The No-ghost Theorem of the BRST Quantized Bosonic String

Fritiof Wallentin
LMU, Munich

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Abstract

This master's thesis proves the No-ghost theorem for the BRST quantized bosonic String. In the style of [10], the *bc*-ghost structure is identified with the semi-infinite cohomology of the Virasoro algebra. The BRST complex of the bosonic String is identified as a relative subcomplex of this full complex. This complex differs from the one from the heuristic treatments. It is here shown that the latter unavoidably possesses a physically undesirable inner product, while the former does not. The physical subspaces of the respective BRST complexes are however isomorphic as vector spaces, so that they contain the same physical states.

In this framework the No-ghost theorem is proved given that the vanishing theorem for the complex holds. The vanishing theorem is then proved to hold in the cases of non-zero space-time momentum.

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Contents

1 Introduction

In String theory the strings are viewed as fundamental objects on which no point along the String is different from any other. In light of this we would like to view strings as unparameterized objects propagating in some space-time (**the target space**), i.e. embeddings of $\mathbb{R} \times \mathbb{S}^1$ into the target space. However, such embeddings are technically involved, especially with regard to quantization. One instead finds it easier to describe parameterized strings, parametrizations from $\mathbb{R} \times \mathbb{S}^1$ to the target space, parameterizing the **world-sheet** of the propagating String, and instead force the physics of the strings to be invariant under reparametrizations. The physics is described by the Polyakov action and the reparametrizations are actions of the diffeomorphisms of the unit circle, $\mathrm{Diff}(\mathbb{S}^1)$. For the quantized bosonic String this symmetry is enforced on the state space of the String, $\mathcal{F}^{\mathcal{M}}$, via the representation of the Lie algebra, $\mathbb{W}$ (the **Witt algebra**), of $\mathrm{Diff}(\mathbb{S}^1)$. This Lie algebra is generated by elements $\{L_m\}_{m\in\mathbb{Z}}$ subjected to the relation

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{1}$$

Naively, one enforces this symmetry by defining the **physical states** to be those that satisfies

$$L^{\mathcal{M}}{}_m \psi = 0, \quad \forall m \in \mathbb{Z}, \tag{2}$$

where $L^{\mathcal{M}}{}_m$ denotes the representation of L_m on $\mathcal{F}^{\mathcal{M}}$. However, as is well known from any textbook on String theory, this does not work. When representing $\mathbb{W}$ on $\mathcal{F}^{\mathcal{M}}$ one gets an anomaly, the corresponding relation to (1) in the representation reads

$$[L^{\mathcal{M}}{}_m, L^{\mathcal{M}}{}_n] = (m-n)L^{\mathcal{M}}{}_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n}, \tag{3}$$

where D is the dimension of the target space. That is, we get a projective representation of $\mathbb{W}$. As a consequence, condition (2) would yield a trivial physical state space. This since for any $m \in \mathbb{Z}$,

$$\begin{aligned} 0 &= 2mL^{\mathcal{M}}{}_0\psi \\ &= [L^{\mathcal{M}}{}_m, L^{\mathcal{M}}{}_{-m}]\psi + \frac{D}{12}(m^3-m)\psi \\ &= \frac{D}{12}(m^3-m)\psi \end{aligned}$$

which yields $\psi = 0$. On top of this, the (naive) state space of the quantized String contains negative norm states (**ghost**) coming from the $(-+\cdots+)$-signature of the metric on the target space. These ghosts violate the probabilistic interpretation of the amplitudes, and hence need to be removed. In the String theory litterature [1] there are various ways of quantizing the String, most notably Covariant, Light-cone and BRST quantization; circumventing the problems arising from the quantum anomaly and the ghosts, all requiring $D = 26$ for consistency. The most modern of these approaches is the BRST method, which is the one we will focus on here.

The general picture of the BRST method is something like this [1]: We consider a physical system whose state space is $\mathcal{F}^{\mathcal{M}}$ in which we have symmetry described by a Lie algebra $\mathfrak{g}$[1]. Let $\{T_n\}_{n=1}^N$ be a basis of $\mathfrak{g}$ such that

$$[T_m, T_n] = \sum_k f_{mn}^k T_k, \tag{4}$$

the f_{mn}^k's being structure constants. The BRST method introduces extra degrees of freedom: $\{c^m\}_{m=1}^N$, the **ghosts**, and $\{b_m\}_{m=1}^N$, the **anti-ghosts**[2], transforming in the coadjoint representation of $\mathfrak{g}$. Collectively these fields are usually called the *bc*-ghosts. Moreover, they satisfy the following anti-commutation relations:

$$\begin{array}{ccccc} \{c^m, b_n\} & = & \delta_{m-n} & & \\ \{c^m, c^n\} & = & \{b_m, b_n\} & = & 0 \end{array}, \tag{5}$$

$(c^m)^* = c^m$ and $(b_m)^* = b_m$. The *bc*-ghost are such that they commute with operators on $\mathcal{F}^{\mathcal{M}}$. The *bc*-ghosts are indeed a version of the canonical anti-commutation relations (CAR) algebra, all but the choice of involution, and they can indeed be represented as creation/annihilation operators on a fermionic Fock space $\mathcal{F}^{\mathcal{G}}$[3]. However, this involution (as will be seen in section 4.3 for the case of String theory in particular) will amount in an indefinite hermitian form on $\mathcal{F}^{\mathcal{G}}$ with respect to which this involution corresponds to taking the adjoint.

One defines the **ghost number operator**,

$$N_{\mathcal{G}} = \sum_m c^m b_m, \tag{6}$$

counting the number of ghost excitations (the **ghost number**) of its eigenvectors, which indeed exhaust a basis of $\mathcal{F}^{\mathcal{G}}$, and hence defines a grading on $\mathcal{F}^{\mathcal{G}}$. We consider the whole space

$$C(\mathfrak{g}, \mathcal{F}^{\mathcal{M}}) := \mathcal{F}^{\mathcal{M}} \otimes \mathcal{F}^{\mathcal{G}},$$

[1] We will at here implicitly suppose $\dim(\mathfrak{g}) < \infty$. Even though we will later on consider an infinite dimensional Lie algebra, the Virasoro algebra.

[2] Not to be confused with 'ghost' in the sense of negative norm states.

[3] One that can be constructed independently from $\mathcal{F}^{\mathcal{M}}$ because the *bc*-ghosts commute with all operators on $\mathcal{F}^{\mathcal{M}}$

and on this introduce the operator[4], the **BRST operator**,

$$Q = \sum_m c^m T_m - \frac{1}{2} \sum_{m,n,k} f^k_{mn} c^m c^n b_k. \tag{7}$$

It is straight forward to show that this operator raises the ghost number by 1 (has degree 1) and is hermitian. By a direct calculation, utilizing the Jacobi identity and the structure of the Lie algebra, one can show that $Q^2 = 0$ (**nilpotency**).

Now let C^k denote the subspace of $C(\mathfrak{g}, \mathcal{F}^{\mathcal{M}})$ of vectors of ghost number k. $\psi \in C^k$ is defined as **BRST invariant**[5] if

$$Q\psi = 0. \tag{8}$$

Clearly, because of Q's nilpotency, any state $Q\psi$, where $\psi \in C(\mathfrak{g}, \mathcal{F}^{\mathcal{M}})$, is nilpotent. Such vectors are known as **BRST exact**.[6] Because,

$$\|Q\psi\|^2 = \left\langle \psi, Q^2\psi \right\rangle = 0,$$

the BRST exact states are not expected to contribute to our physical theory. Hence one defines en equivalence relation,

$$\psi \sim \psi' :\Longleftrightarrow \exists \phi : \psi = \psi' + Q\phi;$$

Hence two equivalent states yield the same amplitude, pointing at their physical equivalence. Since Q is of degree 1, for any $Q\psi$ of ghost number k, ψ must be of ghost number $k-1$. So we are really considering equivalence classes[7],

$$H^k(\mathfrak{g}, \mathcal{F}^{\mathcal{M}}) := \frac{\text{Ker } Q : C^k \to C^{k+1}}{\text{Im } Q : C^{k-1} \to C^k}.$$

Now, $\psi \in C^0$ if and only if ψ is annihilated by all the b_m's.[8] Hence every BRST invariant $\psi \in C^0$ is satisfies

$$Q\psi = \sum_m c^m T_m \psi = 0. \tag{9}$$

[4]For mathematicians, this operator is know as the Chevalley-Eilenberg differential, which computes the cohomology of the $\mathfrak{g}$, with values in a representation of $\mathfrak{g}$. (See [13])

[5]**Cocycles** for mathematicians.

[6]**Coboundaries** for mathematicians.

[7]**Cohomology classes**, for mathematicians.

[8]The non-trivial part of this statement follows from the following: Since $\mathcal{F}^{\mathcal{G}}$ is a Fock space built by the bc-ghosts, $N_{\mathcal{G}}\psi$ can only vanish if its terms vanishes identically. Hence

$$0 = c^m b_m \psi = (1 - b_m c^m)\psi,$$

which implies $\psi = b_m c^m \psi$. So that $b_m\psi = 0$ indeed.

Furthermore, for any $\psi \in C^0$,

$$\psi = \{c^m, b_m\}\psi = b_m c^m \psi,$$

for any m. So for any non-zero $\psi \in C^0$, $c^m\psi \neq 0$ for all m. Furthermore, $\mathcal{F}^{\mathcal{G}}$ being a Fock space generated by the bc-ghosts, means that all terms of (9) are linearly independent, and hence each must vanish separately. So (9) amounts to

$$T_m\psi = 0, \ \forall m,$$

which are exactly the invariant states we, as far as physics goes, are interested in. In here lies the allure of BRST quantization, setting $H^0(\mathfrak{g}, \mathcal{F}^{\mathcal{M}})$ as the physical Hilbert space.

There is however a potential caveat lurking here. $\mathcal{F}^{\mathcal{G}}$ is unavoidably[9] an indefinite inner product space[10]. For example:

- Let $d_m := c^m - b_m$. Then

  $$\{d_m, d_m^*\} = -2, \tag{10}$$

  where the left side clearly defines a positive operator on any Hilbert space in which * corresponds to the adjoint, while the right-hand side clearly is a negative operator.

- As seen before,

  $$\|Q\psi\|^2 = 0$$

  for all ψ. Hence a positive definite inner product gives $Q = 0$. For which the BRST method would be pointless.

Hence $C(\mathfrak{g}, \mathcal{F}^{\mathcal{M}})$ has an indefinite inner product. But we need $H^0(\mathfrak{g}, \mathcal{F}^{\mathcal{M}})$ to be a proper Hilbert space. So we need to make sure that the inner product on $H^0(\mathfrak{g}, \mathcal{F}^{\mathcal{M}})$ is positive definite. That this indeed is so is a result known as the **No-ghost theorem**.

(3) really defines a representation of the Virasoro algebra, $\mathbb{V}$. Hence, for the quantized String, we are dealing with the $\mathbb{V}$ as our symmetry algebra. $\mathbb{V}$ is of infinite dimension, hence we need to regulate the formally infinite sums in the construction of $N_{\mathcal{G}}$ and Q. Something which is done by a proceedure known as *normal ordering*. As a consequence of this regulating, as is well-known, $Q^2 = 0$ if and only if $D = 26$.

The structure of this thesis is as follows: In section 2 we review the quantized String, presenting its algebraic operator structure ('First quantization') and then

[9] Given the involutive properties of the bc-ghost fields. Properties which have to hold for unitarity of the S-matrix.

[10] Otherwise known as a **Krein space**.

constructing its conventional Fock space representation (Second quantization). The main result of this section is that the Fock representation is a Verma module of the Virasoro algebra for space-time momentums different from zero. In section 4 we present the general structure of a BRST complex, in the framework of which we will be able to formulate the No-ghost theorem into a form suitable for String theory. We will in this section also reconnect with the 'textbook' BRST treatment of the bosonic String, pointing out caveats in transforming its BRST complex into the form of the generic structure previously presented. In section 5 the mathematical structure of the BRST complex of the bosonic String is explicitly constructed, here in a way avoiding the caveats lurking in the textbook treatments and putting it into the form of the generic BRST complex. We then prove the No-ghost theorem for the bosonic String given that the vanishing theorem of the cohomology holds. Lastly we prove that the vanishing theorem indeed holds for the BRST complex of the bosonic String.

The methods and results presented in this thesis are largely influenced by [11] and [10]. The interested reader is therefore refereed to these resources for additional/complimentary material.

2 The Quantized Bosonic String

We establish the following convention: Let $\mathbb{R}^{1,D-1}$ denote the D-dimensional Minkowski space, on which we choose the conventions

$$\eta^{\mu\nu} = \mathrm{diag}(- + \cdots +).$$

We denote space-time indices by Greek letters, i.e. $\mu = 0, 1, \ldots, D-1$, and apply the Einstein summation convention.

