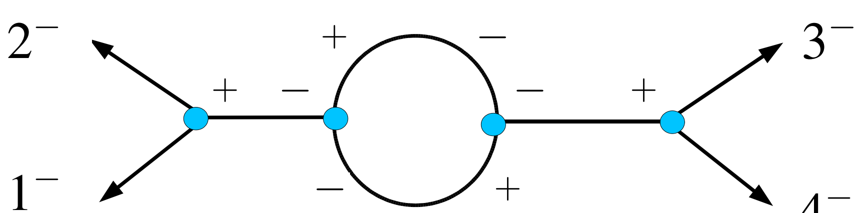
- N=4 and N=1 SUSY contributions are zero
- Only need to consider a complex scalar running in the loop
  - technically simpler
- Use dimensional regularisation (DR): 3pt MHV vertices 4-dim'l (spinor brackets) while propagators are D-dim'l

MHV diagrams for the <---->





+ perms



+ perms

bubbles give vanishing contribution

- Answer $= \dfrac{\langle 1\,2\rangle\,\langle 3\,4\rangle}{[1\,2]\,[3\,4]}\, K_4$

$$K_4 = -\varepsilon(1-\varepsilon)\, I_4^{D=8-2\varepsilon} \xrightarrow[\varepsilon\to 0]{} -\frac{1}{6}$$

Finiteness of the all-minus amplitude

- In (DR): $L_D = L_4 + L_{-2\varepsilon}$
- $$L_D^2 = L_4^2 + L_{-2\varepsilon}^2 := L_4^2 - \mu^2$$
- finite, non-zero result due to incomplete cancellations of (inverse) propagators
- $$\frac{L_4^2}{L_D^2} = \frac{L_4^2 - \mu^2 + \mu^2}{L_D^2} = 1 + \frac{\mu^2}{L_D^2}$$
-
- Finiteness comes as an ϵ/ϵ-effect (smells like an anomaly)
- All-minus amplitude obtained from MHV diagrams, but all-plus amplitude remains mysterious

Where is the all-plus amplitude?

- Go back to the path integral!
- Mansfield's approach:
    - LC quantisation of Yang-Mills: $A^{-}=0$
    - integrate out A^{+} (no derivatives w.r.t. LC time x^{-})
- Schematically the action becomes (Scherk-Schwarz)

$$S = S^{-+} + S^{--+} + S^{-++} + S^{--++}$$

Change of Field Variables

- Change variables in path integral $A_z \,, A_{\bar{z}} \;\longrightarrow\; B_+ \,, B_-$

  such that the S^{-++} term is eliminated

  $$(S^{-+} + S^{-++})[A_z, A_{\bar{z}}] \;=\; S^{-+}[B_+, B_-]$$

  LHS is SDYM action

- Require transformation to be canonical:

  $\Rightarrow$ Jacobian of the transformation is **1**

- Substitute

  $$A_z \sim B_+ + B_+^2 + B_+^3 + \cdots$$

  $$A_{\bar{z}} \;\sim\; B_-(1 + B_+ + B_+^2 + B_+^3 + \cdots)$$

  into $(S^{--+} + S^{--++})[A_z, A_{\bar{z}}]$

Coefficients depend only on:

$p_+^i \,, p_z^i \,, p_{\bar{z}}^i$

Change of Field Variables cont'd

- Result: $S[B_+, B\_] = S^{-+} + S^{--+} + S^{--++} + S^{--+++} + \ldots$
- This reproduces the infinite set of MHV vertices!
- Equivalence Theorem (ET)
  - We can equivalently calculate amplitudes with B field insertions
    - Greens functions of B fields are different from those of A fields, BUT
    - on-shell S-matrix elements are identical (modulo wavefuntion renormalisation)

Not so fast!

- At the same time we have mapped Self-Dual Yang-Mills to a free theory...

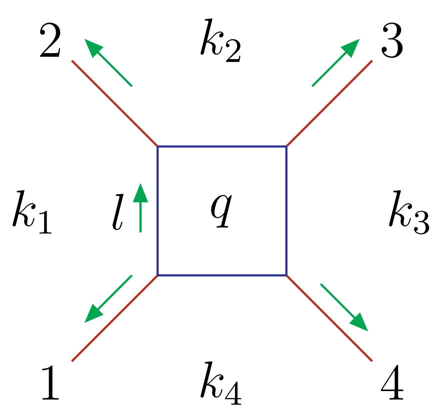
... and eliminated the all-plus amplitude!

