Master's Thesis

Buildings, Amalgams and Reductive Linear Algebraic Groups

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Abstract

The theory of Bruhat and Tits associates to a reductive linear algebraic group over a local field an affine building, which is an abstract simplicial complex together with a regular geometric realization and an action of the group of rational points on it. C. Soulé then proceeded to show that inside this building, if one uses as a valued field the formal laurent series $k((t^{-1}))$ and puts suitable hypotheses on the group, one can find a simplicial fundamental domain for the action of the $k[t]$-valued points, which yields together with generalizations of facts concerning group actions on trees, due to J.-P. Serre, an amalgamation decomposition of the group of points valued in that polynomial ring. The main part of this thesis focuses on the construction of this fundamental region and the presentation of the notions needed in order to prove it.

Contents

1 Introduction

In mathematics there are two recurring paradigms to solve a puzzle, in particular, as a first attempt, one can go ahead and prove (or disprove) a statement directly or by computation, whereas a second more general approach is finding an abstract description of the question and developing a language or framework that disentangles the problem and makes a solution evident. As an example one may look at the proof H. Nagao provides in showing that the unit group of the two by two matrices with entries in a polynomial ring is not finitely generated. In [Nag59] he proceeds by deriving a presentation of the said group as an amalgam, which he shows by direct computation. Similarly Y. Ihara goes about demonstrating that torsion-free, discrete subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ are free. In [Iha66] he even states that: "The problem is nothing but a purely combinatorial one". J.-P. Serre, who prominently featured both of these theorems in his book «Trees» [Ser80], would agree, however his approach in proving these differs radically from the original authors and can be attributed to the second paradigm we identified. He defines an action of the groups involved on special circuit-free graphs, namely trees, and deduces from their construction the existence of a sub-tree which poses as a fundamental domain with respect to that group operation. From this datum the conclusions of Nagao and Ihara are quickly derived consequences. As a matter of fact in his book Serre provides a dictionary of properties of group actions on graphs and links them back to implications on the acting group.
This approach allows for generalizations on several vectors. For example the choice of trees as the central combinatorial objects in the theory is a limiting factor, explicitly it prohibits the generalization of the statements to a variety of higher dimensional groups. Soulé lifted this in [Sou73] by proving a theorem linking group actions on abstract simplicial complexes that admit a fundamental domain to an amalgamation decomposition of that group into stabilizers of the vertices of the fundamental region under suitable geometric hypotheses.[1] The main focus of this thesis however will be another celebrated theorem due to Soulé, in which he proves the existence of a fundamental domain inside the Bruhat-Tits building, an abstract simplicial complex that comes associated to a reductive linear algebraic group defined over a valuation field. Linear algebraic groups are certain group objects in a subcategory of the affine schemes over a field and allow in an analogous manner as the Lie groups do the association of a unique Weyl structure in the form of a Dynkin diagram, which gives means to characterise them. Over valuation fields one can use this structure to define the Bruhat-Tits building as a simplicial complex, whose geometric realization is built up from vector spaces and which comes equipped with a well-defined metric as well as a an action of the rational points of the linear algebraic group via isometries. All this facilitates the use of the first mentioned theorem by Soulé in the case that one is able to obtain a suitable fundamental domain, the presentation of an existence proof of which is the main part of this thesis.

We shall quickly describe how we intent to present above mentioned statements and relations. In the subsequent chapter concepts, that occur naturally in the realm of linear algebraic groups and were historically not founded before them, are introduced. We prefer to introduce them at an earlier stage as this shortens the presentation of linear algebraic groups and since their coverage require the least amount of preliminaries. Among the notions included are root and Coxeter systems, buildings and Tits systems. A reader familiar with these objects is encouraged to start with chapter three, in which we outline the theory of reductive linear algebraic groups by introductorily covering the case over an algebraically closed field and then proceeding with the general

[1] In particular Soulé demands that the geometric realization of the complex is connected and simply connected and the one of the fundamental domain is connected. N. Chebotarev relaxed these assumptions in [Che82] by introducing higher dimensional analogues of a combinatorial version of the fundamental group of a complex.

situation. The presentation thereof is already geared towards the immediately ensuing discussion of the construction of the Bruhat-Tits building, where we progress, in the spirit of chapter two, again in a more abstract fashion by boiling the theory of linear algebraic groups down to several axioms that give rise to the construction. The fourth part covers the proof of the existence of a fundamental domain in the associated Bruhat-Tits building of a linear algebraic group subject to some simplification assumptions. This is due to B. Margaux [Mar09], who generalized Soulé's first version of this theorem which was only applicable to split groups. In the fifth chapter we strive to apply the presented material in order to deduce two generalizations of Nagao's theorem, i.e. we will obtain two amalgamation decompositions of the rational points of a linear algebraic group over a polynomial ring. These results are again due to C. Soulé and B. Margaux. To encapsulate one can start reading this thesis in any of the chapers two to four depending on the level of previous knowledge. Also check the following *leitfaden*:

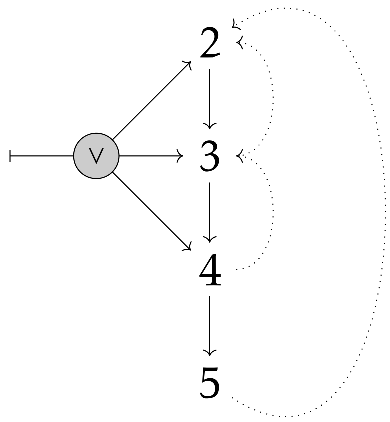

Figure 1.1: Leitfaden on how to read this thesis.

2

Abstract Fundamentals

This chapter introduces concepts, that evolve around linear algebraic groups without using their definition explicitly, among which are root and Tits systems and the theory of buildings. Historically these objects were not formalized before the theory of linear algebraic groups had been introduced, however since their presentation does not require a lot of preliminaries, such as algebraic geometry, they are a good vantage point. Secondly this chapter shall serve, especially for the advanced reader, as a dictionary for the underlying abstract structures, that occur throughout this thesis.

The sections in this part are built in a way, that they introduce a specific concept, give a significant example clarifying the definition and maybe add some important facts. There will be no proofs, but we shall be precise on where to find them. The passages depend loosely on one another and gain complexity as the chapter progresses.

2.1 Root Systems

Root systems are finite configurations of elements in a real vector space, which exhibit certain symmetries and are subject to finiteness conditions. Their regular and bounded structure facilitate their characterisation, which in turn enables one, through strong connections to Lie groups and as we will see later to linear algebraic groups, to deduce a classification of these types of groups. Their definition reads as follows:

Definition 2.1.1 ([Bou02, VI.§1.1. Def. 1])**.** Let V be a real vector space and $R \subseteq V$. Then R is called a **root system** in V, if and only if the subsequent three conditions are met:

(RS_{I}) R is finite, does not contain the zero vector and generates V. In particular V is finite dimensional.

($\mathrm{RS}_{\mathrm{II}}$) For all $a \in R$, there is an element $a^\vee \in V^*$ in the dual V^* of V such that $\langle a, a^\vee \rangle = 2$ holds, where we used the following notation for the evaluation map:

$$\begin{aligned} V \times V^* &\to \mathbb{R} \\ (v, w^\vee) &\mapsto \left\langle v, w^\vee \right\rangle := w^\vee(v). \end{aligned}$$

Moreover we demand that the **reflection**, which is given by

$$\begin{aligned} s_{a,a^\vee} : V &\to V \\ v &\mapsto v - \left\langle v, a^\vee \right\rangle \cdot a, \end{aligned}$$

maps R into R.

($\mathrm{RS}_{\mathrm{III}}$) For all $a \in R$, it holds that $\langle R, a^\vee \rangle \subseteq \mathbb{Z}$.

The elements of R are called **roots** and the dimension of V is called the **rank** of the root system.

This definition is made meaningful by the fact that, for a finite subset R of a real finite-dimensional vector space V, there is, for every $a \in R$, at most one reflection s of V such that $s(a) = -a$ and $s(R) = R$ [Bou02, VI.§1.1. Lemma 1]. Hence one sees that $s_{a,a^\vee}$ in the definition is uniquely determined by a, which is the reason we will write $s_a := s_{a,a^\vee}$ from now on. Note that these are linear involutions, as $s_a \circ s_a = \mathrm{id}_V$ holds.

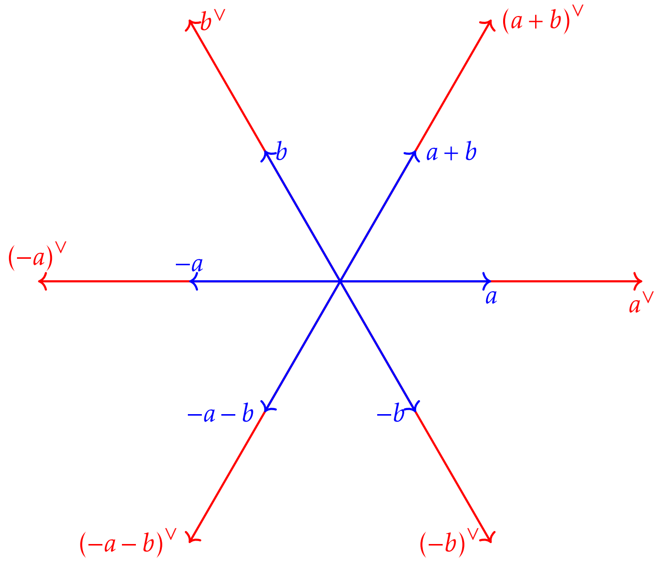

Figure 2.1: The root system A_2.

Definition 2.1.2. Let R be a root system in the vector space V. In the following $A(R)$ denotes the group of linear automorphisms, that stabilize the set R, and $W(R)$ the subgroup of $A(R)$, that is generated by all s_a, with $a \in R$. $W(R)$ is called the **Weyl group** of R.

Since the roots generate the vector space V, one obtains a well-defined injection of $A(R)$ into the group of permutations of R. Thus one can conclude, that $A(R)$ and $W(R)$ are finite.
One can also see that a reflection s_a fixes a hyperplane of codimension one in V, namely the kernel of the functional $a^\vee$. Fixing any positive-definite, symmetric, bilinear form $(\cdot,\cdot)'$ on V, we can define, by averaging, another such form $(\cdot,\cdot)$ that is invariant under the action of $W(R)$ [Spr98, 7.1.7]:

$$(x,y) := \sum_{w \in W(R)} (w(x), w(y))', \quad \forall x,y \in V.$$

In the following we will identify V and its dual V^* by means of such a positive-definite, symmetric, bilinear form $(\cdot,\cdot)$, except when we explicitly state otherwise. One can then show (cf. [Bou02, VI.§ 1.1. Lemma 2]) that, for every $a \in R$, the corresponding **dual root** (also **coroot**) $a^\vee$ is of the form:

$$a^\vee = \frac{2a}{(a,a)}.$$

Plugging this into the formula for s_a, one obtains the usual form of a reflection on a vector space ($a \in R, v \in V$):

$$s_a(v) = v - 2\frac{(a,v)}{(a,a)}a.$$

In this form one immediately sees that the codimension one subspace $\ker(a^\vee)$, which is fixed by s_a, is given by the hyperplane of vectors orthogonal to a.

Example 2.1.3. In figure 2.1 one can see an example of a root system in $V = \mathbb{R}^2$. If one identifies V with the complex plane, the roots are given by the sixth roots of unity and the coroots are scaled versions thereof. One may check (RS_I) and (RS_{II}) by a quick glance at the picture. (RS_{III}) can also be checked by a look into 2.1, since by the definition of the reflection one has to show that for two roots $c,d \in R$

$$s_c(d) - d \subseteq \mathbb{Z}.c$$

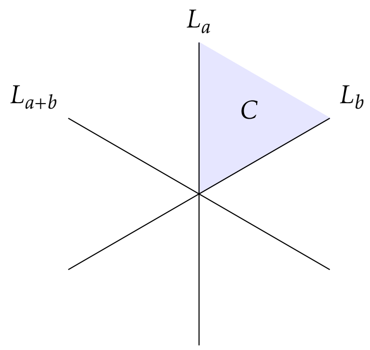

Figure 2.2: Hyperplanes of A_2 dividing $\mathbb{R}^2$ into six conically formed chambers.

holds, which follows by a bit of «puzzling» of the vectors. We observe that the three reflections s_a, s_b and s_{a+b} generate $W(R)$. Since $s_{a+b} = s_a \circ s_b \circ s_a$ holds, only s_a and s_b suffice as generators for $W(R)$. With a bit more calculation one obtains the following abstract presentation:

$$W(R) \cong \left\langle \tilde{s}_a, \tilde{s}_b \;\middle|\; \tilde{s}_a^2 = \tilde{s}_b^2 = (\tilde{s}_a \tilde{s}_b)^3 = \mathrm{e} \right\rangle$$

This is a specialization of the result that Weyl groups are Coxeter groups, which are special groups we will properly introduce later on.

Suppose, that the vector space V is a direct sum of a finite family of vector spaces $(V_i)_{1 \le i \le r}$ and that for every $1 \le i \le r$, there is a root system R_i in V_i. By setting $R := \bigcup_{i=1}^r R_i$ one finds that R is indeed a root system in V and that the coroots $R^\vee$ are given as $\bigcup_{i=1}^r R_i^\vee$, where $R_i^\vee$ are the coroots corresponding to R_i extended to V. R is then called the **direct sum** of the root systems $(R_i)_{1 \le i \le r}$. For every $i \in \{1, \ldots, r\}$ one observes that for another index $j \neq i$, the kernel of a coroot $a^\vee \in R_i^\vee$ contains V_j. Moreover one can note that multiples of a root $a \in R_i$ do not leave V_i. Thus one concludes that s_a viewed as a reflection of V operates non-trivially only on V_i, which leads to the conclusion that one has:

$$W(R) \cong \prod_{i=1}^r W(R_i).$$

A root system, that is not the direct sum of two or more root systems, is called **irreducible**. One sees that the above example of A_2 is an irreducible root system. This discussion is based on [Bou02, VI.§ 1.2].

If we restrict ourselves to a pair of roots a and b in R, then we observe that the numbers of choices one can make for $\langle a, b^\vee \rangle$ and $\langle b, a^\vee \rangle$ are limited. This follows, since

$$\left\langle a, b^\vee \right\rangle \left\langle b, a^\vee \right\rangle = 4 \frac{(a,b)}{(b,b)} \frac{(b,a)}{(a,a)} = 4 \cos\left(\sphericalangle(a,b)\right) \le 4,$$

and $\langle a, b^\vee \rangle$ and $\langle b, a^\vee \rangle$ need to be integers by $(\mathrm{RS}_{\mathrm{III}})$. If one assumes that b is a multiple of a, i.e. that $b = \lambda a$ for $\lambda \in \mathbb{R}^\times$ holds, then from

$$\left\langle a, (\lambda a)^\vee \right\rangle = 2 \frac{(a, \lambda a)}{(\lambda a, \lambda a)} = \frac{2}{\lambda} \quad \text{and} \quad \left\langle \lambda a, a^\vee \right\rangle = 2\lambda,$$

one can gather that the possibilities for λ are only $\left\{\pm\frac{1}{2}, \pm 1, \pm 2\right\}$. A root $a \in R$ is called **divisible**, if and only if $\frac{\lambda}{2}$ is also in R, and **multipliable**, if and only if λ and 2λ are in R. The set of roots, that are not divisible (also **indivisible**) will be denoted by R_{nd}. A root system with $R = R_{\mathrm{nd}}$ is called **reduced**.

Lemma 2.1.4 ([Bou02, VI.§ 1.4. Prop. 13])**.** *Let R be an irreducible root system in V. The set of indivisible roots R_{nd} forms an irreducible, reduced root system in V, with Weyl group $W(R_{\mathrm{nd}}) = W(R)$.*

For the following discussion we fix a root system R in its vectors space V. Denote by L_a, for a root $a \in R$, the hyperplane, that is fixed by its associated reflection s_a. We collect these planes in the set $\mathcal{H}$. If we look at the space

$$V \setminus \bigcup_{L \in \mathcal{H}} L,$$

one sees that the connected components of this space are open cones with peak at the origin. We will call these components **chambers**. Check figure 2.2 for a picture on the chambers associated to A_2. Since one cuts the chambers out along the fixed hyperplanes $\mathcal{H}$ and by ($\mathrm{RS_{II}}$), one sees that the action of $W(R)$ on V maps chambers to chambers. Let C be certain chamber in V. The hyperplanes that are directly involved in the shaping process of C, i.e. those hyperplanes $L \in \mathcal{H}$ whose span of $L \cap \overline{C}$ is of codimension one in V are called the **walls** of C. In figure 2.2 for example the walls of C are exactly L_a and L_b. The following theorem collects a few of the powerful results concerning chambers and walls.

Theorem 2.1.5 ([Bou02, VI.§ 1.5. Thm. 2])**.** *(a) The Weyl group $W(R)$ acts simply transitively on the set of chambers, i.e. if one fixes a chamber C, for every chamber C', there is a unique $w \in W(R)$, such that $C' = w.C$.*

(b) C is an open cone.

(c) Denote by $L_1, \dots, L_r$ the walls of C in $\mathcal{H}$. For every index $1 \leq i \leq r$, there is a unique root a_i, which lies on the same side of the hyperplane L_i as C.

(d) The roots $B(C) := \{a_1, \dots, a_r\}$ form a basis of V and the elements in C are given by the $x \in V$ with $(a_i, x) > 0$ for all $i = 1, \dots, r$.

(e) The s_{a_i}, with $1 \leq i \leq r$, suffice in generating $W(R)$.

Part (c) gives another useful viewpoint on roots, in that one may see them as closed half-spaces such that two opposite such roots intersect exactly in the wall they are separated by.

The basis $B(C)$ of V, that comes with choosing a chamber C has also nice properties, when put in relation to the root system. In fact any root $a \in R$ can be expressed as a linear combination with integer coefficients of the same sign, which motivates the term **basis** of R for $B(C)$. One calls the set of roots, that can be expressed by a linear combination of $B(C)$ with non-negative integer coefficients **positive roots**, denoted by $R^+(C)$ or just R^+, and vice versa, those that are written with non-positive coefficients **negative roots**, denoted by $R^-(C)$ or simply R^-. The sets of positive and negative roots are disjoint (cf. [Bou02, VI.§ 1.6. Thm. 3]). The set of roots, that are positive with respect to a particular choice of a chamber, will be referred to as a **system of positive roots**. One finds the following equivalent characterisation:

Proposition 2.1.6 ([Bou02, VI.§ 1.7. Cor. 1])**.** *Let $P \subseteq R$ be a subset of the root set R. If P and $-P$ form a partition of R and for every two roots a and b in P, with $a+b \in R$, one has $a+b \in P$, then P is a system of positive roots and in particular there is a chamber C such that $P = R^+(C)$.*

Remark 2.1.7. A subset of roots $P \subseteq R$ such that for every two $a, b \in P$, with $a+b \in R$, $a+b \in P$ holds, will be called a **closed** subset of roots. If moreover $P \subseteq R^+$ is true, for some system of positive roots R^+, one says P is **positively closed**.

2.2 Coxeter Groups and Coxeter Complexes

In this section we will present how one can abstractly characterise the situation of a group being generated by reflections, as are for example the Weyl groups from above. Hence in the following we will fix a pair (W, S), where W is a group and S is a set of generators of W. We assume that all elements in S have order 2, since we regard them as reflections and as such applying them twice should yield the initial situation.

From the case of Weyl groups we see that there is a one-to-one correspondence between reflections and walls and by theorem 2.1.5.(c) it can be observed that roots correspond to different sides of walls. Let R be a root system in a vector space V. By a small calculation one observes that the reflection associated with a root $s_a(b)$, for $a, b \in R$, is given by $s_a \circ s_b \circ s_a$. And one knows that $s_a(a) = -a$ holds for all roots $a \in R$. This motivates the following abstract definition:

Definition 2.2.1 ([AB08, pp. 65-66])**.** Let there be a pair (W, S) of a group W generated by S and the elements of S all have order two. Suppose the following condition is given:

(A) Let T be the set of conjugates of S in W. There is an action of W on $T \times \{\pm 1\}$ such that a generator $s \in S$ acts by $(t \in T,\ \epsilon \in \{\pm 1\})$

$$(t,\epsilon) \mapsto \begin{cases} (sts,\epsilon) & \text{if } t \neq s, \\ (t,-\epsilon) & \text{if } t = s. \end{cases}$$

Then we call (W,S) a **Coxeter system**, with **Coxeter group** W.

It can be proven, and the discussion preceding the definition shall convince one that, for every root system R, the Weyl group $W(R)$ together with the set $S := \{s_a \mid a \in B\}$, where B is a basis of R, forms a Coxeter system [Bou02, VI.§ 1.5. Thm. 2].
The Coxeter systems (W,S) stemming from a root system R in V all exhibit finite W. The chambers in V were cones and if one projects these cones onto the unit sphere in V, one obtains a finite number of patches decomposing the sphere. This derives from the fact that all walls that give rise to the reflections in W intersect in the origin and are of finite number. In fact it can be shown that all finite Coxeter systems can be realized as such finite reflection groups, which dissect the sphere [AB08, Cor. 2.68]. Thus one calls the finite Coxeter systems **spherical**.
However not all Coxeter systems are of spherical type. Another important class of Coxeter systems (for us) will be the **affine** ones. They are given by groups generated by affine reflections, i.e. flips alongside walls, that do not need to meet the origin.

Let us return to the abstract case of a Coxeter system (W,S). We will give some equivalent definitions of Coxeter systems, that will show that (W,S) also has nice combinatorial properties. An element $w \in W$ can always be written as a word $s_1 \dots s_r$, where $s_1,\dots,s_r$ are all in the set S. Let us set $l(w)$ to be the natural number r such that $s_1 \dots s_r$ is a word of minimal length representing w. We call the number $l(w)$ the **length** of w with respect to S and a decomposition of length $l(w)$ a **reduced** presentation for w. Observe that the only element in W with length zero is the neutral element. One finds the following equivalent conditions:

Theorem 2.2.2 ([AB08, Thm. 2.49])**.** *Let (W,S) be a pair of a group W generated by S and the elements in S have order two in W. Consider the following conditions:*

(D) *Let w be a non-neutral element in W, with presentation $s_1 \dots s_m$, where $m > l(w)$. Then there are indices $1 \le i < j \le m$, such that $w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_m$ holds, where the hat indicates that an element is omitted. This is called the **d̲eletion condition**.*

(E) *Given $w \in W$, non-neutral, $s \in S$ and fixing a reduced decomposition $s_1 \dots s_r$ for w, one has either $l(ws) = l(w) + 1$ or there is an index $1 \le i \le r$ such that*

$$w = ss_1 \dots \hat{s}_i \dots s_r.$$

*This is called the **e̲xchange condition**.*

(F) *For $w \in W$ and $s,t \in S$ such that one has $l(sw) = l(w) + 1$ and $l(wt) = l(w) + 1$, either $l(swt) = l(w) + 2$ or else $swt = w$ holds. This is called the **f̲olding condition**.*

(C) *W admits the presentation $\left\langle S \,\middle|\, (st)^{m(s,t)} = 1 \right\rangle$, where $m(s,t)$ is the order of the product st in W. This is called the **C̲oxeter condition**.*

*One then finds, that the conditions **(A)**, **(D)**, **(E)**, **(F)** and **(C)** are all equivalent.*

The properties **(D)**, **(E)** and **(F)** are especially useful, if one needs to perform calculations inside a Coxeter group, whereas condition **(C)** gives a way to write down Coxeter groups by choosing $\mathbb{N}^{\ge 2}$-entries for the (possibly infinite) matrix m. The matrix m is sometimes called the **Coxeter matrix** of (W,S), while $\operatorname{card}(S)$ is termed the **rank** of the Coxeter system.

Interlude: Abstract Simplicial Complexes

Before we go further, we need the definition of an abstract simplicial complex. Recall that a simplex in topology is a regular geometric object, that is constructed in the following way: For every natural number $n \in \mathbb{N}$, we think of the standard n-simplex as the convex hull of the standard base in $\mathbb{R}^{n+1}$. Thus we obtain a point, a line, a triangle, a tetrahedron, etc. as the first few standard

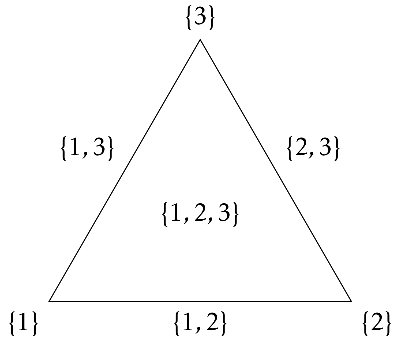

Figure 2.3: Standard 2-simplex.

simplices. In this way we see that an n-simplex always consists of $\binom{n}{m}$ m-simplices, for every $m < n$.

We will demonstrate this fact for the standard 2-simplex, i.e. the triangle (see figure 2.3). Naming the corners first by the numbers $1,2,3$ and proceeding to name higher dimensional simplices by the set of points they are spanned by, one immediately checks the assertion. By continuing with this example we see that the basic geometric objects in the standard 2-simplex can be identified with subsets of the set $\mathcal{V} := \{1,2,3\}$, which are the vertices of the triangle. Moreover we see, that for an inclusion of two such represented substructures, one has an incidence of one object in another one. For example, the point $\{1\}$ lies on the edge $\{1,2\}$. This shall motivate the following definition of an abstract simplicial complex:

Definition 2.2.3 ([AB08, p. 661])**.** Let $\mathcal{I}$ be a non-empty, partially ordered set. Suppose the following two conditions hold:

(a) Any two elements $A, B \in \mathcal{I}$ have a greatest lower bound $A \cap B$.

(b) For any $A \in \mathcal{I}$, the poset $\mathcal{I}_{\leq A}$ of elements less than or equal to A is isomorphic to the poset of the subsets of $\{1,\ldots,r\}$ for some natural number $r \in \mathbb{N}$.

Then $\mathcal{I}$ is called an **abstract simplicial complex**.

Remark 2.2.4. We note that in the above definition we added the empty set to the sub-simplices of a standard simplex, which does not really correspond to anything in the picture 2.3. However using the definition one can immediately see that $\mathcal{I}$ contains a smallest element, which we will call the **empty** simplex [AB08, A.3], because it corresponds to the empty set in $\mathcal{I}_{\leq A}$ for every $A \in \mathcal{I}$.

The number r, that is associated to every element $A \in \mathcal{I}$ will be called the **rank** of A and $r-1$ its **dimension**. The elements in $\mathcal{I}$ of rank one are called the **vertices** of $\mathcal{I}$. For two simplices $A, B \in \mathcal{I}$, with $A \leq B$, one says that A is a **face** of B.

An important construction in a simplicial complex $\mathcal{I}$ is the one of the **link** of a simplex $A \in \mathcal{I}$ and denoted by $\mathrm{lk}_{\mathcal{I}}(A)$. This is defined as the poset of simplices $B \in \mathcal{I}$, which are joinable to A, i.e. A and B have a common upper bound in $\mathcal{I}$, but disjoint, i.e. they are such that $A \cap B$ is the empty simplex. From this definition it is immediate that $\mathrm{lk}_{\mathcal{I}}(A)$ is again a simplicial complex and moreover one can prove that it is isomorphic (as a poset) to $\mathcal{I}_{\geq A}$ via:

$$\begin{aligned} \mathrm{lk}_{\mathcal{I}}(A) &\to \mathcal{I}_{\geq A} \\ B &\mapsto A \cup B. \end{aligned}$$

Note that $A \cup B$ is the least upper bound of A and B and it exists, because A and B have an upper bound, for $B \in \mathrm{lk}_{\mathcal{I}}(A)$.

Simplicial complexes will form the underlying structure in buildings. However before we are able to introduce those, we will give a more elaborate example of a simplicial complex, that links back to Coxeter and root systems.

Example: Coxeter Complexes

Let (W,S) be a Coxeter system. From this part on we will assume that S is finite. For any subset $J \subseteq S$ we see by condition **(C)** that the subgroup W_J of W, that is generated only by the elements of J, together with J itself is a Coxeter system as well. Such subgroups will be called **standard subgroups** and for any element $w \in W$ we will call a coset of the form wW_J a **standard coset**. The deletion condition implies the following fact about standard subgroups:

Proposition 2.2.5 ([AB08, Prop. 2.13])**.** *The map that sends a subset $J \subseteq S$ to its standard subgroup W_J is an isomorphism of partially ordered sets from the power set of S to the set of standard subgroups of W, where both sets are thought of as ordered by inclusion. The inverse is given by $W' \mapsto W' \cap S$.*

This motivates us to define the following.

Definition 2.2.6. Let (W,S) be a Coxeter system. Set $\mathcal{A}(W,S)$ to be the partially ordered set of standard cosets in W, ordered by reverse inclusion. By this we mean that, for two elements $A, B \in \mathcal{A}(W,S)$, $A \leq B$ holds, if and only if $A \supseteq B$ as subsets of W is true. $\mathcal{A}(W,S)$ is called the **Coxeter complex** associated with (W,S).

Note that $\mathcal{A}(W,S)$ has a natural action of W on it: If one fixes an element wW_J, with $w \in W$ and $J \subseteq S$, the translation by $w' \in W$ shall be given by $w'wW_J$, which is again a standard coset in W.

It can be shown, by using proposition 2.2.5 and the above recorded properties of Coxeter systems that $\mathcal{A}(W,S)$ is in fact a simplicial complex, which has several nice properties. For example the maximal elements in $\mathcal{A}(W,S)$ are all of the same of maximal rank $\operatorname{card}(S)$. Such a maximal simplex will be called a **chamber** in $\mathcal{A}(W,S)$. Note that the chambers in $\mathcal{A}(W,S)$ are all of the form $w\langle\varnothing\rangle$, which we abbreviate plainly as w, with $w \in W$.

One is able to introduce a neighbouring relation on the set of chambers of $\mathcal{A}(W,S)$ as follows: Two chambers, given by w and w', are called **adjacent**, if and only if $w = w's$ holds, for some $s \in S$. This is equivalent to saying, that the rank $(\operatorname{card}(S)-1)$-simplex, given by $wW_{\{s\}}$ is a face of w and w', i.e. contains both of them. Note that $W_{\{s\}} = \{\mathrm{e}, s\}$ holds, for all $s \in S$.

Using this adjacency one finds that any two chambers w and w' in $\mathcal{A}(W,S)$ are connected by a path, i.e. that there are chambers $w = w_1, \ldots, w_r = w'$, such that w_i is adjacent to w_{i+1}, for all $1 \leq i < r$. This is equivalent to saying, that the group element $w^{-1}w'$ can be written as an S-combination.