2.1 'First' quantization

As is known from any textbook on String theory, on the field configurations X satisfying the equations of motion stemming from the Polyakov action one can perform an oscillator mode expansion, yielding, in the conformal gauge, the oscillators:

$$\{\alpha^\mu_m\}_{m\in\mathbb{Z}\setminus\{0\};\mu=0,\ldots,D-1}, \quad \{x^\mu\}_{\mu=0,\ldots,D-1} \quad \text{and} \quad \{p^\mu\}_{\mu=0,\ldots,D-1}, \tag{11}$$

satisfying the Poisson relations:

$$\begin{array}{rcl} \{\alpha^\mu_m, \alpha^\nu_n\}_{\text{P.B}} & = & im\eta^{\mu\nu}\delta_{m+n} \\ \{x^\mu, p^\nu\}_{\text{P.B}} & = & \eta^{\mu\nu} \\ \{x^\mu, \alpha^\nu_m\}_{\text{P.B}} & = & \{p^\mu, \alpha^\nu_m\}_{\text{P.B}} \quad = \quad 0 \end{array} . \tag{12}$$

Since The field configurations are required to be real-valued[11], we also require

$$\begin{array}{rcl} (\alpha^\mu_m)^* & = & \alpha^\mu_{-m} \\ (p^\mu)^* & = & p^\mu \\ (x^\mu)^* & = & x^\mu \end{array} , \tag{13}$$

where * denotes complex conjugation[12].

The canonical quantization is performed by promoting the oscillator modes (11) to operators[13], imposing the Poisson structure (12) to a commutator structure,

$$\{\cdot,\cdot\}_{\text{P.B}} \mapsto i[\cdot,\cdot].$$

and promoting the complex conjugation relations (13) to an involution. In our investigations the x^μ's will not appear, hence we ignore it. We choose to denote

[11] $\mathbb{R}^D$-valued, really.

[12] It will later also denote the adjoint.

[13] At this abstract level, in which no space on which the act has been specified, they should really just be viewed as part of some associative $*$-algebra.

$\alpha_0^\mu := p^\mu$, considering our oscillator algebra to be

$$\{\alpha_m^\mu\}_{m\in\mathbb{Z};\ \mu=0,\ldots,D-1},$$

so that the commutation relations can be more concisely written

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}. \tag{14}$$

2.2 Second quantization

For each of the oscillator algebras corresponding respectively to $\mu = 1, \ldots, D-1$, we construct the canonical Fock space representation, $\mathcal{F}_{p^\mu}$. That is, for each $p^\mu \in \mathbb{R}$, we define a **vacuum vector**, Ω_{p^μ}, such that

$$\begin{array}{rcll} \alpha_m^\mu \Omega_{p^\mu} & = & 0 & \forall m \in \mathbb{N}, \\ \alpha_0^\mu \Omega_{p^\mu} & = & p^\mu \Omega_{p^\mu}. & \end{array}$$

For each μ we hence have a Fock space,

$$\mathcal{F}_{p^\mu} := \text{span}\left\{A^\mu_{k_1\cdots k_n}\Omega_{p^\mu} | k_i \in \mathbb{N}\cup\{0\},\ \forall i = 1,\ldots,n\right\},$$

where

$$A^\mu_{k_1\cdots k_n}\Omega_{p^\mu} := (\alpha^\mu_{-1})^{k_1}\cdots(\alpha^\mu_{-n})^{k_n}\Omega_{p^\mu}. \tag{15}$$

These monomials (15) constitute a basis for $\mathcal{F}_{p^\mu}$. By Proposition 2.2 in [12], $\mathcal{F}_{p^\mu}$ carries a unique positive-definite Hermitian form under which the oscillator algebra is unitarily represented.[14]

This Hermitian form is given by

$$\left\langle A^\mu_{k_1\cdots k_m}\Omega_{p^\mu}, A^\mu_{l_1\cdots l_n}\Omega_p\right\rangle := \delta_{m-n}\prod_{i=1}^n \delta_{k_i-l_i}k_i!. \tag{16}$$

For the $\mu = 0$ algebra however, constructing the Fock representation in this way does not give a contravariant representation. As is seen from the following calculation,

$$\left\langle \alpha^0_{-m}\Omega_{p^\mu}, \alpha^0_{-m}\Omega_{p^\mu}\right\rangle = \left\langle \Omega_{p^\mu}, [\alpha^0_m, \alpha^0_{-m}]\Omega_{p^\mu}\right\rangle = -1, \tag{17}$$

and hence cannot be set to 1.

However, considering instead the alternate involution on the $\mu = 0$-mode

[14] Let (π, V) be a representation of an involutive algebra $\mathcal{A}$, in which V is equipped with a non-degenerate hermitian form. The representation V is said to be **contravariant** if

$$\pi(A)^\dagger = \pi(A^*), \text{ for any } A \in \mathcal{A},$$

where † denotes the adjoint with respect to the hermitian form on V. If in addition, the hermitian form is positive-definite, then the representation is **unitary**.

oscillator algebra defined by

$$\alpha^{0}_{m}{}^{\circ} := -\alpha^{0}_{-m}, \tag{18}$$

for which we hence get the normal CCR algebra, and hence a contravariant representation. We can then implement the original involution * by: First defining the operator $\mathbb{O}$ through its anticommutation relations

$$\{\mathbb{O}, \alpha^0_m\} = 0 \quad \text{and} \quad \mathbb{O}\Omega_p = \Omega_p.$$

and linearly extending.[15] Clearly $\mathbb{O}$ is Hermitian[16] and squares to the unit. Secondly defining a hermitian form

$$\langle \cdot, \cdot \rangle_{\mathbb{O}} := \langle \cdot, \mathbb{O}(\cdot) \rangle ,$$

with respect to which it follows that the $\mu = 0$-mode oscillator algebra with its original involution is contravariantly represented,

$$\begin{aligned}
\left\langle v, \alpha^0_m w \right\rangle_{\mathbb{O}} &= \left\langle v, \mathbb{O}\alpha^0_m w \right\rangle \\
&= \left\langle v, -\alpha^0_m \mathbb{O} w \right\rangle \\
&= \left\langle \left(-\alpha^0_m\right)^{\circ} v, \mathbb{O} w \right\rangle \\
&= \left\langle \alpha^0_{-m} v, w \right\rangle_{\mathbb{O}} .
\end{aligned}$$

However , as (17) shows, this inner product $\langle \cdot, \cdot \rangle_{\mathbb{O}}$ is indefinite.

From this, we can now define the Fock space of the quantized bosonic String at $\mathbf{p} \in \mathbb{R}^D$, $\mathcal{F}^{\mathcal{M}}{}_{\mathbf{p}}$.

$$\mathcal{F}^{\mathcal{M}}{}_{\mathbf{p}} := \bigotimes_{\mu=0}^{D-1} \mathcal{F}_{p^\mu},$$

where p^μ denotes the μ:th component of $\mathbf{p}$[17], $\mathcal{F}_{p^\mu}$ is an abbreviation for $(\mathcal{F}_{p^\mu}, \langle \cdot, \cdot \rangle)$ for each $\mu = 1 \ldots, D-1$, and $\mathcal{F}_{p^0}$ is an abbreviation for $(\mathcal{F}_{p^0}, \langle \cdot, \cdot \rangle_{\mathbb{O}})$. Notice that $\mathcal{F}^{\mathcal{M}}{}_{\mathbf{p}}$ inherits the indefiniteness of its Hermitian form from $\mathcal{F}_{p^0}$. We will adopt the notation where $\{\alpha^\mu_m\}$ denotes the CCR operators of the $\mathcal{F}_{p^\mu}$ Fock space. We will also always denote the adjoint operation by *, it being clear from context which one we mean. It is straight forward to show that $\mathcal{F}^{\mathcal{M}}{}_{\mathbf{p}}$ indeed defines a covariant representation of the oscillator mode algebra of the quantized bosonic String. We abbreviate

$$\Omega_{\mathbf{p}} := \otimes_{\mu=0}^{D-1} \Omega_{p^\mu}.$$

[15] It is straight forward to check that this indeed defines $\mathbb{O}$ uniquely, since two such would necessarily be zero on all monomial vectors.

[16] Taking the *-adjoint gives back the same defining properties

[17] Interpreted as the D-momentum, as usual.

A general monomial vector in α_m^μ's of $\mathcal{F}^{\mathcal{M}}{}_{\mathbf{p}}$ is abbreviated

$$A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}\Omega_{\mathbf{p}} := \prod_{\mu=0}^{D-1}\prod_{n=1}^{N}\left(\alpha_{-n}^\mu\right)^{k_n^\mu}\Omega_{\mathbf{p}}. \tag{19}$$

For completeness, the resulting inner product on $\mathcal{F}^{\mathcal{M}}{}_{\mathbf{p}}$, derived from (16), is given by

$$\left\langle A_{\{k_1^\mu\}\cdots\{k_M^\mu\}}\Omega_p, A_{\{l_1^\nu\}\cdots\{l_N^\nu\}}\Omega_p\right\rangle := \delta_{M-N}\prod_{\mu,\nu=0}^{D-1}\prod_{i=1}^{M}\delta_{k_i^\mu - l_i^\nu}\eta^{\mu\nu}k_i^\mu!. \tag{20}$$

We notice that eventhough this hermitian form is indefinite, it is not degenerate. This since it comes from a non-degenerate inner product and since $\mathbb{O}$ is bijective.

We construct a grading, $\deg^\mu$, on each $\mathcal{F}_{p^\mu}$. We set $\deg^\mu\Omega_{p^\mu} \in \mathbb{R}$ and define

$$\deg^\mu A^\mu_{k_1\cdots k_N}\Omega_{\mathbf{p}} := \sum_{n=1}^{N} nk_n + \deg^\mu\Omega_{p^\mu}. \tag{21}$$

Let $\mathcal{F}_{p^\mu}{}^j$ denote the subspace of $\mathcal{F}_{p^\mu}$ consisting of vectors of $\deg = j$. On each subspace Let $p(i)$ denote the number of partitions of the (positive) integer i. Then

$$\dim\, \mathcal{F}_{p^\mu}{}^j = p\left(j - \deg^\mu\Omega_{p^\mu}\right). \tag{22}$$

$\mathcal{F}_{p^\mu}$ can be written as a decomposition of finite dimensional subspaces,

$$\mathcal{F}_{p^\mu} = \bigoplus_j \mathcal{F}_{p^\mu}{}^j.$$

Note furthermore that these subspaces are mutually orthogonal. We may extend this grading to all of $\mathcal{F}^{\mathcal{M}}{}_{\mathbf{p}} = \bigotimes_{\mu=0}^{D-1}\mathcal{F}_{p^\mu}$ by defining

$$\deg\ := \sum_{\mu=0}^{D-1}\deg^\mu, \tag{23}$$

and $\deg\,\Omega_{\mathbf{p}} = \sum_{\mu=0}^{D-1}\deg^\mu\Omega_{p^\mu}$. This amounts to

$$\deg\, A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}\Omega_p := \sum_{\mu=0}^{D-1}\sum_{n=1}^{N} nk_n^\mu + \deg\,\Omega_{\mathbf{p}}.$$

We will later on be interested in the **q-dimension**[18] of $\mathcal{F}^{\mathcal{M}}$, (formally)

[18]Which later will be connected to the character of the oscillator representation

defined as

$$\dim_q \mathcal{F}^{\mathcal{M}} := \sum_j \dim\, \mathcal{F}^{\mathcal{M}j} q^j.$$

From which we get

$$\begin{aligned}
\dim_q \mathcal{F}^{\mathcal{M}} &= \sum_{j_0,\dots,j_{D-1}} \dim\, \mathcal{F}_{p^\mu}{}^{j_0} \cdots \dim\, \mathcal{F}_{p^\mu}{}^{j_{D-1}} q^{j_0+\cdots+j_{D-1}} \\
&= \prod_{\mu=0}^{D-1} \sum_{j_\mu} \dim\, \mathcal{F}_{p^\mu}{}^{j_\mu} q^{j_\mu} \\
&= \prod_{\mu=0}^{D-1} \sum_{i\geq 0} \dim\, \mathcal{F}_{p^\mu}{}^{i+\deg^\mu \Omega_{p^\mu}} q^{i+\deg^\mu \Omega_{p^\mu}} \\
&= q^{\deg\, \Omega_{\mathbf{p}}} \prod_{\mu=0}^{D-1} \sum_{i\geq 0} \dim\, \mathcal{F}_{p^\mu}{}^{i+\deg^\mu \Omega_{p^\mu}} q^i.
\end{aligned}$$

In light of (22), we hence get

$$\begin{aligned}
\dim_q \mathcal{F}^{\mathcal{M}} &= q^{\deg\, \Omega_{\mathbf{p}}} \left(\sum_{i\geq 0} p(i) q^i \right)^D \\
&= q^{\deg\, \Omega_{\mathbf{p}}} \prod_{n\in\mathbb{N}} (1-q^n)^{-D},
\end{aligned} \tag{24}$$

by applying the generating function for $p(i)$.