- Potential sources / resolutions of the problem
  - Subtleties with regularisation: perform D-dim'l Mansfield transform or use 4D regulator (see in a moment)
  - Violations of the Equivalence Theorem (Ettle-Fu-Fudger-Mansfield-Morris: use D-dim'l Mansfield transform)
  - Jacobian: use non-canonical transformation (see later)

One Solution

(AB, Spence, Travaglini, Zoubos)

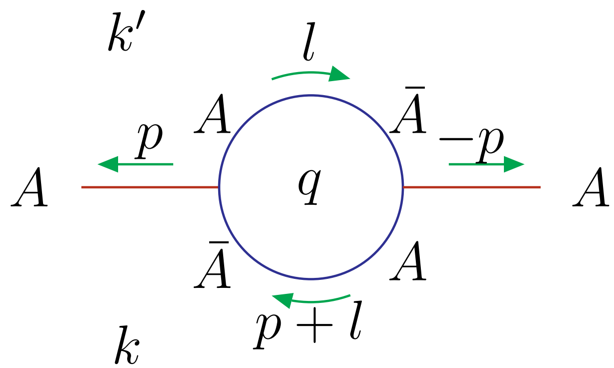
- Use LC worldsheet friendly regularisation of Thorn
  - inherently 4 dimensional so it is twistor space and spinor-helicity friendly as well
  - Regulator:
    δ is sent to zero at the end $$\exp(-\delta \sum_{i=1}^{n} \mathbf{q}_i^2)$$
  - q are loop region momenta $$\mathbf{q}^2 = 2 q_z q_{\bar{z}}$$



E.g.

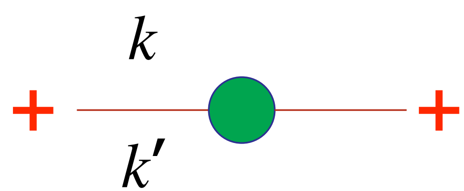
$$l = q - k_1, p_1 = k_1 - k_4, \ldots$$

- Regularisation generates Lorentz violating interactions
  - E.g. the following diagram produces a ++ interaction



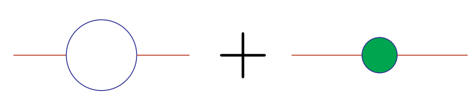
$\Rightarrow$ <++> is non-zero!

  - cancel with appropriate (finite) ++ counterterm



counterterm ~ $\frac{g^2 N}{12\pi^2}\left((k_{\bar{z}})^2 + (k'_{\bar{z}})^2 + k_{\bar{z}} k'_{\bar{z}}\right)$

-
  - so that



$+$ $=0$

- Counterterm Lagrangian:

$$\mathcal{L}_{\rm CT} = -\frac{g^2 N}{12\pi^2}\int_\Sigma \mathrm{d}^3p\,\mathrm{d}^3p'\,\delta(p+p')\ A^i{}_j(p')((k^i_{\bar{z}})^2 + (k^j_{\bar{z}})^2 + k^i_{\bar{z}}k^j_{\bar{z}})A^j{}_i(p)$$

All-plus amplitudes from Lorentz violating counterterms

- Let’s first look at 4-point case: in D=4 naively combining all diagrams the loop-integrand vanishes
-
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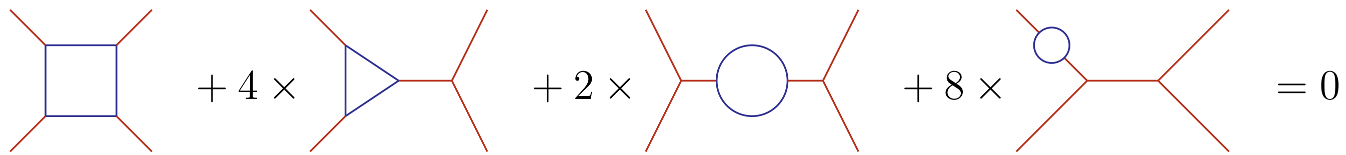
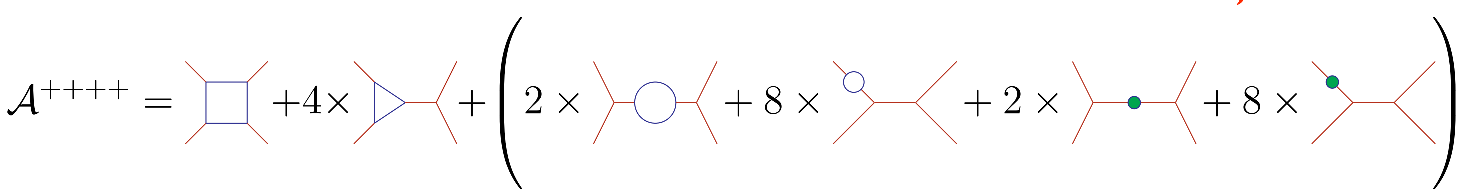
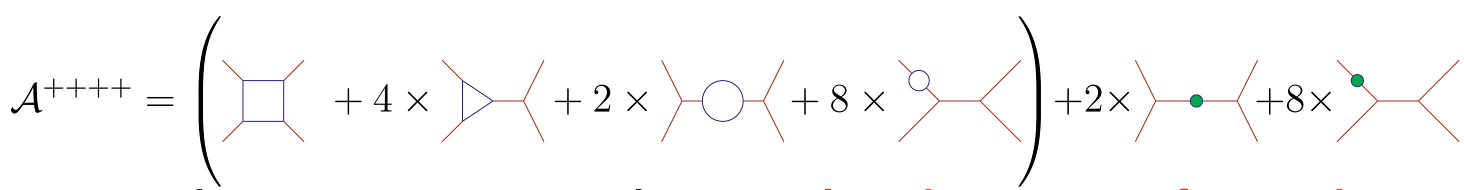