Moreover one can **colour** the simplicial complex $\mathcal{A}(W,S)$. By this it is meant that one is able to assign to every vertex an element of a finite set I such that the vertices of a chamber are mapped bijectively onto I. In the example of the standard 2-simplex above we «coloured» the vertices with the numbers $\{1,2,3\}$.
The vertices in a Coxeter complex can be written in the form $wW_{S\setminus\{s\}}$, with $w \in W$ and $s \in S$. One observes that setting $s \in S$ as the colour of $wW_{S\setminus\{s\}}$ fulfils the definition, since the vertices for a given chamber w are exactly the ones that can be written as $wW_{S\setminus\{s\}}$.
We note that if one is able to colour a simplicial complex, it is customary to term the set of colours of the vertices the **type** of a simplex. This definition in the context of Coxeter complexes yields that the type of a simplex wW_J, with $w \in W$ and $J \subseteq S$, is $S \setminus J$. One immediately observes that the action of W on $\mathcal{A}(W,S)$ preserves the type. This discussion is based on [AB08, Thm. 3.5].

Lastly we note that for a root system R in a vector space V and a chamber C in V, we have a one-to-one correspondence of chambers in $\mathcal{A}(W(R), B(C))$ and chambers in V, given by $w \mapsto w(C)$ [Bou02, VI.§ 1.5. Thm. 2]. One sees that this map is $W(R)$-equivariant.

Example 2.2.7. We will illustrate this in the case of the root system A_2. We already know that for the choice of a fundamental chamber C as in figure 2.2 the basis $B(C)$ of A_2 is given by $\{a,b\}$ (also see example 2.1.3) and that the Weyl group is given by

$$W(A_2) = \{\mathrm{e},\ s_a,\ s_b,\ s_as_b,\ s_bs_a,\ s_as_bs_a\},$$

where we omitted the composition sign for better readability. The Coxeter complex we then obtain is visualised in figure 2.4, where we abbreviated the standard subgroup W_J by $\langle J\rangle$, with $J \subseteq S$. Also note that $s_as_bs_a = s_bs_as_b$ holds. The fact that the chamber corresponding to the longest element in $W(A_2)$ sits directly opposite to the fundamental chamber e is not a coincidence, but a general fact of Coxeter complexes associated with finite Coxeter groups [AB08, Prop. 1.77]. One can also observe that two vertices of the same colour are not connected by an edge, which resembles the fact that the Coxeter complex is colourable. Also we left the minimal simplex $\langle s_a, s_b\rangle$ out of figure 2.4.

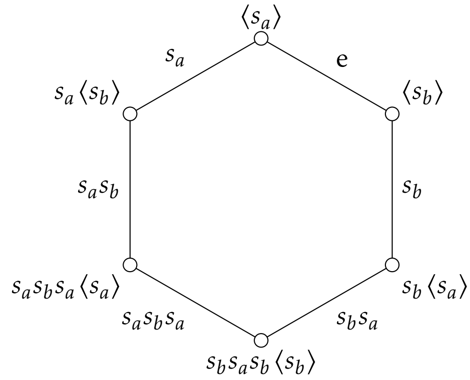

Figure 2.4: The Coxeter complex $\mathcal{A}(W(A_2), B(C))$.

2.3 Buildings

The previous sections laid the ground work for the definition of buildings, which we give in this part. Their construction is reminiscent of the one of manifolds. For those one collects charts, which are homeomorphic maps from a part in the manifold to open balls in the euclidean space, which are collected in a set called the atlas of the manifold. Thus a manifold has the property of being locally euclidean [Lee13, pp. 2-3]. In a similar sense buildings are formed to be «locally Coxeter». The following definition specifies this in more detail.

Definition 2.3.1 ([AB08, pp. 173–175])**.** Let $\mathcal{I}$ be a simplicial complex and let $\mathscr{A}$ be a collection of subcomplexes of $\mathcal{I}$ whose union is $\mathcal{I}$. An element in $\mathscr{A}$ is called an **apartment**.

(B0) Each apartment $\mathcal{A} \in \mathscr{A}$ is a Coxeter complex.

(B1) For any two simplices $A, B \in \mathcal{I}$, there is an apartment $\mathcal{A} \in \mathscr{A}$ containing both of them.

(B2) If $\mathcal{A}$ and $\mathcal{A}'$ are two apartments containing A and B, then there is an isomorphism $\mathcal{A} \to \mathcal{A}'$ fixing every simplex in $\mathcal{A} \cap \mathcal{A}'$.

If the above three conditions hold, then $\mathcal{I}$ is called a **building**.

This definition has quite strong consequences of which we will record a few here. First we note that by taking twice the minimal simplex in **(B2)** one sees that all apartments are isomorphic, a fortiori it can be shown that the Coxeter matrices of the apartments are all the same [AB08, Prop. 4.7]. That means in particular that the chambers in an apartment are also chambers in the whole building $\mathcal{I}$. As hinted above the adjacency relation in a Coxeter complex can be expressed by two chambers having a common codimension one face. By applying this definition to $\mathcal{I}$ then, together with **(B1)**, one can check that all chambers in a building are connected.

Buildings are, as are their apartments, colourable simplicial complexes, which is shown by using the fact for apartments and lifting it to the case of buildings via the axioms [AB08, Prop. 4.6]. Thus to determine the type of a simplex, it suffices to know its type in an apartment.
The following proposition asserts that the links in buildings behave extraordinarily well. It follows mainly by a similar fact about links in Coxeter complexes.

Proposition 2.3.2 ([AB08, Prop. 4.9])**.** *Let $\mathcal{I}$ be a building and $\mathcal{A}$ its set of apartments. Moreover fix a simplex $A \in \mathcal{I}$. Then $\mathrm{lk}_{\mathcal{I}}(A)$ is a building and its apartments are given by $\mathrm{lk}_{\mathcal{A}}(A)$, for every $\mathcal{A} \in \mathscr{A}$.*

A strength of the theory of buildings is the existence (and uniqueness) of retractions. Let C be a chamber in $\mathcal{I}$ and $\mathcal{A}$ an apartment containing it. A **retraction** onto $\mathcal{A}$ centered at C is a simplicial map $\rho_{\mathcal{A},C} : \mathcal{I} \to \mathcal{A}$, that sends chambers to chambers, fixes C pointwise, i.e. fixes every vertex in C, and maps every apartment containing C isomorphically onto $\mathcal{A}$. The importance of retractions stems from the fact, that they reduce problems to the case of Coxeter complexes, which is easier as one has to deal with the structure of a specific Coxeter group, of which we already noted that they are well-behaved. The following proposition collects some properties of retractions to further convince the reader of their significance. We note that the **distance** between two chambers C and D is the length of a minimal path connecting them. One can show that it suffices to look at paths in apartments containing both C and D and thus we know, by the Coxeter structure, that the distance is just the length $l(w^{-1}w')$, where $w \in W$ corresponds to the chamber C and $w' \in W$ to the chamber D in an apartment $\mathcal{A}(W,S)$ containing C and D.

Proposition 2.3.3 ([AB08, Prop. 4.39]). *(a) For every chamber C and apartment $\mathcal{A}$ containing it a retraction $\rho_{\mathcal{A},C}$ exists and it is the unique simplical map, mapping chambers to chambers and preserving the distance from C.*

(b) For every face $A \leq C$, one finds $\rho^{-1}(A) = \{A\}$, i.e. C is fixed ***pointwise****.*

(2.3.4). Buildings are often times characterised by the geometric type of their Coxeter systems. This can also be visualized, since to any building $\mathcal{I}$ one can construct its **geometric realization**, written formulaic $|\mathcal{I}|$, in the following way:
Denote by $\mathcal{V}$ the set of vertices of $\mathcal{I}$. We want to put a topology on the space of functions $\mathcal{V} \to [0,1]$, denoted by $[0,1]^{\mathcal{V}}$. For a finite subset $V \subseteq \mathcal{V}$ this can be done by identifying $[0,1]^{V}$ with the compact space $[0,1]^{\mathrm{card}(V)}$. By taking the direct limit one then defines a topology on $[0,1]^{\mathcal{V}}$, i.e a subset $S \subseteq [0,1]^{\mathcal{V}}$ is open (closed) in $[0,1]^{\mathcal{V}}$, if and only if the restriction $S_{\restriction V}$ for every finite $V \subseteq \mathcal{V}$ is open (closed) in $[0,1]^{V}$. We will define $|\mathcal{I}|$ as a subspace of $[0,1]^{\mathcal{V}}$ by letting ourselves being guided by the form of the geometric standard simplices. A function $\lambda : \mathcal{V} \to [0,1]$ is in $|\mathcal{I}|$, if and only if its support is the vertex set of a simplex $A \in \mathcal{I}$ and it holds that

$$\sum_{v \in \mathcal{V}} \lambda(v) = 1.$$

Note that this construction works for arbitrary abstract simplicial complexes and if one takes for example the geometric realization of a single simplex, one obtains the geometric standard simplex from above again. One is able to analyse the homotopy type of buildings quite generally, which we will record here:

Theorem 2.3.5 ([AB08, Thm. 4.127]). *Let $\mathcal{I}$ be a building. If $\mathcal{I}$ is* ***spherical****, i.e. its apartments are finite Coxeter complexes of rank n, then $|\mathcal{I}|$ has the homotopy type of a bouquet of $(n-1)$-spheres, where there is one sphere for every apartment in $\mathcal{I}$. If $\mathcal{I}$ is non-spherical, then $|\mathcal{I}|$ is contractible.*

Lastly we remark that one can show that if $\mathcal{I}$ is an **affine** building, i.e. if its apartments are affine Coxeter complexes, then one finds the geometric realizations of their apartments to be vector spaces [AB08, 10.2.1]. Thus they are also sometimes referred to as **euclidean** buildings.
In chapter three we will construct a spherical building and an affine building associated with a linear algebraic group. But before we can begin this task, we will need to present some facts about groups acting on buildings.

2.4 Tits systems

Let $\mathcal{I}$ be a building with apartment set $\mathscr{A}$ and let G be a group acting on $\mathcal{I}$ that operates in a type preserving manner. We will also assume in the following that the G-action maps apartments to apartments. It turns out that transitivity on chambers does not yield a useful theory [AB08, p. 295], however the following property does:

Definition 2.4.1 ([AB08, Def. 6.1]). A G-action on $\mathcal{I}$ is said to be **strongly transitive**, if and only if G acts transitively on the set of pairs $(\mathcal{A},C)$, where $\mathcal{A}$ is an apartment and C is a chamber in $\mathcal{A}$.

Thus we will in the following fix an apartment $\mathcal{A}$ and a chamber C contained in it. The terms **fundamental apartment** for $\mathcal{A}$ and **fundamental chamber** for C are customary. We will record a few results concerning G and the subsequent subgroups:

$$\begin{aligned} B &:= \{g \in G \mid gC = C\}, \\ N &:= \{g \in G \mid g\mathcal{A} = \mathcal{A}\} \quad \text{and} \\ T &:= \{g \in G \mid g \text{ fixes } \mathcal{A} \text{ pointwise }\}. \end{aligned} \tag{2.1}$$

By the last condition is meant that any $g \in T$ fixes all the vertices in $\mathcal{A}$. Since one can show that any type preserving automorphism of a Coxeter complex is given by an action of an element in the associated Coxeter group [AB08, Prop. 3.32], one sees that there is a homomorphism $f : N \to W$, where (W,S) is the Coxeter system associated with $\mathcal{A}$. In particular one sees that T is, as kernel of f, a normal subgroup in N. One can also deduce that f is surjective, yielding $N/T \cong W$, and $T = B \cap N$ [AB08, p. 296]. Strongly transitive group actions also have consequences on the structure of the group itself. We note a first such statement.

Proposition 2.4.2 ([AB08, Prop. 6.3])**.** *Let G be a group that acts strongly transitive on a building $\mathcal{I}$. With the notation as above one arrives at:*

$$G = BNB.$$

More precisely, if one chooses for every $w \in W$ a representative $\tilde{w} \in N$, then one gets the following disjoint union of double cosets [AB08, p. 304]:

$$G = \bigsqcup_{w \in W} B\tilde{w}B.$$

This is known as the ***Bruhat decomposition****.*

In the following we will be concerned with the following partially ordered set

$$\mathcal{I}(G,B) := \left\{ Q \leq G \;\middle|\; \exists\, g \in G : \; gBg^{-1} \subseteq Q \right\},$$

where the ordering shall be given by the reverse inclusion. Conjugation shall define an action of G on $\mathcal{I}(G,B)$. The elements in $\mathcal{I}(G,B)$ are called **parabolic subgroups** of G. A consequence of the Bruhat decomposition is that for every parabolic subgroup $P \leq G$, there is an element $g \in G$ and a subset $J \subseteq S$ such that

$$P = g\left(\bigsqcup_{w \in W_J} B\tilde{w}B \right) g^{-1}.$$

The converse is true as well, i.e. every subgroup of this form is parabolic, and the ones of the form $\bigsqcup_{w \in W_J} B\tilde{w}B$ are called **standard** parabolic subgroups. This suggests that $\mathcal{I}(G,B)$ might be a colourable simplicial complex and even more is true:

Theorem 2.4.3. *$\mathcal{I}(G,B)$ is a building (see [AB08, Cor. 6.47, Prop. 6.52]), with apartments $g.\mathcal{A}'$, where [AB08, p. 320]*

$$\mathcal{A}' := \left\{ nPn^{-1} \;\middle|\; n \in N, \; B \subseteq P \right\}$$

and $g \in G$ holds. Furthermore G acts strongly transitively on $\mathcal{I}(G,B)$. If the original building $\mathcal{I}$ is ***thick****, i.e. if every codimension one simplex is contained in at least three chambers, then there is a canonical isomorphism between $\mathcal{I}$ and $\mathcal{I}(G,B)$ [AB08, Thm. 6.56].*

This is an interesting result, but not as commonly used as its opposite direction. By analysing the assumptions that are needed to derive the Bruhat decomposition, one arrives at the following notion, which demands some familiar facts concerning G and its subgroups we recorded initially:

Definition 2.4.4 ([AB08, Def. 6.55])**.** Let there be a group G together with a pair of subgroups B and N. Suppose B and N generate G, the intersection $T := B \cap N$ is normal in N and that the group $W := N/T$ admits a set of generators S such that the following conditions are met (again we denote by tilde a lifting of W into N):

(BN1) For $s \in S$ and $w \in W$, one has $\tilde{s}B\tilde{w} \subseteq B\tilde{s}\tilde{w}B \cup B\tilde{w}B$.

(BN2) For $s \in S$, $\tilde{s}B\tilde{s}^{-1} \nsubseteq B$ is true.

Then one calls the quadruple (G,B,N,S) a **Tits system**.

The following theorem will be the basis of the constructions of the buildings we will present in chapter three.

Theorem 2.4.5 ([AB08, Thm. 6.56])**.** *Let (G,B,N,S) be a Tits system. Then $\mathcal{I}(G,B)$ is a thick building with a strongly transitive G-action on it, given by conjugation, such that B is the stabilizer of a fundamental chamber and N stabilizes a fundamental apartment and acts transitively on the chambers of that apartment.*

Remark 2.4.6. Unlike as in the initial definition of N in (2.1), the N of the Tits system (G,B,N,S) does not need to be the full stabilizer of the fundamental apartment. To make this the case, it turns out that one needs to impose the condition

$$T = \bigcap_{w \in W} \tilde{w} B \tilde{w}^{-1},$$

where we set $T := B \cap N$ as usual. A Tits system fulfilling this additional criterion is called **saturated** [AB08, Def. 6.57].

3

Reductive Linear Algebraic Groups and Bruhat-Tits Buildings

In this section we seek to give a quick review of the theory of reductive linear algebraic groups in the context of the Bruhat-Tits framework, which associates to such algebraic groups an affine building, i.e. a building whose apartments can be realized as vector spaces or more generally as affine spaces. Our introduction of algebraic groups is geared towards a reader, who is already familiar with the subject, as we will be rather brief and informal in our definitions. Our main sources for the contents of this chapter will be the standard text books on algebraic groups, namely Springer [Spr98], Borel [Bor91] and the new book by Milne [Mil17]. The general structure of this part is aligned to [Abr94].

We will start out by giving some facts about the absolute theory, i.e. the theory of reductive linear algebraic groups over an algebraically closed field, combined with the most important definitions, and then we proceed to lay out the general case afterwards. The third part in this chapter covers the theory of Bruhat and Tits.

3.1 Reductive Groups

We will commence this passage by fixing the most important notions and definitions. Therefore let us fix for the remainder of this section an algebraically closed field $\mathbf{k}$ and a sub-field thereof, denoted by k.

3.1.1 Basic Definitions

Algebraic groups will be group objects in a suitable sub-category of the scheme-category. The subsequent definition sets the stage for the geometric objects, that we intend to focus on in the following.

Definition 3.1.1. Let X be a separated scheme of finite type over the field k, which is absolutely reduced, i.e. its extension of scalars $X_{\mathbf{k}}$ is reduced. We will call such schemes k-**varieties** and omit mentioning k, if $k = \mathbf{k}$ holds. A scheme morphism $\phi : X \to Y$ over $\mathrm{Spec}(k)$, between two k-varieties X and Y, will be called a k-**morphism**, if and only if it is of finite type.
Let B be a k-algebra and X a k-variety. We denote by $X(B)$ the scheme morphisms $\mathrm{Spec}(B) \to X$, defined over k. They will be called the B-**valued points** in X, and if $B = k$ holds the k-**rational points** in X.

Let X be a k-variety.

Remarks 3.1.2. (a) In the remainder of this thesis we will disregard non-closed points in the k-varieties we examine. Let us subsume the closed points of X in X^{cl}. Then, since X is of finte type over k, there is a bijection

$$\begin{aligned}\tau_X &\to \tau_{X^{\mathrm{cl}}}\\ U &\mapsto U \cap X^{\mathrm{cl}},\end{aligned}$$

where we denoted by τ the corresponding topologies (cf. [GW10, Prop. 3.35]). However our choice of neglecting the non-closed points is also unproblematic in another sense. Suppose one introduces a notion analogous to schemes, called **ultra-schemes**, by using as an affine model, not prime ideal spectra, but maximal ideal spectra. It can be shown, that there is a categorical equivalence between ultra-schemes and schemes, if one works with algebras of finite type over a field (cf. [GD71, Appendice]). Moreover the points in ultra-schemes are in one-to-one correspondence to the closed points in the corresponding scheme in the usual sense. Finally we note that our sources, also chose to neglect non-closed points in their discussions.

(b) The mapping of a k-algebra B to the set $X(B)$, is functorial in B. Since, suppose there is a k-algebra map $B \to B'$, then one can write the following correspondence, if we denote by ϕ the morphism $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$:

$$\begin{aligned} X(B) &\to X(B') \\ x &\mapsto x \circ \phi. \end{aligned}$$

There is even a special form of Yoneda's lemma accompanying this. Let there be two k-varieties X and Y, and let there be a natural transformation between the functors of points of X and of Y, restricted to the case of finitely generated k-algebras. Then there is a unique k-morphism corresponding to this natural transformation and, vice versa, to every k-morphism there is a natural transformation of the point functors (cf. [Mil17, A.d]). We will refer to this equivalence as the *functorial approach*.

(c) Let $k \subseteq k'$ be a field extension. By viewing $X(k')$ as a subset of the topological space X^{top}, through evaluation with the generic point in $\mathrm{Spec}(k')$, one puts a topology on the k'-valued points.

(d) There are k-varieties, that have no k-rational points. As an example one may consider the $\mathbb{R}$-variety $X := \mathrm{Spec}\left(\mathbb{R}[X,Y]/(X^2+Y^2+1)\right)$.

(e) In the absolute case, i.e. if $k = \mathbf{k}$ holds, there is a one-to-one correspondence of k-rational points $X(k)$ and points (by which we mean closed points) in X.

Next we introduce the basic group objects, in the category of k-varieties, that we want to study. Note that the diagrams in the following may seem complicated, but they derive, after a close look, of the corresponding diagrams for abstract groups.

Definition 3.1.3. Let G be an affine k-variety, together with k-morphisms

$$m : G \times_k G \to G, \quad \mathrm{e} : \mathrm{Spec}(k) \to G \quad \text{and} \quad \mathrm{inv} : G \to G$$

such that the following diagrams commute:

(**Ass**)
$$\begin{tikzcd}
(G\times G)\times G \arrow[r, "m\times \mathrm{id}"] \arrow[d, "\mathrm{can.}"', "\cong"] & G\times G \arrow[dr, "m"] & \\
 & & G \\
G\times(G\times G) \arrow[r, "\mathrm{id}\times m"] & G\times G \arrow[ur, "m"'] &
\end{tikzcd}$$

(**Un**)
$$\begin{tikzcd}
G\times \mathrm{Spec}(k) \arrow[r, "\mathrm{id}\times \mathrm{e}"] \arrow[dr, "\mathrm{can.}"', "\cong"] & G\times G \arrow[d, dashed, "m"] & \mathrm{Spec}(k)\times G \arrow[l, "\mathrm{e}\times \mathrm{id}"'] \arrow[dl, "\mathrm{can.}", "\cong"'] \\
 & G &
\end{tikzcd}$$

(**Inv**)
$$\begin{tikzcd}
G \arrow[r, "\begin{pmatrix}\mathrm{inv}\\ \mathrm{id}\end{pmatrix}"] \arrow[d] & G\times G \arrow[d, dashed, "m"] & G \arrow[l, "\begin{pmatrix}\mathrm{id}\\ \mathrm{inv}\end{pmatrix}"'] \arrow[d] \\
\mathrm{Spec}(k) \arrow[r, "e"] & G & \mathrm{Spec}(k). \arrow[l, "e"']
\end{tikzcd}$$

Then we will call G a k-**linear algebraic group**. As it was before, we will omit mentioning k, if $k = \mathbf{k}$ holds.

Remark 3.1.4. Note, that since the valued points correspondence is functorial, for every k-algebra B, there is the structure of an abstract group on $G(B)$, whenever this set is non-empty. If we have k algebraically closed, we even have the chance to obtain a group structure on the set of points G^{top}, by the above remarks. We will use this identification often in the next subsection.

Example 3.1.5. (a) We will bring the structure of a k-linear algebraic group on the affine k-variety $\operatorname{Spec}\left(k\left[T,T^{-1}\right]\right)$, by defining the morphism m, e and inv, via their corresponding k-algebra homomorphisms.

$$m \leftrightsquigarrow \begin{pmatrix} k\left[T,T^{-1}\right] \to k\left[T,T^{-1}\right] \otimes k\left[T,T^{-1}\right] \\ T \mapsto T \otimes T \end{pmatrix},$$
$$\mathrm{e} \leftrightsquigarrow \begin{pmatrix} k\left[T,T^{-1}\right] \to k \\ T \mapsto 1 \end{pmatrix} \text{ and}$$
$$\mathrm{inv} \leftrightsquigarrow \begin{pmatrix} k\left[T,T^{-1}\right] \to k\left[T,T^{-1}\right] \\ T \mapsto T^{-1} \end{pmatrix}.$$

A quick checks shows, that the diagrams (**Ass**), (**Un**) and (**Inv**) are commuting. In the following we refer to this k-algebraic group by $\mathbb{G}_{\mathrm{m}}$. Also note, that the B-valued points for a k-algebra B are given by

$$\mathbb{G}_{\mathrm{m}}(B) = B^{\times},$$

i.e. by the multiplicative group of units in B.

(b) As a generalization thereof we will also briefly touch on the definition of the general linear group as an algebraic group. Over $\mathbf{k}$ one can immediately see, since the determinant is a polynomial in the entries, that GL_n is a variety and as multiplication and inverse of matrices are also polynomial one obtains a linear algebraic group. In the relative case, this is a bit more difficult. However what one can gather from the algebraically closed situation is that one needs to put the structure of an algebraic group on the k-variety $\operatorname{Spec}(A)$, with

$$A := k\left[T_{11}, T_{12}, \ldots, T_{nn}, \det{}^{-1}\right],$$

where det is the determinant of the matrix $(T_{ij})_{1\leqslant i,j\leqslant n}$. Again we will do this by giving the dual k-algebra homomorphisms

$$m \leftrightsquigarrow \begin{pmatrix} A \to A \otimes_k A \\ T_{ij} \mapsto \sum_{k=1}^{n} T_{ik} \otimes T_{kj} \end{pmatrix}, \quad \mathrm{e} \leftrightsquigarrow \begin{pmatrix} A \to k \\ T_{ij} \mapsto \delta_{ij} \end{pmatrix} \quad \text{and} \quad \mathrm{inv} \leftrightsquigarrow \begin{pmatrix} A \to A \\ T_{ij} \mapsto \left(T^{-1}\right)_{ij} \end{pmatrix},$$

where δ denotes the Kronecker delta and $(T^{-1})_{ij}$ is the ij-entry of the inverse of the matrix $(T_{ij})_{1\leqslant i,j\leqslant n}$, which is, by Cramer's rule, the quotient of a polynomial in the entries of that matrix and det. The properties (**Ass**), (**Un**) and (**Inv**) can be checked by essentially the same calculations one has to perform to show GL_n is an abstract group. Moreover it can be shown that the B-valued points of GL_n are exactly the invertible $n\times n$-matrices with entries in B. To see that this is actually a generalization of the above, we note $\mathrm{GL}_1 = \mathbb{G}_{\mathrm{m}}$.

Remark 3.1.6. The property of being linear in definition 3.1.3 refers to the fact that a k-linear algebraic group G can be embedded into GL_n, for some $n \in \mathbb{N}^+$ [Bor91, Prop. 1.10]. This means in particular that there is a k-morphism $\phi : G \to \mathrm{GL}_n$, which is a closed immersion and fulfils the k-**homomorphism** property

$$\phi \circ m_G = m_{\mathrm{GL}_n} \circ (\phi \times_k \phi).$$

Any such k-homomorphism $\phi : G \to \mathrm{GL}_n$ is called a k-**representation** and moreover **faithful**, if ϕ is a closed immersion.

3.1.2 Reductive Groups over Algebraically Closed Fields

In this section we will recall the theory of linear algebraic groups over algebraically closed fields. Since we have the structure of an abstract group on the points of a linear algebraic group G, as $G^{\mathrm{top}} = G(\mathbf{k})$ holds, we are able to carry some notions, like «normal subgroup» or «solvable group», of the theory of groups over to G, by demanding, that the group of points fulfils the hypotheses of the notion. First we discuss some simplicity assumptions:

Definition 3.1.7 ([Bor91, 11.21])**.** Let G be a connected linear algebraic group.

(a) G is called **almost simple**, if and only if for every closed, connected and normal subgroup N of G, one has either $N = G$ or $N = \{e\}$.[1]

(b) $\mathrm{R}(G) :=$ (maximal, closed, connected, solvable, normal subgroup of G) is called the **radical** of G. G is said to be **semi-simple**, if $\mathrm{R}(G)$ is trivial.

(c) $\mathrm{R}_u(G) := \mathrm{R}(G)_u = \{g \in \mathrm{R}(G) \mid g \text{ unipotent}\}$ is the **unipotent radical** of G. It is the maximal closed, connected, unipotent, normal subgroup of G. G is called **reductive** if $\mathrm{R}_u(G)$ is trivial.

Remark 3.1.8. A group element $g \in G$ is called unipotent, if and only if there is a faithful representation $\phi : G \to \mathrm{GL}_n$ such that $\phi(g)$ is a unipotent matrix, i.e. if there is $m \in \mathbb{N}$, with $(\phi(g) - \mathrm{id})^m = 0$. One can then show that under any representation the image of g is a unipotent matrix [Mil17, 9.19]. The set of unipotent elements in G form a closed subgroup of G [Mil17, 9.22] and thus also a linear algebraic group.

We will denote the connected component of a linear algebraic group G, containing the identity e, by G^0 as it is customary in the literature [Bor91, 1.2]. For H a closed subgroup of G, denote by $\mathrm{Z}_G(H)$, the **centraliser** of H in G, i.e. the set of group elements that commute with all of H, which is a closed subgroup of G again (cf. [Bor91, 1.7]). If $H = G$, then we write $\mathrm{Z}(G) := \mathrm{Z}_G(G)$ and call it the **center** of G. We record the following about reductive and semi-simple groups:

Proposition 3.1.9. *Let G be a connected linear algebraic group.*

(a) If G is reductive, then $\mathrm{R}(G) = \mathrm{Z}(G)^0$ *is a* ***torus****, i.e. isomorphic to* $\mathbb{G}_{\mathrm{m}}^l$*, with* $l \in \mathbb{N}^+$*, the commutator subgroup* $[G, G]$ *is semi-simple and the intersection* $\mathrm{R}(G) \cap [G, G]$ *is finite. G can be presented as the almost-direct product* $\mathrm{R}(G) \tilde{\times} [G, G]$ *(cf.[Bor91, 14.2]).*

(b) If G is moreover semi-simple, there are finitely many minimal connected, closed, normal subgroups $N_1, \ldots, N_r$ *in G such that* $G = N_1 \tilde{\times} \ldots \tilde{\times} N_r$ *holds (cf. [Bor91, 14.10]).*

Root systems for Reductive Groups

Let G be a linear algebraic group G and let T be a **torus** in G, i.e. a closed subgroup, that is isomorphic to $\mathbb{G}_{\mathrm{m}}^l$. We fix the **adjoint representation**, i.e. the homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, given by conjugation on points, where $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of G (cf. [Bor91, 3.5, 3.13]), which is a **k**-vector space. Since for tori it can be shown, that any linear representation is diagonalizable [Spr98, Thm. 3.2.3] (and hence also $\mathrm{Ad}_{\restriction T}$), we have that $\mathfrak{g}$ decomposes as the direct sum of eigenspaces, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{a \neq 0} \mathfrak{g}_a$, where we have set:

$$\mathfrak{g}_0 := \{x \in \mathfrak{g} \mid (\mathrm{Ad}(t))(x) = x \ \forall t \in T\} = \mathrm{Lie}(\mathrm{Z}_G(T)),$$
$$\mathfrak{g}_a := \{x \in \mathfrak{g} \mid (\mathrm{Ad}(t))(x) = a(t)x \neq 0\}, \quad \text{for some } 0 \neq a \in \mathrm{X}^*(T) := \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}}).$$

The elements of the set $\{0\} \cup R(G, T)$ are called the **weights** of T with respect to Ad, whereas the elements in $R(G, T) := \{a \in \mathrm{X}^*(T) \mid a \neq 0 \wedge \mathfrak{g}_a \neq 0\}$ are dubbed the **roots** of T (cf. [Bor91, 8.17]). The roots and weights are homomorphisms of linear algebraic groups from T to $\mathbb{G}_{\mathrm{m}}$, which will be referred to as **(rational) characters** and denoted by $\mathrm{X}^*(T)$.
The following properties of **maximal tori**, i.e. sub-tori of a linear algebraic group, that are of maximal dimension, are important in the theory of algebraic groups.