For ease of notation we from now on simply denote $\Omega_{\mathbf{p}}$ by Ω_p.

Remark 2.2.1. It is well worth noticing that $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathbb{O}}$ both are indefinite, since

$$\|\alpha_0 \cdot \alpha_0 \Omega_p\|^2 = p^2,$$

which may take negative values, or even zero.

3 The Virasoro Algebra and Its Representations

We will here present the necessary prerequisite regarding the Virasoro algebra and some of its representations.

3.1 The Witt algebra

We begin by considering $\mathit{Vect}(\mathbb{S}^1)$, the Lie algebra of vector fields over $\mathbb{S}^1$. The elements of $\mathit{Vect}(\mathbb{S}^1)$ are of the form $f(\theta)\frac{d}{d\theta}$, where f is any real-valued periodic function on $\mathbb{S}^1$, i.e. $f(\theta+2\pi) = f(\theta)$. The Lie bracket of $\mathit{Vect}(\mathbb{S}^1)$ is

$$\left[f(\theta)\frac{d}{d\theta}, g(\theta)\frac{d}{d\theta}\right] = (fg' - f'g)\frac{d}{d\theta},$$

where $'$ denotes the derivative. A basis (over $\mathbb{R}$) is given by

$$\tfrac{d}{d\theta}, \qquad \sin(n\theta)\tfrac{d}{d\theta}, \qquad \cos(n\theta)\tfrac{d}{d\theta}, \quad \text{where} \quad n \in \mathbb{N}.$$

The Witt algebra, $\mathbb{W}$, is the complex Lie algebra which is the complex extension of $\mathit{Vect}(\mathbb{S}^1)$. Equivalently,

$$\mathbb{W} = \mathrm{span}_{\mathbb{C}}\left\{\frac{d}{d\theta}, \sin(n\theta)\frac{d}{d\theta}, \cos(n\theta)\frac{d}{d\theta}\right\}_{n\in\mathbb{N}}.$$

We may hence construct the basis consisting of vectors

$$l_n := i\exp(in\theta)\tfrac{d}{d\theta} = -z^{n+1}\tfrac{d}{dz} \quad \text{where} \quad n \in \mathbb{N},$$

and $z := \exp(i\theta)$. From which we get the following commutation relations:

$$[l_m, l_n] = (m-n)l_{m+n}, \tag{25}$$

as a simple calculation shows.

The relevance of $\mathit{Vect}(\mathbb{S}^1)$ to String theory is summarized in the following remark.

Remark 3.1.1. The importance of $\mathit{Vect}(\mathbb{S}^1)$ for String theory comes from the fact that $\mathit{Vect}(\mathbb{S}^1)$ can be considered as the Lie algebra of the group of diffeomorphisms of the unit circle, $\mathit{Diff}(\mathbb{S}^1)$, i.e. repraparametrizations, under which we require our physical theory to be invariant. As mentioned in the introduction, the Strings are considered as elements $X \in \mathrm{Map}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{1,D-1})$[a] and their physics is described by the Polyakov action [1], which we consequently require to be invariant under the action of $\mathit{Diff}(\mathbb{S}^1)$.

[a] Referred to as the **configuration space**.

We will need an involution on $\mathbb{W}$. We define an anti-linear map * on $\mathbb{W}$ by defining

$$l_m^* := l_{-m} \tag{26}$$

and anti-linearly extending it to all of $\mathbb{W}$. A simple calculation shows that * defines an involution[19] on $\mathbb{W}$, i.e.

$$[l_m, l_n]^* = [l_{-n}^*, l_{-m}^*].$$

> **Remark 3.1.2.** This involution is the direct abstraction of the corresponding relation of the Virasoro generators known from classical bosonic String [2], which stem from requiring the String action to be real-valued.

3.2 The Virasoro algebra

The Virasoro algebra, $\mathbb{V}$, is the central extension[20] of the Witt algebra (25). That is, we add a central element c to $\mathbb{W}$

$$\mathbb{V} := \mathbb{W} \oplus \mathbb{C}c,$$

and modify the Lie algebraic structure as

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + c \cdot A(m,n), \\ [L_m, c] &= 0, \end{aligned}$$

for any $m, n \in \mathbb{Z}$, and where $A(m,n) : \mathbb{Z}^2 \to \mathbb{C}$ is some anti-symmetric function[21]. It turns out [12] that one can without loss of generality pick

$$A(m,n) = \frac{1}{12} m(m^2-1)\delta_{m+n},$$

which is also conventionally done.

3.2.1 Reparametrization invariance of the quantized string

Classically, the Witt algebra generators are represented through the oscillator modes as

$$L_m := \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_k \cdot \alpha_{m-k}. \tag{27}$$

These are in the quantized String canonically quantized, i.e. turned into operators on a Hilbert space. However, at this stage an ambiguity arises. One must

[19] A means of 'taking adjoints'
[20] The central extension is indeed unique [12].
[21] Known as the **quantum anomaly** in Sring theory.

specify an order of the α's in each term $\alpha_k \alpha_{m-k}$. This is done with respect to the ${\mathcal{F}^{\mathcal{M}}}_{\mathbf{p}}$-representation, moving the annihilation operators to the right, so that the seemingly infinite sum of (27) becomes finite on any monomial vector. Specifically, the L_m's are quantized to operators, ${L^{\mathcal{M}}}_m$, on ${\mathcal{F}^{\mathcal{M}}}_{\mathbf{p}}$ defined by

$$
{L^{\mathcal{M}}}_m := \frac{1}{2} \sum_{k \in \mathbb{Z}} : \alpha_k \cdot \alpha_{m-k} :, \tag{28}
$$

where $: \cdots :$ denotes the ordering prescription[22]

$$
: \alpha^{\mu}_m \alpha^{\nu}_n ::= \begin{cases} \alpha^{\mu}_m \alpha^{\nu}_n & \text{if} \quad n \geq m \\ \alpha^{\nu}_n \alpha^{\mu}_m & \quad \text{otherwise} \end{cases}
$$

It is a standard result that this set of operators $\{{L^{\mathcal{M}}}_m\}$ only yield a projective representation of The Witt algebra.

Theorem 3.1. The set of operators $\{{L^{\mathcal{M}}}_m\}_{m \in \mathbb{Z}}$ yields a representation of the Virasoro algebra such that $c^{\mathcal{M}} = D \cdot I$, i.e.

$$
[{L^{\mathcal{M}}}_m, {L^{\mathcal{M}}}_n] = (m-n){L^{\mathcal{M}}}_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n}. \tag{29}
$$

In the proof of this theorem we utilize the following lemma:

Lemma 3.2.1.

$$
[{L^{\mathcal{M}}}_m, \alpha^{\mu}_n] = -n\alpha^{\mu}_{m+n}.
$$

Proof. We have

$$
[\alpha_k \cdot \alpha_m, \alpha^{\mu}_n] = m\alpha^{\mu}_k \delta_{m+n} + k\alpha^{\mu}_m \delta_{k+n}. \tag{30}
$$

Since the order of the two α_i's differ by an operator proportional to the identity only, the normal ordering prescription drops out in the commutator,

$$
[: \alpha_k \cdot \alpha_{m-k} :, \alpha^{\mu}_n].
$$

[22]**normal ordering**

So by (30) we hence get[23]

$$
\begin{aligned}
[L^{\mathcal{M}}{}_m, \alpha_n^\mu] &= \frac{1}{2}\sum_k [\alpha_k \cdot \alpha_{m-k}, \alpha_n^\mu] \\
&= \frac{1}{2}\sum_k \big((m-k)\alpha_k^\mu \delta_{m-k+n} + k\alpha_{m-k}^\mu \delta_{k+n}\big) \\
&= -n\alpha_{m+n}^\mu,
\end{aligned}
$$

where the last step follows from simply performing the summation. □

Now we present the proof of Theorem 3.1.

Proof. By Lemma 3.2.1,

$$
[L^{\mathcal{M}}{}_m, \alpha_n \cdot \alpha_k] = -n\alpha_{m+n} \cdot \alpha_k - k\alpha_n \cdot \alpha_{m+k}. \tag{31}
$$

We may again ignore the normal ordering prescription. By (31) we hence get

$$
\begin{aligned}
&[L^{\mathcal{M}}{}_m, L^{\mathcal{M}}{}_n] \\
&= \frac{1}{2}\sum_k [L^{\mathcal{M}}{}_m, \alpha_k \cdot \alpha_{n-k}] \\
&= \frac{1}{2}\sum_k \left(-k\alpha_{m+k} \cdot \alpha_{n-k} - (n-k)\alpha_k \cdot \alpha_{m+n-k}\right) \\
&= -\frac{1}{2}\sum_k (k-m)\alpha_k \cdot \alpha_{m+n-k} - \frac{1}{2}\sum_k (n-k)\alpha_k \cdot \alpha_{m+n-k} \\
&= (m-n)\frac{1}{2}\sum_k \alpha_k \cdot \alpha_{m+n-k} \\
&= (m-n)\frac{1}{2}\sum_{k \le (m+n)/2} \alpha_k \cdot \alpha_{m+n-k} + (m-n)\frac{1}{2}\sum_{k > (m+n)/2} \alpha_k \cdot \alpha_{m+n-k} \\
&= (m-n)L^{\mathcal{M}}{}_{m+n} + Dm\delta_{m+n}\sum_{k>0} k.
\end{aligned}
$$

We emphasize that at every step in the above all sums are finite since we are implicitly acting on monomial vectors. Furthermore, the last sum is a scalar only non-zero when $m + n = 0$. We denote its value by $A(m, n)$, and proceed to calculate it. By this symmetry we may without loss of generality assume $m \ge 1$. For $m \ge 0$,

$$
L^{\mathcal{M}}{}_m \Omega_p = 0.
$$

[23]Where it is understood, here and in the proceeding, the this is acting on some vector in the domain of $L^{\mathcal{M}}{}_m$, so that the sum indeed is finite.

Hence,

$$\begin{aligned}
A(m,n) &= \delta_{m+n} \left\langle \Omega_p, \left([L^{\mathcal{M}}{}_m, L^{\mathcal{M}}{}_{-m}] - 2mL^{\mathcal{M}}{}_0\right) \Omega_p \right\rangle \\
&= \delta_{m+n} \left\langle \Omega_p, L^{\mathcal{M}}{}_m L^{\mathcal{M}}{}_{-m} \Omega_p \right\rangle \\
&= \frac{1}{2}\delta_{m+n} \sum_{-m\le k\le 0} \left\langle \Omega_p, L^{\mathcal{M}}{}_m \alpha_k \cdot \alpha_{m-k} \Omega_p \right\rangle \\
&= \frac{1}{2}\delta_{m+n} \sum_{-m\le k\le 0} \left\langle \Omega_p, \left(-k\alpha_{m+k} \cdot \alpha_{-m-k} + (m+k)\alpha_k \cdot \alpha_{-k}\right) \Omega_p \right\rangle \\
&= \frac{1}{2}\delta_{m+n} \sum_{-m\le k\le 0} \left\langle \Omega_p, -k\alpha_{m+k} \cdot \alpha_{-m-k} \Omega_p \right\rangle \\
&= \frac{1}{2}D\delta_{m+n} \sum_{-m\le k\le 0} \left\langle \Omega_p, \left(-k(m+k)\right) \Omega_p \right\rangle \\
&= \frac{1}{2}D\delta_{m+n} \sum_{k=1}^{m} (m-k)k,
\end{aligned}$$

where: in the third step we removed those terms in the sum of $L^{\mathcal{M}}{}_{-m}$ which vanish; in the fourth step, commuted $L^{\mathcal{M}}{}_m$ through to the right, applying relation (31), and in the sixth step commuted α_{m+k} through to the right. Now, since

$$\begin{aligned}
\sum_{k=1}^{m} k &= \frac{1}{2}m(m+1) \\
\sum_{k=1}^{m} k^2 &= \frac{1}{6}m(m+1)(2m+1),
\end{aligned} \tag{32}$$

we get

$$\begin{aligned}
A(m,n) &= \delta_{m+n}\frac{D}{2}\left(m\frac{1}{2}m(m+1) - \frac{1}{6}m(m+1)(2m+1)\right) \\
&= \delta_{m+n}\frac{D}{12}m\left(3m(m+1) - (m+1)(2m+1)\right) \\
&= \delta_{m+n}\frac{D}{12}m\left(m^2-1\right).
\end{aligned}$$

This finishes the proof. $\square$

We refer to the $\{L^{\mathcal{M}}{}_m\}$ representation of $\mathbb{V}$ as the **oscillator representation**.