- However we need to include the counterterms, so
-



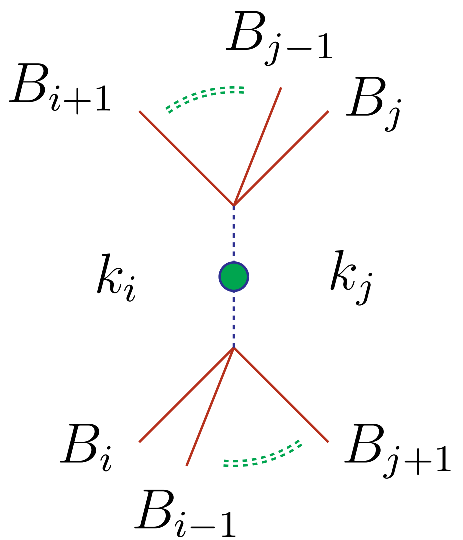
- by construction the counterterms cancel all bubbles, but importantly we can rewrite this result also as:
-



- parenthesis is zero, so the amplitude comes from the counterterm alone (Chakrabarti-Qiu-Thorn)

All-plus amplitudes continued

- Proposal: Mansfield transform applied to counterterm generates all-plus amplitudes



Reminder: A=A(B) holomorphic
A is positive helicity gluon
Equivalence Theorem: A $\rightarrow$ B

- Explicit calculation for 4-point agrees with known answer
- for n>4 all soft and collinear limits work correctly, dependence on region momenta disappears

Generic amplitudes

- There are further Lorentz violating counterterms in Thorn's regularisation, e.g. <-->, <+-->, <-++>
- Applying Mansfield's transforms to these terms will lead to new contributions to MHV vertices, we expect that these account for the missing rational terms
- E.g. the <-++> counterterm will generate an infinite series of single-minus vertices. Together with other contributions this should reproduce the known one-loop amplitudes with helicities <-++....++> (work in progress)

A Second Solution

(AB-Spence-Travaglini; related work Feng-Huang)

- Investigate non-canonical field redifinitions
- Start again with SDYM, simpler but contains all-plus amplitude
- Chalmers-Siegel action for SDYM

$$\mathcal{L}_{CS} = \bar{A}(\Box A + ig^2[\partial_+ A\,,\partial_{\bar{z}} A])$$

- Holomorphic change of variables

$$B(A) = A + ig^2 \Box^{-1}[\partial_+ A\,,\partial_{\bar{z}} A]$$

$$\bar{B} = \bar{A}$$

Singular on the mass shell

- Non-trivial Jacobian: $\mathcal{J} = \det_{x,y}(\delta A(x)/\delta B(y))$

Holomorphic transformation cont'd

- Illustrate the effects of the Jacobian and Equivalence Theorem (ET) in a toy model

$$\mathcal{L} = \bar{A}(-\Box A + \lambda A^2)$$

- Redefine the fields: $$-\Box A + \lambda A^2 = -\Box B$$ $$\bar{A} = \bar{B}$$

- Solve perturbatively $$A = A(B)$$
  - Contributions from violation of ET

- All-plus amplitude produced by nontrivial Jacobian!

$$\mathcal{J} = \exp\left[-\mathrm{Tr}\log(1 - 2\lambda\Box^{-1}A)\right]$$

Holomorphic transformation cont'd

- Contributions from the Jacobian & ET give the correct all-plus amplitude (and other finite one-loop amplitudes) - at least in our toy model
- New vertices generated by the holomorphic change of variables are more complicated than MHV vertices
- This is expected to work for generic helicity configurations

Summary

- Simplicity of scattering amplitudes $\Leftrightarrow$ Geometry in Twistor Space
- Novel, powerful tools to calculate amplitudes
  - MHV diagrams, on-shell recursion relations, (generalised) unitarity in 4 and D dimensions
- Common idea: (re)use only on-shell quantities!

- MHV diagrams: provide a new diagrammatic method to calculate scattering amplitudes at tree and one-loop level in super Yang-Mills
- Progress in non-supersymmetric Yang-Mills
  - all-minus/all-plus helicity amplitudes, ...
  - Mansfield transform combined with different regulators
  - Holomorphic field redefinition
- Mansfield gave a systematic, Lagrangian derivation of MHV diagrams
  - Along the same lines alternative diagrammatic rules could be derived, even for theories with massive particles