Proposition 3.1.10. *Let G be a connected, reductive linear algebraic group and T a torus in G.*

(a) $\mathrm{Z}_G(T)$ *is connected and reductive.* *[Bor91, Cor. 11.12]*

(b) If T is maximal, then $\mathrm{Z}_G(T) = T$. *[Bor91, Cor. 13.17.2]*

(c) The maximal tori in G are conjugate. *[Bor91, Cor. 11.3]*

For a reductive group G and a maximal torus T, one calls the dimension of T the **rank** of G and the dimension of T minus the dimension of the central torus $\mathrm{R}(G)$ (cf. 3.1.9.(a)), the **semi-simple rank** of G:

$$\mathrm{rk}(G) := \dim T \quad \text{and} \quad \mathrm{rk}_{\mathrm{s}}(G) := \dim T - \dim(\mathrm{R}(G)).$$

[1] The notion of simplicity or almost simplicity varies from author to author. For example, what we defined here is called *quasi-simple* in [Spr98, 8.1.12.(4)]. For a quick survey of the different notions of simplicity check [Mil17, p. 399].

Furthermore we set $W(G,T)$ to be the group $\mathrm{N}_G(T)/T$, where $\mathrm{N}_G(T)$ is the closed subgroup of G, which contains all the elements that normalise T (cf. [Bor91, 1.7]). By [Spr98, Cor. 3.2.9] $W(G,T)$ is a finite subgroup and one can even show, that $W(G,T)$ is a Coxeter group [Spr98, Thm. 8.3.4].

The normaliser $\mathrm{N}_G(T)$ operates on the characters $\mathrm{X}^*(T)$ via

$$(n.\chi)(t) = \chi\left(n^{-1}tn\right),$$

with $n \in \mathrm{N}_G(T)$, $\chi \in \mathrm{X}^*(T)$ and $t \in T$ [Hum75, p. 151]. Let $a \in R(G,T)$ be a root and $n \in \mathrm{N}_G(T)$, then by using the definition of $\mathfrak{g}_a$ one can calculate:

$$\mathrm{Ad}(t).\mathrm{Ad}(n).v = \mathrm{Ad}(n\, n^{-1}tn).v = \mathrm{Ad}(n).\mathrm{Ad}(\underbrace{n^{-1}tn}_{\in T}).v = a(n^{-1}tn)\cdot \mathrm{Ad}(n).v = (n.a)(t)\cdot \mathrm{Ad}(n).v.$$

This readily implies that $\mathrm{Ad}(n).v \in \mathfrak{g}_{n.a}$ and a fortiori

$$\mathrm{Ad}(n)(\mathfrak{g}_a) = \mathfrak{g}_{n.a}. \tag{3.1}$$

Thus one obtains an action of $W(G,T)$ on $R(G,T)$, which is a restriction of its action on $\mathrm{X}^*(T)$. The following is to justify the name of the elements of $R(G,T)$:

Proposition 3.1.11 ([Bor91, Thm. 13.18, Thm. 14.8]). *Let G be a connected, reductive linear algebraic group and T be a maximal torus. Define the following subspace of the characters:*

$$X_s = \{\chi \in \mathrm{X}^*(T) \mid \chi(\mathrm{R}(G)) = 1\} \supseteq R(G,T).$$

Then $R(G,T)$ is a reduced root system in $V := X_s \otimes_{\mathbb{Z}} \mathbb{R}$, with Weyl group $W(G,T)$, and the action of the Weyl group agrees with the action of $W(G,T)$ introduced above. Moreover the rank of the root system $R(G,T)$ agrees with $\mathrm{rk}_s(G)$.

Remarks 3.1.12. (a) The above action of $W(G,T)$ on $\mathrm{X}^*(T)$, yields a linear representation of the Weyl group $W(G,T)$ in $\mathrm{GL}(V)$. As generating reflections one can take the elements $(N \cap \mathrm{Z}_G(T_a))/T$, with $T_a = \ker(a)^0$, for some $a \in R(G,T)$ (cf. [Bor91, 13.18]).
(b) If G is moreover semi-simple, i.e. $\mathrm{R}(G) = 1$, then one has that $X_s = \mathrm{X}^*(T)$ and $V = \mathrm{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

For a general root system R inside a vector space V, one can introduce the **root lattice** $Q(R) := \langle R \rangle_{\mathbb{Z}}$ and the **weight lattice** $P(R) := \{\lambda \in V \mid \langle \lambda, a^\vee \rangle \in \mathbb{Z},\ \forall a \in R\}$. A quick analysis yields, that for linear algebraic groups, the group of characters lies between these two extremes (the semi-simplicity in the following is just a simplification):

Lemma 3.1.13 ([Spr98, 8.1.11]). *Let G be a connected, semi-simple linear algebraic group and T a maximal torus in it. Then one has the following chain of inclusions:*

$$Q(R(G,T)) \subseteq \mathrm{X}^*(T) \subseteq P(R(G,T)).$$

With respect to this lemma the following notation is customary:

$$\begin{aligned} G \text{ is } \textbf{adjoint} &\iff \mathrm{X}^*(T) = Q(R(G,T)),\\ G \text{ is } \textbf{simply-connected} &\iff \mathrm{X}^*(T) = P(R(G,T)). \end{aligned}$$

A well-known result, which is due to Chevalley [Che58], states that semi-simple linear algebraic groups are readily determined by their character group and their root system. More precisely, one has the following pair of theorems. Firstly the *isomorphism theorem*:

Theorem 3.1.14 ([Spr98, Thm. 9.6.2]). *Let (G,T) and (G',T') be two connected, semi-simple linear algebraic groups with maximal tori. Denote by V and V', the vector spaces, which the root systems $R(G,T)$ and $R(G',T')$ span. Let $f : V' \xrightarrow{\sim} V$ be an isomorphism of vector spaces such that $f(R(G',T')) = R(G,T)$ and $f(\mathrm{X}^*(T')) = \mathrm{X}^*(T)$ holds. Then there is an isomorphism $\phi : G \to G'$, with $\phi(T) = T'$, that induces f.*

And secondly the *existence theorem*:

Theorem 3.1.15 ([Spr98, Thm. 10.1.1]). *Let R be a reduced root system in a vector space V and denote by Λ a lattice in V such that $Q(R) \subseteq \Lambda \subseteq P(R)$ holds. Then there is a semi-simple linear algebraic group G with maximal torus T, such that the character group $\mathrm{X}^*(T)$ is given by Λ and its root system $R(G,T)$ is R.*

Remark 3.1.16. Since the reduced, irreducible root systems are classified, which is for example exposed in [Bou02, Ch. VI, Thm. 4.2.3], and irreducibility of a root system of a semi-simple linear algebraic group is equivalent to the group being almost simple [Spr98, 8.1.12.(4)], one thus obtains a characterisation of all connected, semi-simple, simply-connected, almost simple linear algebraic groups in terms of their root system.

In the following we turn our attention towards asserting that every reductive linear algebraic group, possesses a *root datum* in the sense of Bruhat and Tits (cf. [BT72, (6.1.1)]). This is a notion that is central in the construction of the associated euclidean building to such a group, which we will see later in this chapter. Therefore we will need another important example of algebraic groups, which we will readily give in a more general setting:

Example 3.1.17. We will put the structure of a k-linear algebraic group on $\operatorname{Spec}(k[T])$, by again defining the group structure via their corresponding k-algebra homomorphisms:

$$m \leftrightsquigarrow \begin{pmatrix} k[T] \to k[T] \otimes_k k[T] \\ T \mapsto T \otimes 1 + 1 \otimes T \end{pmatrix}$$

$$\mathrm{e} \leftrightsquigarrow \begin{pmatrix} k[T] \to k \\ T \mapsto 0 \end{pmatrix} \text{ and}$$

$$\mathrm{inv} \leftrightsquigarrow \begin{pmatrix} k[T] \to k[T] \\ T \mapsto -T \end{pmatrix}.$$

Again the checks of (**Ass**), (**Inv**) and (**Un**) are immediate. By the universal property of the polynomial ring one also checks that, for every k-algebra B, the group of B-valued points is given by the additive group on B itself. The so obtained k-linear algebraic group will be denoted by $\mathbb{G}_\mathrm{a}$.

We will start by recalling the following statement about the existence of special subgroups, which follows from an analysis of rank one subgroups.

Proposition 3.1.18. *Let G be a connected, reductive group and T a maximal torus contained in it.*

(a) For every root $a \in R(G,T)$ there is a uniquely determined, connected, unipotent subgroup U_a, that is invariant under conjugation by T and which has $\operatorname{Lie}(U_a) = \mathfrak{g}_a$ (cf. [Bor91, 13.18]). There is an isomorphism $U_a \simeq \mathbb{G}_\mathrm{a}$ (cf. [Spr98, Thm. 3.4.9]) and in particular $\mathfrak{g}_a$ is one dimensional.

(b) For every isomorphism $x_a : \mathbb{G}_\mathrm{a} \to U_a$, with $a \in R(G,T)$, it holds that:

$$t x_a(\lambda) t^{-1} = x_a(a(t)\lambda), \quad \forall t \in T, \lambda \in \mathbf{k}.$$

(c) For every $n \in \mathrm{N}_G(T)$ and $a \in R(G,T)$ we have $nU_a n^{-1} = U_{n.a}$

Remark. (b) follows from the statement $\operatorname{Lie}(U_a) = \mathfrak{g}_a$ and (c) derives from the uniqueness statement of (a) and (3.1). ☐

The subgroups U_a, for a root $a \in R(G,T)$ will be called **root subgroups**. The next proposition summarizes the analysis of the relations of the root subgroups among themselves.

Proposition 3.1.19 ([Bor91, 14.4-14.5])**.** *Let G be a connected, reductive linear algebraic group and $T \subseteq G$ a maximal torus.*

*(a) Let a and b be two linearly independent roots and fix $\lambda, \mu \in \mathbf{k}$. For every root $c \in R(G,T)$ fix an isomorphism $x_c : \mathbb{G}_\mathrm{a} \to U_c$. Then **Chevalley's commutation relations** hold:*

$$[x_a(\lambda), x_b(\mu)] = \prod_{p,q \in \mathbb{N}^+,\ pa+qb \in R(G,T)} x_{pa+qb}\left(c_{a,b;p,q} \lambda^p \mu^q\right).$$

*One may choose the isomorphisms $(x_a)_{a \in R(G,T)}$ in such a way, that all **structure constants** $c_{a,b;p,q}$ are in $\mathbb{Z}.1_\mathbf{k}$ (cf. [Spr98, 9.2.5, 9.5.3]).*

(b) For every system of positive roots R^+ in $R(G,T)$ let U_{R^+} be the subgroup of G generated by the U_a, with $a \in R^+$. Then one finds that U_{R^+} is a connected, unipotent, subgroup of G, which is invariant under conjugation by T. Moreover there is the following isomorphism of affine varieties given by the product map

$$U_{a_1} \times \cdots \times U_{a_r} \xrightarrow{\sim} U_{R^+},$$

where $\{a_1, \ldots, a_r\}$ is any ordering of the roots in R^+. This can be extended to any positively closed subset $P \subseteq R^+$.

(c) *Let R_1^+ and R_2^+ be two systems of positive roots in $R(G,T)$. Then one has $TU_{R_1^+} \cap TU_{R_2^+} = TU_{R_1^+ \cap R_2^+}$.*

The proof of 3.1.11 is based on, as we hinted before already, an analysis of rank one subgroups. [Spr98, Thm. 7.2.4] tells us that such groups are isomorphic to SL_2 or its projective version PSL_2. One observes that the diagonal matrices in SL_2 form a maximal torus and that the corresponding root system is given by $\{\pm(e_1 - e_2)\}$ in the one dimensional vector space $\mathbb{R}^2/(e_1 + e_2)$. One derives then that the root subgroups are given by

$$U_a = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbf{k} \right\} \quad \text{and} \quad U_{-a} = \left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbf{k} \right\},$$

with $a := e_1 - e_2$. The subsequent proposition is based on the following observation in the matrix case ($\lambda \in \mathbf{k}$):

$$\underbrace{\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}}_{x_{-a}(\lambda)} = \underbrace{\begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}}_{x_a(\lambda^{-1})} \underbrace{\begin{pmatrix} -\lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}}_{\in T} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=: n_a} \underbrace{\begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}}_{x_a(\lambda^{-1})}. \tag{3.2}$$

Proposition 3.1.20 ([Bor91, 13.18, 14.12]). *Let G be a connected, reductive linear algebraic group, T a maximal torus in G and $a \in R(G,T)$ a root. Define $T_a := (\ker a)^0$ and $G_a := \mathrm{Z}_G(T_a)$. Then we have:*

(a) *G_a is a reductive linear algebraic group of semi-simple rank one.*

(b) $\mathrm{Lie}(G_a) = \mathrm{Lie}(T) \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ *and* $G_a = \langle U_{-a} \cup U_a \cup T \rangle$.

(c) *For $W_a := (G_a \cap \mathrm{N}_G(T))/T$, one has* $\mathrm{card}(W_a) = 2$. *If one takes a non-trivial element $1 \neq s_a \in W_a$ together with a lift $n_a \in G_a \cap \mathrm{N}_G(T)$, it holds that $G_a = B_a \cup B_a n_a B_a$, with $B_a = TU_a$, is true.*

As a consequence of (c), we note $U_{-a} \setminus \{\mathrm{e}\} \subseteq B_a n_a B_a = U_a (T n_a) U_a$, which makes the connection to (3.2) more evident.
Now we are in a position to write down the axioms which Bruhat and Tits distilled from the the system of root subgroups of a connected reductive linear algebraic group. This is the first step in defining the affine building associated with such a group and has to be generalized later on to linear algebraic groups over an arbitrary field.

Proposition 3.1.21 ([BT72, Exemples (6.1.3)]). *Let G be a connected, reductive linear algebraic group together with a maximal torus T. Denote by R the root system $R(G,T)$ and fix a system of positive roots R^+ in it. Then $(T, (U_a)_{a \in R})$ is a* ***generating root datum****, i.e. the following conditions are fulfilled:*

(**DR0**) $G = \langle T, U_a \mid a \in R \rangle$;

(**DR1**) *T, U_a ($a \in R$) are subgroups of G and all $U_a \neq \{\mathrm{e}\}$;*

(**DR2**) $[U_a, U_b] \subseteq \left\langle U_{pa+qb} \,\middle|\, p,q \in \mathbb{N}^+,\ pa+qb \in R \right\rangle \quad \forall a,b \in R,\ b \neq -a$;

(**DR3**) *(only concerns non-reduced root systems; to be discussed later)*;

(**DR4**) *For all $a \in R$ there are right cosets $M_a := Tn_a$, such that $U_{-a} \setminus \{\mathrm{e}\} \subseteq U_a M_a U_a$;*

(**DR5**) *$nU_bn^{-1} = U_{s_a(b)}$, $\quad \forall a,b \in R$, $n \in M_a$ and s_a being the reflection corresponding to a;*

(**DR6**) $U^{\pm} := \left\langle U_a \,\middle|\, a \in R^{\pm} \right\rangle \Rightarrow TU^+ \cap U^- = \{\mathrm{e}\}$.

Remarks 3.1.22. (a) Setting $N := \langle M_a \mid a \in R \rangle$ one can show (cf. [BT72, (6.1.2)]), that N normalises T, i.e. that $N \subseteq \mathrm{N}_G(T)$ holds.

(b) Furthermore one is able to deduce that the Weyl group is already determined by the M_a:

$$W(G,T) = \left\langle \bigcup_{a \in R} \underbrace{(\mathrm{N}_G(T) \cap (G_a \setminus T))}_{M_a} \right\rangle.$$

It follows from the fact that the Weyl group associated with the root system R agrees with the Weyl group $W(G,T)$.

(c) From (**DR5**) one deduces, that there is an epimorphism ${}^v\nu : N \to W(R)$, that maps the elements in M_a to the reflection s_a of the root system R. The superscript v in this regard stand for «**v**ectorial», since, as we will see later on, there will be an affine version of such a map as well. The kernel of this map is given by $TU^+ \cap N$, which is T (cf. [Abr94, Lemma 2]).

Just by using the above abstract properties, Bruhat and Tits checked the following:

Proposition 3.1.23 ([BT72, Prop. 6.1.12])**.** *If one sets $B := TU^+$ and G is generated by T and the U_a (cf.* (**DR0**)*), then (G,B,N,S) is a saturated Tits system with Weyl group $W(R)$, where S is the set of reflections corresponding to the roots in the base associated with R^+.*

Thus there is a strongly transitive action of the group G on the building $\mathcal{I}(G,B)$, whose simplices can representated by conjugates of the standard parabolic subgroups $P_I = B\tilde{W}_I B$, for I being a subset of S, W_I being the subgroup of W generated by I and tilde denotes its lift to N.
The notion of an (abstract) parabolic subgroup comes from its namesake in the realm of linear algebraic groups, where a closed subgroup $P \leq G$ is called **parabolic**, if and only if G/P is a projective variety. One can show that, if one puts the context of linear algebraic groups on the situation of proposition 3.1.23, the parabolic subgroups of G and the elements of $\mathcal{I}(G,B)$ agree [Bor91, 14.16], i.e. the abstract parabolic subgroups of G with respect to B coincide with the parabolic subgroups of G.

3.1.3 Reductive Groups over Arbitrary Fields

In the following we assume that G is a k-linear algebraic group. We will also use the notions, that we introduced in the absolute case for k-groups, by demanding that a property is fulfilled by the base changed absolute group. Notationally this is made up for by a k prefix if an object is concerned, e.g. a k-torus is a k-group, that is a torus after changing base to $\mathbf{k}$.

Relative Root Systems and Weyl-Groups

In the general case tori need not be as well-behaved as they are in the absolute case. The following definition introduces one nice property a k-torus might have. Below, we cite some equivalent formulations of the same feature.

Definition 3.1.24 ([Bor91, 8.2])**.** A k-torus S is called k**-split**, if and only if there is an isomorphism $S \cong (\mathbb{G}_\mathrm{m})^l$, defined over k, for some positive integer l.

$\iff$ Every character of S is defined over k. Recall that a character is a homomorphism of $\mathbf{k}$-groups from $S_\mathbf{k}$ to $\mathbb{G}_\mathrm{m}$. One can also look at homomorphism of k-groups from S to $\mathbb{G}_\mathrm{m}$. Comparing the pull-back of those with the absolute characters, this assertion simply states that every absolute character is a pull-back of a relative (over k defined) one.

$\iff$ Every k-linear representation $S \to \mathrm{GL}(V)$ is diagonalisable over k. This is the generalisation of the statement we used to obtain the decomposition of the Lie algebra into eigenspaces.

Remarks 3.1.25. (a) The second equivalence is one reason why maximal k-split tori take the role of the maximal tori in the relative theory. This is what we were initially referring to by *nice property*.
(b) Not every k-torus is k-split. An example may be SO_2, whose rational points over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ are given by

$$\left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \middle| \, x^2 + y^2 = 1, \; x, y \in \mathbb{K} \right\}.$$

Over $\mathbb{C}$, which is algebraically closed, SO_2 is isomorphic to GL_1, via the map $(x,y) \mapsto x + iy$, with inverse $z \mapsto \left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$. However over the real numbers GL_1 and SO_2 cannot be isomorphic, since $\mathrm{SO}_2(\mathbb{R})$ does contain exactly two elements of order 2 and three of order 3, as opposed to $\mathrm{GL}_1(\mathbb{R})^l$, which contains 2^l elements of any given even finite order and one element for every odd one.

The other extreme case that occurs for k-tori, is covered by the following definition.

Definition 3.1.26 ([Bor91, 8.15])**.** A k-torus S is called **anisotropic**, if and only if it has no non-trivial k-split k-subtorus.

$\iff$ $\mathrm{X}^*(S)_k := \{\chi \in \mathrm{X}^*(S) \mid \chi \text{ defined over } k\} = \{0\}$.

The following asserts that the general case of a k-torus is a mixture of the both already described cases.

Lemma 3.1.27 ([Bor91, 8.15]). *Let S be a k-torus. Then there is a maximal k-split k-subtorus S_d of S, as well as a maximal anisotropic k-subtorus S_a of S. It holds, that* $\operatorname{card}(S_d \cap S_a)$ *is finite and that S is the almost direct product of S_d and S_a.*

The following fact, which is related to the fact that eigenvalues of semi-simple endomorphisms are separable over k, will be important in the proof of Soulé's theorem.

Theorem 3.1.28 ([Bor91, 8.11]). *Every k-torus splits over a finite separable extension of k.*

In the absolute case it posed no trouble defining linear algebraic groups from group theoretic objects, such as centralisers, as long as one could show that they are closed subsets of the original algebraic group. In the relative case difficulties may arise. For example, let X be an affine k-variety and let $\mathbf{Y}$ be a sub-variety of $X_{\mathbf{k}}$. $\mathbf{Y}$ can be equivalently given by a radical ideal $\mathbf{I}$ inside the $\mathbf{k}$-algebra $\mathbf{k}[X]$ of $X_{\mathbf{k}}$. When transferring this to the relative situation we identify two problems:

1. The radical ideal of $(\mathbf{I} \cap k[X])\,\mathbf{k}[X]$, where $k[X]$ is the k-algebra associated to X, may disagree with $\mathbf{I}$, which suggests that the closed k-subscheme, defined by $\mathbf{I} \cap k[X]$, is not closely related to $\mathbf{Y}$.

2. Even if the above case is fulfilled, it may be that $(\mathbf{I} \cap k[X])\,\mathbf{k}[X]$ is not radical, which would be equivalent to saying that the closed k-subscheme, defined by $\mathbf{I} \cap k[X]$, is not absolutely reduced.

Both of these can be avoided, if we say that $\mathbf{Y}$ is **defined** over k, if and only if there is a k-subvariety Y of X, such that $Y_{\mathbf{k}}$ agrees with $\mathbf{Y}$ and the inclusion of $\mathbf{Y}$ in $X_{\mathbf{k}}$ is a pull-back of the inclusion of Y in X. From the above discussion we see that such a structure is unique in the affine case.

Proposition 3.1.29. *Let G be a connected k-linear algebraic group.*

(a) If S is a k-torus, then $\mathrm{Z}_{G_{\mathbf{k}}}(S_{\mathbf{k}})$ and $\mathrm{N}_{G_{\mathbf{k}}}(S_{\mathbf{k}})$ are defined over k [Spr98, Prop. 13.3.1]. We denote their corresponding k-structures by $\mathrm{Z}_G(S)$ and $\mathrm{N}_G(S)$.

(b) There is a maximal torus $\mathbf{T}$ of $G_{\mathbf{k}}$, that is defined over k. If G is moreover reductive (i.e. $G_{\mathbf{k}}$ is reductive), then $\mathrm{Z}(G_{\mathbf{k}})$ (and thus $\mathrm{Z}(G_{\mathbf{k}})^0$) is defined over k [Bor91, 18.2].

Corollary 3.1.30 ([Abr94, p. 15]). *Let S be a k-torus in G. Then there is a maximal torus $\mathbf{T}$ in $G_{\mathbf{k}}$, which is defined over k and contains $S_{\mathbf{k}}$.*

Proof. From the first part of the above proposition it follows that $\mathrm{Z}_G(S)$ is a k-group. One also knows that $\mathrm{Z}_G(S)$ is connected, by [Bor91, 11.12]. Because the statement is true in the absolute case, we know that $S_{\mathbf{k}}$ is contained in a maximal torus $\mathbf{T}'$ of $G_{\mathbf{k}}$. Thus one has $\mathbf{T}' \subseteq \mathrm{Z}_{G_{\mathbf{k}}}(S_{\mathbf{k}})$ and all maximal tori of $\mathrm{Z}_{G_{\mathbf{k}}}(S_{\mathbf{k}})$ are maximal tori of $G_{\mathbf{k}}$. By the second assertion of the above proposition $\mathrm{Z}_{G_{\mathbf{k}}}(S_{\mathbf{k}})$ contains a maximal torus $\mathbf{T}$, which is defined over k. From the maximality of $\mathbf{T}$, we deduce that $S_{\mathbf{k}}\mathbf{T} = \mathbf{T}$, which implies that $S \subseteq T$, with T being the k-structure corresponding to $\mathbf{T}$. □

Theorem 3.1.31 ([Bor91, 20.9]). *Let G be a connected, reductive k-linear algebraic group. Then the maximal k-split k-tori are conjugated by elements in $G(k)$.*

By the following proposition we assert that maximal k-tori, which are k-split, are very similar to their absolute counterparts, when it comes to their associated root subgroup structure.

Proposition 3.1.32. *Let G be a connected, reductive k-linear algebraic group and T a maximal k-torus in G, which is k-split. We will use the abbreviations $R = R(G,T)$ and $W := W(G,T)$. (Note that these always correspond to the absolute structures, i.e. $R(G,T)$ means $R(G_{\mathbf{k}}, T_{\mathbf{k}})$, etc.).*

(a) The root subgroups $\mathbf{U}_a$ of $G_{\mathbf{k}}$ are defined over k [Bor91, 18.6-18.7]. If we denote their k-structures by U_a, then the isomorphisms $\mathbf{x}_a : \mathbf{U}_a \to \mathbb{G}_{\mathrm{a}}$, are pull-backs from k-isomorphisms $x_a : U_a \to \mathbb{G}_{\mathrm{a}}$.

(b) $W = \mathrm{N}_G(T)(k)/T(k)$ [Bor91, 21.15].

(c) $G(k) = U^+(k)\mathrm{N}_G(T)(k)U^+(k)$ (ibid.), where U^+ is the k-subgroup of G generated by the U_a, with $a \in R^+$, for R^+ being a system of positive roots in R.

One can conclude, that the system $(T(k), (U_a(k))_{a\in R})$ is a generating root datum. We also note that in the situation, in which G has a maximal k-torus, that is k-split, G is called k-**split** as well.

We proceed to the general theory of k-linear algebraic groups, i.e. a situation, where special assumptions on the maximal tori are dropped. We will however introduce the simplification of semi-simplicity, in order to get rid of the uninteresting central torus.

(3.1.33). In the following discussion, let G be a connected, semi-simple k-linear algebraic group S a maximal k-split k-subtorus and T a maximal k-torus, containing S. For the characters, roots and Weyl group, with respect to S, we introduce the following notation:

$$ {}_kX := \mathrm{X}^*(S), \quad {}_kR := R(G,S) \quad \text{and} \quad {}_kW := \mathrm{N}_G(S)/\mathrm{Z}_G(S). $$

From the characterisation of k-split tori, together with the fact, that the adjoint representation is defined over k [Bor91, 3.13], we obtain a decomposition of the Lie algebra of G (again note, that $\mathrm{Lie}(G) := \mathrm{Lie}(G_{\mathbf{k}})$), given by:

$$ \mathfrak{g} := \mathrm{Lie}(G) = \mathrm{Lie}(\mathrm{Z}_G(S)) \oplus \bigoplus_{a\in {}_kR} \mathfrak{g}_a. $$

We also introduce the abbreviations $X := \mathrm{X}^*(T)$, $R := R(G,T)$ and $W := W(G,T)$. The inclusion $S_{\mathbf{k}} \hookrightarrow T_{\mathbf{k}}$ induces an injection

$$ \begin{aligned} j : X &\to {}_kX \\ \chi &\mapsto \chi\restriction_{S_{\mathbf{k}}}. \end{aligned} $$

Remark 3.1.34 ([Abr94, p. 17]). ${}_kR \overset{(1)}{\subseteq} j(R) \overset{(2)}{\subseteq} {}_kR \cup \{0\}$

(1) Since S and T are commuting (as they are inside the commutative algebraic group T), for every $a' \in {}_kR$, the eigenspace $\mathfrak{g}_{a'}$ is not only invariant under S, but also under T. Thus there is a decomposition of $\mathfrak{g}_{a'}$ into eigenspaces with respect to T:

$$ \mathfrak{g}_{a'} = \bigoplus_{a\in\eta(a')\subseteq R} \mathfrak{g}_a, \quad \text{with } \eta(a') := \{a \in R \mid j(a) = a'\} \neq \varnothing. $$

(2) For $a \in R$ one has a non-empty eigenspace $\mathfrak{g}_a$, on which also S operates in virtue of $a' := j(a) \in {}_kR$. Thus one has $\mathfrak{g}_a \subseteq \mathfrak{g}_{a'}$, which implies $a' = 0$ or $a' \in {}_kR$.

The following proposition states that ${}_kR$, which has been up to now only a set of special characters that give rise to certain eigenspaces of $\mathfrak{g}$, is also a root system.

Proposition 3.1.35 ([Bor91, 21.2, 21.6]). *${}_kR$ is a (in general non-reduced) root system in ${}_kV = {}_kX \otimes_{\mathbb{Z}} \mathbb{R}$, with Weyl group ${}_kW$. In this context the root system ${}_kR$ is dubbed the* ***relative root system****, whereas R is called the* ***absolute root system****.*

Remark 3.1.36. Note that in comparison to proposition 3.1.11 one does not have to introduce a subspace ${}_kX_s$, since we additionally imposed semi-simplicity.

Recall that for non-reduced root systems, one can define the set of non-divisible and non-multipliable roots:

$$ {}_kR_{\mathrm{nd}} := \left\{ a \in {}_kR \;\middle|\; \frac{a}{2} \notin {}_kR \right\} \quad \text{and} \quad {}_kR_{\mathrm{nm}} := \{a \in {}_kR \mid 2a \notin {}_kR\}. $$

We will also need the following notation of the positive multiples of a non-divisible root $a \in {}_kR_{\mathrm{nd}}$:

$$ (a) := \{\lambda a \in {}_kR \mid \lambda \in \mathbb{N}^+\} = \begin{cases} \{a\} & \text{if } a \in {}_kR_{\mathrm{nm}} \\ \{a, 2a\} & \text{otherwise.} \end{cases} $$

We will carry this notation to the decomposition of eigenspaces of $\mathfrak{g}$, by defining $\mathfrak{g}_{(a)} = \bigoplus_{b\in(a)} \mathfrak{g}_b$. Thus one has:

$$ \mathfrak{g} = \mathrm{Lie}(\mathrm{Z}_G(S)) \oplus \bigoplus_{a\in {}_kR_{\mathrm{nd}}} \mathfrak{g}_{(a)}. $$

The following proposition gives the result of the analysis of the chain ${}_kR \subseteq j(R) \subseteq {}_kR \cup \{0\}$ in the context of positive root systems and their respective bases.