We notice that

$$L^{\mathcal{M}}{}_m = \sum_{\mu=0}^{D-1} \eta_{\mu\mu} L^{\mathcal{M}\mu}{}_m,$$

where (with no implicit summation of μ)

$$L^{\mathcal{M}\mu}{}_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\alpha_k^\mu \alpha_{m-k}{}^\mu.$$

Where each set $\{L^{\mathcal{M}\mu}{}_m\}$ defines a representation on $\mathcal{F}_{p^\mu}$ with $c^\mu := \pi(c) = I$.

3.2.2 The oscillator representation

Notice that

$$L^{\mathcal{M}}{}_0 = \frac{1}{2}\alpha_0^2 + \sum_{k\in\mathbb{N}} \alpha_{-k}\cdot\alpha_k, \tag{33}$$

hence

$$L^{\mathcal{M}}{}_0 A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}\Omega_p = \left(\sum_{\mu=0}^{D-1}\sum_{l=1}^{N} l\cdot k_l^\mu + \frac{1}{2}p^2\right) A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}\Omega_p. \tag{34}$$

Consider the grading (23), setting deg $\Omega_p = \frac{1}{2}p^2$, we see that $L^{\mathcal{M}}{}_0$ 'measures' the degree deg, and hence the decomposition of $\mathcal{F}^{\mathcal{M}}$ into finite dimensional and orthogonal eigenspaces of $L^{\mathcal{M}}{}_0$,

$$\mathcal{F}^{\mathcal{M}} = \bigoplus_j \mathcal{F}^{\mathcal{M}}{}_j. \tag{35}$$

Now, assuming that $\mathcal{F}^{\mathcal{M}}$ contains a subrepresentation, U, of $\mathbb{V}$, it to can be decomposed into finite dimensional spaces,

$$U = \bigoplus_j U_j, \tag{36}$$

where $U_j := U \cap \mathcal{F}^{\mathcal{M}j}$. Let $U_j^\perp$ denote the orthogonal complement of U_j[24] and define

$$U^\perp = \bigoplus_j U_j^\perp. \tag{37}$$

Since $L^{\mathcal{M}}{}_m U \subseteq U$ for all m[25], we get

$$\begin{aligned}\left\langle U, L^{\mathcal{M}}{}_m U^\perp\right\rangle &= \left\langle L^{\mathcal{M}}{}_{-m} U, U^\perp\right\rangle \\ &= 0.\end{aligned}$$

Hence $U^\perp$ defines a subrepresentation as well. Continuing in this matter we

[24]Since $\mathbb{O}$ is bijective it does not matter whether the orthogonal complement is taken with respect to $\langle\cdot,\cdot\rangle$ or $\langle\cdot,\cdot\rangle_{\mathbb{O}}$.

[25]Since is by assumption U is a subrepresentation

eventually get an orthogonal decomposed of $\mathcal{F}^{\mathcal{M}}$ into irreducible representations of $\mathbb{V}$.

We construct the following subrepresentation

$$\mathcal{F}^{\mathcal{M}\prime} = \text{span}\ \left\{ \prod_{k=1}^{N} L^{\mathcal{M}\mu_k}_{-i_k} \Omega_p : 0 < i_1 < \cdots < i_N \right\} \cup \{\Omega_p\}.$$

This is clearly a **highest weight representation** of weight $(D, p^2/2)$, i.e.

$L^{\mathcal{M}}{}_m \Omega_p = 0$ for all $m \in \mathbb{N}$,

$L^{\mathcal{M}}{}_0 \Omega_p = \frac{p^2}{2} \Omega_p$ and

$c^{\mathcal{M}} \Omega_p = D\Omega_p$.

Supposing $\mathcal{F}^{\mathcal{M}\prime}$ to be reducible, then, as seen before, it can be decomposed into orthogonal subrepresentations $\mathcal{F}^{\mathcal{M}\prime} = U \oplus U^{\perp}$. Each subrepresentation can be decomposed into the eigenspaces of $L^{\mathcal{M}}{}_0$ as in (37). Since Ω_p spans the eigenspace of eigenvalue $p^2/2$, that eigenspace can only belong to one of the subrepresentations, U say.[26] But this means that $\mathcal{F}^{\mathcal{M}\prime} = U$, contradicting the assumption. Hence $\mathcal{F}^{\mathcal{M}\prime}$ is irrreducible. A direct calculation furthermore shows that the defining vectors of $\mathcal{F}^{\mathcal{M}\prime}$ to be mutually orthogonal, and hence indeed constitute a basis.

Remark 3.2.1. If $p = 0$, then $v := \alpha^{\mu}_{-1} \Omega_p$ satisfies:

$L^{\mathcal{M}}{}_m v = 0$ for all $m \in \mathbb{N}$,

$L^{\mathcal{M}}{}_0 v = v$ and

$c^{\mathcal{M}} v = Dv$.

Hence v generates highest weight representation which is orthogonal to $\mathcal{F}^{\mathcal{M}\prime}$[a], and hence is different from it.[b]

[a] In particular,

$$\begin{aligned}\left\langle v, L^{\mathcal{M}}{}_{-1}\Omega_p \right\rangle &= \left\langle L^{\mathcal{M}}{}_1 v, \Omega_p \right\rangle \\ &= \langle \alpha^{\mu} \Omega_p, \Omega_p \rangle = p^{\mu} \|\Omega_p\|^2.\end{aligned}$$

[b] It will in fact turn out that this is the reason that the vanishing theorem, which is essential in our proof of the no-ghost theorem, does not hold for $p = 0$.

The **graded $\mathbb{V}$-modules**, V, are those which permits a decomposition, a **grading**,

$$V = \bigoplus_{\lambda} V_{\lambda}$$

[26] Since the inner product is non-degenerate, Ω_p cannot be orthogonal to itself.

such that

$$\mathbb{V}_m : V_\lambda \to V_{\lambda+m}.$$

If, furthermore, V is such that every dim $V_\lambda < \infty$. We may consider the following quantities[27]:

Definition 3.1. ***q*-character** of V is defined as

$$\mathrm{Ch}_q V := \sum_\lambda q^\lambda \cdot \dim\, V_\lambda. \tag{38}$$

If in addition V has a hermitian form, $\langle\cdot,\cdot\rangle$, with respect to which the V_λ's are mutually orthogonal, then we define the ***q*-signature** of V as

$$\mathrm{Sgn}_q V := \sum_\lambda q^\lambda \cdot \mathrm{Sgn}\, V_\lambda, \tag{39}$$

where Sgn $V_{-\lambda}$ (the **signature**) is the difference between the number positive signs and negative signs in diagonalized form of the restriction of $\langle\cdot,\cdot\rangle$ to $V_{-\lambda}$.

The grading deg on $\mathcal{F}^{\mathcal{M}}$ is inherited by $\mathcal{F}^{\mathcal{M}\prime}$ by defining ${\mathcal{F}^{\mathcal{M}}}'_j := \mathcal{F}^{\mathcal{M}\prime} \cap {\mathcal{F}^{\mathcal{M}}}_j$. Hence

$$\mathrm{Ch}_q \mathcal{F}^{\mathcal{M}\prime} = q^{p^2/2} \sum_i q^i \dim\, {\mathcal{F}^{\mathcal{M}}}'_{i+p^2/2}. \tag{40}$$

By inspection of the definition of the q-dimension, (24), we see that

$$\mathcal{F}^{\mathcal{M}\prime} = \mathcal{F}^{\mathcal{M}} \quad \Longleftrightarrow \quad \mathrm{Ch}_q \mathcal{F}^{\mathcal{M}\prime} = \dim_q \mathcal{F}^{\mathcal{M}}. \tag{41}$$

By the orthogonality of eigenspaces of ${L^{\mathcal{M}}}_0$ in $\mathcal{F}^{\mathcal{M}\prime}$, we get

$$\mathrm{Ch}_q \mathcal{F}^{\mathcal{M}\prime} = \mathrm{Tr}_{\mathcal{F}^{\mathcal{M}\prime}}(q^{{L^{\mathcal{M}}}_0}) \tag{42}$$

We may orthogonally decompose $\mathcal{F}^{\mathcal{M}\prime}$ as

$$\mathcal{F}^{\mathcal{M}\prime} = \bigotimes_{\mu=0}^{D-1} \bigotimes_{n\in\mathbb{N}} S_n^\mu, \tag{43}$$

where each S_n^μ is the space built from ${L^{\mathcal{M}}}^\mu_{-n}$, i.e. is spanned by vectors $\left({L^{\mathcal{M}}}^\mu_{-n}\right)^k \Omega_p$[28]. Hence, by applying that the trace is multiplicative over ten-

[27] Which may be considered as formal quantities, or for $|q|$ small enough.

[28] Or, rather their normalized counterparts.

sor products,

$$\mathrm{Tr}_{\mathcal{F}^{\mathcal{M}\prime}}\left(q^{L^{\mathcal{M}}{}_0}\right) = q^{\frac{p^2}{2}} \prod_{\mu=0}^{D-1} \prod_{n\in\mathbb{N}} \mathrm{Tr}_{S_n^\mu}\left(q^{L^{\mathcal{M}}{}^\mu_0}\right) \tag{44}$$

$$= q^{p^2/2} \prod_{\mu=0}^{D-1} \prod_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} q^{nk} \tag{45}$$

$$= q^{p^2/2} \prod_{n\in\mathbb{N}} (1-q^n)^{-D}, \tag{46}$$

where the second step follows from noting that nk is the $L^{\mathcal{M}}{}^\mu_0$-eigenvalue of $\left(L^{\mathcal{M}}{}^\mu_{-n}\right)^k \Omega_p$, and last equality follows from applying the Maclaurin series for $(1-x)^{-1}$. So by (24) and statement (41) and we arrive at the following result:

Result 3.1. $\mathcal{F}^{\mathcal{M}}$ is a **Verma module**[a] of $\mathbb{V}$ if and only if $p \neq 0$.

[a] A highest weight representation in which the vectors $\prod_{k=1}^N L^{\mathcal{M}}{}^{\mu_k}_{-i_k}\Omega_p$ are linearly independent.

For later convenience we calculate the q-signature of $\mathcal{F}^{\mathcal{M}}$. We notice that

$$\mathrm{Sgn}\ \mathcal{F}^{\mathcal{M}\prime}{}_{i+p^2/2} = \mathrm{Tr}_{\mathcal{F}^{\mathcal{M}\prime}{}_{i+p^2/2}} \mathbb{O}, \tag{47}$$

I.e. taking the trace over $\mathbb{O}$ with respect to $\langle\cdot,\cdot\rangle$ yields the signature of $\langle\cdot,\cdot\rangle_{\mathbb{O}}$. Let $\mathcal{A}^\mu_m$ denote the Fock space generated by α^μ_{-m}. Then

$$\mathcal{F}^{\mathcal{M}} = \bigotimes_{\mu=0}^{D-1} \bigotimes_m \mathcal{A}^\mu_m.$$

Hence, by again applying that the trace is multiplicative over tensor products,

$$\begin{aligned}
\mathrm{Sgn}_q(\mathcal{F}^{\mathcal{M}}) &= \mathrm{Tr}_{\mathcal{F}^{\mathcal{M}}}\left(q^{L^{\mathcal{M}}{}_0}\mathbb{O}\right) \\
&= q^{\frac{p^2}{2}} \prod_{m\in\mathbb{N}} \mathrm{Tr}_{A^0_m}\left(\mathbb{O} q^{\alpha^0_{-m}\alpha_{m0}}\right) \prod_{\mu=1}^{D-1}\prod_{n\in\mathbb{N}} \mathrm{Tr}_{A^\mu_n}\left(q^{\alpha^\mu_{-m}\alpha_{m\mu}}\right) \\
&= q^{\frac{p^2}{2}} \prod_{m\in\mathbb{N}} \left(\sum_{k=0}^\infty (-1)^k q^{km}\right) \prod_{n\in\mathbb{N}} \left(\sum_{k=0}^\infty q^{kn}\right)^{1-D} \\
&= q^{\frac{p^2}{2}} \prod_{n\in\mathbb{N}} (1+q^n)^{-1}(1-q^n)^{1-D},
\end{aligned} \tag{48}$$

where the last equality follows from application of the identity

$$\begin{aligned}(1+q^n)^{-1} &= \frac{d}{dq}\log(1+q^n) \\ &= \frac{d}{dq}\log\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{q^{kn}}{k}\right) \\ &= \sum_{k=0}^{\infty}(-1)^k q^{kn},\end{aligned}$$

where such an expansion of the logarithm holds for sufficiently small q.