Proposition 3.1.37 ([Bor91, 21.8]). *For every system of positive roots ${}_kR^+$ in ${}_kR$, there is a system of positive roots R^+ in R such that*

$$ {}_kR^+ \subseteq j(R^+) \subseteq {}_kR^+ \cup \{0\} $$

and a fortiori for the respective bases ${}_kB$ of ${}_kR^+$ and B of R^+

$$ {}_kB \subseteq j(B) \subseteq {}_kB \cup \{0\} $$

holds.

Galois Action on Characters Let $\mathbf{X}$ be an affine variety. Then we know, since $\mathbf{X}$ is of finite type over $\mathbf{k}$, that there is an affine space $\mathbb{A}^m$, with $m \in \mathbb{N}^+$, of which $\mathbf{X}$ is a closed subvariety of. Hence $\mathbf{X}$ is equivalently given by an ideal $\mathbf{I}$ in $\mathbf{k}[T_1, \dots, T_m]$. If we have an extension of fields $k \subseteq \tilde{k} \subseteq \mathbf{k}$, with $k \subseteq \tilde{k}$ being finite and Galois, and $\mathbf{X}$ being defined over $\tilde{k}$, then we know that its $\tilde{k}$-structure is given by $\tilde{I} := \mathbf{I} \cap \tilde{k}[T_1, \dots, T_m]$. If we take an element $\sigma \in \mathrm{Gal}(\tilde{k}/k)$, there is an action of σ on the $\tilde{k}$-algebra $\tilde{k}[T_1, \dots, T_m]$, through the action of σ on the coefficients of a polynomial. Thus we obtain an ideal $\tilde{I}^\sigma$, or equivalently a $\tilde{k}$-variety $\tilde{X}^\sigma$, and we denote by $\tilde{X}$ the $\tilde{k}$-structure of $\mathbf{X}$. One sees that $\tilde{X}$ has moreover a k-structure, if and only if $\tilde{X}^\sigma = \tilde{X}$ holds, for all $\sigma \in \mathrm{Gal}(\tilde{k}/k)$. In this case $\tilde{I}$ is left invariant by the action of the Galois group, which readily implies that there is a well-defined action of $\mathrm{Gal}(\tilde{k}/k)$ on the $\tilde{k}$-algebra $\tilde{k}[\tilde{X}]$ of $\tilde{X}$.

We apply this to the situation of a k-torus T. By 3.1.28 there is a finite Galois extension $k \subseteq \tilde{k}$, such that $T_{\tilde{k}}$ is a $\tilde{k}$-split $\tilde{k}$-torus. Since T is defined over k, there is an action of the Galois group $\mathrm{Gal}(\tilde{k}/k)$ on the $\tilde{k}$-algebra $\tilde{k}[T]$ of $T_{\tilde{k}} =: \tilde{T}$. Via restriction[2] of this action, one obtains an operation on the characters $\mathrm{X}^*(T)$. One finds:

Lemma 3.1.38 ([Bor91, 8.11]). *T is split over k, if and only if* $\mathrm{X}^*(T)$ *is invariant under* $\mathrm{Gal}(\tilde{k}/k)$.

Finally we will examine this Galois action in the context of semi-simple k-groups. The following is due to [Abr94, pp. 25-26]. Recall the definitions and assumptions on G, S and T from (3.1.33) above and let $k \subseteq \tilde{k}$ be a finite Galois extension that splits T. We want to see that $R^\sigma = R$ holds, for all $\sigma \in \mathrm{Gal}(\tilde{k}/k)$. By 3.1.32 for every $a \in R$, there are $\tilde{k}$-isomorphisms $\tilde{x}_a : \mathbb{G}_\mathrm{a} \to \tilde{U}_a$ and we know, that $T^\sigma = T$ operates on $\tilde{U}_{a^\sigma}$ by multiplication with a^σ (cf. 3.1.18.(b)), for every $\sigma \in \mathrm{Gal}(\tilde{k}/k)$. Again since T is fixed by $\mathrm{Gal}(\tilde{k}/k)$, T operates on $\mathrm{Lie}(\tilde{U}_a^\sigma)$ also by multiplication with a^σ. From this we deduce $\mathrm{Lie}(\tilde{U}_a^\sigma) = \mathfrak{g}_{a^\sigma}$ and $\tilde{U}_a^\sigma = \tilde{U}_{a^\sigma}$ (cf. 3.1.18). Hence we obtain that $\mathfrak{g}_{a^\sigma} \neq \emptyset$, i.e. $a^\sigma \in R$, for all $\sigma \in \mathrm{Gal}(\tilde{k}/k)$ and $a \in R$.

Consequences. (a) There is a homomorphism $\mathrm{Gal}(\tilde{k}/k) \to \mathrm{Stab}_{\mathrm{GL}(X \otimes_\mathbb{Z} \mathbb{R})}(R)$.

(b) $\sigma s_a \sigma^{-1} = s_{a^\sigma}$, for $a \in R$, $\sigma \in \mathrm{Gal}(\tilde{k}/k)$ and the reflections s_a and s_{a^σ} associated with a and a^σ.

(c) For a system of positive roots R^+ with associated basis B, $(R^+)^\sigma$ is a system of positive roots with associated basis B^σ.

(d) Since S is k-split, we have for every character χ of T: $j(\chi^\sigma) = j(\chi)$.

Unipotent and Parabolic Subgroups in G

In the following we present a way, how 3.1.19.(b) can be generalized to semi-simple k-linear algebraic groups. Thus take G to be a connected, semi-simple k-linear algebraic group, with maximal k-split k-torus S and maximal k-torus T, containing S, together with a finite Galois extension $k \subseteq \tilde{k}$, that splits T (cf. (3.1.33)). We then have the following generalization of the above proposition.

Proposition 3.1.39 ([Bor91, 20.3]). *Let $R' \subseteq R^+$ be a positively closed set of roots, which is* $\mathrm{Gal}(\tilde{k}/k)$*-invariant, i.e. $R'^\sigma = R'$. Then the closed subgroup $\mathbf{U}_{R'}$ of $G_\mathbf{k}$ from proposition 3.1.19.(b) is defined over k. We will denote this k-subgroup by $U_{R'}$.*

[2] By the assumption of splitness, the (absolute) characters arise as pull-backs of relative characters over $\tilde{k}$. One obtains the action of $\mathrm{Gal}(\tilde{k}/k)$ by viewing the relative characters as special elements in $\tilde{k}[T]$. Milne calls such elements *group-like* [Mil17, 4.g]. They fulfil $\mu(a) = a \otimes a$, where $\mu : \tilde{k}[T] \to \tilde{k}[T] \otimes_{\tilde{k}} \tilde{k}[T]$ is the $\tilde{k}$-algebra homomorphism corresponding to the group multiplication $m : \tilde{T} \otimes_{\tilde{k}} \tilde{T} \to \tilde{T}$.

Let ${}_kR^+$ be a system of positive roots for ${}_kR$ and choose a compatible system of positive roots R^+ for R. In the following, we will be concerned about subsets of absolute roots, that come from a subsets of relative roots in ${}_kR^+$. Fixing such a subset $R' \subseteq {}_kR^+$, we then set:

$$\eta(R') := j^{-1}(R') \cap R \subseteq R^+.$$

We note that by $j(\chi^\sigma) = j(\chi)$, for all $\chi \in X$ and $\sigma \in \mathrm{Gal}(\tilde{k}/k)$, we have that $\eta(R')$ is $\mathrm{Gal}(\tilde{k}/k)$-invariant. Moreover if R' is positively closed in ${}_kR$, then $\eta(R')$ is positively closed in R. In this case, one obtains an even stronger result than the one in proposition 3.1.39.

Proposition 3.1.40 ([Bor91, 21.20])**.** *Let $R' \subseteq {}_kR$ be a positively closed subset of ${}_kR$. Then $U_{\eta(R')}$ is k-isomorphic to an affine k-space.*

We fix the following notation: For $R' \subseteq {}_kR$ closed, set $U_{R'} := U_{\eta(R')}$ and for a relative root $a' \in {}_kR$, we set $U_{a'} := U_{(a')}$, where we recall $(a') = \{\lambda b \in {}_kR \mid \lambda \in \mathbb{N}^+\}$. The latter will be called the **relative root subgroups**.

Remarks 3.1.41. (a) If $a' \in {}_kR$ is such that $(a') = \{a', 2a'\}$, then $\{a'\}$ is not a closed set in ${}_kR$ and hence in general $\eta(\{a'\})$, will not be closed in R. Thus one does not define $U_{a'}$ to be $U_{\eta(\{a'\})}$.

(b) $\mathrm{Lie}(U_{a'}) = \bigoplus_{b \in R, j(b) \in (a')} \mathfrak{g}_b = \bigoplus_{b' \in (a')} \mathfrak{g}_{b'} =: \mathfrak{g}_{(a')}$, for $a' \in {}_kR$.

(c) If a' is a relative root, with $(a') = \{a'\}$. Then for $a, b \in R$, with $j(a) = j(b) = a'$, we have that $j(a+b) = 2a'$, of which we know that it is not in ${}_kR$. Thus by $j(R) \subseteq {}_kR \cup \{0\}$, $j(a+b)$ has to be zero. If we suppose that $a+b$ were in R, then by closedness of $\eta(\{a'\})$, $j(a+b) = a'$, which is a contradiction, as roots are always non-zero. Thus $a+b \notin R$ holds, from which we deduce by Chevalley's commutation relations that $U_{(a')}$ is abelian. (Note that a k-linear algebraic group is said to be abelian, if and only if its absolute counterpart is).

The relative root subgroups possess a characterisation similar to those of the absolute ones (cf. 3.1.18):

Lemma 3.1.42 ([Bor91, 21.9])**.** *Let $a' \in {}_kR$ be a relative root. Then $U_{a'}$ is the uniquely determined closed, connected, unipotent and $\mathrm{Z}_G(S)$-invariant k-subgroup of G, having*

$$\mathfrak{g}_{(a')} = \bigoplus_{a \in \eta((a'))} \mathfrak{g}_a$$

as a Lie algebra.

Chevalley's commutation relations have the following direct consequence for the relative root subgroups:

Lemma 3.1.43 ([Abr94, Lemma 8, Korollar])**.** *Let a' and b' be two relative roots in ${}_kR$ such that $b' \notin \left\{-\frac{1}{2}a', -a', -2a'\right\}$. Then one finds:*

$$[U_{a'}, U_{b'}] = \left\langle U_{pa'+qb'} \;\middle|\; pa' + qb' \in {}_kR,\ p, q \in \mathbb{N}^+ \right\rangle$$

and analogously for rational points:

$$[U_{a'}(k), U_{b'}(k)] = \left\langle U_{pa'+qb'}(k) \;\middle|\; pa' + qb' \in {}_kR,\ p, q \in \mathbb{N}^+ \right\rangle.$$

As we have expressed above and lemma 3.1.42 reinforced, the centralisers of maximal k-split tori play a similarly central role in the relative situation, as the maximal tori do in the absolute one. Note, that by proposition 3.1.31, these centralisers are conjugated by rational elements. For what follows we fix the abbreviation $Z := \mathrm{Z}_G(S)$ and additionally $N := \mathrm{N}_G(S)$. The Lie algebra of Z is given by

$$\mathrm{Lie}(Z) = \mathrm{Lie}(T) \oplus \bigoplus_{a \in R^0} \mathfrak{g}_a,$$

where we set $R^0 := R(Z, T)$. It can be shown that R^0 is given by $R \cap \ker(j)$ (cf. [Abr94, p. 35]), where $j : \mathrm{X}^*(T) \to \mathrm{X}^*(S)$, is the restriction homomorphism. This implies furthermore that $Z = T\left\langle U_a \;\middle|\; a \in R^0 \right\rangle$, since $[U_a, S]$ is trivial, for $a \in R^0$.

Fix a system of positive roots ${}_kR^+$ for ${}_kR$. As a byproduct of the proof of 3.1.37, one may assume that there is not only a compatible system of positive roots R^+, but also a system of positive roots $(R^0)^+$, which is contained in R^+. We define the subgroups $U := U_{R^+ \setminus (R^0)^+}$ and $P := U \rtimes Z$ (which is a semi-direct product, since Z normalises U by 3.1.42). We remark some facts concerning these:

(a) $U = U_{{}_kR}$, is defined over k by means of proposition 3.1.39.

(b) P is a k-**parabolic subgroup** (cf. [Bor91, 21.11]), i.e. a closed subgroup such that $G_{\mathbf{k}}/P_{\mathbf{k}}$ is a projective variety. It is defined over k, because Z is defined over k by proposition 3.1.29.

(c) We recall that for any k-parabolic subgroup P', a **Levi subgroup** L of P' is a closed subgroup of P' such that $P' = \mathrm{R}_u(P) \rtimes L$ holds. Every parabolic subgroup $\mathbf{P}''$ of $G_{\mathbf{k}}$ has a Levi subgroup [Bor91, 14.18], that is defined over k, if $\mathbf{P}''$ is [Bor91, 20.5]. It can be shown that $U = \mathrm{R}_u(P)$ holds, and thus Z is a Levi subgroup of P.

Proposition 3.1.44 ([Bor91, 20.6]). *Let Q be a k-parabolic subgroup, with Levi subgroup L. The k-torus $\mathrm{R}(L)$ (radical to be taken in Q) shall be decomposed, as given by lemma 3.1.27, as $\mathrm{R}(L) = \mathrm{R}(L)_d \mathrm{R}(L)_a$. Then one has:*

(a) $\mathrm{N}_G(\mathrm{R}(L)_d) \cap Q = \mathrm{Z}_G(\mathrm{R}(L)_d) = L$.

(b) G contains a proper k-parabolic subgroup, if and only if it has a non-trivial k-split k-subtorus.

(c) Q is a minimal k-parabolic subgroup, if and only if $\mathrm{R}(L)_d$ is a maximal k-split k-torus.

Remark 3.1.45. If $P = U \rtimes Z$ as above, then we have $L = Z$ and $S \subseteq \mathrm{R}(Z)_d$, from which we deduce by the maximality of S, that $S = \mathrm{R}(Z)_d$ holds. From part (c) of the above proposition, we thence conclude that P is a minimal k-parabolic subgroup.

The minimal k-parabolic subgroups take the role of the Borel subgroups, i.e. the maximal closed, connected, solvable subgroups of the absolute theory. As it holds in the algebraically closed case, there is a uniqueness statement concerning minimal k-parabolic subgroups.

Proposition 3.1.46 ([Bor91, 20.9]). *(a) Two minimal k-parabolic subgroups in G are conjugated by an element in $G(k)$.*

(b) If Q_1 and Q_2 are two k-parabolic subgroups, that are conjugated by an element in $G(\mathbf{k})$, then they are also conjugated by an element in $G(k)$.

(3.1.47) ([Bor91, 21.11]). For the following discussion, we fix a system of positive roots ${}_kR^+$ and thus also a basis ${}_kB$, of ${}_kR^+$, $U = U_{{}_kR^+}$ and $P = U \rtimes Z$. For a subset $I \subseteq {}_kB$, we write $[I]$ for the $\mathbb{Z}$-span of I in ${}_kR$. Then we form $U_I := U_{{}_kR^+ \setminus [I]}$, as well as

$$S_I := \left(\bigcap_{a \in I} \ker(a) \right)^0 \quad \text{and} \quad L_I := \mathrm{Z}_G(S_I).$$

We then set $P_I := U_I \rtimes L_I$. A careful analysis yields that the relative root system of L_I is given by $R(L_I, S) = [I]$ and in particular L_I is generated by $\mathrm{Z}_G(S)$ and the relative root subgroups U_a, with $a \in [I]$. We recall that P_I is a k-parabolic subgroup, that contains the minimal k-parabolic subgroup $P = P_\varnothing$. Furthermore the following is true:

Proposition 3.1.48 ([Bor91, 21.12]). *(a) The mapping*

$$\begin{aligned} \mathscr{P}({}_kB) &\to \{Q \mid P \subseteq Q \subseteq G,\ Q \ \textit{is } k\textit{-parabolic}\} \\ I &\mapsto P_I \end{aligned}$$

is bijective, where $\mathscr{P}$ denotes the power set.

(b) If Q is a k-parabolic subgroup, then there is $g \in G(k)$ and $I \subseteq {}_kB$ such that Q is conjugated under g to P_I.

Definition 3.1.49. The P_I, or by (a) equivalently the k-parabolic subgroups containing P, are called the **standard** k-**parabolic subgroups** in G.

We will now commence checking the existence of a root datum associated with G, but, as it was the case in the absolute theory, before doing so a treatment of rank one subgroups is useful:

Proposition 3.1.50 ([Bor91, 21.2]). *Let $a \in {}_kR_{\mathrm{nd}}$ be a relative, non-divisible root. Set $S_a := \ker(a)^0$ and $L_a := \mathrm{Z}_G(S_a)$. Then we have:*

(a) The relative root system of L_a is given by $R(L_a, S) = (a) \cup (-a)$.

(b) $\mathrm{Lie}(L_a) = \mathrm{Lie}(Z) \oplus \mathfrak{g}_{(a)} \oplus \mathfrak{g}_{(-a)}$, $L_a = Z\langle U_a \cup U_{-a}\rangle$.

(c) *There is an element $n_a \in N(k)$, such that $N \cap L_a = Z \sqcup Zn_a$. The element n_a operates as the reflection on $\mathrm{X}^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ which is given by the root a.*

(d) $\langle n_a \mid a \in {}_kR_{\mathrm{nd}}\rangle Z = N$.

Consequence. $N/Z \cong N(k)/Z(k) \cong W({}_kR)$.

Now set $s_a := n_a Z(k) \in W({}_kR)$, for all $a \in {}_kR_{\mathrm{nd}}$. If $2a$ is in ${}_kR$, then define $s_{2a} := s_a$. We collect the simple relative reflections in the set ${}_k\overline{B} := \{r_a \mid a \in {}_kB\}$. One then finds a generalization of proposition 3.1.23:

Proposition 3.1.51 ([Bor91, 21.15]). *For every connected, semi-simple k-linear algebraic group G, with the above definitions, one finds:*

(a) $G(k) = U(k)N(k)U(k)$.

(b) $\left(G(k), P(k), N(k), {}_k\overline{B}\right)$ *is a Tits-System, with Weyl group* $W({}_kR)$.

Remarks 3.1.52. (a) Statement (a) of the above proposition may be derived as a consequence of (b), but is proven independently in [Bor91].

(b) The propositions 3.1.48 and 3.1.51 imply that the following mapping

$$\begin{aligned}\{Q \mid P \subseteq Q \subseteq G,\ Q \text{ is } k\text{-parabolic}\} &\to \left\{Q' \,\middle|\, P(k) \subseteq Q' \subseteq G(k)\right\}\\ P_I &\mapsto P(k)\tilde{W}_I P(k)\end{aligned}$$

is bijective (also see [Bor91, 21.16]), where W_I is the subgroup of $W({}_kR)$, generated by the simple reflections in $I \subseteq {}_k\overline{B}$, and the tilde indicates that we take a lift in $N(k)$. This means in particular that there is a one to one correspondence between k-parabolic subgroups in G and (abstract) parabolic subgroups of $G(k)$ with respect to $P(k)$. Thus there is a well-defined **spherical building** $\mathcal{B}(G)$ associated with G, in which the simplices are given by the k-parabolic subgroups of G and the face relation is the reverse inclusion. In that regard the choice of a maximal k-split k-torus only fixes a fundamental apartment and by singling out a minimal k-parabolic subgroup, containing that torus, one merely determines a fundamental chamber, the building itself however exists independently.

The following is a consequence of 3.1.51.(a) only.

Corollary 3.1.53 ([Abr94, p. 38]). $L_a(k) = U_a(k)\{\mathrm{e}, n_a\}U_a(k)$.

Now we are in the position to put the presented results together and introduce a root datum in the general case.

Theorem 3.1.54 ([Abr94, Satz 27]). *Let G be a connected, semi-simple k-linear algebraic group. With the notations as above one finds that*

$$\left(Z(k), (U_a(k))_{a \in {}_kR}\right)$$

is a generating (corresponds to (DR0)*) root datum, i.e. it holds that:*

(DR0) $G(k) = \langle Z(k), U_a(k) \mid a \in {}_kR\rangle$;

(DR1) $Z(k), U_a(k) \leq G(k);\ U_a(k) \neq \{\mathrm{e}\},\ \forall a \in {}_kR$;

(DR2) $[U_a(k), U_b(k)] \subseteq \left\langle U_{pa+qb}(k) \,\middle|\, pa+qb \in {}_kR; p,q \in \mathbb{N}^+\right\rangle, \quad \forall a,b \in {}_kR, b \neq -\frac{1}{2}a, -a, -2a$;

(DR3) $a, 2a \in {}_kR \implies U_{2a}(k) \subsetneq U_a(k)$;

(DR4) $U_{-a}(k) \setminus \{\mathrm{e}\} \subseteq U_a(k)Z(k)n_aU_a(k), \quad \forall a \in {}_kR$;

(DR5) $nU_b(k)n^{-1} = U_{s_a(b)}(k), \quad \forall a,b \in {}_kR;\ n \in Z(k)n_a$;

(DR6) $Z(k)U^+(k) \cap U^-(k) = \{\mathrm{e}\}, \quad$ *for* $U^{\pm}(k) := \left\langle U_a(k) \,\middle|\, a \in {}_kR^{\pm}\right\rangle$.

Sketch. DR0: This follows from $G(k) = U(k)N(k)U(k)$ 3.1.51.(a), after remarking that $N(k) = \langle n_a \mid a \in {}_kR\rangle Z(k)$ 3.1.50.(d) and $n_a \in Z(k)\langle U_a(k) \cup U_{-a}(k)\rangle$ hold (cf. (DR4)).

DR1: As U_a is isomorphic as a k-variety to some affine space (cf. Proposition 3.1.40), we have $U_a(k) \neq \{e\}$.

DR2: This is identical to lemma 3.1.43.

DR3: The inclusion is clear, as $2a \in (a)$, and furthermore we have $\mathfrak{g}_{(a)} \neq \mathfrak{g}_{(2a)}$ implying $U_a \neq U_{2a}$ and $U_a(k) \neq U_{2a}(k)$.

DR4: Start first with a non-divisible root $a \in {}_kR_{\mathrm{nd}}$. Corollary 3.1.53 implies that we have

$$U_{-a}(k) \subseteq U_a(k)\{e, n_a\}\, U_a(k).$$

Since on the other hand $U_{-a}(k) \cap Z(k)\, U_a(k) = \{e\}$ (cf. (DR6)) one arrives at the claim, keeping in mind lemma 3.1.42.

If a and $\frac{1}{2}a$ lie in ${}_kR$, we set $L'_a := Z\langle U_a \cup U_{-a}\rangle \subseteq L_{\frac{1}{2}a}$. The essential properties of $L_{\frac{1}{2}a}$, that were composed in proposition 3.1.50, remain true for L'_a, but are proved in other ways (cf. [BT65, 3.4, 3.13]). In particular L'_a is a connected, reductive k-linear algebraic group, with relative root system $\{-a, a\}$ and $P'_a = U_a \rtimes Z$ is a minimal k-parabolic subgroup in L'_a. An application of an analogue to corollary 3.1.53 again yields the claim.

DR5: Z stabilizes all U_c, with $c \in {}_kR$, by lemma 3.1.42, and thus $Z(k)$ stabilizes $U_c(k)$. From $\mathrm{Ad}(n_a)\mathfrak{g}_b = \mathfrak{g}_{s_a(b)}$, which follows from the fact for absolute roots, and the uniqueness statement in lemma 3.1.42, one deduces $n_a U_b n_a^{-1} = U_{s_a(b)}$, for a, b relative roots, which implies the claim.

DR6: $P^+ = ZU^+$ and $P^- = ZU^-$ are opposite k-parabolic subgroups in G (cf. [Bor91, 14.21]). Hence they intersect in $P^+ \cap P^- = Z$. Thus one has $P^+ \cap U^- = Z \cap U^- = \{e\}$. This implies in particular the statement for rational points. Note that we only have to prove this for one system of positive roots ${}_kR^+$ and it follows for all such systems in ${}_kR$ by the transitivity of the Weyl group action on the chambers of ${}_kR$ [BT72, (6.1.2).(11)].

□

Remark 3.1.55. In the original definition of a root datum (fr.: ***d****onnée* ***r****adicielle*), given by Bruhat and Tits in [BT72, (6.1.1)], the n_a are part of the definition and explicitly belong to the root datum. However it is then subsequently shown that the root datum is already determined by the information contained in what is called a root datum here [BT72, (6.1.2).(9)].

3.2 Bruhat-Tits Buildings

This section is devoted to the analysis of linear algebraic groups over valued fields. By theorem 3.1.54, one has an abstract structure on the rational points of the relative root subgroups. The idea, that we will develop in the following, is to carry the valuation we have on the field over to the root subgroups, which in the split case are isomorphic to our field, in order to filter the root subgroups. By this process, we will obtain another Tits system and a geometric structure on which the rational points of our group act. The presentation follows the material covered in [BT72, 6.-7.], however we adapted some of the notation.

3.2.1 Valuation of Root Data

In this part we will present, by an introductory example, the axiomatics behind Bruhat and Tits' valued root data. Therefore we fix the following notation before proceeding:

- K a discretely valued and complete field, with
- ω the non-trivial valuation of K.
- $\mathcal{O}$ the valuation ring $\{x \in K \mid \omega(x) \geq 0\}$,
- $\mathfrak{m}$ the maximal ideal in $\mathcal{O}$, i.e. $\pi\mathcal{O}$, with uniformizing element $\omega(\pi) = 1$ and
- k residue field $\mathcal{O}/\mathfrak{m}$.

Example 3.2.1. Let $\underline{G}$ be a connected, semi-simple K-linear algebraic group, that has a maximal K-torus $\underline{T}$, which is also K-split. We abbreviate the root system $R(\underline{G}, \underline{T})$ by R. We then set $G := \underline{G}(K)$, $T := \underline{T}(K)$, and $U_a := \underline{U}_a(K)$, for all root subgroups $\underline{U}_a$, associated with an $a \in R$. We also fix, by 3.1.32, isomorphisms $x_a : \mathbb{G}_a \to \underline{U}_a$, for every $a \in R$, conforming to the following:

1) The structure constants $c_{a,b;p,q}$ of Chevalley's commutation relations all reside inside $\mathbb{Z}.1_K$ (cf. 3.1.19.(a)).
2) For every $a \in R$, there are homomorphisms $\varepsilon_a : \mathrm{SL}_2 \to \langle U_a \cup U_{-a} \rangle$ such that

$$\varepsilon_a(\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}) = x_a(\mu) \quad \text{and} \quad \varepsilon_a(\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}) = x_{-a}(\mu),$$

for all $\mu \in K$ hold. This follows from the way the isomorphisms x_a are constructed, which can be found, e.g. in [Mil17, 21.11].

Let there be a root $a \in R$. Via the isomorphisms x_a, we are able to introduce the following filtration of the rational points of the root subgroups ($l \in \mathbb{R}$):

$$U_{a,l} := \{x_a(\lambda) \mid \lambda \in K,\ \omega(\lambda) \geq l\}.$$

Another perspective on this is to define a valuation on U_a by means of the isomorphisms x_a:

$$\begin{aligned} \varphi_a : U_a &\to \mathbb{R} \cup \{\infty\} \\ x_a(\lambda) &\mapsto \omega(\lambda). \end{aligned}$$

In this way one observes the equality $U_{a,l} = \varphi_a^{-1}([l,\infty])$. The root datum $(T,(U_a)_{a\in R})$ together with the maps $\varphi := (\varphi_a)_{a\in R}$ forms a prime example of a valued root datum. In the following we discuss some properties, that link the root datum axioms to the system of maps φ. This shall be the build-up to a definition and since we are analysing a special case, not all the properties are realised.

V0: $\mathrm{card}(\varphi_a(U_a)) \geq 3$, for all $a \in R$. This is true in our example, as 0, ∞ and a value in between those two are always in the image of φ_a, as ω is non-trivial.

V1: $U_{a,l} = \varphi_a^{-1}([l,\infty]) \leq U_a$ and $U_{a,\infty} = \{\mathrm{e}\}$ hold.

From (DR5) we know that for two roots $a, b \in R$, and $n \in M_a := Tn_a$, we have $nU_bn^{-1} = U_{s_a(b)}$. Fixing a single root a and $n \in M_a$, we want to deduce a relation between $\varphi_{-a}(u)$ and $\varphi_a(nun^{-1})$, with $u \in U_{-a}^{\times} := U_{-a} \setminus \{\mathrm{e}\}$. Consider the following commuting diagram

$$\begin{array}{ccc} \mathbb{G}_{\mathrm{a}} & \xrightarrow{x_{-a}} & \underline{U}_{-a} \\ \downarrow{\scriptstyle \eta_n} & & \downarrow{\scriptstyle \kappa_n} \\ \mathbb{G}_{\mathrm{a}} & \xrightarrow{x_a} & \underline{U}_a, \end{array}$$

where κ_n denotes the conjugation by n and $\eta_n := x_a^{-1} \circ \kappa_n \circ x_{-a}$. Since η_n is an isomorphism of $\mathbb{G}_{\mathrm{a}}$, on rational points it is necessarily given as multiplication by some unit $\lambda(n) \in K^{\times}$. Hence we get, for all $\mu \in K$

$$x_a(\lambda(n)\mu) = nx_{-a}(\mu)n^{-1}, \tag{3.3}$$

which implies for the valuations φ

$$\varphi_{-a}(x_{-a}(\mu)) - \varphi_a(nx_{-a}(\mu)n^{-1}) = -\omega(\lambda(n)).$$

One sees, that the equation (3.3) is mainly taking place inside the rational points of SL_2. Explicitly it is related to the following formula:

$$\begin{pmatrix} 1 & -\lambda^2\mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}$$

V2: For all $a \in R$, $n \in M_a$ the function $\varphi_{-a}(u) - \varphi_a(nun^{-1})$ is constant for all choices $u \in U_{-a}^{\times}$.

V3: $[U_{a,l}, U_{b,m}] \subseteq \left\langle U_{pa+qb,pl+qm} \;\middle|\; p,q \in \mathbb{N}^+,\ pa+qb \in R \right\rangle$, for all roots a and b, with $b \notin \mathbb{R}^{<0}a$. This follows from a quick glance at Chevalley's commutation relations ($\lambda, \mu \in K$)

$$[x_a(\lambda), x_b(\mu)] = \prod_{p,q\in\mathbb{N}^+,\ pa+qb\in R} x_{pa+qb}\left(c_{a,b;p,q}\lambda^p\mu^q\right),$$

by noting that the structure constants are of value zero by our choice.

We will also use the following already presented matrix formula, which is related to (DR4), to deduce an additional property of φ. One finds that in $\mathrm{SL}_2(K)$ the subsequent identity holds ($\lambda \in K$):

$$\begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\lambda^{-1} & 1 \end{pmatrix}$$

If one applies ε_a to this relation, one obtains $x_{-a}(-\lambda^{-1})x_a(\lambda)x_{-a}(-\lambda^{-1}) \in M_a$, see also e.g. [Spr98, 8.1.4]. On the other hand for $u \in U_a^\times$, one obtains with (DR4) that there is a decomposition $u = x'yx''$, with $x', x'' \in U_{-a}$ and $y \in M_{-a} = M_a$. One can actually derive via abstract reasoning involving only the DR-axioms, that this decomposition is unique (see [BT72, (6.1.2).(2)]). Together with the formula derived by the matrix calculus, we motivated the following property:

V5: $u'uu'' \in M_a,\ u \in U_a,\ u', u'' \in U_{-a} \implies \varphi_{-a}(u') = -\varphi_a(u)$.

V4: $a, 2a \in R \implies \varphi_{2a} = (2\varphi_a)_{\restriction U_{2a}}$. This is only needed, if the root system, which lies under the root datum, that is supposed to be valued, is non-reduced.

This concludes our discussion of axioms for valuations of root data. We define in accordance to Bruhat and Tits [BT72, (6.2.1)]:

Definition 3.2.2. A family of functions $\varphi := (\varphi_a : U_a \to \mathbb{R} \cup \{\infty\})_{a\in R}$ is called a **valuation** of the root datum $(T, (U_a)_{a\in R})$, if and only if it fulfils the conditions (V0) to (V5). Moreover φ is called **discrete**, if and only if $\varphi_a(U_a^\times)$ is discrete in $\mathbb{R}$, and **integral**, if and only if $\varphi_a(U_a^\times) = \mathbb{Z}$, for all $a \in R$, holds.

According to this nomenclature in the above example we introduced an integral valuation of the root datum $(T, (U_a)_{a\in R})$ that is associated to a connected, semi-simple, K-split K-linear algebraic group $\underline{G}$, with maximal K-split K-torus $\underline{T}$.
We will see in the proof of Soulé's theorem that, under some special assumptions on the field K, a larger class than only the root data of K-split linear algebraic groups can be equipped with a valuation.

3.2.2 Construction of the Building

In the following we will show how a valued root datum gives rise to an affine building associated to it. This will be done in a more abstract fashion, i.e. we will fix a valuation of a root datum, without assuming a linear algebraic group underlying it. Therefore we first introduce the following notation:

V	a n-dimensional real vector space,
R	a (not necessarily reduced) root system in V,
${}^vW := W(R)$	the Weyl group associated to R (the v stands for vectorial, in contrast to the affine Weyl group that we will define below),
$(\cdot,\cdot)$	${}^vW$-invariant scalar product in V,
R^+	a system of positive roots in R and
B	the basis for R^+ in R.

Furthermore we will assume that $\mathcal{S} := (T, (U_a)_{a\in R})$ is an abstract root datum and that $\varphi := (\varphi_a)_{a\in R}$ is a valuation of it. On top of that we will use the following definitions:

$$\left.\begin{aligned} U_a^\times &:= U_a \setminus \{\mathrm{e}\}, \\ \Theta_a &:= \varphi_a(U_a^\times), \\ M_{a,l} &:= M_a \cap \left(U_{-a}\varphi_a^{-1}(\{l\})U_{-a}\right),\ l \in \Theta_a, \end{aligned}\right\}, \quad \forall a \in R \quad ^3$$

$$N := \langle T \cup M_a \mid a \in R \rangle,$$

$$N^0 := \left\langle M_{a,l} \;\middle|\; a \in R,\ l \in \Theta_a \right\rangle.$$

By the remarks 3.1.22 there is a group homomorphism ${}^v\nu : N \twoheadrightarrow {}^vW$, with kernel T, that maps M_a to the reflection associated with a. We know that ${}^vW$ can be viewed as a finite reflection group of V, i.e. as a subset of the orthogonal group $O(V)$ associated with V and $(\cdot,\cdot)$ (cf. chapter two). The next step in the definition of the affine building will be to define an action of N^0 on V via

[3] We join Margaux in using Θ_b as a notation for the group of values, since the notation of Γ_b, comming from [BT72], is conflicting with the choice of names for our fixers Γ_x later on, which derives of Soulé's notation.

affine transformations. Therefore we will need an identification of V and a set of valuations of $\mathcal{S}$.[4]

Lemma 3.2.3 ([BT72, (6.2.5)]). *(a) For every $v \in V$ define a system of maps $\varphi + v$ by setting*

$$(\varphi + v)_a(u) := \varphi_a(u) + (a, v), \quad \forall a \in R,\ u \in U_a.$$

Then $\varphi + v$ is a valuation for the root datum $\mathcal{S}$ and $A := \{\varphi + v \mid v \in V\}$ is an affine space.

(b) Let ψ be a valuation of $\mathcal{S}$ and $n \in N$. Then

$$(n \cdot \psi)_a(u) = \psi_{{}^v\nu(n)^{-1}(a)}(n^{-1}un),$$

for $a \in R$ and $u \in U_a$ defines a valuation $n \cdot \psi$ of $\mathcal{S}$ and the following compatibility results can be obtained:

$$\begin{aligned} (n_1 n_2) \cdot \psi &= n_1 \cdot (n_2 \cdot \psi), && \forall n_1, n_2 \in N, \\ n \cdot (\varphi + v) &= n \cdot \varphi + {}^v\nu(n)(v), && \forall n \in N,\ v \in V. \end{aligned}$$

(c) For every $t \in T \subseteq N$, there is a $v \in V$, such that $t \cdot \varphi = \varphi + v$ (cf. [BT72, (6.2.10)]).

(d) For every $m \in M_{a,l}$, with $a \in R$, $l \in \Theta_a$, it holds that $m \cdot \varphi = \varphi - la^\vee$, where we recall $a^\vee = \frac{2a}{(a,a)}$ (cf. [BT72, (6.2.7)]).

Consequence ([BT72, (6.2.10)]). There is a homomorphism $\nu : N \to V \rtimes O(V)$, which maps T to a translation given by an element in V and every $M_{a,l}$, with $a \in R$ and $l \in \Theta_a$, to the affine reflection $s_{a,l}$, that acts with respect to the affine hyperplane $L_{a,l} := \{v \in V \mid (a, v) + l = 0\}$. In particular one obtains that the concatenation of ν with the projection onto $O(V)$ yields back ${}^v\nu$. Note that in this case we identified V and A by choosing φ as an origin for the affine space.

À propos origin: One can show by using lemma 3.2.3 (also see [BT72, (6.2.15)]), that there is a $v \in V$ such that $0 \in {}^{\varphi+v}\Theta_a$ holds, for all $a \in R$, where the superscript $\varphi + v$ means that we view Θ_a with respect to this valuation. A valuation of root data of this kind is called **special**.

Remarks 3.2.4. (a) If φ is discrete and special, then Θ_a is an additive subgroup of $\mathbb{R}$ isomorphic to $\mathbb{Z}$, for all $a \in R$ [BT72, (6.2.16)].

(b) If φ is special, then $\Theta_a = \Theta_{w(a)}$ holds, for all $a \in R$ and $w \in {}^vW$ [BT72, (6.2.14)].

In what follows, we will need some more notation.

$$\begin{aligned} \alpha_{a,l} &:= \{v \in V \mid (a, v) + l \geq 0\}, \quad a \in R,\ l \in \mathbb{R} \text{ and} \\ L_{a,l} &:= \partial \alpha_{a,l} := \{v \in V \mid (a, v) + l = 0\}. \end{aligned}$$

For $a \in R$ and $l \in \Theta_a$, one calls $\alpha_{a,l}$ an **affine root** and $L_{a,l}$ an **affine wall**. One comprises the affine roots in the set σ. In addition to these definitions, we also fix the subsequent abbreviations:

$$\begin{aligned} \hat{W} &:= \nu(N), \\ W &:= \nu(N^0) \trianglelefteq \hat{W}, \\ H &:= \ker(\nu), \\ N' &:= \nu^{-1}(W) = HN^0, \quad T' := T \cap N' \text{ and} \\ G' &:= H \langle U_a \mid a \in R \rangle. \end{aligned}$$

Lemma 3.2.5 ([BT72, (6.2.11)]). *$\mathcal{S}' := (T', (U_a)_{a \in R})$ is a generating root datum in G' and $\varphi = (\varphi_a)_{a \in R}$ is a valuation of it. Moreover we have $N' = G' \cap N$.*

Remark 3.2.6. The purpose of G' is that the affine Tits system, that we are going to construct in the following, will a priori only generate G'. In general $G' \neq G$, where $G := T \langle U_a \mid a \in R \rangle$, will be true, however we will see that under the special assumption, that $\mathcal{S}$ comes from a simply-connected linear algebraic group, one has actually equality. It remains to note that the affine building that will correspond to the affine Tits system that we construct will come with a G action in any case.

[4] Note that in the original version of the theory [BT72, 6.2] one identified the dual V^* with a set of valuations. However by the use of an invariant bilinear form these two approaches coincide.

(3.2.7). In the following we restrict ourselves to a discrete and special φ. By part (a) of remark 3.2.4, we then see that there are $e_a \in \mathbb{R}^+$, such that $\Theta_a = \mathbb{Z}e_a$, for all $a \in R$. From lemma 3.2.3, we observe that $W = \langle s_{a,l} \mid a \in R,\ l \in \Theta_a \rangle$ is an affine reflection group, corresponding to the locally finite system of hyperplanes given by $\mathcal{H} := \{L_{a,l} \mid a \in R,\ l \in \Theta_a\}$, where $s_{a,l}$ is the reflection with respect to the affine hyperplane $L_{a,l}$. Thus one knows that W is the affine reflection group associated with a unique reduced root system R' [Bou02, VI.§ 2.5. Prop. 8] (also [BT72, (6.2.22)]). More precisely we obtain:

Lemma 3.2.8. *One finds* $R' = \left\{ \frac{1}{e_a} a \,\middle|\, a \in R_{\mathrm{nd}} \right\}$ *and:*

(a) $W(R') = W(R) = {}^vW$. *[Bou02, VI.§ 1.4. Prop. 13]*

(b) $Q(R'^{\vee}) \rtimes W(R') = W_{\mathrm{aff}}(R') = W$, *where* $W_{\mathrm{aff}}(R')$ *is generated by the reflections along the hyperplanes*

$$\{v \in V \mid (a, v) + l = 0\},$$

with $a \in R'$ *and* $l \in \mathbb{Z}$. *[BT72, (6.2.19)]*

(c) $\nu(T') = \left\langle la^{\vee} \,\middle|\, a \in R,\ l \in \Theta_a \right\rangle_{\mathbb{Z}} = \left\langle l'a'^{\vee} \,\middle|\, a \in R', l' \in \mathbb{Z} \right\rangle_{\mathbb{Z}} = Q(R'^{\vee})$. *[BT72, (6.2.20)]*

(d) $\nu(T) \subseteq \{v \in V \mid (v, a') \in \mathbb{Z},\ \forall a' \in R'\} = P(R'^{\vee})$. *Hence* $\hat{W}/W \hookrightarrow P(R'^{\vee})/Q(R'^{\vee})$. *(*ibid.*)*

Remark 3.2.9. For an integral φ, one has $R' = R$ and $W = W_{\mathrm{aff}}(R')$ by definition. If $e_a = e$, for all $a \in R$, then one has $R' = \frac{1}{e}R$ and $W \cong W_{\mathrm{aff}}(R_{\mathrm{nd}})$. This is for example the case, if R is irreducible and all roots in R are of the same length.

Construction via a Fundamental Chamber

The system of hyperplanes $\mathcal{H}$ yields a cell decomposition of V into «polysimplices», by using the affine hyperplanes to slice the space V (compare to 2.2). W is a Coxeter group and its associated Coxeter complex is equivalent to this (affine) chamber system of V, i.e. to the set of cells of maximal dimension together with a neighbouring relation. If R is irreducible, the link between this Coxeter complex and the cell decomposition of V is even stronger: $\mathcal{H}$ decomposes V into simplices and the resulting simplicial complex is isomorphic to the Coxeter complex $\mathcal{A}(W,S)$, where S denotes the reflections with respect to the walls of a singled out (affine) chamber C in V. Note that the walls of C are exactly those walls L, such that $\dim(\mathrm{span}(L \cap \overline{C}))$ is of codimension one in V. This discussion is due to [Abr94, p. 57].

Remark 3.2.10. In the situation of 3.2.7 one can construct C and S for an irreducible root system R as follows: One chooses a basis $B' := \left\{a'_1, \ldots, a'_n\right\}$ of R' and sets $L_i := L_{a'_i,0}$, for the walls associated to this basis running through the origin. Let a' be the highest root with respect to this basis, i.e. the positive root such that for every other positive root b', it holds that $a' - b'$ is given by $\sum_{i=1} m_i a'_i$ and all $m_i \geq 0$ (cf. [AB08, p. 528]). Setting $L_0 := L_{-a',1}$, one obtains the closure of C in V as (recall that the affine roots are affine half-spaces of V)

$$\alpha_{-a',1} \cap \bigcap_{i=1}^{n} \alpha_{a'_i,0}$$

and as a set of walls of C one gets $\{L_i \mid 0 \leq i \leq n\}$. See also figure 3.1 for a visualisation of this process in the case of SL_3. Note that since φ is special to every hyperplane there is a parallel one that meets the origin.

The vector space V (more precisely one should say A) and its cellular decomposition forms the geometric realization of the fundamental apartment of the affine building that we seek to construct. We will actually give to ways of forming the geometric representation of the affine building associated to $\mathcal{S}$ and φ, firstly by giving an affine Tits system and realising it, and secondly by using a G-action and glueing the transformed parts together. For both we need to say, which subgroups $\hat{P}_x$ in G (resp. G') fix a point $x \in V$ of the fundamental apartment. This is the motivation for the following definition:

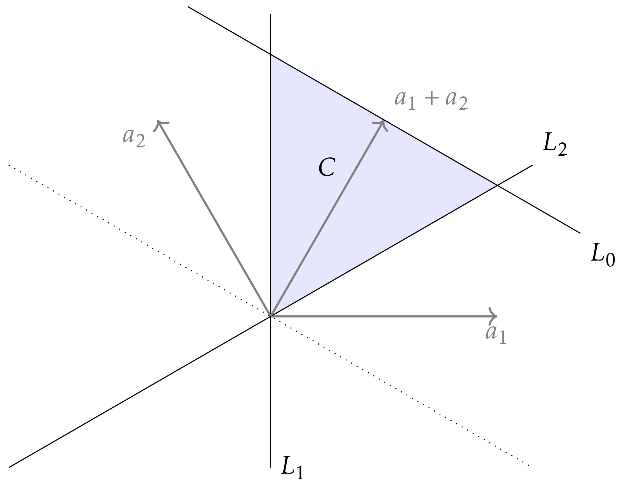

Figure 3.1: Example construction of the fundamental chamber and its walls in the case of SL_3.

Definition 3.2.11. Let $\alpha = \alpha_{a,l} \in \sigma$ and $\varnothing \neq \Omega \subseteq V$. Then we set:

$$U_\alpha := U_{a,l},$$
$$U_\Omega := \left\langle U_\beta \;\middle|\; \beta \in \sigma,\ \Omega \subseteq \beta \right\rangle, \qquad P_\Omega := U_\Omega H, \qquad N_\Omega := N \cap P_\Omega,$$
$$\hat{N}_\Omega := \{n \in N \mid \nu(n)(x) = x, \forall x \in \Omega\}, \qquad \hat{P}_\Omega := P_\Omega \hat{N}_\Omega = U_\Omega \hat{N}_\Omega.$$

If Ω consists just of a point $x \in V$, we introduce the abbreviations $U_x := U_{\{x\}}, \ldots, \hat{P}_x := \hat{P}_{\{x\}}$.

Remarks 3.2.12. (a) We see that U_Ω and P_Ω only depend on the convex hull of Ω in $\mathcal{A}(W,S)$, which is (cf. [BT72, (7.1.2)]):

$$\bigcap_{\alpha \in \sigma,\ \Omega \subseteq \alpha} \alpha.$$

(b) For $n \in N$ it holds that $nP_\Omega n^{-1} = P_{\nu(n)(\Omega)}$ and $n\hat{P}_\Omega n^{-1} = \hat{P}_{\nu(n)(\Omega)}$ is true [BT72, (7.1.8)]. We also have $N_\Omega \subseteq \hat{N}_\Omega$ (*ibid.*).

(c) The difference between the hatted objects and the unhatted ones disappears, if G is already generated by H and the U_a, for $a \in R$, i.e. if $G = G'$ holds (cf. [BT72, (7.1.10)]). In particular, we have:

$$W = \hat{W}, \quad P_\Omega = \hat{P}_\Omega \quad \text{and} \quad N_\Omega = \hat{N}_\Omega.$$

This case occurs for example, if the root datum $\mathcal{S}$ comes from a simply-connected linear algebraic group (see [BT72, (7.1.10)] and [BT84, 5.2.9]).

Take an (affine) chamber C in the vector space V. C is for reducible R an open poly-simplex, i.e. a product of open simplices, and for irreducible R an open simplex. Denote as above by S the set of reflections corresponding to the walls of C. Then one has the following result from Bruhat-Tits theory:

Theorem 3.2.13 ([BT72, 6.5] and [Abr94, Satz 37])**.** *(a) (G', P_C, N, S) is a Tits system with affine Weyl group $W_{\mathrm{aff}}(R')$. G operates (in general not in a type-preserving manner) via conjugation on the associated affine building*

$$\mathcal{I}(G', P_C) = \left\{ Q \leq G' \;\middle|\; \exists g' \in G',\ s.t.\ gP_c g^{-1} \subseteq Q \right\}.$$

Note that G' operates on the above building in a type-preserving way.

(b) If R is moreover irreducible, then one can identify the geometric realization of the standard apartment $\mathcal{A} := \left\{ Q \leq G' \;\middle|\; \exists n' \in N',\ s.t.\ n'P_C n'^{-1} \subseteq Q \right\} = \mathcal{A}(W,S)$ with V. The affine continuation of the action of N on $\mathcal{A}$ on the realization $|\mathcal{A}|$ agrees with the one given by ν. For $\varnothing \neq \Omega \subseteq V$, one finds, that the fixer of Ω in G' is P_Ω and that the fixer of Ω in G is $\hat{P}_\Omega$.

(3.2.14). For completeness and since it will be useful in the proof of Soulé's theorem, we give another definition of the U_Ω. Thus suppose $\varnothing \neq \Omega \subseteq V$, then we define the function:

$$\begin{aligned} f_\Omega : R &\to \mathbb{R} \cup \{\infty\} \\ a &\mapsto \inf\{l \in \mathbb{R} \mid \Omega \subseteq \alpha_{a,l}\} \end{aligned}$$

Then we have $U_\Omega = \langle U_{a,f(a)} \mid a \in R\rangle$. The following assertions are useful in some calculations [BT72, (6.4.9)]:

$$\begin{aligned} U_a \cap U_\Omega &= U_{a,f_\Omega(a)} . U_{2a,f_\Omega(2a)} =: U_{f_\Omega,a}, \quad \forall a \in R \quad , \\ U^\pm \cap U_\Omega &= \prod_{a \in R^\pm_{\mathrm{nd}}} U_{f_\Omega,a} \quad \text{and} \\ U_\Omega &= (U^+ \cap U_\Omega)(U^- \cap U_\Omega)(N \cap U_\Omega). \end{aligned}$$

As we already hinted at above, we will give another way to obtain a geometric representation of the affine building associated to $\mathcal{S}$ and φ without reference to the Tits system. The following is based on [Abr94, pp. 61-62] and [BT72, 7.4]. We will solely rely on lemma 3.2.3 and the definitions made in 3.2.11. Since we want the groups $\hat{P}$ to be the fixers, we will introduce the following equivalence relation on $G \times V$ ($g, h \in G$ and $v, w \in V$):

$$(g,v) \sim (h,w) \iff \exists n \in N, \text{ s.t. } w = \nu(n)(v) \wedge g^{-1}hn \in \hat{P}_v.$$

The motivation for such a relation is that we want to have $(g,v) \sim (h,v) \iff g^{-1}h \in \hat{P}_v$ and on the other hand $(hn,v) \sim (h, \nu(n)(v))$, for all $g,h \in G$, $v \in V$ and $n \in N$.
We set $\mathcal{I} := G \times V / \sim$ as a set. There is a natural G-action on $\mathcal{I}$ via $h \cdot [(g,v)] = [(hg,v)]$, for all $h,g \in G$ and $v \in V$. One also finds a natural embedding of V into $\mathcal{I}$, by setting:

$$\begin{aligned} V &\hookrightarrow \mathcal{I} \\ v &\mapsto [(\mathrm{e},v)]. \end{aligned}$$

By use of this identification and action, we see that $[(g,v)] = g \cdot v$, for all $g \in G$ and $v \in V$, holds. Moreover we have, by construction, that the fixer of a point $v \in V$ is P_v in G' and $\hat{P}_v$ in G (cf. [BT72, (7.4.4)]). The stabiliser (resp. fixer) of the fundamental apartment V is given by N (resp. H) [BT72, (7.4.10)]. This can be generalised to arbitrary subsets of V as well.
An apartment in $\mathcal{I}$ shall be a subset of the form $g \cdot V$, with $g \in G$, and a chamber is as above a translate $g \cdot C$ of a fundamental (affine) chamber C in V. One finds the following statements, which are well-known from the theory of buildings.

Lemma 3.2.15. *(a) Two chambers of $\mathcal{I}$ are contained in a common apartment [BT72, 7.4.18.(i)].*

(b) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two apartments, whose intersection contains a chamber. Then there is a $g \in G$ such that $\mathcal{A}_2 = g \cdot \mathcal{A}_1$ and every point of the intersection $\mathcal{A}_1 \cap \mathcal{A}_2$ is fixed by g [BT72, (7.4.19)].

This enables one to use retractions in order to lift the special metric V comes with, which is derived from $(\cdot,\cdot)$, to a well-defined metric in all of $\mathcal{I}$. Thus the action of G on any apartment is given by affine isometries. More precisely one obtains:

Proposition 3.2.16 ([BT72, (7.4.20)]). *(a) There is a well-defined metric $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^{\geq 0}$ such that $d(g \cdot x, g \cdot y) = d(x,y)$ holds, for all $g \in G$ and $x,y \in \mathcal{I}$. Since the root datum valuation φ is discrete, $\mathcal{I}$ is a complete metric space [BT72, (7.5.1)].*

(b) Let $x,y \in \mathcal{I}$ and define the following set

$$[xy] := \{z \in \mathcal{I} \mid d(x,y) = d(x,z) + d(z,y)\}.$$

Then $[xy]$ is contained in any apartment that contains both x and y and for $\mathcal{A}'$ being such an apartment, $[xy]$ coincides with the affine segment in $\mathcal{A}'$ joining x and y.

(c) The metric space $\mathcal{I}$ is contractible.

Definition 3.2.17. The above introduced metric space $(\mathcal{I}, d)$ is called the **Bruhat-Tits building** associated with $(\mathcal{S}, \varphi)$. By theorem 3.2.13.(b) if R is an irreducible root system, one is able to identify the so constructed $\mathcal{I}$ with the realization of $\mathcal{I}(G', P_C)$. Whenever this is the case and it is clear from the context, we will use $\mathcal{I}$ to refer to the simplicial complex and its realization.

Remark 3.2.18. The constructions leading up to proposition 3.2.16 can be carried out in the case, where φ is non-discrete, if one adjusts the notions of chamber and cell suitably. Thus one obtains in that case a contractible metric space $(\mathcal{I}, d)$ associated to $(\mathcal{S}, \varphi)$, which is however non-complete in general. The bulk of the work done in [BT72] is concerned with this case.

4
Soulé's Theorem

In the previous chapters we have introduced two types of walls, namely vectorial and affine ones. A situation in which both of these come together can be created by fixing a connected, semi-simple K-split K-linear algebraic group $\underline{G}$ over a valued field K together with a maximal K-split K-torus $\underline{T}$. Then we see that the walls associated with the roots in $X^*(\underline{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ form a subset of the affine walls in the fundamental apartment of the Bruhat-Tits building associated with the valued root datum defined in example 3.2.1, after choosing the origin appropriately. Thus the vectorial chambers are cones, which are tiled by affine chambers. The essence of Soulé's theorem is that such a cone in the Bruhat-Tits building, which will also be called a **sector**, is a fundamental domain for the action of a subgroup of $\underline{G}(K)$ with suitable assumptions on the field K.

We intend to give a proof of this, due to Margaux [Mar09], in the present chapter. Therefore we firstly fix some notation, motivate a few of the hypotheses and precisely state the theorem. After that we show a series of preliminary assertions, which will also be useful in some applications, in two separate sections and conclude the proof of the main statement thereafter.

4.1 Setting the Stage

This section is split into two parts. The first subsection introduces some necessary notation and contains the construction of the Bruhat-Tits building associated with a non-split group, whereas in the second part Soulé's theorem is stated.

4.1.1 Notation and Basic Construction

Let k be a commutative field and set $K' := k(t^{-1})$. This is a discrete valuation field, with valuation given by

$$\begin{aligned} \omega' : K'^{\times} &\to \mathbb{Z} \\ \frac{f}{g} &\mapsto \deg(g) - \deg(f). \end{aligned}$$

Let K denote the completion of K' with respect to the metric induced by the valuation ω', which is also given as the field of formal power series $k((t^{-1}))$ (cf. [AM16, 10.]). Furthermore let $\omega : K^{\times} \to \mathbb{Z}$ be the induced valuation on K, which we extend to K by setting $\omega(0) = \infty$. We fix the abbreviations A for the polynomial ring $k[t]$ and $\mathcal{O}$ for the valuation ring associated with K and ω, for which we sometimes write $k[[t^{-1}]]$ as well.

Let $\underline{G}$ be a connected, semi-simple, simply-connected and almost simple linear k-algebraic group. We quickly run through these assumptions to decipher, what we try to achieve by imposing them. The premises of connectedness and semi-simplicity enable us to regard the roots in the same sense, that we defined above, while the hypothesis of almost simplicity guarantees that the systems of roots associated to $\underline{G}$ are irreducible (cf. 3.1.16), which is needed in order to identify the two geometric realizations that we have given for the Bruhat-Tits building (cf. 3.2.13.(b)). Finally simply-connectedness reduces the complexity of the Bruhat-Tits theory as we saw above (cf. 3.2.12.(c)).

The construction of the Bruhat-Tits building in this general case will also come from a valued root datum, where the valuation is defined via a descent from the split case. Thus we fix as in (3.1.33) a maximal k-split k-torus $\underline{S}$, a maximal k-torus $\underline{T}$ containing $\underline{S}$ and let $\tilde{k}$ be a minimal finite Galois extension of k splitting $\underline{T}$. In the following we will use tilde to denote objects that come from a base-change by $\tilde{k}$, i.e. we set

$$\tilde{A} := \tilde{k}[t], \quad \tilde{\mathcal{O}} := \tilde{k}[[t^{-1}]], \quad \tilde{K} := \tilde{k}((t^{-1}))$$

and for the algebraic group structures as well. Note that the Galois groups $\mathrm{Gal}(\tilde{K}/K)$ and $\mathrm{Gal}(\tilde{k}/k)$ are canonically isomorphic and will be in the following denoted by Σ [BT84, 5.1].
In this chapter the relative root system will be the more common one, which is the reason, we fix the abbreviation $R := R(\underline{G}, \underline{S})$. The absolute roots will be referred to by $\tilde{R}$, which is testament to the fact, that $\underline{T}$ splits over $\tilde{k}$ and thus the absolute roots $R(\underline{G}, \underline{T})$ are given as pull-backs of the roots defined over $\tilde{k}$. Furthermore we fix a system of positive roots R^+ in R and a compatible system of positive roots $\tilde{R}^+$ in $\tilde{R}$ (cf. 3.1.37). The corresponding bases will be addressed by B and $\tilde{B}$.
Next we address the root subgroups. For an absolute root $\tilde{a} \in \tilde{R}$ we denote by $\underline{\tilde{U}}_{\tilde{a}}$ the absolute root subgroup defined over $\tilde{K}$ associated with $\tilde{a}$ (cf. 3.1.32.(a)). For a relative root $a \in R$ we fix the notation $\underline{U}_a$ for its associated relative root subgroup defined over K (cf. 3.1.40). By lemma 3.1.42 we know that over the field $\tilde{K}$ we have the following isomorphism of $\tilde{K}$-varieties, which is given by the product morphism ($a \in R$):

$$\underline{\tilde{U}}_a \cong \prod_{\tilde{a}\in\tilde{R},\ j(\tilde{a})=a} \underline{\tilde{U}}_{\tilde{a}} \times \prod_{\tilde{a}\in\tilde{R},\ j(\tilde{a})=2a} \underline{\tilde{U}}_{\tilde{a}}.$$

We will now turn our attention towards root data and valuations. Therefore we introduce the convention that dropping the underline of a scheme structure makes a transition to the corresponding rational points. This means in particular for the root subgroups:

$$\tilde{U}_{\tilde{a}} := \underline{\tilde{U}}_{\tilde{a}}\left(\tilde{K}\right), \quad U_a := \underline{U}_a(K) \quad \text{and} \quad \tilde{U}_a := \underline{\tilde{U}}_a\left(\tilde{K}\right) \quad \text{, for all } \tilde{a} \in \tilde{R} \text{ and } a \in R.$$

Recall that by 3.1.32, there is a generating root datum on $\tilde{G}$ given by

$$\left(\tilde{T}, \left(\tilde{U}_{\tilde{a}}\right)_{\tilde{a}\in\tilde{R}}\right),$$

which admits a discrete and special valuation $\tilde{\varphi}$ that is given as in 3.2.1. The Bruhat-Tits building $\tilde{\mathcal{I}}$ corresponding to $\tilde{G}$ and $\tilde{K}$ is then defined via the constructions given in 3.2.2 with respect to this valued root datum.
There is also a root datum associated with G, which is given by

$$(\mathrm{Z}_G(S), (U_a)_{a\in R}),$$

where we used $\mathrm{Z}_G(S) := \mathrm{Z}_{\underline{G}}(\underline{S})(K)$. In order to define a valuation for this root datum we will use the abstract theory that Bruhat and Tits provide to descend the valuation $\tilde{\varphi}$. To do that we introduce the following filtration of the relative root subgroups over $\tilde{K}$ ($a \in R$ and $m \in \mathbb{R}$):

$$\tilde{U}_{a,m} := \prod_{\tilde{a}\in\tilde{R},\ j(\tilde{a})=a} \tilde{U}_{\tilde{a},m} \times \prod_{\tilde{a}\in\tilde{R},\ j(\tilde{a})=2a} \tilde{U}_{\tilde{a},m}, \tag{4.1}$$

where the $\tilde{U}_{\tilde{a},m}$ are defined through the valuation $\tilde{\varphi}$, by $\tilde{U}_{\tilde{a},m} := \tilde{\varphi}_a^{-1}([m,\infty])$. By regarding U_a as the subgroup of $\tilde{U}_a$, which is fixed by the action of the Galois group Σ, for every $a \in R$, one is able to define the following system of maps:

$$\left(\begin{aligned} \varphi_a : U_a &\to \mathbb{R} \cup \{\infty\} \\ u &\mapsto \sup\left\{m \in \mathbb{R} \;\middle|\; u \in \tilde{U}_{a,m}\right\} \end{aligned}\right).$$

In [BT84, 5.1] it is shown that this definition makes the system $\varphi = (\varphi_a)_{a\in R}$ a discrete and special valuation for the above root datum of G over K.[1] By (4.1) one sees that the group of values $\Theta_b := \varphi_b(\mathrm{U}_b \setminus \{\mathrm{e}\})$ is either given as $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$. Whenever $2a \notin R$ we set $\mathrm{U}_{2a} = 1$. Finally we define analogously to the above filtration, a filtration $(\mathrm{U}_{a,m})_{m\in\Theta_a}$ of U_a, for $a \in R$.