We summarize

Result 3.2.

$$\mathrm{Ch}_q\mathcal{F}^{\mathcal{M}} = q^{p^2/2}\prod_{n\in\mathbb{N}}(1-q^n)^{-D} \tag{49}$$

$$\mathrm{Sgn}_q(\mathcal{F}^{\mathcal{M}}) = q^{\frac{p^2}{2}}\prod_{n\in\mathbb{N}}(1+q^n)^{-1}(1-q^n)^{1-D}. \tag{50}$$

Remark 3.2.2. So why are we forced to consider the Virasoro algebra instead of the Witt algebra as the symmetry algebra of the quantum String? Why do we not disregard the oscillator representation and find another anomaly free representation?

The answer follows from the fact that the Witt algebra has no non-trivial unitary highest weight representations [16]. We do need our symmetry algebra to be a highest representation since L_0 is thought as corresponding to the energy of our system, and as such should be bounded from below for the physically relevant representations. We do furthermore need our representation to be unitary, since we need to preserve the involutive properties of our operators inherited from our classical theory, requiring the action to be real-valued. Hence, the Virasoro algebra is forced upon us as the symmetry algebra of the quantum String.

Additionally, a priori, symmetries need only be projective upon quantization. Since even if the symmetry group only is projectively represented, i.e.

$$U^*U\psi = e^{i\phi}\psi,$$

this poses no problem since $e^{i\phi}\psi$ is physically equivalent to ψ. So neither in this regard does the occurence of the Virasoro algebra a priori pose any problems.

4 The Generic BRST Complex

In this section we present the generic BRST complex, done with more mathematical care than in the introduction. The main goal is to (re)formulate the No-ghost theorem into a form more suitable for proving it. It is into this category of generic BRST complexes which we wish categorize the BRST complex of the bosonic String. We will in subsection 4.3 investigate the conventional BRST complex of the bosonic String, from which we will conclude that this form falls short on fitting the properties of the generic BRST complex. There is however a fix for this, which will be presented in section 5.

This section, all but for subsection 4.3, is highly inspired by chapter 5 in [11]. Subsection 4.3 is in turn highly inspired by the corresponding material presented in [1] and [3].

Note that this is no motivation of BRST quantization. The generic BRST complex merely represents the general mathematical properties of a such. A short motivation for its use in physics can be found in the introduction (section 1) and more thuroughly in the litterature, [1], [3] and [8] for example.

4.1 The generic BRST complex

The generic quantum **BRST complex** is a triplet $(\mathcal{F}, N_{\mathcal{G}}, Q)$, where $\mathcal{F}$ is a Fock space graded by the eigenvalues of the skew-self-adjoint operator $N_{\mathcal{G}}$ (the **the Ghost number operator**) and Q is a nilpotent self-adjoint operator (the **the BRST operator**). Specifically, the grading is

$$\mathcal{F} = \bigoplus_g \mathcal{F}_g,$$

where

$$\mathcal{F}_g = \{\psi \in \mathcal{F} : N_{\mathcal{G}}\psi = g\psi\},$$

i.e. the eigenspace of $N_{\mathcal{G}}$ corrsponding the the eigenvalue g. A generic eigenvalue of $N_{\mathcal{G}}$ (a **ghost number**) is denoted by g and the set of g's is assumed to be a discrete subset of $\mathbb{R}$. We denote the Fock vacuum by Ω and assume it to be BRST invariant, i.e. $Q\Omega = 0$.

Remark 4.1.1. As we remarked in the introduction, we unavoidably get an indefinite metric[a] from imposing these involution properties of the above operators. This is easily seen from the following calculation

$$\|Q\psi\|^2 = \langle \psi, Q^2\psi \rangle = 0.$$

So unless the metric is indefinite, Q must be trivial.

Furthermore, in light of $N_{\mathcal{G}}$ being skew-self-adjoint but with real-valued eigenvalues,

$$g^2 \langle \psi_h, \phi_g \rangle = \langle \psi_h, {N_{\mathcal{G}}}^2 \phi_g \rangle = \langle (-)N_{\mathcal{G}}\psi_h, N_{\mathcal{G}}\psi_g \rangle = -hg \langle \psi_h, \phi_g \rangle \tag{51}$$

for every $\phi_g \in \mathcal{F}_g$ and $\psi_h \in \mathcal{F}_h$. So the only non-zero inner products of eigenvectors of $N_{\mathcal{G}}$ are those with opposite eigenvalues. It is also because of the indefiniteness of the inner products that $N_{\mathcal{G}}$ may have non-pure imaginary eigenvalues.

The purpose of the No-ghost theorem is to prove that this indefinite inner product is positive on our space of physical states. As a consequence of (51), this can only be the case at at ghost number zero. But the physical vectors will indeed in the end be identified as such vectors.

[a]Hence $\mathcal{F}$ is strictly speaking not a Fock space.

We also suppose there is a further structure on $\mathcal{F}$, a further grading induced by a self-adjoint operator Λ on $\mathcal{F}$ that commutes with both $N_{\mathcal{G}}$ and Q.[29] The set of eigenvalues of Λ is by assumption discrete. We denote a generic eigenvalue of Λ by λ. By assumption we have the decomposition of $\mathcal{F}$:

$$\mathcal{F} = \bigoplus_{\lambda} \mathcal{F}(\lambda) \quad \text{with} \quad \dim \mathcal{F}(\lambda) < \infty \tag{52}$$

for any λ, where $\mathcal{F}(\lambda)$ denotes the eigenspace associated to the eigenvalue λ. This decomposition will allow us the work on finite subspaces on which calculations can be performed without worrying about divergences, but in the end still allow is the collate the results to the whole complex.

Remark 4.1.2. Applied to the bosonic String, the role of Λ will be played by the total lever number operator ${L^{\mathcal{T}}}_0$. ${L^{\mathcal{T}}}_0$ may be physically interpreted as the Hamiltonian of the BRST complex (matter and ghost), suggesting a more general interpretation of Λ as an energy operator.

We denote the restriction of Q to $\mathcal{F}_g$ by Q_g. It is assumed that Q raises the

[29]As is considered in [11], we may consider a set of such operators Λ. But for our purposes one such is enough.

ghost number by one, i.e.

$$Q_g : \mathcal{F}_g \to \mathcal{F}_{g+1}.$$

Since Q is nilpotent we must have $Q_{g+1}Q_g = 0$. We hence have the following sequence

$$\cdots \to \mathcal{F}_{g-1} \underset{Q_{g-1}}{\longrightarrow} \mathcal{F}_g \underset{Q_g}{\longrightarrow} \mathcal{F}_g \to \cdots .$$

Just as in the introduction, for every g, we define subspaces

$$\begin{aligned} \text{Ker } Q_g &= \{\psi \in \mathcal{F}_g | Q\psi = 0\} \\ \text{Im } Q_{g-1} &= \{Q\psi | \psi \in \mathcal{F}_{g-1}\} \end{aligned},$$

The elements of Ker Q are referred to as **BRST cocycles**[30] and those of Im Q are referred to as **BRST coboundaries**[31]. Clearly, since Q is nilpotent, Im $Q_{g-1} \subseteq$ Ker Q_g. Hence the quotient space

$$H^g(\mathcal{F}) := \frac{\text{Ker } Q_g}{\text{Im } Q_{g-1}}$$

makes sense for every g. The **BRST cohomology** is defined

$$H(\mathcal{F}) = \bigoplus_g H^g(\mathcal{F}).$$

The elements of $H(\mathcal{F})$ are referred to as **BRST cochains**. The **physical space** of the BRST complex, $\mathcal{H}_{\text{phys}}$ is defined as $H^0(\mathcal{F})$[32].

Since Λ commutes with both Q and $N_{\mathcal{G}}$, we can decompose the BRST complex into subcomplexes,

$$\cdots \to \mathcal{F}_{g-1}(\lambda) \underset{Q^\lambda_{g-1}}{\longrightarrow} \mathcal{F}_g(\lambda) \underset{Q^\lambda_g}{\longrightarrow} \mathcal{F}_g(\lambda) \to \cdots ,$$

where Q^λ_g denotes the restriction of Q onto $\mathcal{F}_g(\lambda)$. Hence,

$$H^g(\mathcal{F}) = \bigoplus_\lambda H^g_\lambda(\mathcal{F}),$$

where $H^g_\lambda(\mathcal{F}) = H^g(\mathcal{F}(\lambda))$. Since every such sub-cohomology is finite dimensional, we may take the convenient route of calculating on each such, and later collate this to the full Complex.

[30] or **BRST invariant**.

[31] or **BRST exact**.

[32] Motivated by the observation that in the heuristic picture this space consists of the invariant vectors, as in the introduction. Also, because of Remark 4.1.1, $H^0(\mathcal{F})$ is the only option for a No-ghost theorem to hold for this kind of BRST complex.

4.2 The No-ghost theorem of the generic BRST complex

We here present the reformulation of the No-ghost theorem in the context of the generic BRST complex. This reformulation is not only useful for proving the No-ghost theorem for the bosonic String but has been applied for the Superstring as well [11].

It is important to note that there is no proof of the No-ghost theorem for the generic BRST complex. It typically has to be proven on a case-by-case basis.

4.2.1 The Decomposition theorem

The decomposition theorem lets us identify the BRST cohomology as a certain subspace of Ker Q, eventhough this really is a subquotient, i.e. it lets us pick certain representatives of each BRST cochain. This identification is essential in reformulating the No ghost theorem.

As already mentioned, beacause of the involutive properties of our operators Q and $N_{\mathcal{G}}$, $\mathcal{F}$ is equipped with a indefinite hermitian form, $\langle \cdot, \cdot \rangle$. Q is set to be self-adjoint since in BRST quantization physical states differing by coboundaries,

$$\psi_{\text{phys}} = \phi_{\text{phys}} + Q\xi,$$

are supposed to be physically equivalent, i.e.

$$\|\psi_{\text{phys}}\|^2 = \|\phi_{\text{phys}} + Q\xi\|^2,$$

which holds for Q self-adjoint and nilpotent. However, the cohomology is an algebraic construction, independent of the choice of inner product. After all, in its construction all that is used is that Q is nilpotent, a property independent of the inner product. We are hence free to work with any inner product on $\mathcal{F}$ we want. It will prove convenient to work with a positive definite inner product on $\mathcal{F}$, so we construct a such.

We construct this inner product by first constructing an operator J. First pick a pseudo-orthonormal basis of $\mathcal{F}$. Then define J to be:

- the identity on the subspace, $\mathcal{F}_+$, spanned by the norm 1 basis vectors, and
- minus the identity on $\mathcal{F}_-$, the complementary subspace to $\mathcal{F}_+$.

We then define a positive definite hermitiain form as

$$\langle \cdot, \cdot \rangle_J := \langle \cdot, J(\cdot) \rangle \,, \tag{53}$$

This implies that J is self-adjoint and unitary.[33] However, such a decomposition of $\mathcal{F}$ is not unique, hence neither is J. But all such inner products (53) are related by a unitary transformation and the specific form does not matter for our considerations. In the end we are interested in applying this inner product for calculating traces, hence our intent is invariant under which specific such positive definite inner product we pick.

As shown in (51), the inner product of $N_{\mathcal{G}}$ eigenvectors was non-zero only for eigenvectors of opposite ghost number. Since the new inner product is positive definite, it must be that the norm of each non-zero eigenvector of $N_{\mathcal{G}}$ is positive. Hence

$$J|_{\mathcal{F}_g} : \mathcal{F}_g \to \mathcal{F}_{-g}. \tag{54}$$

Adjoints between the different inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_J$ need not agree. Since

$$\begin{aligned} \langle \cdot, A(\cdot) \rangle_J &= \langle \cdot, JA(\cdot) \rangle \\ &= \left\langle J^2 A^* J(\cdot), \cdot \right\rangle \\ &= \langle JA^* J(\cdot), J(\cdot) \rangle \\ &= \langle JA^* J(\cdot), \cdot \rangle_J , \end{aligned}$$

the notion of 'adjoint' of an operator A with respect to the different inner products agrees if and only if A commutes with J. In particular, Q is not J-self-adjoint, which follows from (54),

$$QJ|_{\mathcal{F}_g} : \mathcal{F}_g \to \mathcal{F}_{-g+1} \quad \text{and} \quad JQ|_{\mathcal{F}_g} : \mathcal{F}_g \to \mathcal{F}_{-g-1},$$

so they cannot commute. However $Q^\dagger$ (The J-adjoint of Q) has similar properties as Q. It is still nilpotent,

$$(Q^\dagger)^2 = (JQJ)^2 = JQ^2J = 0,$$

and since

$$\begin{aligned} Q^\dagger \mathcal{F}_g &= JQJ\mathcal{F}_g \\ &= JQ\mathcal{F}_{-g} \\ &= J\mathcal{F}_{-g+1} \\ &= \mathcal{F}_{g-1}, \end{aligned}$$

it has ghost number -1. We are hence completely justified in discussing the

[33] With respect to both inner products.

differential complex

$$\cdots \to \mathcal{F}_{g+1} \xrightarrow[Q^\dagger_{g+1}]{} \mathcal{F}_g \xrightarrow[Q^\dagger_g]{} \mathcal{F}_{g-1} \to \cdots .$$

We, similarly as for the Q case, define the cohmology in the $Q^\dagger$ case as

$$H(\mathcal{F};Q^\dagger) = \bigoplus_g H^g(\mathcal{F};Q^\dagger).$$

For now, we denote the BRST complex by $H(\mathcal{F};Q)$ to differentiate it from $H(\mathcal{F};Q^\dagger)$.