[1] In [BT84, 5.1] it is used that both K and $\tilde{K}$ fulfil the properties of the Hensel lemma. Moreover it is supposed that $\tilde{k}$ shall be perfect. This is not necessary in our case, since by completeness of K the extension $K \subseteq \tilde{K}$ is étale in the sense of [BT84, Def. 1.6.2] and G splits over $\tilde{K}$.

By using the construction given in 3.2.2 we obtain a building $\mathcal{I}$ corresponding to $\underline{G}$ and K. The properties of the descent imply in particular that there is an isometric embedding (recall that the Bruhat-Tits building comes equipped with a metric) $\iota : \mathcal{I} \to \tilde{\mathcal{I}}$, which fulfils $\iota(\varphi) = \tilde{\varphi}$ and $\iota(g.x) = g.\iota(x)$, for $g \in \underline{G}(K)$ and $x \in \mathcal{I}$ [BT72, 9.1.17]. Moreover we have that the fundamental apartments in $\mathcal{I}$ and in $\tilde{\mathcal{I}}$, which are respectively given by

$$\mathcal{A} := \varphi + \mathrm{X}^*(\underline{S}) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad \tilde{\mathcal{A}} := \tilde{\varphi} + \mathrm{X}^*(\underline{T}) \otimes_{\mathbb{Z}} \mathbb{R},$$

are such that $\iota(\mathcal{A})$ is the subset of $\tilde{\mathcal{A}}$, which is fixed under the action of the Galois group Σ. In fact one is able to extend the action of Σ on the characters $\mathrm{X}^*(\underline{T})$ to an isometric action (cf. [BT84, 4.2.12]) on $\tilde{\mathcal{I}}$ and then ι identifies $\mathcal{I}$ with $\tilde{\mathcal{I}}^{\Sigma}$. The isometry ι in particular entails that the Weyl group invariant scalar products, we assume on the vector spaces lying under the fundamental apartments $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are compatible under ι. We set the origin of $\mathcal{A}$ to φ and $\tilde{\varphi}$ for $\tilde{\mathcal{A}}$.

4.1.2 Statement of the Theorem

We define a sector, i.e. a translate of a vectorial chamber, in $\mathcal{A}$ by setting

$$\mathcal{Q} := \varphi + \underbrace{\{v \in \mathrm{X}^*(\underline{S}) \otimes_{\mathbb{Z}} \mathbb{R} \mid (a, v) \geq 0,\ \forall a \in B\}}_{D:=}. \tag{4.2}$$

In the ensuing proof we will also use the abbreviation Γ referencing the A-valued points in $\underline{G}$, i.e. $\Gamma := \underline{G}(A) = \underline{G}(k[t])$. The theorem we seek out to show in this chapter reads as follows:

Theorem 4.1.1 ([Mar09, Thm. 2.1]). *$\mathcal{Q}$ is a simplicial fundamental domain for the action of Γ on $\mathcal{I}$, i.e. any simplex in $\mathcal{I}$ is equivalent under Γ to a unique simplex in $\mathcal{Q}$.*

We proceed to carry out the proof of 4.1.1 as in [Mar09] by splicing in parts of the proof of the split case, which are due to [SW79]. The general outline of the proof did not change much from the transition from Soulé to Margaux, it did however gain complexity and necessity of reference to the explicit theory of Bruhat and Tits.
The structure of the proof below is as follows: We start out by analysing, in a purely group theoretic way, the isotropy groups of points in $\mathcal{Q}$. This is trailed by a careful examination of the links of certain vertices in the sector $\mathcal{Q}$, after which we complete the proof.

4.2 Description of the Isotropy Group Γ_x of a Point x of $\mathcal{Q}$ in Γ

Although the title already gives away the most important definition, let us be more precise. Let $\Omega \subseteq \mathcal{Q}$ be an arbitrary subset in the sector $\mathcal{Q}$, then we denote by Γ_Ω the fixer of all points in Ω inside Γ. Let there be an analogous definition for fixers inside $\tilde{\Gamma}$. These two definitions are related by the property of Galois descent, and one sees that

$$\Gamma_\Omega = \left(\tilde{\Gamma}_{\iota(\Omega)}\right)^{\Sigma} \tag{4.3}$$

holds. Finally if Ω is reduced to a point $x \in \mathcal{Q}$, we write Γ_x instead of $\Gamma_{\{x\}}$.

Example 4.2.1. We know, from the construction of $\tilde{\mathcal{I}}$ and simply-connectedness (see 3.2.12.(c)), that the fixer of $\tilde{\varphi}$ in $\tilde{G}$ is given as the product $\tilde{P}_{\tilde{\varphi}} = \tilde{U}_{\tilde{\varphi}}.\tilde{H}$, where $\tilde{H}$ is the fixer of the fundamental apartment $\tilde{\mathcal{A}}$ and $\tilde{U}_{\tilde{\varphi}}$ is defined in 3.2.11. Since $\tilde{\varphi}$ is special, we observe that

$$\tilde{U}_{\tilde{\varphi}} = \left\langle \tilde{U}_{\tilde{a},0} \mid \tilde{a} \in \tilde{R} \right\rangle$$

holds, and by [BT84, 5.2.1] we have $\tilde{H} = \underline{T}(\tilde{\mathcal{O}})$. From $\tilde{U}_{\tilde{a},0} = \underline{U}_{\tilde{a}}(\tilde{\mathcal{O}})$ one then sees that the fixer of $\tilde{\varphi}$ in $\underline{G}(\tilde{K})$ is given as $\underline{G}(\tilde{\mathcal{O}})$. Thus one has $\tilde{\Gamma}_{\tilde{\varphi}} = \underline{G}(\tilde{\mathcal{O}}) \cap \tilde{\Gamma} = \underline{G}(\tilde{k})$. And since k is the sub-field of $\tilde{k}$ fixed by the Galois group Σ, one has $\Gamma_\varphi = \underline{G}(k)$.

For rest of this section fix a point $x \in \mathcal{Q} \setminus \{\varphi\}$.

4.2.1 Equivalence of the Isotropy group of a Vertex and its Ray

In here we will show, that $\Gamma_x = \Gamma_{[x[}$ holds, where $[x[$ is the ray, emanating from x in the direction of $\overrightarrow{\varphi x}$. This will follow from (4.3) as soon as we establish the assertion in the case that G is split.

The split case

We will assume that $\underline{G}$ is split in this paragraph. By remark 3.1.6 we know, that there is a faithful rational representation $\underline{G} \to \mathrm{GL}_n$, for some non-zero natural number n. By analysing the restricted representation $\underline{T} \to \mathrm{GL}_n$ one sees by [Bor91, 8.2] that there is a conjugation of GL_n such that the image of $\underline{T}$ lies in D_n, i.e. the diagonal matrices in GL_n. We will embed $\underline{G}$ furthermore into the special linear group, by defining a homomorphism $\mathrm{GL}_n \to \mathrm{SL}_{n+1}$, that is a closed immersion via the functorial approach. Let R be a finite type k-algebra, then we see that the following group homomorphism is natural in R:

$$\begin{aligned} \mathrm{GL}_n(R) &\to \mathrm{SL}_{n+1}(R) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix} \end{aligned}$$

This is a closed immersion, since the image of GL_n in SL_{n+1} is characterized by setting the components T_{ij}, with one index being $n+1$ and the other in the range from 1 to n, to zero, which is a polynomial condition.

In summarization we found a faithful embedding $\underline{G} \to \mathrm{SL}_{n+1}$ such that the image of the maximal split torus $\underline{T}$ lies in the diagonal matrices. From the theory of Bruhat and Tits we thus know (precisely by [BT72, 9.1.19.c)]), that there is a a unique injection $\mathcal{I} \to \mathcal{I}'$, from the Bruhat-Tits building associated to $\underline{G}$ and K into the one associated to SL_{n+1} and K, which is compatible with the action of G on $\mathcal{I}$ and $\mathcal{I}'$ such that the fundamental apartment $\mathcal{A}$ associated to $\underline{T}$, is mapped into the one associated to the diagonal matrices in $\mathcal{I}'$. Furthermore it follows from the theory that the embedding $\mathcal{I} \to \mathcal{I}'$ multiplies distances by a fixed constant. From this one derives immediately that it suffices to show the claim in the case that $\underline{G} = \mathrm{SL}_{n+1}$, considering that $\underline{G}(A) = \Gamma = \mathrm{SL}_{n+1}(A) \cap \underline{G}(K)$ holds.

So we assume $\underline{G} = \mathrm{SL}_{n+1}$ and $\underline{T}$ is given by the diagonal matrices in SL_{n+1}. In this case it follows from [BT72, Cor. 10.2.9] that for a subset Ω of the standard apartment $\mathcal{A}$ for an element $g \in \underline{G}(K)$ to be in the fixer of Ω of the action of $\underline{G}(K)$ on the building $\mathcal{I}$, it is a necessary and sufficient condition to fulfil:

$$\omega(g_{ij}) \geq \sup\left\{y_i - y_j \,\middle|\, y \in \Omega\right\} \quad \forall\ 1 \leq i,j \leq n+1,$$

where the g_{ij} are the matrix components of g and by y_i we refer to the coordinates of a vector $y \in \mathrm{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, after a suitable identification with $(e_1 + \cdots + e_{n+1})^{\perp} \subseteq \mathbb{R}^{n+1}$ with respect to the standard scalar product. Hence we have the following for the fixer of a point $x \in \mathcal{Q}$ and its ray:

$$\hat{P}_x = \left\{g \in \underline{G}(K) \,\middle|\, \omega(g_{ij}) \geq x_i - x_j\right\}, \quad \hat{P}_{[x[} = \left\{g \in \underline{G}(K) \,\middle|\, \begin{matrix} g_{ij} = 0 & \text{if } x_i - x_j > 0 \\ \omega(g_{ij}) \geq x_i - x_j & \text{if } x_i - x_j \leq 0 \end{matrix}\right\}. \tag{4.4}$$

By intersecting with $\underline{G}(A)$ we see, since $\omega(A \setminus \{0\})$ is non-positive, that $\Gamma_x = \Gamma_{[x[}$ holds.

4.2.2 Decomposing the Isotropy Group of a Vertex

We return to the general case. Firstly we recall the definition of the isotropy group of the ray $[x[$ in $G = \underline{G}(K)$. Therefore we note that, by [BT84, 5.2.1], the fixer of the standard apartment $\mathcal{A}$ in G is given as $\mathrm{Z}_{\underline{G}}(\underline{S})(\mathcal{O})$ and we will from now on denote it by H. Furthermore we recall the definition of the subgroup $U_{[x[}$ of U in terms of the function (cf. 3.2.14):

$$\begin{aligned} f_{[x[} : R &\to \mathbb{R} \cup \{\infty\} \\ a &\mapsto \inf\{s \in \mathbb{R} \mid (a,y) + s \geq 0,\ \forall y \in [x[\}. \end{aligned}$$

$U_{[x[}$ is then generated by the $U_{a,m}$, for $a \in R^+$ and $m \geq f_{[x[}(a)$. Since the elements of the ray $[x[$ are just scaled versions of x itself, one sees that the function $f_{[x[}$ admits the following presentation into three cases ($a \in R$):

$$f_{[x[}(a) = \begin{cases} \infty & \text{if } (a,x) < 0, \\ 0 & \text{if } (a,x) = 0, \\ -(a,x) & \text{if } (a,x) > 0. \end{cases}$$

From this one may observe that the root subgroups that contribute to $U_{[x[}$ can be separated into the following three sorts:

(1) $\mathrm{U}_{a,0} = \underline{U}_a(\mathcal{O})$, for $a \in R^+$ and $(a,x) = 0$;

(2) $\mathrm{U}_{a,0} = \underline{U}_a(\mathcal{O})$, for $a \in R^-$ and $(a,x) = 0$;

(3) $\mathrm{U}_{a,m}$, for $a \in R^+$, for $(a,x) > 0$ and $m \in \Theta_a$, such that $m \geq -(a,x)$.

The following lemma gives an analysis of the roots associated with these three sorts:

Lemma 4.2.2 ([Mar09, Lemma 2.2]). *Define $I_x := \{a \in B \mid (a,x) = 0\}$. Then one has the following:*

$$[I_x] \cap R^+ = \{a \in R^+ \mid (a,x) = 0\}, \tag{4.5}$$

$$[I_x] \cap R^- = \{a \in R^- \mid (a,x) = 0\} \text{ and} \tag{4.6}$$

$$R^+ \setminus [I_x] = \{a \in R^+ \mid (a,x) > 0\}. \tag{4.7}$$

Proof. Let there be $a \in [I_x]$, then one knows that a is a linear combination of elements in $[I_x]$, which implies $(a,x) = 0$. Thus in particular we know $[I_x] \cap R^+ \subseteq \{a \in R^+ \mid (a,x) = 0\}$. Starting on the right-hand-side, with $a \in R^+$ such that $(a,x) = 0$, we employ that B forms a basis of the root system R. Thus a can be written as $\sum_{b \in B} n_b b$, where the n_b are non-negative integers. By the definition of $\mathcal{Q}$ (4.2) we see that $(b,x) \geq 0$ holds, for all $b \in B$. Thus from $\sum_{b \in B} n_b(b,x) = 0$ one can gather that a is a linear combination of elements in $[I_x]$. This shows (4.5) and (4.6) follows analogously.

To obtain (4.7), we note that again by $x \in \mathcal{Q}$,

$$\{a \in R^+ \mid (a,x) \neq 0\} = \{a \in R^+ \mid (a,x) > 0\}$$

holds and thus the lemma is proven. □

Define $U^+_{[x[}$ to be the subgroup of $U_{[x[}$ generated by the sorts **(1)** and **(3)** and $U^-_{[x[}$ the one generated by **(2)** and **(3)**. Recall that (in the simply-connected case) the isotropy group of $[x[$ in $G(K)$ is given by

$$P_{[x[} = U_{[x[}.H.$$

The following corollary marks the starting point of a few statements concerning inclusions of root subgroups and fixers of points in $\mathcal{Q}$. They are essential to what follows and to the application of Soulé's theorem. One may want to refresh some facts about standard parabolic subgroups and their Levi subgroups recalled in 3.1.47.

Corollary 4.2.3. *The following inclusions hold*

$$U_{[x[} \subseteq \left(U_{[x[} \cap \underline{U}_{I_x}(K)\right) \rtimes \underline{L}_{I_x}(\mathcal{O}) \subseteq \underline{P}_{I_x}(K).$$

Proof. (4.5) and (4.6) imply, that the subgroups of $U_{[x[}$ of sorts **(1)** and **(2)** are in $\underline{L}_{I_x}(\mathcal{O})$. This follows from the fact that $\underline{L}_{I_x}$ is generated by $\mathrm{Z}_{\underline{G}}(\underline{S})$ and the root subgroups $\underline{U}_b$, with $b \in [I_x]$. The subgroups of $U_{[x[}$ of sort **(3)** are contained in $\underline{U}_{I_x}(K)$ by (4.7). This shows the first inclusion. The second inclusion is clear since $\underline{L}_{I_x}(\mathcal{O}) \subseteq \underline{L}_{I_x}(K)$ holds. □

The following lemma covers more of such inclusions that will be used in the following. The proof is quite involved, however it also conveys how to handle the subgroups $U_{[x[}$ and its subgroups $U^{\pm}_{[x[}$. An analysis of their structure will be integral for the subsequent proofs.

Lemma 4.2.4 ([Mar09, Lemma 2.3]). *With the above notation the following inclusions holds:*

$$\underline{L}_{I_x}(\mathcal{O}) \subseteq P_{[x[} \subseteq \underline{U}_{I_x}(K) \rtimes \underline{L}_{I_x}(\mathcal{O}) \subseteq \underline{P}_{I_x}(K); \tag{4.8}$$

$$\underline{U}_{I_x}(K) \cap P_{[x[} \subseteq U^+_{[x[}; \tag{4.9}$$

$$\bigcup_{z \in [1,\infty[} \left(U^+_{[zx[} \cap \underline{U}_{I_x}(K)\right) = \underline{U}_{I_x}(K). \tag{4.10}$$

Proof. I_x will be abbreviated by I for course of this proof.
(4.8): Since $\underline{S}_I \subseteq \underline{S}$ holds, we have (for example by the functorial approach) $\mathrm{Z}_{\underline{G}}(\underline{S}) \subseteq \mathrm{Z}_{\underline{G}}(\underline{S}_I) = \underline{L}_I$. Together with $U_{[x[} \subseteq \underline{U}_I(K) \rtimes \underline{L}_I(\mathcal{O})$, which is due to Corollary 4.2.3, we get that

$$P_{[x[} = U_{[x[}.H = U_{[x[}.\mathrm{Z}_{\underline{G}}(\underline{S})(\mathcal{O}) \subseteq \underline{U}_I(K) \rtimes \underline{L}_I(\mathcal{O}).$$

This settles the middle inclusion in (4.8) and the last inclusion is clear.

We proceed to show $\underline{L}_I(\mathcal{O}) \subseteq P_{[x[}$. To this end, we need to analyse the structure of the standard Levi subgroup $\underline{L}_I$.Since as noted already above the root system of $\underline{L}_I$ is given by $[I]$, we know that the unipotent radical of the standard parabolic subgroup associated to the basis I of $[I]$ is $\underline{V}_I$, which is generated by the $\underline{U}_b$, with $b \in [I] \cap R^+$. The unipotent radical associated to the opposite parabolic in $\underline{L}_I$, here denoted by $\underline{V}_I^-$, is generated by the U_b, with $b \in [I] \cap R^-$, as one can see for example by [Bor91, Prop. 14.21]. We define the **big cell**, an open subset in $\underline{L}_I$, by $\underline{\Omega} := \underline{V}_I^- \times_k \mathrm{Z}_{\underline{G}}(\underline{S}) \times_k \underline{V}_I$ and obtain by a well-known fact that[2]

$$\bigcup_{g \in \underline{V}_I(k)} g.\underline{\Omega} = \underline{L}_I$$

holds. From this one deduces furthermore that after recalling $H = \mathrm{Z}_{\underline{G}}(\underline{S})(\mathcal{O})$,

$$\underline{L}_I(\mathcal{O}) = \underline{V}_I(k).\underline{\Omega}(\mathcal{O}) = \underline{V}_I(k).\underline{V}_I^-(\mathcal{O}).H.\underline{V}_I(\mathcal{O})$$

is true. We shall briefly sketch on how to arrive at this claim. Therefore fix a morphism of schemes $\mathrm{Spec}(O) \to \underline{L}_I$. It is known that a scheme morphism from the spectrum of a local domain $\mathcal{O}$ into a scheme $\underline{L}_I$ is given by two (not necessarily closed) points z and y, with y in the closure of z in $\underline{L}_I$ such that

$$\begin{array}{ccc} \mathcal{O}_{\underline{Z},y} & \subseteq & \mathcal{O} \\ \cap\!\!| & & \cap\!\!| \\ \kappa(z) & \subseteq & K \end{array}$$

holds, where $\underline{Z}$ is the reduced induced subscheme associated with $\overline{\{z\}}$ in $\underline{L}_I$, $\kappa(z)$ is the function field of z in Z and the inclusion $\mathcal{O}_{\underline{Z},y} \subseteq \mathcal{O}$ is dominating, i.e. the image of the maximal ideal of $\mathcal{O}_{\underline{Z},y}$ is contained in the maximal ideal of $\mathcal{O}$.[3] Since the translates of $\underline{\Omega}$ cover $\underline{L}_I$, we then find $g \in \underline{V}_I(k)$ such that $y \in g\underline{\Omega}$ and as $g\underline{\Omega}$ is open in $\underline{L}_I$ the generization z of y needs to be in $g\underline{\Omega}$ as well. Thus one can define a morphism $\mathrm{Spec}(\mathcal{O}) \to g\underline{\Omega}$ through which $\mathrm{Spec}(O) \to \underline{L}_I$ factors.
Since $\underline{V}_I(k) \subseteq \underline{V}_I(\mathcal{O})$, and $\underline{V}_I(\mathcal{O})$ and $\underline{V}_I^-(\mathcal{O})$ are in $U_{[x[}$, as they form the subgroups of sort **(1)** and **(2)**, we finally arrive at $\underline{L}_I(\mathcal{O}) \subseteq P_{[x[}$.

(4.9): Will use the theory of Bruhat and Tits to show the even stronger claim $\underline{U}(K) \cap P_{[x[} = U^+_{[x[}$. Recall that $\underline{U} = \underline{U}_\varnothing$ and thus by definition $\underline{U}_I(K) \subseteq \underline{U}(K)$, which together with the claim implies (4.9). Recall the direction D of our sector $\mathcal{Q}$, whose definition is given in (4.2). Now from Bruhat-Tits theory one knows, by [BT72, 7.1.4],

$$P_{[x[} \cap \underline{U}(K) = U_{[x[+D},$$

where $U_{[x[+D}$ is similarly given as $U_{[x[}$, namely by means of the function

$$\begin{aligned} f_{[x[+D} : R &\to \mathbb{R} \cup \{\infty\} \\ a &\mapsto \inf\{s \in \mathbb{R} \mid (a,y) + s \geq 0 \ \forall y \in [x[+D\}. \end{aligned}$$

Since the ray $[x[$ is contained in $x+D$, we have $[x[+D = x+D$ as subset of $\mathcal{A}$ and as the definitions of $f_{[x[+D}$ and in turn $U_{[x[+D}$ only depend on $[x[+D$ as a set, one has $U_{[x[+D} = U_{x+D}$ and $f_{[x[+D} = f_{x+D}$. Using this simplification we obtain the following explicit form of f_{x+D}, for $a \in R$:

$$f_{x+D}(a) = \begin{cases} -(a,x) & \text{if } a \in R^+ \\ \infty & \text{if } a \notin R^+ \end{cases}$$

This comes from the fact, that $a \in R^+$ is characterised by $(a,y) \geq 0$ for all $y \in D$, which is true by the very definition of D (cf. (4.2)) and since D is the closed fundamental chamber corresponding to the root system R in $\mathrm{X}^*(\underline{S}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Now U_{x+D} is defined as the subgroup generated by the $U_{b,m}$, with $m \geq f_{x+D}(a)$, for every $a \in R$. By comparing this with the definition of $U^+_{[x[}$ as the subgroup generated by the sorts **(1)** and **(3)**, one sees that $U^+_{[x[} = U_{x+D}$ holds, which implies the claim.

[2]See [DG70, XXVI.4.3.6] and [DG70, Thm. XXVI.5.1].

[3]This can be proven as in [Har77, II. Lemma 4.4] where it is additionally assumed that $\mathcal{O}$ is a valuation ring of K. This is true in our case, however upon inspection of the proof of [Har77, II. Lemma 4.4], one sees that it is only required that $\mathcal{O}$ is a local domain with field of fractions K.

(4.10): First we note that $\subseteq$ trivially holds. Thus we need to argue for equality. The definition of $\underline{U}_I$ states that it is the subgroup generated by the $\underline{U}_a$, with $a \in R^+ \setminus [I]$. From (4.7) we furthermore know that such an a fulfils $(a,x) > 0$. In the proof of (4.9), one could see, that $U^+_{[zx[}$ is generated by the $U_{a,-(a,x)\cdot z}$, with $a \in R^+$. Thus if one takes a root a, with $(a,x) > 0$, one sees that for z going to ∞, $-a(x)\cdot z$ goes to $-\infty$. Thus

$$\bigcup_{z\in[1,\infty[} \left(U^+_{[zx[} \cap \underline{U}_I(K)\right)$$

contains a filtration of U_a, with $a \in R^+ \setminus [I]$, which shows the claim. □

The above lemma contained important intermediary results, that are needed in the following proposition, on which the proof of Soulé's theorem rests and also parts of the applications.

Proposition 4.2.5 ([Mar09, Prop. 2.5])**.** *We have the following decomposition of the vertex stabiliser groups:*

$$\Gamma_x = \left(\Gamma_x \cap \underline{U}_{I_x}(K)\right) \rtimes \underline{L}_{I_x}(k); \tag{4.11}$$

$$\bigcup_{z\in[1,\infty[} \Gamma_{zx} = \underline{U}_{I_x}(k[t]) \rtimes \underline{L}_{I_x}(k). \tag{4.12}$$

Furthermore if we set for a root $a \in R$

$$m_x(a) := \inf\{m \in \Theta_a \mid m + (a,x) \geq 0\},$$

where Θ_a is the image of the valuation φ_a in $\mathbb{R}$. Then we have the following decomposition

$$\Gamma_x = \left\langle \left(U_{a,m_x(a)}\cdot U_{2a,m_x(2a)}\right) \cap \Gamma \;\middle|\; (a,x) > 0 \right\rangle \rtimes \underline{L}_{I_x}(k). \tag{4.13}$$

Proof. As in the previous proof, we will set $I = I_x$, to simplify the notation.

(4.11): Recall that in order to show that an abstract group C is isomorphic to the semi-direct product $A \rtimes B$, it is necessary and sufficient to find a right-split exact sequence[4]

$$0 \longrightarrow A \longrightarrow C \rightleftarrows B \longrightarrow 0.$$

Thus since by 3.1.47 the standard parabolic subgroup $\underline{P}_I$ is the semi-direct product $\underline{U}_I \rtimes \underline{L}_I$, we have a right-split exact sequence

$$0 \longrightarrow \underline{U}_I(K) \longrightarrow \underline{P}_I(K) \underset{p}{\overset{s}{\rightleftarrows}} \underline{L}_I(K) \longrightarrow 0$$

after applying the functor of K-points [Mil17, p. 50]. If we intersect the first two groups with Γ_x, we readily get

$$0 \to \underline{U}_I(K) \cap \Gamma_x \to \Gamma_x,$$

since $P_{[x[} \cap \Gamma = \Gamma_x$ and $P_{[x[} \subseteq \underline{P}_I(K)$ by (4.8). Thus in order to complete the proof, one needs to find that the image of Γ_x under the projection p is $\underline{L}_I(k)$ and that the restriction of the inclusion s to $\underline{L}_I(k)$ maps to Γ_x.

From (4.8) one sees that the image of $P_{[x[}$ under p is given by $\underline{L}_I(\mathcal{O})$. Furthermore one knows that $p(\underline{P}_I(k[t])) \subseteq \underline{L}_I(k[t])$ holds, which implies that the image of Γ_x inside $\underline{L}_I(K)$ is given as $\underline{L}_I(k[t]) \cap \underline{L}_I(\mathcal{O}) = \underline{L}_I(k)$. Similarly (4.8) implies that the inclusion s maps $\underline{L}_I(k)$ into $\Gamma \cap P_{[x[} = \Gamma_x$. This finishes the proof of (4.11).

(4.12): Let us first note that by the definition of I_x for any positive real number z, it holds that $I := I_x = I_{zx}$. Fixing a number $z \in [1,\infty[$, we know that $U^+_{[zx[} \subseteq P_{[zx[}$ and thus $\underline{U}_I(K) \cap U^+_{[zx[} \subseteq \underline{U}_I(K) \cap P_{[zx[}$ holds. The opposite inclusion follows from (4.9) and intersecting with Γ yields:

$$\underline{U}_I(K) \cap U^+_{[zx[} \cap \Gamma = \underline{U}_I(K) \cap P_{[zx[} \cap \Gamma = \underline{U}_I(K) \cap \Gamma_{zx} = \underline{U}_I(k[t]) \cap \Gamma_{zx}.$$

[4] See [Rot09, Thm. 9.5] and [Rot09, Exc. 9.13].

Applying this to (4.10) leaves us with

$$\bigcup_{z\in[1,\infty[} (\Gamma_{zx} \cap \underline{U}_I(K)) = \Gamma \cap \underline{U}_I(K) = \underline{U}_I(k[t]) \ .$$

Now by (4.11) together with the introductory remark of this paragraph one can conclude the proof of this point.

(4.13): Define the set $V := \left\langle \left(U_{a,m_x(a)}.U_{2a,m_x(2a)}\right) \cap \Gamma \;\middle|\; a \in R,\ (a,x) > 0 \right\rangle$. We will show that $V = \Gamma_x \cap \underline{U}_I(K)$ holds, which implies the claim by (4.11). To obtain $\subseteq$ we note firstly that V is a subgroup of Γ_x, since $U_{a,m_x(a)}.U_{2a,m_x(2a)} \subseteq U_{[x[} \subseteq P_{[x[}$ for all $a \in R$ and $\Gamma_x = P_{[x[} \cap \Gamma$ hold. Recalling (4.7) and the definition of $\underline{U}_I$ we only need to prove that no $a \in R^-$ contributes to V. But this is obvious, since by the definition of $\mathcal{Q}$ in (4.2), $(a,x) \le 0$ holds, for $a \in R^-$.