The respective complexes are fortunately related, as the following Lemma shows.

Lemma 4.2.1. *$H^g(\mathcal{F};Q)$ and $H^{-g}(\mathcal{F};Q^\dagger)$ are canonically isomorphic.*

Proof. Let $[\psi]$ and $[\psi]^\dagger$ denote the equivalence classes of ψ in $H(\mathcal{F};Q)$ respectively $H(\mathcal{F};Q^\dagger)$. Consider the map

$$\tilde{J} : [\psi] \mapsto [J\psi]^\dagger.$$

We need to verify that $\tilde{J}$ indeed is well-defined as a map from $H^g(\mathcal{F};Q)$ to $H^{-g}(\mathcal{F};Q^\dagger)$. For any $\phi \in \mathcal{F}_{g-1}$,

$$Q^\dagger J\phi = JQ\phi \quad \text{and} \quad JQ^\dagger\phi = QJ\phi.$$

Hence,

$$\begin{aligned}
[J(\psi + Q\phi)]^\dagger &= [J\psi + JQ\phi]^\dagger \\
&= [J\psi + Q^\dagger J\phi]^\dagger \\
&= [J\psi]^\dagger.
\end{aligned}$$

So that $\tilde{J}$ indeed is well-defined. Furthermore, $\tilde{J}$ clearly inherits the linearity from J and the fact that it reverses the ghost number. Hence $\tilde{J}$ defines a morphism

$$\tilde{J} : H^g(\mathcal{F};Q) \to H^{-g}(\mathcal{F};Q^\dagger).$$

We can construct the inverse of $\tilde{J}$ by reversing the roles of $H^g(\mathcal{F};Q)$ and $H^{-g}(\mathcal{F};Q^\dagger)$,

$$\tilde{J}^{-1} : [\psi]^\dagger \mapsto [J\psi],$$

which, by similar arguments, is a well-defined morphism,

$$\tilde{J}^{-1} : H^g(\mathcal{F};Q^\dagger) \to H^{-g}(\mathcal{F};Q),$$

which indeed is the inverse of $\tilde{J}$. $\square$

We move on to the **Decomposition theorem**, also referred to as the **Poincaré duality theorem**.

Theorem 4.1. $H^g(\mathcal{F};Q)$ and $H^{-g}(\mathcal{F};Q)$ are canonically isomorphic.

Proof. Since $\langle\cdot,\cdot\rangle_J$ is positive definite we may split $\mathcal{F}$ into an orthogonal direct sum of vector spaces,

$$\mathcal{F} = \text{Im } Q \oplus (\text{Im } Q)^{\perp}. \tag{55}$$

Since

$$\langle\psi, Q\phi\rangle_J = \left\langle Q^\dagger\psi, \phi\right\rangle_J, \tag{56}$$

Ker $Q^\dagger \subseteq (\text{Im } Q)^{\perp}$. (56) similarly also implies that $\left\langle Q^\dagger\psi, \phi\right\rangle_J = 0$ for all $\phi \in \mathcal{F}$ if $\psi \in (\text{Im } Q)^{\perp}$; so that $\psi \in \text{Ker } Q^\dagger$. Hence, $(\text{Im } Q)^{\perp} = \text{Ker } Q^\dagger$.

By (55) we may uniquely write any $\phi \in \text{Ker } Q$ as

$$\phi = Q\psi_1 + \psi_2,$$

where $\psi_1 \in \mathcal{F}$ and $\psi_2 \in \text{Ker } Q^\dagger$. Define the projection

$$h : \phi \in \text{Ker } Q \mapsto \psi_2.$$

Then h defines a linear surjection onto $\mathcal{H} := \text{Ker } Q \cap \text{Ker } Q^\dagger$. Defining

$$\mathcal{H}^g := \mathcal{H} \cap \mathcal{F}_g,$$

we get the decomposition

$$\mathcal{H} = \bigoplus_g \mathcal{H}^g.$$

We define the linear map

$$\tilde{h} : [\phi] \mapsto h\phi.$$

Due to (55), this map is clearly well-defined as a map $H^g(\mathcal{F};Q) \to \mathcal{H}^g$, for every g. It is furthermore injective, since

$$0 = \tilde{h}[\phi] = \psi_2 \iff \phi = Q\psi_1 \iff [\phi] = 0.$$

Hence

$$\tilde{h} : H^g(\mathcal{F};Q) \to \mathcal{H}^g \tag{57}$$

defines a canonical isomorphism .

Now notice that we could just as well have considered the above construction with the roles of $Q^\dagger$ and Q reversed. Hence getting the a canonical isomorphism

$$H^g(\mathcal{F}; Q^\dagger) \to \mathcal{H}^g. \tag{58}$$

These isomorphisms together with Lemma 4.2.1 proves the theorem. □

Remark 4.2.1. The isomorphism between the quotient spaces H^g and $\mathcal{H}^g$ allows us to consistently pick certain representatives of each equivalence class. The inner product on the cohomology is simply the trivially induced one from $\langle\cdot,\cdot\rangle$, and hence the it too is preserved under the canonical isomorphism. In particular, from before $\mathcal{H}_{\text{phys}} := H^0(\mathcal{F}; Q)$; Hence we may canonically identify

$$\mathcal{H}_{\text{phys}} \simeq \mathcal{H}^0, \tag{59}$$

preserving the physical structure given by the inner product. This will be very useful in the reformulation of the No-ghost theorem of the generic BRST complex.

We notice further:

Remark 4.2.2. Since

$$\left\langle Q^\dagger\psi, \phi\right\rangle_J = \langle\psi, Q\phi\rangle_J\,,$$

we get Im $Q^\dagger \perp$ Ker Q. Hence the proof provides us with a decomposition

$$\mathcal{F}_g = \text{Im }Q_{g-1} \oplus \text{Im }Q^\dagger_{g+1} \oplus \mathcal{H}^g. \tag{60}$$

This decomposition is indeed the source of the name of the theorem.

4.2.2 Formulating the No-ghost theorem

The No-ghost theorem states that $(\mathcal{H}_{\text{phys}}, \langle\cdot,\cdot\rangle)$ is a Hilbert space. In particular, that the inner product is positive definite.[34] From Theorem 4.1 we know that J maps $\mathcal{H}^g$ isomorphically to $\mathcal{H}^{-g}$, and in particular leaves $\mathcal{H}^0$ invariant. By construction, $J|_{\mathcal{F}_\pm} = \pm I$.[35] Hence $\mathcal{H}^0$ can be broken into eigenspaces $\mathcal{H}^0_\pm$ corresponding to eigenvalues ± 1 respectively. We have thus reached the following result:

Result 4.1. The No-ghost theorem holds if and only if

$$\mathcal{H}^0 = \mathcal{H}^0_+.$$

[34] The inner product is the one inherited from $\mathcal{F}$, i.e. (with abusive notation) $\langle[\cdot],[\cdot]\rangle := \langle\cdot,\cdot\rangle$.
[35] $\mathcal{F}_+$ denotes the positive definite subspace of $\mathcal{F}$ and $\mathcal{F}_-$ it complement.

At this stage we will take advantage of our operator Λ, whose eigenspaces $\mathcal{F}(\lambda)$ are finite dimensional, first restricting our investigations on these, and then collate the result to all of $\mathcal{F}$. But for notational ease, we will simply denote these eigenspaces by $\mathcal{F}$; only bringing back the distinction at the end.

Because of this, the following relation is justified:

$$\mathrm{Tr}_{\mathcal{H}^0} J = \dim\, \mathcal{H}^0_+ - \dim\, \mathcal{H}^0_-,$$

where we take the trace with respect to $\langle \cdot, \cdot \rangle_J$. By Result 4.1, the No-ghost theorem holds holds if and only if

$$\dim\, \mathcal{H}^0_- = 0.$$

Equivalently, the No-ghost theorem holds if and only if

$$\mathrm{Tr}_{\mathcal{H}^0} J = \dim\, \mathcal{H}^0. \tag{61}$$

Now, let

$$\bigcup_g \{\psi_i^{(g)}\}_{i=1}^{n_g} \tag{62}$$

be a J-orthonormal basis of $\mathcal{F}$ such that each $\{\psi_i^{(g)}\}_{i=1}^{n_g}$ is a basis of the corresponding eigenspace $\mathcal{F}_g$. That is, $n_g = \dim\, \mathcal{F}_g$. Then, for every $g \neq 0$,

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}_g} J &= \sum_{i=1}^{n_g} \left\langle \psi_i^{(g)}, J\psi_i^{(g)} \right\rangle_J \\
&= \sum_{i=1}^{n_g} \left\langle \psi_i^{(g)}, J^2\psi_i^{(g)} \right\rangle \\
&= \sum_{i=1}^{n_g} \left\langle \psi_i^{(g)}, \psi_i^{(g)} \right\rangle \\
&= 0,
\end{aligned}$$

since $\langle \cdot, \cdot \rangle$ is zero for vectors of equal non-zero ghost number[36]. Hence

$$\mathrm{Tr}_{\mathcal{F}} J = \mathrm{Tr}_{\mathcal{F}_0} J.$$

We notice that the left-hand side is the signature of the complex $\mathcal{F}$, so that we get

$$\mathrm{Sgn}\ \mathcal{F} = \mathrm{Tr}_{\mathcal{F}_0} J.$$

[36]See (51).

By decomposition (60),

$$\mathcal{F}_0 = \text{Im } Q_{-1} \oplus \text{Im } Q_1^\dagger \oplus \mathcal{H}^0. \tag{63}$$

Moreover, the terms in the trace of J on Im Q_{-1} and Im $Q_1^\dagger$ are respectively of the form

$$\begin{aligned} \left\langle Q^\dagger \phi, JQ^\dagger \phi \right\rangle_J = \left\langle \phi, QJQ^\dagger \phi \right\rangle_J = \left\langle \phi, Q^2 J\phi \right\rangle_J = 0, \\ \left\langle Q\phi, JQ\phi \right\rangle_J = \left\langle \phi, Q^\dagger JQ\phi \right\rangle_J = \left\langle \phi, JQ^2 \phi \right\rangle_J = 0. \end{aligned}$$

Hence both of them vanish. So we end up with

$$\text{Sgn } \mathcal{F} = \text{Tr}_{\mathcal{H}^0} J. \tag{64}$$

Hence, in light of (61), the No-ghost theorem holds if and only if

$$\text{Sgn } \mathcal{F} = \dim \mathcal{H}^0. \tag{65}$$

If we assume that the **vanishing theorem** for our BRST cohomology holds, i.e.

$$H^{g\neq 0}(\mathcal{F}; Q) = 0, \tag{66}$$

then we have the following (trivial) equality

$$\dim \mathcal{H}^0 = \sum_g (-1)^g \dim \mathcal{H}^g. \tag{67}$$

This relation can be written differently by noting the following: As seen from the proof of Theorem 4.1,

$$\text{Ker } Q_g = \mathcal{H}^g \oplus \text{Im } Q_{g-1},$$

Hence,

$$\dim \mathcal{H}^g = \dim \text{Ker } Q_g - \dim \text{Im } Q_{g-1}. \tag{68}$$

Now, Q_g is a linear map from $\mathcal{F}_g$ to $\mathcal{F}_{g+1}$, so by the Rank-nullity theorem from linear algebra,

$$\dim \text{Im } Q_g + \dim \text{Ker } Q_g = \dim \mathcal{F}_g. \tag{69}$$

This together with (68) then gives

$$\dim \mathcal{H}^g = \dim \mathcal{F}_g - \dim \text{Im } Q_g - \dim \text{Im } Q_{g-1}. \tag{70}$$