To deduce the converse inclusion, we will introduce some simplification steps. Firstly note that it suffices to show

$$\Gamma_x \cap \underline{U}_I(K) \subseteq \left\langle \left(U_{a,m_x(a)}.U_{2a,m_x(2a)}\right) \cap \Gamma \;\middle|\; a \in R,\ (a,x) \ge 0 \right\rangle. \tag{4.14}$$

This can be justified by stating that $\underline{U}_I$ is generated by the $\underline{U}_a$, with $a \in R^+ \setminus [I]$ and those roots are, by (4.10), exactly those roots that fulfil $(a,x) > 0$. Secondly we remark that by (4.9) (intersected with Γ), one has

$$\Gamma_x \cap \underline{U}_I(K) \subseteq \Gamma \cap U^+_{[x[}.$$

Correspondingly we will only show that $\Gamma \cap U^+_{[x[}$ is a subset of the right-hand side of (4.14). Fix by $R^+_{\mathrm{nd}} = \{a_1, \ldots, a_N\}$ an arbitrary ordering of the set of non-divisible positive roots. From proposition 3.1.40 one knows that the product of the $\underline{U}_{a_i}$, with $i = 1, \ldots, N$, is isomorphic to $\underline{U}$ as a variety. Thus one has in particular the following commutative diagram, after applying the functor of points suitably:

$$\begin{array}{ccc} \prod_{i=1}^N \underline{U}_{a_i}(K) & \xrightarrow{\simeq} & \underline{U}(K) \\ \cup\!\shortmid & & \cup\!\shortmid \\ \prod_{i=1}^N \underline{U}_{a_i}(k[t]) & \xrightarrow{\simeq} & \underline{U}(k[t]). \end{array} \tag{4.15}$$

From the theory of Bruhat and Tits (3.2.14) one obtains another bijection given by the product map, namely:

$$\prod_{i=1}^N U_{a_i,m_x(a_i)}.U_{2a_i,m_x(2a_i)} \xrightarrow{\simeq} U^+_{[x[}. \tag{4.16}$$

For every $i = 1, \ldots, N$ we know, by lemma 3.1.42, that $\left(U_{a_i,m_x(a_i)}.U_{2a_i,m_x(2a_i)} \cap \underline{U}(k[t])\right)$ is a subgroup of $\underline{U}_{a_i}(k[t])$. By using the bijections in (4.15) and (4.16) it is furthermore implied that the elements in $\Gamma \cap U^+_{[x[}$ are generated by elements of these subgroups. Since, for every $i = 1, \ldots, N$, the valuation $\varphi_{a_i} : U_{a_i} \to \mathbb{R}$ is non-positive on $\underline{U}_{a_i}(k[t])$, one then deduces that $m_x(a_i) \le 0$, which implies, by

$$m_x(a_i) + (a_i, x) \ge 0,$$

that $(a_i, x) \ge 0$. So we can conclude, that $U^+_{[x[} \cap \underline{U}(k[t])$ is generated by products of elements of $\left(U_{a_i,m_x(a_i)}.U_{2a_i,m_x(2a_i)}\right) \cap \Gamma$, with $(a_i, x) \ge 0$, which is what we reduced our claim to. $\square$

4.3 Group Action on the Link of a Vertex of Type 0

Recall that φ is a special point in the apartment $\mathcal{A}$, which means in particular that to every hyperplane in $\mathcal{A}$, which is a wall of a chamber, there is a parallel version meeting φ. Since $\mathcal{I}$ is colourable as the geometric realization of a building, we assume the colour 0 to be assigned to φ. In this section we strive to examine points in $\mathcal{Q}$ that are of the same type as φ.

Firstly it is useful to find points in $\mathcal{Q}$, which exhibit type 0. We know that the affine Weyl group associated with with the valuation root datum associated with $\mathcal{I}$ acts type-preservingly on the fundamental apartment $\mathcal{A}$. Moreover it is known that for an element s in $\underline{S}(K)$ the operation of s on $\mathcal{A}$ is given by translation with a vector $v_s = \nu(s)$ (cf. 3.2.8), which is given by

$$(a, v_s) = -(\omega \circ a)(s), \quad \forall a \in R. \tag{4.17}$$

In the split case this can be derived from 3.1.18.(b) and carried over to the general one by descent [BT84, 5.1.22]. We also note the following fact from the theory of linear algebraic groups: For every root $a \in R$ (note that a is a K-character, since S is K-split) there is a K-homomorphism $\mathbb{G}_{\mathrm{m}} \to S_K$, which we will denote by ${}^s a^\vee$, such that

$$2\frac{(b,a)}{(a,a)} = -(\omega \circ b \circ {}^s a^\vee)(t)$$

holds for all $b \in R$ [Bor91, 8.6, 8.11]. Thus we see that the translation vector associated with ${}^s a^\vee(t) \in \underline{S}(K)$ is given by the co-root $a^\vee = \frac{2a}{(a,a)}$. This gives another way of understanding 3.2.8.(b), where we noted that the co-root lattice is the lattice of possible translations given by the affine Weyl group. In the following the K-homomorphisms $\mathbb{G}_{\mathrm{m}} \to \underline{S}_K$ will be called the K-**co-characters** of $\underline{S}$ and comprised in the set $\mathrm{X}_*(\underline{S})$. In the subsequent statements we will be concerned with those type 0 vertices in $\mathcal{Q}$ that are translations of φ coming from a co-character $\chi \in \mathrm{X}_*(\underline{S})$, i.e. the $\chi(t).\varphi$. Hence let x be such a point.

Recall that the link $\mathrm{lk}_{\mathcal{I}}(x)$ of x, which is given by the joinable but disjoint simplices with respect to x and which is in in bijection to the poset of simplices having x as a face, is a building (cf. 2.3.2). Moreover it is known that the apartments in $\mathrm{lk}_{\mathcal{I}}(x)$ are given by $\mathrm{lk}_{\mathcal{A}'}(x)$, for every apartment $\mathcal{A}'$ of $\mathcal{I}$, and since the point x is special (because φ is), one has that for every root $a \in R$ there is an associated wall in $\mathrm{lk}_{\mathcal{I}}(x)$. Since the spherical building associated with $\underline{G}$ over k, has the same configuration of walls, their associated Weyl structures agree. That entails in particular, that the Weyl group ${}^vW := \mathrm{N}_{\underline{G}}(\underline{S})(k)/\mathrm{Z}_{\underline{G}}(\underline{S})(k)$ is the Weyl group of $\mathrm{lk}_{\mathcal{I}}(x)$. From the theory of Bruhat and Tits one can conclude even more strongly that the building $\mathrm{lk}_{\mathcal{I}}(x)$ is given as the spherical building associated to $\underline{G}$ over k (cf. [BT84, Prop. 5.1.32.(iv)]), i.e. $\mathcal{B}(\underline{G})$. We will add a few more details to that isomorphism below. Recall that the simplices in $\mathcal{B}(\underline{G})$ are given by the k-parabolic subgroups of $\underline{G}$.

Note that since Γ_x fixes the vertex x, any simplex containing x as a face, is mapped to a simplex containing x as face, since the face relation is stable under the action of $\underline{G}(K)$. An important step in the proof of Soulé's theorem is the study of this action of Γ_x on the link of x. More precisely the aim of this section is the proof of the following lemma, which is central in what follows:

Lemma 4.3.1 ([Mar09, Lemma 2.8])**.** *Let x be a vertex in $\mathcal{Q}$, which comes from a co-character. Then one has $\Gamma_x.(\mathrm{lk}_{\mathcal{I}}(x) \cap \mathcal{Q}) = \mathrm{lk}_{\mathcal{I}}(x)$.*

Before we proceed to proof this lemma, some preliminary results are needed. We will always assume that a point x as in lemma 4.3.1 is given. We note that from [AB08, Prop. 10.31] it follows that there is a unique chamber $\varphi + \mathcal{C}$ in $\mathcal{Q}$, which has φ as a vertex.

Lemma 4.3.2 ([Mar09, Lemma 2.6])**.** *The chambers in $\mathrm{lk}_{\mathcal{I}}(x) \cap \mathcal{Q}$ are the $x + w\mathcal{C}$, with $w \in {}^vW$ satisfying $I_x \subseteq w.R^+$.*

Proof. As above, we set $I := I_x$. A chamber in the apartment $\mathrm{lk}_{\mathcal{A}}(x)$ in $\mathrm{lk}_{\mathcal{I}}(x)$, is given, since $\mathrm{lk}_{\mathcal{A}}(x)$ is a Coxeter complex with Weyl group ${}^vW$, as $x + w.\mathcal{C}$, for some $w \in {}^vW$. Fix an element $y \in \mathcal{C}$ and suppose that $x + w.\mathcal{C} \subseteq \mathcal{Q}$ holds, then by (4.2), one has

$$(a, x + w.y) \geq 0, \text{ for all } a \in B.$$

Since we chose the scalar product $(\cdot,\cdot)$ invariant under the action of the Weyl group, we get moreover:

$$(a, x) + (w^{-1}.a, y) \geq 0, \text{ for all } a \in B. \tag{4.18}$$

Thus for $a \in I$, i.e. $(a,x) = 0$, one has $(w^{-1}.a, y) \geq 0$, which implies that $w^{-1}.a$ is a positive root by 2.1.5.

Conversely take $w \in {}^vW$ satisfying $I \subseteq w.R^+$. We will prove (4.18) by a quick case distinction. First assume, that a is in I. Then one knows that $w^{-1}.a$ is a positive root and as such, since $\phi + \mathcal{C}$ is in $\mathcal{Q}$, we have $(w^{-1}.a, y) \geq 0$ (cf. (4.2)), for a point y in the open simplex $\mathcal{C}$. Since $(a,x) = 0$, this shows (4.18). If one assumes $a \in B \setminus I$, by (4.7), we have $(a,x) > 0$. Thus for $0 < \epsilon < 1$ small enough, one gets $(a, x + \epsilon y) \geq 0$, i.e. $x + \epsilon y \in \mathcal{Q}$ for some $y \in \mathcal{C}$, from which one concludes $x + w.\mathcal{C} \subseteq \mathcal{Q}$. □

(4.3.3). The above lemma provides us with a technical tool to further dissect the structure of $\mathrm{lk}_{\mathcal{I}}(x)$. However before we continue we will add some more details to the isomorphism of buildings

$$l_x : \mathrm{lk}_{\mathcal{I}}(x) \to \mathcal{B}(\underline{G}).$$

Recall that we assume x to be given as $\lambda(t).\varphi$ for some co-character $\lambda \in \mathrm{X}_*(\underline{S})$. Hence we see in particular that the isotropy groups of x and φ are related as $P_x = g_\lambda P_\varphi g_\lambda^{-1}$, where we set $g_\lambda := \lambda(t)$. Furthermore we already know by example 4.2.1 that the fixer of $\tilde{\varphi}$ in $\underline{G}(\tilde{K})$ is given by $\underline{G}(\tilde{\mathcal{O}})$. From the descent it follows immediately that the fixer of φ in $\underline{G}(K)$ is given by $\underline{G}(\mathcal{O})$. With the theory of Bruhat and Tits one can show that the fixer of a simplex $F \in \mathrm{lk}_{\mathcal{I}}(x)$ gives rise to a subgroup $g_\lambda^{-1} P_F g_\lambda \subseteq \underline{G}(\mathcal{O})$, which can be identified with an element of $\mathcal{B}(\underline{G})$ [BT84, 5.1.32] after forgetting the $\mathcal{O}$-structure. We set the image $l_x(F)$ to be that element. Moreover one sees that the chamber $x + \mathcal{C}$ corresponds under l_x to the minimal k-parabolic subgroup $\underline{P} \times_K \mathrm{Spec}(k)$, which we will, by abuse of notation, also denote by $\underline{P}$. One also finds that the fundamental apartment $\mathrm{lk}_{\mathcal{A}}(x)$ corresponds to the apartment corresponding to $\underline{S}$ (cf. 3.1.52.(b)). From now on we will implicitly assume the buildings $\mathrm{lk}_{\mathcal{I}}(x)$ and $\mathcal{B}(\underline{G})$ to be identified via l_x.

The next statement asserts which groups act in such a way that $\mathrm{lk}_{\mathcal{I}}(x) \cap \mathcal{Q}$ already fills out the apartment $\mathrm{lk}_{\mathcal{A}}(x)$ or the building $\mathrm{lk}_{\mathcal{I}}(x)$. This is a brings us closer to proving 4.3.1.

Lemma 4.3.4 ([Mar09, Lemma 2.7]). *Let I be a subset of the simple roots B and let ${}^vW_I$ be the subgroup of ${}^vW$, that is generated by the reflections associated with the elements of I. We fix the notation C for the fundamental chamber of $\mathcal{B}(\underline{G})$ corresponding to the minimal k-parabolic subgroup $\underline{P}$. Denote by $\mathcal{A}_I$, the collection of the $w.C$ and its adherences, for $w \in {}^vW$ satisfying $I \subseteq w.R^+$. Then we have:*

$$ {}^vW_I.\mathcal{A}_I = \mathrm{lk}_{\mathcal{A}}(x); \tag{4.19}$$

$$ \underline{P}_I(k).\mathcal{A}_I = \mathrm{lk}_{\mathcal{I}}(x) = \mathcal{B}(G). \tag{4.20}$$

Proof. (4.19): The proof goes by induction on the number of elements in I. Suppose $I = \varnothing$, then $\mathcal{A}_I = \mathrm{lk}_{\mathcal{A}}(x)$, since every $w \in {}^vW$ fulfils $I \subseteq w.R^+$ and ${}^vW$ is transitive on the chambers of an apartment. For the inductive step suppose that a decomposition $I = I' \cup \{b\}$, with $b \notin I'$, is given. Let $w.C$ be an arbitrary chamber in $\mathrm{lk}_{\mathcal{A}}(x)$. We want to show that $w.C$ is equivalent under ${}^vW_I$ to a chamber in $\mathcal{A}_I$. By the inductive hypothesis, we may assume that there is a $w' \in {}^vW_{I'}$ such that $w'w.C \in \mathcal{A}_{I'}$. That means in particular that $I' \subseteq w'.w.R^+$ holds. Suppose the root b is in $w'.w.R^+$, then $I \subseteq w'.w.R^+$ and, since ${}^vW_{I'} \subseteq {}^vW_I$ holds, we are done in this case. Suppose the other case in which $-b \in w'.w.R^+$ holds. As b is in I, the associated reflection s_b is in ${}^vW_I$ and $s_b(b) = -b$ is true. This implies that $b \in s_b.w'.w.R^+$. For a root $b' \in I' \subseteq w'.w.R^+$, one obtains $s_b(b') \in w'.w.R^+$. This stems from the fact that the reflection s_b permutes the positive roots not proportional to b [Bou02, VI.§ 1.6. Cor.1] and the set $w'.w.R^+$ is a system of positive roots, as the Weyl group acts simply transitive on those 2.1.5.(a). This completes the proof of the first point, as now $b' \in s_b.w'.w.R^+$ holds.

(4.20): We will show that any chamber C' in $\mathrm{lk}_{\mathcal{I}}(x) \cong \mathcal{B}(\underline{G})$ is equivalent under $\underline{P}_I(k)$ to a chamber in $\mathcal{A}_I$. Suppose that C' corresponds to a k-parabolic subgroup $\underline{P}'$ of $\underline{G}$. Then by [Bor91, Prop. 20.7.(i)] $\underline{P}_I \cap \underline{P}'$ contains the centraliser of a maximal k-split k-torus. By [Bor91, Prop. 20.5] any two such are conjugated by a unique rational point of the unipotent radical $\underline{U}_I(k)$ of $\underline{P}_I$ (cf. [Bor91, 21.11]). Thus there is $u \in \underline{U}_I(k)$ such that $u\underline{S}u^{-1} \subseteq \underline{P}_I \cap \underline{P}'$, and in particular $\underline{S} \subseteq u^{-1}\underline{P}'u$. This implies that $u^{-1}\underline{P}'u$ corresponds to a chamber C'' in the fundamental apartment of $\mathcal{B}(\underline{G})$. By (4.19) we already know that there is $w \in {}^vW_I$, with $w.C'' \in \mathcal{A}_I$. Since we know that ${}^vW_I$ is the Weyl group $\mathrm{N}_{\underline{L}_I}(\underline{S})(k)/\mathrm{Z}_{\underline{L}_I}(\underline{S})(k)$ of $\underline{L}_I$ with respect to $\underline{S}$ (see 3.1.47 and 3.1.50), one deduces the existence of an element $n \in \mathrm{N}_{\underline{L}_I}(\underline{S})(k)$ such that $nu^{-1}\underline{P}'un^{-1}$ corresponds to a chamber, that lies in $\mathcal{A}_I$. As $\underline{G}(k)$ acts by conjugation on the simplices of the building $\mathcal{B}(\underline{G})$ and $\underline{L}_I \subseteq \underline{P}_I$, this concludes the proof. □

After the above, we are in the position to finish the proof of lemma 4.3.1 by utilising the above statements.

Proof of 4.3.1. We ought to prove the equation $\Gamma_x.(\mathrm{lk}_{\mathcal{I}}(x) \cap \mathcal{Q}) = \mathrm{lk}_{\mathcal{I}}(x)$. The inclusion $\subseteq$ is evident, thus we will demonstrate the converse direction. From our discussion of the isomorphism l_x together with lemma (4.19) we only need to prove that the image of Γ_x under the conjugation map

$$\begin{aligned} \gamma : P_x &\to G(\mathcal{O}) \\ p &\mapsto g_\lambda^{-1} p g_\lambda \end{aligned}$$

contains $\underline{P}_I(k)$, where the notation is as in (4.3.3). In order to proof this, we will use the decomposition of Γ_x as the semi-direct product in (4.11) from lemma 4.2.5. Set $I = I_x$ for convenience. Since for $a \in I$, $(a,x) = 0$ holds, we have with (4.17)

$$0 = (a,x) = -(\omega \circ a \circ \lambda)(t). \tag{4.21}$$

It is known that $(a \circ \lambda)(t)$ needs to be a $\mathbb{Z}$-power of t [Bor91, 8.11] and thus by (4.21) we can conclude that $\lambda(t) = g_\lambda \in \ker(a)$ holds. Moreover since $\mathbb{G}_{\mathrm{m}}$ is connected, $\lambda(\mathrm{e}) = \mathrm{e}$ and $\lambda(t) = g_\lambda$ holds, we even have

$$g_\lambda \in \left(\bigcap_{a\in I} \ker(a)\right)^0 = \underline{S}_I.$$

Since the Levi subgroup $\underline{L}_I$ is the centraliser of $\underline{S}_I$ we have thus shown that $\underline{L}_I(k)$ is in the image of $\gamma_{\restriction_{\Gamma_x}}$.

We know that $\underline{P}_I = \underline{U}_I \rtimes \underline{L}_I$ holds, so the proof is complete, if we show that $\underline{U}_I(k)$ is part of the image of $\gamma_{\restriction_{\Gamma_x}}$ as well, or equivalently, if $g_\lambda \underline{U}_I(k) g_\lambda^{-1} \subseteq \Gamma_x$ holds. In a first reduction step we claim that in order to proof this it suffices to show that $g_\lambda \underline{U}(k) g_\lambda^{-1} \subseteq \Gamma$. Since

$$\underline{U}(k) \subseteq \underline{G}(\mathcal{O}) = P_\phi$$

holds, we have $g_\lambda \underline{U}(k) g_\lambda^{-1} \subseteq P_x$, which together with the claim implies that $g_\lambda \underline{U}_I(k) g_\lambda^{-1} \subseteq \Gamma_x = P_x \cap \Gamma$ is true. The second reduction step we will use is that it suffices to show $g_\lambda \underline{U}(k) g_\lambda^{-1} \subseteq \Gamma$ over $\tilde{k}$, i.e. that we show $g_\lambda \underline{\tilde{U}}(\tilde{k}) g_\lambda^{-1} \subseteq \tilde{\Gamma}$. This follows from the fact, that the groups without tilde are defined via descent and that, as g_λ is the image of t of a co-character of $\underline{S}$, it is invariant under the action of the Galois group Σ. The reduction of the claim is implied by the fact that morphisms defined over k are Σ-equivariant [Bor91, AG.14.3] together with equality $t^\sigma = t$ for all $\sigma \in \Sigma$. Hence we will show the inclusion for every $\tilde{U}_a$, with $a \in B$. In order to that, we will use that the product map induces a bijection (cf. 3.1.42)

$$\prod_{\tilde{a}\in\tilde{R},\ j(\tilde{a})=a} \tilde{U}_{\tilde{a}} \cdot \prod_{\tilde{a}\in\tilde{R},\ j(\tilde{a})=2a} \tilde{U}_{\tilde{a}} \xrightarrow{\simeq} \tilde{U}_a,$$

which was already stated in the beginning of this chapter. If we use $\tilde{x}_{\tilde{a}}$ to denote the isomorphisms from $\mathbb{G}_{\mathrm{a}}(\tilde{K})$ to $\tilde{U}_{\tilde{a}}$ (cf. 3.1.32.(a)), we obtain the following for $\tilde{a} \in \eta((a))$ and $s \in \tilde{k}$

$$g_\lambda \tilde{x}_{\tilde{a}}(s) g_\lambda^{-1} = \begin{cases} \tilde{x}_{\tilde{a}}((a \circ \lambda)(t)s) & \text{if } j(\tilde{a}) = a, \\ \tilde{x}_{\tilde{a}}((2a \circ \lambda)(t)s) & \text{if } j(\tilde{a}) = 2a. \end{cases} \tag{4.22}$$

By using (4.17) and the already mentioned property that $(\epsilon a \circ \lambda)(t)$ is a power of t we see that $(\epsilon a \circ \lambda)(t) = t^{(\epsilon a, v_{g_\lambda})}$, with $\epsilon \in \{1,2\}$, holds and as $\varphi + v_{g_\lambda} = x \in \mathcal{Q} \subseteq \tilde{\mathcal{Q}}$, we have with (4.2) the inequality $(\epsilon a, v_{g_\lambda}) \geq 0$ for all $a \in B$. Thence (4.22) shows $g_\lambda \tilde{x}_{\tilde{a}}(s) g_\lambda^{-1} \in \tilde{\Gamma}$, which concludes our proof of lemma 4.3.1. □

4.4 Completion of the Proof

This last section is devoted to the completion of the proof of Soulé's theorem 4.1.1. In particular it remains to be proven that every point in the building $\mathcal{I}$ is equivalent to a unique point in $\mathcal{Q}$ under the action of Γ. Thus there will be two parts, a first one, that is concerned with the uniqueness statement, and a second one, that deals with the existence assertion of the theorem.

4.4.1 Two distinct points in $\mathcal{Q}$ are not equivalent under Γ

Firstly we will show that any two points of $\mathcal{Q}$ are not equivalent under Γ, which implies the desired uniqueness property of (4.1.1). However this will follow, if one considers the case of $\underline{G}$ split first. Thus assume that two distinct points in $\tilde{\mathcal{Q}}$ are not equivalent, where $\tilde{\mathcal{Q}}$ denotes the analogue of $\mathcal{Q}$ in the building $\tilde{\mathcal{I}}$. Since the points in $\mathcal{I}$ are the points in $\tilde{\mathcal{I}}$, that are fixed by the action of the Galois group Σ, two distinct points in $\mathcal{Q}$, which lie thus also in $\tilde{\mathcal{Q}}$, will not be equivalent under the action of $\Gamma \subseteq \tilde{\Gamma}$. This shows, that it suffices to take care of the split case.

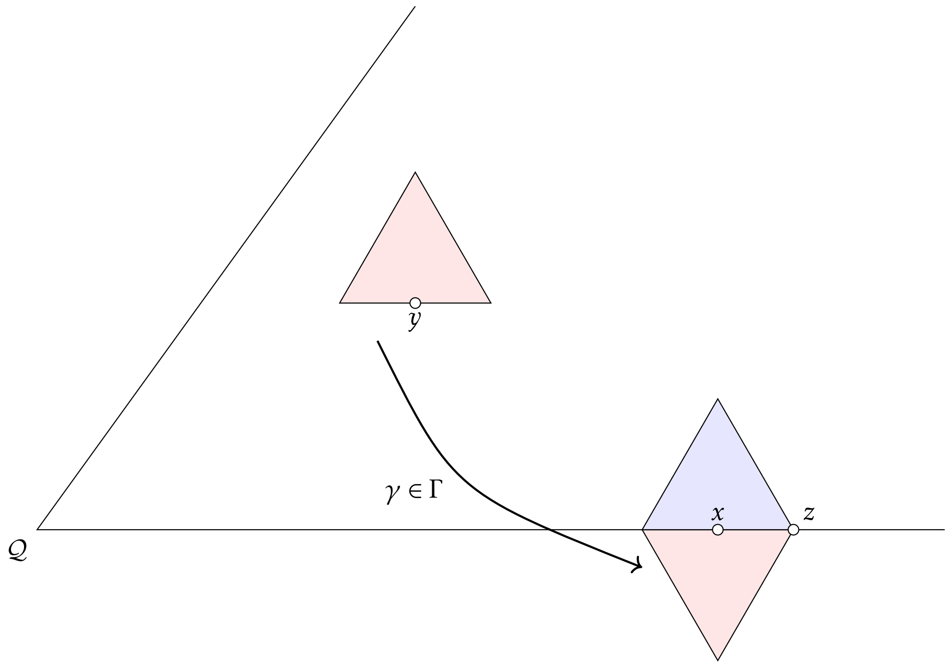

Figure 4.1: Two points in $\mathcal{Q}$ equivalent under Γ, which are not in open chambers.

The split case

In this paragraph, we will assume, as we already did once above, that $\underline{G}$ is split. The unique injection of buildings $\mathcal{I} \to \mathcal{I}'$, where $\mathcal{I}'$ is the euclidean building associated to some special linear group SL_{n+1}, multiplies the distances of points by a constant factor. We will follow the argument in [SW79, 1.3].

If two points x and y in $\mathcal{Q}$ are equivalent under Γ, then two chambers containing them are also equivalent under Γ. This follows immediately, if the points lie inside the open chambers. If they do not lie not in an open chamber, but either one is residing on a face of a chamber, then, for example transporting y onto x, transports the chamber containing y to a chamber, that shares a non-empty face with the chamber containing x in its adherence. Hence there is a vertex $z \in \mathcal{Q}$ lying on this common face. Then, by lemma 4.3.1, it follows that these two chambers are equivalent under Γ. This situation is described in the picture 4.1.

It is known that any two chambers in the apartment $\mathcal{A}$ are equivalent under the action of the Weyl group W associated with the affine building $\mathcal{I}$. In the affine case the Weyl group W decomposes as the semi-direct product $Q \rtimes {}^vW$ (cf. 3.2.8), where Q is the co-root lattice, which corresponds to translations, and a finite (spherical) Weyl group ${}^vW$, which is here given by the Weyl group associated to $\underline{G}$ and $\underline{S}$ over k. As $\underline{G}(k) \subseteq \Gamma$, Γ contains representatives of the whole linear Weyl group ${}^vW$. Thus we may assume in the following that there are two points in $\mathcal{Q}$, which are equivalent under the action of an element of Γ and by a translation. Recall that the embedding of $\underline{G}$ into SL_{n+1} was in such a way that the maximal torus $\underline{T}$, whose action gives rise to the translations, is mapped to the diagonal matrices in SL_{n+1}. Hence we can transfer this situation to the case of SL_{n+1}, where explicit formulas can be used.

Assume that $\gamma \in \mathrm{SL}_{n+1}(k[t])$ and τ is a diagonal matrix in $\mathrm{SL}_{n+1}(K)$ such that $\tau^{-1}x = \gamma x = y$. Hence we have $\tau\gamma \in \hat{P}_x$, which implies by (4.4) that

$$\omega(\tau_{ii}\gamma_{ij}) \geq x_i - x_j, \quad \forall\ 1 \leq i, j \leq n+1. \tag{4.23}$$

In the following we will also make use of the following formula, taken from [BT72, 10.2.5.(ii)], which follows from the fact that $\tau y = x$ and is related to (4.17):

$$x_j - x_i = y_j - y_i + \omega(\tau_{jj}) - \omega(\tau_{ii}). \tag{4.24}$$

Suppose that there is one index $i_0 \in \{1, \ldots, n+1\}$ such that $\omega(\tau_{i_0 i_0}) < 0$ and also fix the indices i and j such that $1 \leq j \leq i_0 \leq i \leq n+1$ and $\gamma_{ij} \neq 0$ hold. From (4.23) it then follows, as $\omega(\gamma_{ij}) \leq 0$, that

$$x_i - \omega(\tau_{ii}) \leq x_j.$$

Using this inequality, together with (4.24), where we replace j by i_0, we obtain:

$$0 \le y_{i_0} - y_i + x_j - x_{i_0} + \omega(\tau_{i_0 i_0}) \overset{\omega(\tau_{i_0 i_0})<0}{<} \left(y_{i_0} - y_i\right) + \left(x_j - x_{i_0}\right). \tag{4.25}$$

We note, that in the explicit formulas taken from [BT72, 10.2], the choice of a basis of the root system for SL_{n+1} implicitly has been made. By fixing the vector space $\mathbb{R}^{n+1}$ and its associated standard basis $(e_m)_{1\le m\le n+1}$ together with the standard scalar product on it, one identifies $\mathrm{X}^*(\mathrm{SL}_{n+1}) \otimes_{\mathbb{Z}} \mathbb{R}$ with the subspace $(e_1 + \cdots + e_{n+1})^{\perp}$, as already noted above. Then the basis for the root system of SL_{n+1} used by Bruhat and Tits is given by the elements $e_l - e_{l+1}$, with l running over $1, \ldots, n$. Furthermore the Weyl group invariant scalar product is chosen to be induced by the standard one.
Since x and y are elements of $\mathcal{Q}$, we then see that $y_{i_0} - y_i \le 0$ and $x_j - x_{i_0} \le 0$ holds (recall $1 \le j \le i_0 \le i \le n+1$). Thus we obtain a contradiction from (4.25), which leads us to deduce that for all $i = 1, \ldots, n+1$, it holds that $\omega(\tau_{ii}) \ge 0$. And thus we see that, since

$$0 = \omega(1) = (\omega \circ \det)(\tau) = \sum_{i=1}^{n} \omega(\tau_{ii}),$$

$\omega(\tau_{ii}) = 0$ for all $i = 1, \ldots,\ n+1$ is true. Hence $\tau \in \mathrm{SL}_{n+1}(k)$ and by the formula for translation vectors given by torus elements (4.17) we see that the translation given by τ is trivial. Hence we can conclude $x = y$.