Which plugged into (67), performing the alternating sum for the two latter terms, yields

$$\dim\, \mathcal{H}^0 = \sum_g (-1)^g \dim\, \mathcal{F}_g.$$

But the right-hand side is just the character of the graded complex $\mathcal{F}$, so

$$\dim\, \mathcal{H}^0 = \text{Ch}\ \mathcal{F}.$$

By again considering the basis (62), we get

$$\begin{aligned}
\text{Tr}_{\mathcal{F}}(-1)^{N_{\mathcal{G}}} &= \sum_{g\in\mathbb{Z}} \sum_{i=1}^{N_g} \left\langle \psi_i^{(g)}, (-1)^{N_{\mathcal{G}}} \psi_i^{(g)} \right\rangle_J \\
&= \sum_{g\in\mathbb{Z}} (-1)^g N_g \\
&= \sum_{g\in\mathbb{Z}} (-1)^g \dim\, \mathcal{F}_g = \text{Ch}\ \mathcal{F},
\end{aligned}$$

where the last equality is the definition of the character. Hence,

$$\text{Ch}\ \mathcal{F} = \text{Tr}_{\mathcal{F}}(-1)^{N_{\mathcal{G}}}. \tag{71}$$

Again making it explicit that we are working on the finite dimensional eigenspaces $\mathcal{F}(\lambda)$, we have obtained: If the vanishing theorem holds for our BRST (sub)complex $\mathcal{F}(\lambda)$, then the No-ghost theorem holds if and only if

$$\text{Ch}\ \mathcal{F}(\lambda) = \text{Sgn}\ \mathcal{F}(\lambda).$$

Collating to all of $\mathcal{F}$, Result 4.1 may be reformulated as:

Result 4.2. If the vanishing theorem holds for our BRST complex, then the No-ghost theorem holds if and only if

$$\sum_\lambda q^\lambda \text{Ch}\ \mathcal{F}(\lambda) = \sum_\lambda q^\lambda \text{Sgn}\ \mathcal{F}(\lambda) \tag{72}$$

as a formal power series.

Remark 4.2.3. We make contact with the notation which will be applied in the coming sections. For the left-hand side of (72), we have:

$$\begin{aligned}\sum_\lambda q^\lambda \mathrm{Ch}\ \mathcal{F}(\lambda) &= \sum_\lambda q^\lambda \sum_g (-1)^g \dim\ \mathcal{F}_g(\lambda) \\ &= \sum_g (-1)^g \mathrm{Ch}_q \mathcal{F}_g \\ &= \mathrm{Tr}_{\mathcal{F}} (-1)^{N_{\mathcal{G}}} q^\Lambda,\end{aligned}$$

where the trace is with respect to $\langle \cdot, \cdot \rangle_J$ and we have made use of the fact that the eignespaces $\mathcal{F}_g(\lambda)$ are mutually orthogonal.

For the right-hand side of (72), we have:

$$\begin{aligned}\sum_\lambda q^\lambda \mathrm{Sgn}\ \mathcal{F}(\lambda) &= \mathrm{Sgn}_q \mathcal{F} \\ &= \mathrm{Tr}_{\mathcal{F}} J q^\Lambda.\end{aligned}$$

For future reference:

Remark 4.2.4. The decomposition (68) furthermore respects the Λ-grading, i.e.

$$\mathcal{F}_g(\lambda) = \mathrm{Im}\ Q_{g-1}(\lambda) \oplus \mathrm{Im}\ Q_{g-1}(\lambda) \oplus \mathcal{H}^g(\lambda).$$

Hence, again by performing the alternating sum,

$$\sum_g (-1)^g \dim\ \mathcal{F}_g(\lambda) = \sum_g (-1)^g \dim\ \mathcal{H}^g(\lambda).$$

So by applying our definition of the q-character[a] we hence get

$$\sum_g (-1)^g \mathrm{Ch}_q \mathcal{F}_g = \sum_g (-1)^g \mathrm{Ch}_q H^g(\mathcal{F}; Q). \tag{73}$$

A result referred to as the the **Euler-Poincaré principle**.

[a]Definition 3.1

4.3 Reconnecting with the bosonic String

We will here connect the formalism of the previous section with the typical textbook treatments of the BRST complex of the bosonic String, as found in [1] and [3]. It will become apparent that one cannot just straight forwardly translate it into the blue print of the generic BRST complex. However, with a few tweaks it is indeed translatable. The key lies in considering certain subcomplexes of the 'full' BRST complex known as 'relative subcomplexes'. We will hint towards these subcomplexes in this section, but their proper introduction will be in section 5. Because of this we will here pay no extra attention to the formally infinite sums appearing in the various operators introduced, instead dealing with those in section 5, where the BRST complex of the bosonic String is properly introduced.

4.3.1 The ghost Fock space

We begin by constructing the *bc*-ghost Fock space, $\mathcal{F}^{\mathcal{G}}$. The *bc*-ghost oscillator modes $\{b_m\}_{m\in\mathbb{Z}} \cup \{c_m\}_{m\in\mathbb{Z}}$ satisfies[37]:

$$\begin{aligned} \{b_m, c_n\} &= \delta_{m+n}, & b_m^* &= b_{-m}, \\ \{b_m, b_n\} &= \{c_m, c_n\} = 0, & c_n^* &= c_{-n} \end{aligned} \tag{74}$$

of which $\{b_m\}_{m\in\mathbb{N}\cup\{0\}} \cup \{c_n\}_{n\in\mathbb{N}}$ serves as annihilation operators and the remaining as creation operators. The **vacuum** thus defined is denote ω_0. We furthermore set $\langle \omega_0, c_0\omega_0\rangle = 1$, from which, by enforcing the involutions of (74), all inner products can be calculated.[38] We have thus generated the ***bc*-ghost Fock space**,

$$\mathcal{F}^{\mathcal{G}} = \mathrm{Span}_{\mathbb{C}} \left\{ G^B_C \omega_0 : B, C \in 1_\infty \right\}, \tag{75}$$

where

$$1_\infty := \left\{ (k_1, k_2, \cdots) : k_i \in \{0,1\} \ \forall i \text{ and } \sum_{i\in\mathbb{N}} k_i < \infty \right\}$$

and

$$G^B_C := \left(\prod_{m\in\mathbb{N}} b_{-m}^{B_m} \right) \left(\prod_{n\in\mathbb{N}} c_{-n+1}^{C_n} \right). \tag{76}$$

It can be shown that these vectors satisfy

$$\left\langle G^B_C \omega_0, G^{\widetilde{B}}_{\widetilde{C}} \omega_0 \right\rangle = \delta_{1-C_1-\widetilde{C}_1} (-1)^{N_B N_{\widetilde{B}}} \delta_{B-T\widetilde{C}} \delta_{\widetilde{B}-TC}, \tag{77}$$

where $TC = (C_2, C_3, \ldots)$, $\delta_B := \prod_{i\in\mathbb{N}} \delta_{B_i}$ and $N_B := \sum_{i\in\mathbb{N}} B_i$.[39]

We make the following important remark:

[37] Referred to as the *bc*-ghost algebra.

[38] A more general construction of such an inner product will be discussed in section 5.

[39] As mentioned in the introduction (see (10)), the inner product on $\mathcal{F}^{\mathcal{G}}$ is indefinite. (77) makes this fact even more apparent. So strictly speaking it is not a Fock space.

Remark 4.3.1. Given the involutions in (74), setting $\langle\omega_0, \omega_0\rangle = 1$ leads to a contradiction. This since

$$\begin{aligned}\langle\omega_0, \omega_0\rangle &= \langle\omega_0, \{b_0, c_0\}\omega_0\rangle \\ &= \langle\omega_0, b_0 c_0 \omega_0\rangle \\ &= \langle b_0\omega_0, c_0\omega_0\rangle = 0,\end{aligned}$$

where the last equality holds since b_0 annihilates the vacuum.

4.3.2 Ghost Virasoro representation

$\mathcal{F}^{\mathcal{G}}$ carries a representation of $\mathbb{V}$, $\{L^{\mathcal{G}}{}_m\}_{m\in\mathbb{Z}}$, where

$$L^{\mathcal{G}}{}_m = \sum_{n\in\mathbb{Z}} (n-m) : c_{-n} b_{m+n} : -\delta_m \tag{78}$$

and the normal ordering prescription is

$$: b_k c_l := \begin{cases} b_k c_l, & k < 0 \\ -c_l b_k, & k \geq 0 \end{cases} . \tag{79}$$

Clearly, the $L^{\mathcal{G}}{}_m$'s are such that $L^{\mathcal{G}}{}_m{}^* = L^{\mathcal{G}}{}_{-m}$.

The following theorem shows that it indeed is a representation.

Theorem 4.2.

$$[L^{\mathcal{G}}{}_m, L^{\mathcal{G}}{}_n] = (m-n) L^{\mathcal{G}}{}_{m+n} + \frac{-26}{12}(m^3 - m)\delta_{m+n}.$$

We will not prove this here, but instead leave the proof of this for the slightly different setting of section 5.

We also have the following (the proof of which we also leave for section 5):

Proposition 4.3.1.

$$[L^{\mathcal{G}}{}_m, b_n] = (m-n) b_{m+n} \tag{80}$$

$$[L^{\mathcal{G}}{}_m, c_n] = -(n+2m) c_{m+n}. \tag{81}$$

4.3.3 BRST complex

Definition 4.1. The BRST complex of the bosonic String is $(\mathcal{F}, N_{\mathcal{G}}, Q)$, where $\mathcal{F} = \mathcal{F}^{\mathcal{M}} \otimes \mathcal{F}^{\mathcal{G}}$,

$$N_{\mathcal{G}} = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{m \in \mathbb{N}} (c_{-m} b_m - b_{-m} c_m) \tag{82}$$

and

$$Q = \sum_{m \in \mathbb{Z}} L^{\mathcal{M}}{}_m c_{-m} - \frac{1}{2} \sum_{m,n \in \mathbb{Z}} (m - n) : b_{m+n} c_{-m} c_{-n} : - c_0. \tag{83}$$

The ghost numbering of the bosonic string differs from that of the generic BRST complex.

Proposition 4.3.2. The defining basis of $\mathcal{F}^{\mathcal{G}}$, (75), is an eigenbasis of $N_{\mathcal{G}}$. Specifically,

$$N_{\mathcal{G}} G^B_C \omega_0 = \left(N_C - N_B - \frac{1}{2} \right) G^B_C \omega_0, \tag{84}$$

where $N_K = \sum_{i \in \mathbb{N}} K_i$

Proof. A standard calculation shows that

$$[b_{-n} c_n, b_m] = \delta_{m+n} b_m,$$
$$[b_{-n} c_n, c_m] = -\delta_{m-n} c_m.$$

Which by insertion yields:

$$[N_{\mathcal{G}}, b_m] = \frac{1}{2} ([c_0 b_0, b_m] - [b_0 c_0, b_m]) + \sum_{n \in \mathbb{N}} ([c_{-n} b_n, b_m] - [b_{-n} c_n, b_m]) \tag{85}$$

$$= -\delta_m b_m + \sum_{n \in \mathbb{N}} (-\delta_{m-n} b_m - \delta_{m+n} b_m) \tag{86}$$

$$= -\sum_{n \in \mathbb{Z}} \delta_{m-n} b_m \tag{87}$$

$$= -b_m \tag{88}$$

and, through an analogous calculation,

$$[N_{\mathcal{G}}, c_m] = c_m. \tag{89}$$

From these we get:

$$\left[N_{\mathcal{G}}, \prod_{m \in \mathbb{N}} b_{-m}^{B_m} \right] = -N_B \prod_{m \in \mathbb{N}} b_{-m}^{B_m}$$

and

$$\left[N_{\mathcal{G}}, \prod_{n\in\mathbb{N}} c_{-n+1}^{C_n}\right] = N_C \prod_{n\in\mathbb{N}} c_{-n+1}^{C_n},$$

which can be verified via an induction argument. We insert these to get

$$\begin{aligned}[N_{\mathcal{G}}, G_C^B] &= \left[N_{\mathcal{G}}, \prod_{m\in\mathbb{N}} b_{-m}^{B_m}\right] \left(\prod_{n\in\mathbb{N}} c_{-n+1}^{C_n}\right) \\ &\quad + \left(\prod_{m\in\mathbb{N}} b_{-m}^{B_m}\right) \left[N_{\mathcal{G}}, \prod_{n\in\mathbb{N}} c_{-n+1}^{C_n}\right] \\ &= (N_C - N_B) G_C^B.\end{aligned}$$

We hence get

$$\begin{aligned}N_{\mathcal{G}} G_C^B \omega_0 &= [N_{\mathcal{G}}, G_C^B]\omega_0 + G_C^B N_{\mathcal{G}} \omega_0 \\ &= \left(N_C - N_B - \frac{1}{2}\right) G_C^B \omega_0.\end{aligned}$$

□

That is: the eigenvectors are those corresponding to monomials in the creation operators of $\mathcal{F}$, and the ghost number is the number of ghost creation operators minus the number of antighost creation operators minus 1/2.