4.4.2 Any point in $\mathcal{I}$ is equivalent to a point in $\mathcal{Q}$

We will now conclude the proof of theorem 4.1.1 in two final steps. Let us first consider points in $\mathcal{I}$, that are of the same type as ϕ. To that end, we define the subgroup M of $\underline{S}(K)$, which is generated by the images of $\lambda(t)$, where the λ run over the co-characters of $\underline{S}$, and the semigroup M_+ inside M, which is generated by those images $\lambda(t)$, where λ is a co-character satisfying $(a, v_{g_\lambda}) \ge 0$, for all $a \in B$. A result by Gille [Gil94, II.3.4.2], which is a generalization of a result due to Raghunathan [Rag94, Thm. 3.4], gives the following decomposition

$$\underline{G}(K) = \Gamma.M.\underline{G}(\mathcal{O})\ .$$

As we already used above, there is a representative for every element of the linear Weyl group in Γ. Since the linear Weyl group acts in such a way that every co-character $\lambda \in \mathrm{X}_*(\underline{S})$ is equivalent to a co-character λ' fulfilling $(a, v_{g_{\lambda'}}) \ge 0$ for all $a \in B$, ${}^vW.M_+ = M$ holds[5] and one can refine Raghunathan's theorem from above to:

$$\underline{G}(K) = \Gamma.M_+.\underline{G}(\mathcal{O})\ . \tag{4.26}$$

Since the building $\mathcal{I}$ is given by a Tits System 3.2.13, $\underline{G}(K)$ acts in a type preserving manner on $\mathcal{I}$ (cf. 2.4.5). Together with the fact that the subgroup $\underline{G}(\mathcal{O})$ is the stabiliser of φ in $\underline{G}(K)$, one can derive that there is a bijection of $\underline{G}(K)/\underline{G}(\mathcal{O})$ to the set of points that have the same type as φ. Using (4.26), we see that every point of type 0 is equivalent under Γ to a point in $M_+.\phi$, which are, by definition, points in $\mathcal{Q}$. Hence every point of the same type of φ in $\mathcal{I}$ is conjugate to a point in $\mathcal{Q}$.

For the general case, let there be a point $y \in \mathcal{I}$. Then there is a chamber $\mathcal{C}$ in $\mathcal{I}$, in whose adherence y lies in. From the theory of buildings, one knows that $\mathcal{C}$ contains a unique vertex x' of type 0. By the above x' is conjugated by an element $\gamma_2 \in \Gamma$ to a point x in $\mathcal{Q}$. Hence the chamber $\gamma_2.\mathcal{C}$ lies in the link of x. By lemma 4.3.1 one knows that by conjugating further with an element $\gamma_1 \in \Gamma_x$, the chamber $\gamma_1.\gamma_2.\mathcal{C}$ will lie in $\mathcal{Q}$. Thus y is conjugated via $\gamma_1.\gamma_2$ to a point in $\mathcal{Q}$, from which we conclude the proof of theorem 4.1.1.

[5] This follows from a dual version [Bou02, VI.§ 1.1. Prop. 2] of 2.1.5.(a) by taking into account, how the Weyl group ${}^vW$ operates on the characters and co-characters [Bor91, 21.2-21.6].

5
Applications

In this chapter we will apply the theorem proven above in a particular way, hence, after some preparations, we will pick up the assumptions and a share of the notation introduced in 4.1. We will use the fact that the action of $\underline{G}(k[T])$ on the affine building $\mathcal{I}$ admits a simplicial fundamental domain, to obtain an amalgamation decomposition of $\underline{G}(k[T])$ in terms of isotropy subgroups of points and cones. In order to facilitate this transition, we will interweave some background information about group actions on the geometric realizations of simplicial complexes with suitable properties in an intermittent manner. The following section follows [Sou73] closely and proofs therein will be rather sketchy.

5.1 Another Theorem due to Soulé

Let Σ be a set, denote by $\mathrm{F}(\Sigma)$ the free group, with base Σ, and let $\mathscr{R}$ be a normal subgroup in $\mathrm{F}(\Sigma)$. Define the group G to be the quotient $\mathrm{F}(\Sigma)/\mathscr{R}$ and note that any group G can be presented in such a way.
Suppose moreover that there is a not necessarily disjoint decomposition of Σ as $\bigcup_{i\in I}\Sigma_i$, with $\Sigma_i \subseteq \Sigma$, for i in some index set I. In addition we assume that $\mathscr{R}$ is generated as a normal subgroup by a family of elements r_α, for α running over the index set A, such that each r_α belongs to one of the free subgroups $\mathrm{F}(\Sigma_i)$ of $\mathrm{F}(\Sigma)$. Denote by Σ' the disjoint union of the Σ_i and let φ_i be the canonical injection from $\mathrm{F}(\Sigma_i)$ into $\mathrm{F}(\Sigma')$. Then one can prove the following:

Proposition 5.1.1. *The group G can be equivalently presented by using the elements of Σ' as generators and subjecting them to relations, given by:*

$$\begin{aligned} &\varphi_i(\sigma) = \varphi_j(\sigma) && \text{, for all } \sigma \in \Sigma_i \cap \Sigma_j, \text{ with } i,\ j \in I \text{ arbitrary;} \\ &\varphi_i(r_\alpha) = e && \text{, for every } r_\alpha \in \mathrm{F}(\Sigma_i). \end{aligned}$$

Proof. By using the universal property of free groups [Bou89, I.§ 7.5. Prop. 8] one can prove that there are mutually inverse maps going from a presentation with Σ to a presentation with Σ' and vice versa. □

Let X be a non-empty, Hausdorff topological space, on which a group G acts via homeomorphisms. Furthermore suppose that there is an open subset U inside X such that $X = \bigcup_{g\in G} g(U)$[1]. Denote by Σ the set of $g \in G$ such that $g(U)$ meets U and by x_g the image of g in $\mathrm{F}(\Sigma)$. Furthermore let A be the set of pairs (g,h) of elements in G such that $U \cap g(U) \cap gh(U)$ is non-empty. We quote the following theorem of Macbeath [Mac64, Thm. 1][2].

Theorem 5.1.2. *If U is path-connected and X simply-connected, then Σ together with the relations $x_g x_h = x_{gh}$, for all $(g,h) \in A$, give a presentation of G.*

We seek to apply this theorem in the following to a special situation. However first, we will reference an important concept, that will occur often in this chapter. For a family of subgroups $(H_i)_{i\in I}$ of a group H, we wish to consider the direct limit of the family

$$\left\{ H_i \cap H_j \;\middle|\; i,j \in I \right\},$$

[1] In [Mac64] U is then called a G-covering of X.

[2] In [Mac64] it is more generally assumed that G is a topological group and that its action on X is continuous. We have no need for that kind of generality and specialise the theorem to the case, where G is equipped with the discrete topology.

where we assume that the only transition maps are the inclusions $H_i \cap H_j \subseteq H_i$ and $H_i \cap H_j \subseteq H_j$, for $i, j \in I$. We call this limit the **sum of the $(H_i)_{i\in I}$ amalgamated over their intersections** (cf. [Ser80, pp. 91-92]).

Corollary 5.1.3. *We keep the notations and assumptions of theorem 5.1.2. Suppose moreover that there is a family of subgroups $(G_i)_{i\in I}$ such that*

$$\Sigma = \bigcup_{i\in I} G_i \quad \textit{and} \quad A = \bigcup_{i\in I} (G_i \times G_i).$$

Then G is the sum of the G_i amalgamated over their intersections.

Proof. Apply proposition 5.1.1 by considering Σ as the union of the sets $G_i \cap G_j$ for all pairs $(i,j) \in I^2$. One obtains for G a presentation which can be identified with the inductive limit of the groups $G_i \cap G_j$ relative to the canonical injections $G_i \cap G_j \to G_i$. (Also note that the roles of A in 5.1.1 and here agree). ☐

In the rest of this section let G_M denote the stabiliser of a point $M \in X$ in the group G.

Lemma 5.1.4. *Recall the notations and assumptions from 5.1.2. Let there be a set V' inside U, which has the following property: If two points M_1 and M_2 in V' fulfil $g(M_1) = M_2$, for some $g \in G$, then $M_1 = M_2$ holds. Denote by V the union of transforms of V' under G. Suppose moreover that for all $g_1, g_2 \in G$*

$$U \cap g_1(U) \cap g_2(U) \cap V = V' \cap g_1(V') \cap g_2(V')$$

holds and this set is non-empty, if and only if $U \cap g_1(U) \cap g_2(U)$ is non-empty. Then G is the sum of the G_M, $M \in V'$, amalgamated upon their intersections.

Proof. The lemma follows, if one applies corollary 5.1.3 to the family $(G_M)_{M\in V'}$. ☐

Recall the definition of the realization $|\mathcal{I}|$ of an abstract simplicial complex $\mathcal{I}$ with respect to its vertex set $\mathcal{V}$ as a suitable subspace of $[0,1]^{\mathcal{V}}$ (2.3.4). Suppose moreover that the group G acts on the simplicial complex $\mathcal{I}$ in such a way that G operates naturally on $|\mathcal{I}|$, i.e. $g \in G$ gives rise to the topological map

$$\begin{aligned} |\mathcal{I}| &\to |\mathcal{I}| \\ \lambda &\mapsto \lambda(g \cdot \ldots). \end{aligned}$$

Assume that the action of G on $\mathcal{I}$ admits as a simplicial fundamental domain a sub-complex $\mathcal{I}'$ of $\mathcal{I}$ and denote by $\mathcal{V}'$ the set of its vertices. The following theorem is due to Soulé [Sou73].

Theorem 5.1.5. *Suppose that the realisation of $\mathcal{I}$ is connected and simply-connected and the realization of $\mathcal{I}'$ is path-connected. Then G is the sum of the G_M, $M \in \mathcal{V}'$, amalgamated over their intersections.*

Proof. We define the open subset U in $|\mathcal{I}|$, formed by the $\lambda \in |\mathcal{I}|$ that fulfil $\lambda(M) < \frac{1}{\operatorname{supp}(\lambda)}$ for all $M \notin \mathcal{V}'$. We note that in U there are no vertices other than those already in $\mathcal{V}'$ and that U deformation retracts to $|\mathcal{I}'|$. Besides that we have $|\mathcal{I}| = \bigcup_{g\in G} g(U)$, since $\mathcal{I}'$ is a simplicial fundamental domain. Finally the theorem follows by the result of lemma 5.1.4, of which we are left to check that if $U \cap g_1(U) \cap g_2(U)$ is non-empty, with $g_1, g_2 \in G$, one finds a vertex in this set. Thus suppose there are $\lambda_0, \lambda_1, \lambda_2 \in U$ such that $\lambda_0 = g_1.\lambda_1 = g_2.\lambda_2$ holds. If λ_0 is a vertex, one is already done, hence assume the contrary. This implies that λ_0 and therefore λ_1 and λ_2 as well lie in open simplices spanned by the vertices in their respective support and these simplices are mapped to each other via g_1 and g_2. For $i = \{0,1,2\}$, since there is $\lambda_i \in U$, we find non-empty faces in these simplices given by the vertex sets

$$\emptyset \neq T_i := \{v \in \mathcal{V} \mid \lambda_i(v) \geq \operatorname{supp}(\lambda_i)\}.$$

Because of the relation of the λ_i and the special form of the action one obtains $T_0 = g_1.T_1 = g_2.T_2$. From the definition of U we derive that the vertices in the T_i all belong to the fundamental domain and thus this equation yields that they are fixed by g_1 and g_2. This completes the proof. ☐

The following corollary to the above statement was the application of Soulé's theorem that we have worked for in this section.

Corollary 5.1.6. *Recall the assumptions and the notation from 4.1. Denote by $\mathcal{V}$ the set of vertices in the fundamental domain $\mathcal{Q}$, then the group $G(k[T])$ can be presented as the sum of the $(\Gamma_x)_{x\in\mathcal{V}}$ amalgamated over their intersections.*

Proof. This follows from 5.1.5 and the main theorem 4.1.1 by noting that $\mathcal{Q}$ is a convex cone, hence path-connected, that the action of $\underline{G}(K)$ extends naturally to the realization 3.2.13.(b) and that the only non-simply-connected buildings are the spherical ones 2.3.5. Note that $|\mathcal{I}|$ is Hausdorff as its topology is the subspace topology of a Hausdorff space. ☐

Since our analysis of the stabilisers of points in $\mathcal{Q}$ has been fruitful, we may also note the following corollary.

Corollary 5.1.7 ([Mar09, Cor. 3.6]). $\underline{G}(k[T]) = \langle \underline{G}(k), \underline{U}(k[T])\rangle$.

Proof. From corollary 5.1.6 one sees that $\underline{G}(k[T])$ is generated by the stabilisers Γ_x, with x running over the vertices in $\mathcal{Q}$. From (4.11) of proposition 4.2.5 we see that Γ_x is generated by $\underline{U}_{I_x}(k[T]) \subseteq \underline{U}(k[T])$ and $\underline{L}_{I_x}(k) \subseteq \underline{G}(k)$. This was what we set out to show. ☐

5.2 A Quick Glance at Simplifications

In this section we seek to cut down the number of amalgamates, that sum to the group $\underline{G}(k[T])$, drastically. We will be rather quick, as many details needed for the proof are burried under a heavy load of building theory. Hence this chapter will contain a few quotes instead of proofs, since the goal is to give a brief overview of the simplification process and keep the thesis at an appropriate length.

In order to reduce the number of amalgamates, one is in need of a stronger version of 5.1.5, which we will quote here:

Theorem 5.2.1 ([Mar09, Prop. 3.2]). *Recall the hypotheses and assumptions of 5.1.5. Assume moreover that the realization $|\mathcal{I}|$ of $\mathcal{I}$ comes equipped with a metric d such that the following points are true:*

A) Any two points x and y in $|\mathcal{I}|$ are linked by a unique geodesic, i.e. a subset that is isometric to a closed interval of the real numbers [AB08, Def. 11.2].

B) For any $x \in |\mathcal{I}|$ there is an open neighbourhood D_x of x in $|\mathcal{I}|$ such that for any simplex F, regarded as a subset of $|\mathcal{I}|$, the following implication is true:

$$D_x \cap F \neq 0 \implies x \in \overline{F}.$$

C) H acts isometrically on $|\mathcal{I}|$.

D) For each simplex F, the stabiliser of F of the simplicial action coincides with the isotropy group (or pointwise stabiliser) of the geometric closure $\overline{F} \subseteq |\mathcal{I}|$.

The above assumptions have the following two consequences:

(i) The group G is the direct limit of the family $(G_M \cap G_N)_{M,N\in\mathcal{V}'}$, with transition maps the inclusions $G_M \cap G_N \to G_M$ and $G_M \cap G_N \to G_N$, whenever M and N belong to a common edge in $\mathcal{I}'$. (Note that this reduces the set of transition maps compared to 5.1.5).

(ii) The group G is the sum of the $(G_x)_{x\in|\mathcal{I}'|}$ amalgamated over their intersections.

One might remark at this point that only (i) contains a simplification, whereas in (ii) even more points are added to the amalgamation process. The purpose of (ii) shall be thought of as «filling in the gaps», as in (i) only vertices are considered and the points on a connecting edge are neglected. By mending this discrimination first we will be able, by use of the subsequent lemma, to part the whole fundamental domain in several rays, that give rise to stabiliser subgroups over which we will be left to amalgamate over. This is just to give an idea of how we intend to proceed.

Lemma 5.2.2 ([Mar09, Lemma 3.4]). *Let $(H_M)_{M\in\Lambda}$ be a family of groups and denote by H the sum of the $(H_M)_{M\in\Lambda}$ amalgamated over their intersections.*

(i) Suppose there is a directed subset $\Lambda' \subseteq \Lambda$, i.e. for every two $M, N \in \Lambda'$ there is $P \in \Lambda'$ such that $H_M \subseteq H_P$ and $H_N \subseteq H_P$ holds. Then the sum of the $(H_M)_{M\in\Lambda'}$ amalgamated over their intersections is canonically isomorphic to the subgroup

$$H' := \bigcup_{M\in\Lambda'} H_M$$

of H.

(ii) Suppose there is a partition $\Lambda = \bigsqcup_{j\in J} \Lambda_j$ of Λ into directed subsets Λ_j. For $j \in J$ denote by H_j the subgroup, that is the subgroup H' of (i) corresponding to Λ_j. Then H is the sum of the $\left(H_j\right)_{j\in J}$ amalgamated over their intersections.

Sketch. (i): This follows directly, after we note that H' is a subgroup, due to the fact that Λ' is a directed subset.

(ii): Let us denote by $\tilde{H}$ the sum of the $(H_j)_{j\in J}$ amalgamated over their intersections. Since the H_j are subgroups of H, their intersections agree in H and thus one obtains a well-defined map $\tilde{H} \to H$. For a converse map, note that for any $M \in \Lambda$, there is a unique $j_M \in J$ such that $M \in \Lambda_{j_M}$. Hence we get for every $M \in \Lambda$ a chain of inclusions:

$$H_M \subseteq H_{j_M} \subseteq \tilde{H}.$$

The inclusions $H_M \hookrightarrow \tilde{H}$ agree as well over their intersections and thence there is $H \to \tilde{H}$. After noting the H_M, with M running over $M \in \Lambda$, generate H and $\tilde{H}$, one easily sees that the so obtained maps are inverse to each other. □

Before we proceed with our simplification matters, we will, as we did before with the first theorem connecting group actions on geometric realizations to amalgamated sums, first note that the theorem at hand is applicable. This may be a bit of a dry business, since we will use quotes to state that the assumptions are in fact fulfilled. However it can point the reader to further reading material on the topic.

Corollary 5.2.3 ([Mar09, Cor. 3.7])**.** *$\underline{G}(k[T])$ is the sum of the family $(\Gamma_x)_{x\in|\mathcal{Q}|}$ amalgamated over their intersections.*

Proof. Since the assumptions of theorem 5.1.5 were already checked in corollary 5.1.6, we quickly run through the remaining assumptions of theorem 5.2.1. Since the geometric realizations of apartments in euclidean buildings are affine spaces, they can be equipped with a metric. By glueing this metric together, one obtains a well-defined metric for the realization of the whole affine building (cf. 3.2.16).

A): We already noted the existence of geodesics in 3.2.16.(b). Moreover it is true that the geometric realization enjoys the CAT(0)-property (cf. [AB08, Thm. 11.16]), i.e. it can be thought of as a non-positive curvature space (cf. p. 550 *ibid.*).

B): This is covered in [BT72, Lemme (2.5.11)]. One finds the radius of such an open ball in an apartment and then lifts the property to the whole building.

C): That $\underline{G}(K)$ acts isometrically was already part of proposition 3.2.16.

D): By the simply-connectedness of $\underline{G}$ the Bruhat-Tits theory yields that the stabilisers of simplices agree with the stabilisers of their individual points (cf. [BT84, Prop. 4.6.32]). The fact that their closure is also fixed by the same stabilisers, is a general fact of buildings (cf. [BT72, Prop. (2.4.13)]).

Now the corollary follows from (ii) of theorem 5.2.1. □

Next we will proceed to give a nicer presentation of $\underline{G}(k[T])$, by using lemma 5.2.2, through a decomposition of $\mathcal{Q}$ into directed sets. For a subset $I \subseteq B$, we define:

$$\mathcal{Q}_I := \{x \in \mathcal{Q} \mid I_x = I\}.$$

As we already noted above one has $I_{zx} = I_x$ for $z > 1$ and $x \in \mathcal{Q}$, from which we deduce that $\mathcal{Q}_I$ has cone form. In the spirit of 4.2.5, we also set:

$$\Gamma_I := \underline{U}_I(k[T]) \rtimes \underline{L}_I(k).$$

The following lemma basically asserts that an application of 5.2.2 will be justified.

Lemma 5.2.4. *Let I be a subset of B.*

(i) Then $\mathcal{Q}_I$ is a directed subset of $\mathcal{Q}$.

(ii) Γ_I is the sum of the $(\Gamma_x)_{x\in\mathcal{Q}_I}$ amalgamated over their intersections.

Proof. (i): Let x and y be points in $\mathcal{Q}_I$ and make the definition $z := x + y$. In order to show that $\mathcal{Q}_I$ is a directed subset, we will show that

$$\Gamma_x \subseteq \Gamma_z \quad \text{and} \quad \Gamma_y \subseteq \Gamma_z$$

hold and that we have $z \in \mathcal{Q}_I$. To see the later, we note that since x and y are in $\mathcal{Q}_I$, they fulfil $(a,x) = (a,y) = 0$, for all $a \in I$ and by (4.7) one obtains $(b,x) > 0$ and $(b,y) > 0$ for all $b \in B \setminus I$. Thus

$$(a,z) = 0 \iff a \in I$$

holds, which shows that $z \in \mathcal{Q}_I$. To understand the following, it might be necessary to reread the notation of 4.2.5. For $b \in R^+ \setminus [I]$, or equivalently $(a,y) > 0$, we have by what we already used that $(b,z) > (b,x)$ holds. This implies $m_z(b) \le m_x(b)$, from which one deduces in turn that

$$U_{b,m_x(b)}.U_{2b,m_x(2b)} \subseteq U_{b,m_z(b)}.U_{2b,m_z(2b)}$$

holds. Since $I_x = I_y$, we then have by (4.13) that $\Gamma_x \subseteq \Gamma_z$ is true. One concludes similarly for Γ_y.

(ii): Since we have already shown that $\mathcal{Q}_I$ is a directed subset, by point (i) of lemma 5.2.2, it suffices that we show

$$\bigcup_{x\in\mathcal{Q}_I} \Gamma_x = \Gamma_I.$$

The decomposition $\Gamma_x = (\Gamma_x \cap \underline{U}_{I_x}(K)) \rtimes \underline{L}_{I_x}(k)$ from (4.11) readily yields $\subseteq$. Hence let there be an element $g \in \Gamma_I$ and $x \in \mathcal{Q}_I$. By the decomposition (4.12), we obtain a $z \in [1,\infty[$ such that $g \in \Gamma_{zx}$ is true. Since $\mathcal{Q}_I$ is a cone, with x also zx is in $\mathcal{Q}_I$, and thus $\supseteq$ follows. We quickly note that for any subset $I \subsetneq B$ there is at least one point $x \in \mathcal{Q} \setminus \{\varphi\}$ such that $I_x = I$ holds, because the elements of B form a basis of the vector space $\mathrm{X}^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. □

Now we are able to puzzle the above lemmata together, to obtain a simplification in the amalgamation presentation of $\underline{G}(k[T])$.

Theorem 5.2.5. *$\underline{G}(k[T])$ is the sum of the $(\Gamma_I)_{I\subseteq B}$ amalgamated over their intersections.*

Proof. By lemma 5.2.4 (i) we can apply point (ii) of lemma 5.2.2 to the partition of $\mathcal{Q}$ given by:

$$\mathcal{Q} = \bigsqcup_{I\subseteq B} \mathcal{Q}_I.$$

Then 5.2.4 (ii) completes the proof of the theorem. □

6
Conclusion

In this thesis we provided a quick introduction in the abstract topics of roots, buildings and groups acting upon buildings and recorded an introductory piece of Bruhat and Tits' theory of constructing an affine building associated with a reductive (we restricted ourselves to the semi-simple case) linear algebraic group over a valuation field. This was preliminary work to give a presentation of a proof of Soulé's theorem, which states that the fundamental sector in the Bruhat-Tits building associated with a simply-connected, almost simple linear algebraic group G and the field of formal laurent series $k((t^{-1}))$ poses as a simplicial fundamental domain of the action of the group of points of the polynomial ring $k[t]$. By use of this theorem and equipped with a statement linking the structure of a fundamental domain to an amalgamation decomposition of the operating group, we recalled two generalizations of Nagao's theorem due to Soulé and Margaux.

In closing we want to give room to another proof of Soulé's theorem which is due to P. Abramenko [Abr96]. It uses the theory of twin buildings instead of buildings. A twin building consists of a pair of buildings (Δ_+, Δ_-) of the same Coxeter type subject to a compatibility relation, that mimics the fact that the fundamental chamber in a spherical building and hence any chamber has an *opposite* (recall for example figure 2.2), i.e. a chamber that is farthest away from it. By generalizing also the notion of a Tits system to the situation of twin buildings, one obtains a way of describing a strongly transitive group action on a twin building. Abramenko then proceeds to analyse this group operation to deduce in which case the stabilizer of a simplex in Δ_- admits a simplicial fundamental domain for its action on Δ_+. From this he could then derive the statement of Soulé's original theorem [SW79, Thm. 1]. The motivation for this abstract discussion was a strongly transitive action of $\underline{G}(k[t, t^{-1}])$ on the product $\Delta := \Delta_+ \times \Delta_-$, where Δ_+ is the Bruhat-Tits building associated with $\underline{G}$ and $k((t^{-1}))$ and Δ_- corresponds to $\underline{G}$ and $k((t))$. In that context Soulé's theorem provided a tool to analyse the stabilizer of a twin simplex $(a_+, a_-) \in \Delta$, which yielded notably that it is given as the product of the individual stabilisers of a_+ and a_-. This sparked a more careful analysis of twin Tits systems whose applications included the above mentioned, combinatorial proof of Soulé's theorem.

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Bibliography

[AB08] P. Abramenko and K. S. Brown. *Buildings*, vol. 248 of *Graduate Texts in Mathematics*. Springer New York, 1. edn., 2008.

[Abr94] P. Abramenko. *Reduktive Gruppen über lokalen Körpern und Bruhat-Tits-Gebäude*, vol. 47 of *Bayreuther Mathematische Schriften*, chap. 1, pp. 1–69. Universität Bayreuth, 1994.

[Abr96] ———. *Group actions on twin buildings*. Bull. Belg. Math. Soc. Simon Stevin, 3(4):pp. 391–406, 1996.
`https://projecteuclid.org:443/euclid.bbms/1105554412`

[AM16] M. F. Atiyah and I. G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Series in Mathematics. Westview Press, student economy edn., 2016.

[Bor91] A. Borel. *Linear Algebraic Groups*, vol. 126 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 2. edn., 1991.

[Bou89] N. Bourbaki. *Algebra I - Chapters 1-3*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg, 2. printing edn., 1989.

[Bou02] ———. *Lie Groups and Lie Algebras - Chapters 4-6*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg, 1. english edn., 2002.

[BT65] A. Borel and J. Tits. *Groupes réductifs*. Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 27(1):pp. 55–150, 1965.

[BT72] F. Bruhat and J. Tits. *Groupes réductifs sur un corps local: I. Données radicielles valuées*. Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 41(1):pp. 5–251, December 1972. ISSN 1618-1913. doi:10.1007/BF02715544.
`https://doi.org/10.1007/BF02715544`

[BT84] ———. *Groupes réductifs sur un corps local: II. Schémas en groupes. Existence d'une donnée radicelle valuée*. Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 60(1):pp. 5–184, December 1984. ISSN 1618-1913. doi:10.1007/BF02700560.
`https://doi.org/10.1007/BF02700560`

[Che58] C. Chevalley. *Classification des Groupes de Lie algébriques (two volumes)*. Secrétariat mathématique, Paris, 1958.

[Che82] N. G. Chebotarev. *The decomposition of groups acting on complexes with a fundamental domain*. Uspekhi Mat. Nauk, 37(226):pp. 191–192, 1982.

[DG70] M. M. R. Demazure and A. Grothendieck. *Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes III*. No. 153 in Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg New York, 1. edn., 1970.

[GD71] A. Grothendieck and J. A. E. Dieudonné. *Eléments de Géometrie Algébrique I*, vol. 166 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin, 1971.

[Gil94] P. Gille. *Torseurs sur la droite affine et R-équivalence*. Doctoral thesis, IUT Orsay, 1994. Retrieved at 06.05.2018.
`http://math.univ-lyon1.fr/homes-www/gille/prepublis/doct.pdf`

[GW10] U. Görtz and T. Wedhorn. *Algebraic Geometry*. Advanced Lectures in Mathematics. Vieweg+Teubner Verlag, 1. edn., 2010.

[Har77] R. Hartshorne. *Algebraic Geometry*, vol. 52 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 1977.

[Hum75] J. E. Humphreys. *Linear Algebraic Groups*, vol. 21 of *Graduate Texts in Mathematics*. Springer New York, 1. edn., 1975. doi:10.1007/978-1-4684-9443-3.
`https://doi.org/10.1007/978-1-4684-9443-3`

[Iha66] Y. Ihara. *On discrete subgroups of the two by two projective linear group over p-adic fields*. Journal of the Mathematical Society of Japan, 18(3):pp. 219–235, 1966. doi:10.2969/jmsj/01830219.
`https://doi.org/10.2969/jmsj/01830219`

[Lee13] J. M. Lee. *Introduction to Smooth Manifolds*, vol. 218 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 2. edn., 2013.

[Mac64] A. M. Macbeath. *Groups of Homeomorphisms of a Simply Connected Space*. Annals of Mathematics, 79(3):pp. 473–488, 1964. ISSN 0003486X. doi:10.2307/1970405.
`http://www.jstor.org/stable/1970405`

[Mar09] B. Margaux. *The structure of the group $G(k[t])$: Variations on a theorem of Soulé*. Algebra & Number Theory, 3(4):pp. 393–409, 2009.

[Mil17] J. S. Milne. *Algebraic Groups*, vol. 170 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1. edn., September 2017.

[Nag59] H. Nagao. *On* GL(2, K[x]). J. Inst. Polytech. Osaka City Univ. Ser. A, 10(2):pp. 117–121, 1959.
`https://projecteuclid.org:443/euclid.ojm/1353054965`

[Rag94] M. S. Raghunathan. *Principal bundles admitting a rational section*. Inventiones mathematicae, 116(1):pp. 409–423, 1994.

[Rot09] J. J. Rotman. *An Introduction to Homological Algebra*. Universitext. Springer-Verlag New York, 2. edn., 2009. doi:10.1007/b98977.
`https://doi.org/10.1007/b98977`

[Ser80] J.-P. Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 1. edn., 1980. doi:10.1007/978-3-642-61856-7.
`https://doi.org/10.1007/978-3-642-61856-7`

[Sou73] C. Soulé. *Groupes opérant sur un complexe simplicial avec domaine fondamental*. Comptes Rendus Acad. Sci. Paris Sér. AB, 276:pp. 607–609, 1973.

[Spr98] T. A. Springer. *Linear Algebraic Groups*. Modern Birkhäuser Classics. Birkhäuser Basel, 2. edn., 1998. doi:10.1007/978-0-8176-4840-4.
`https://doi.org/10.1007/978-0-8176-4840-4`

[SW79] C. Soulé and C. T. C. Wall. *Chevalley groups over polynomial rings*, pp. 359–368. London Mathematical Society Lecture Note Series. Cambridge University Press, 1979. doi:10.1017/CBO9781107325449.022.
`https://doi.org/10.1017/CBO9781107325449.022`

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