$N_{\mathcal{G}}$ and Q are clearly skew- respectively self-adjoint, as they should. The following proposition even shows that Q has ghost number +1.

Proposition 4.3.3.

$$[N_{\mathcal{G}}, Q] = Q \tag{90}$$

Proof. Owing to relation (89) the only non-trivial part is dealing with the term

$$\frac{1}{2} \sum_{m,n\in\mathbb{Z}} (m-n) : b_{m+n} c_{-m} c_{-n} : \tag{91}$$

of Q. But since

$$\begin{aligned}[N_{\mathcal{G}}, b_{m+n} c_{-m} c_{-n}] &= [N_{\mathcal{G}}, b_{m+n}] c_{-m} c_{-n} \\ &\quad + b_{m+n} [N_{\mathcal{G}}, c_{-m}] c_{-n} + b_{m+n} c_{-m} [N_{\mathcal{G}}, c_{-n}] \\ &= b_{m+n} c_{-m} c_{-n}.\end{aligned}$$

and similarly,

$$[N_{\mathcal{G}}, c_{-m} c_{-n} b_{m+n}] = c_{-m} c_{-n} b_{m+n},$$

owing to relations (88-89); We get

$$[N_{\mathcal{G}}, :b_{m+n}c_{-m}c_{-n}:] = \begin{cases} [N_{\mathcal{G}}, b_{m+n}c_{-m}c_{-n}], & m+n<0 \\ [N_{\mathcal{G}}, c_{-m}c_{-n}b_{m+n}], & m+n \geq 0. \end{cases} =: b_{m+n}c_{-m}c_{-n}: .$$

Hence the proof is finished. □

However, what is special for the bosonic String, as is well known, is that Q is nilpotent if and only if the the dimension of the target space is equal to 26. We thus have to fix $D = 26$. We show that this indeed is the case in Section 5 (Corollary 5.1.1).

We define ${L^{\mathcal{T}}}_m := {L^{\mathcal{M}}}_m + {L^{\mathcal{G}}}_m$. These ${L^{\mathcal{T}}}_m$'s then define an anomaly free representation of the Witt algebra, i.e. c is represented as the zero operator. We have the following:

Proposition 4.3.4.

$$[{L^{\mathcal{T}}}_m, N_{\mathcal{G}}] = [{L^{\mathcal{T}}}_m, Q] = 0 \tag{92}$$

for all $m \in \mathbb{Z}$., where the latter holds if and only if $D = 26$.

The proof is left for section 5.[40]

Remark 4.3.2. Proposition 4.3.4 tells us that, every eigenspace of the ghost number operator and even every cohomology $H^g(Q)$ forms a subrepresentations of the Virasoro algebra.

${L^{\mathcal{T}}}_0$ is clearly self-adjoint, so by Proposition 4.3.4 all that is missing in order for it to qualify as a Λ-operator from the generic BRST complex[41] is that its eigenspaces are finite dimensional. In order to show this we notice that we have a mutual eigenbasis of ${L^{\mathcal{G}}}_0$ and ${L^{\mathcal{M}}}_0$ right at hand, and hence of ${L^{\mathcal{T}}}_0$ namely, (75) tensor (19),

$$A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}\Omega_p \otimes G_C^B \omega_0. \tag{93}$$

By Proposition 4.3.1, a direct (although tedious) calculation, shows that

$$[{L^{\mathcal{G}}}_0, G_C^B] = \sum_{n\in\mathbb{N}} n\left(C_{n+1} + B_n\right) G_C^B.$$

Hence,

$${L^{\mathcal{G}}}_0 G_C^B \omega_0 = \left(\sum_{n\in\mathbb{N}} n\left(C_{n+1} + B_n\right) - 1\right) G_C^B \omega_0 \tag{94}$$

[40] Proposition 5.2.1.
[41] See (52).

$L^{\mathcal{G}}{}_0$ counts the 'ghost level'. By Proposition 3.2.1, a direct (although also tedious) calculation, shows that

$$\begin{aligned} & L^{\mathcal{M}}{}_0 A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}\Omega_0 \\ = & \left(\sum_{\mu=0}^{D-1}\sum_{l=1}^{N} l\cdot k_l^\mu + \frac{1}{2}p^2\right) A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}\Omega_0. \end{aligned}$$

$L^{\mathcal{M}}{}_0$ counts the 'matter level'. For each fixed L and p^2,

$$L = \left(\sum_{l=1}^{N} l\cdot k_l + \frac{1}{2}p^2\right) + \left(\sum_{n\in\mathbb{N}} n\left(C_{n+1}+B_n\right) - 1\right)$$

only has a finite number of solutions in the $(N, \{k_l^\mu\}_{l=1}^N, B, C)$'s. Hence each eigenspace of $L^{\mathcal{T}}{}_0$ is finite dimensional. So $L^{\mathcal{T}}{}_0$ has the properties of a Λ-operator.

We furthermore need to define a J-operator of this complex. We define J as

$$\begin{aligned} J\omega_0 &= c_0\omega_0, & \{J, \alpha_m^0\} &= [J, \alpha_m^\mu] = 0, \\ Jc_m &= b_m J, & c_m J &= Jb_m, \end{aligned} \tag{95}$$

for $\mu \neq 0$ and $m \in \mathbb{Z}$, and then linearly extend it to all of $\mathcal{F}$.[42][43]

Theorem 4.3. The relations (95) defines J uniquelly as a self-adjoint unitary operator on $\mathcal{F}$. Moreover, the induced inner product $\langle\cdot,\cdot\rangle_J$ is positive definite. Consequently, the vectors of negative norm have J-eigenvalue -1.

Proof. For any operator J satisfying (95), a direct calculation yields

$$JA_{\{k_1^\mu\}\cdots\{k_N^\mu\}}G_C^B = (-1)^{N_B N_C + (N_C+N_B-1)C_1 + \sum_{l=1}^N k_l^0} A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}G_{T_0B}^{TC} b_0^{C_1} J, \tag{96}$$

where $TC := (C_2, C_3, \ldots)$ and $T_0B := (0, B_1, B_2, \ldots)$. Hence, since any two such operators must agree on any such basis vector, they must also agree on all of $\mathcal{F}$.

By the construction of J, it is apparent that J^2 commutes with every operator $A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}G_C^B$. Since also

$$J^2\omega_0 = Jc_0\omega_0 = b_0 J\omega_0 = b_0 c_0\omega_0 = \omega_0, \tag{97}$$

we must have $J^2 = I$.

Taking the hermitian adjoint of the defining relations (95) gives back the same exact relations. Hence J is self-adjoint.

[42] ω_0 denotes $\Omega_0 \otimes \omega_0$, the vacuum of our complex.
[43] Notice that J restricted to $\mathcal{F}^{\mathcal{M}}$ agrees with $\mathbb{O}$ from section 2.2.

It is straight forward to show that the defining basis (75) is orthonormal with respect to $\langle\cdot,\cdot\rangle_J$. Hence it is positive definite.

For any self-adjoint unitary operator U there is always an eigenbasis right at hand,

$$\{\psi_i \pm U\psi_i\}_i, \tag{98}$$

where $\{\psi_i\}_i$ is any given basis. Now, let $\psi_\pm$ denote an eigenvector of J-eigenvalue ±1. Then

$$\|\psi_\pm\|^2 = \|J\psi_\pm\|^2 = \pm\langle\psi_\pm, J\psi_\pm\rangle = \pm\|\psi_\pm\|_J^2.$$

Hence, those of negative J-eigenvalue correspond to those J-eigenvectors of non-positive norm. Which finishes the proof. □

By Theorem 4.3 we are fully justified in considering J as corresponding to the J-operator of the generic BRST complex.

Remark 4.3.3. We have

$$JA_{\{k_1^\mu\}\cdots\{k_N^\mu\}}G_C^B\omega_0 = \begin{cases} (-1)^{(N_B+1)(N_C+1)+\sum_{l=1}^N k_l^0} A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}G_{T_0B}^{TC}\omega_0, & \text{if } \quad C_1 = 1 \\ (-1)^{N_B(N_C+1)+\sum_{l=1}^N k_l^0} A_{\{k_1^\mu\}\cdots\{k_N^\mu\}}G_{T_1B}^{TC}\omega_0, & \text{if } \quad C_1 = 0 \end{cases}, \tag{99}$$

where $T_1B := (1, B_1, B_2, \ldots)$. From this we get, for both $C_1 = 0$ and 1,

$$N_{\mathcal{G}}JA_{\{k_1^\mu\}\cdots\{k_N^\mu\}}G_C^B\omega_0 = -\left(N_C - N_B - \frac{1}{2}\right)JA_{\{k_1^\mu\}\cdots\{k_N^\mu\}}G_C^B\omega_0. \tag{100}$$

In other words, J takes ghost number g to $-g$. So (54) has been verified explicitly.

Since there is no ghost number 0, the ghost number grading of this bosonic String puts us with odds with defining $\mathcal{H}_{\text{phys}} = H^0(\mathcal{F}; Q)$. Let us investigate when the BRST condition implements the physical state condition from the (old) covariant quantization.

Since $\mathcal{F}^{\mathcal{M}}$ is a Verma module, $Q\psi = 0$ only if ${L^{\mathcal{M}}}_m c_{-m}\psi = 0$ for all $m \neq 0$. Hence, the BRST condition implements the physical state conditions only if

$$c_{-m}\psi \neq 0 \qquad \forall m \in \mathbb{N}. \tag{101}$$

This does however not implement $({L^{\mathcal{M}}}_0 - 1)\psi = 0$. The part of Q containing the c_0's is

$$\left({L^{\mathcal{M}}}_0 + {L^{\mathcal{G}}}_0\right)c_0. \tag{102}$$

Hence we see that $(L^{\mathcal{M}}{}_0 - 1)\psi = 0$ is implemented if $c_0\psi \neq 0$ and $(L^{\mathcal{G}}{}_0 + 1)c_0\psi = 0$. But $[L^{\mathcal{G}}{}_0, c_0] = 0$, hence we require $(L^{\mathcal{G}}{}_0 + 1)\psi = 0$. That is we have required,

$$L^{\mathcal{T}}{}_0\psi = 0. \tag{103}$$

Furthermore, given (101) this means that

$$c_m\psi = b_m\psi = 0 \qquad \forall m \in \mathbb{N}, \tag{104}$$

in order for the last double sum $Q\psi$ to vanish. Vectors ψ satisfying (101), (104) and $c_0\psi \neq 0$, all have ghost number $-1/2$. Hence we identify $\mathcal{H}_{\text{phys}} := H^{-1/2}(\mathcal{F}; Q)$.

Remark 4.3.4. Our physical (76)-basis vectors are such that $c_0\psi \neq 0$. But then it follows that $b_0\psi = 0$.[a] Hence our physical basis vectors are in the subspace of vectors satisfying

$$b_0\psi = L^{\mathcal{T}}{}_0\psi = 0.$$

[a] Since then ψ contains no c_0, so that b_0 just is commuted through, annihilating the vacuum.

Now, Theorem 4.1 still holds, so

$$H^{-1/2}(\mathcal{F}; Q) \simeq H^{1/2}(\mathcal{F}; Q). \tag{105}$$

We hence would need to modify the assumption of the vanishing cohomology to

$$H^{k \neq \pm 1/2}(\mathcal{F}; Q) = 0. \tag{106}$$

But this gives

$$\left((-1)^{-1/2} + (-1)^{1/2}\right) \dim \mathcal{H}^{-1/2} = \sum_g (-1)^g \dim \mathcal{H}^g. \tag{107}$$

But $(-1)^{-1/2} + (-1)^{1/2} = 0$, so we cannot proceed as in the generic case.

There is yet another caveat in this BRST complex:

Remark 4.3.5. Calculation (51) shows that only the ghost eigenvectors of opposite ghost number have non-zero inner product. But ${L^{\mathcal{M}}}_0$ is related to the energy of the String, and it furthermore commutes with $N_{\mathcal{G}}$. Hence we may find simultaneous eigenvectors of $N_{\mathcal{G}}$ and ${L^{\mathcal{M}}}_0$, ψ_g^m, ϕ_g^m.[a] But deeming $H^{-1/2}$ as the physical space, then unavoidably yields a trivial theory, i.e.

$$\|\psi^m_{-1/2}\|^2 = \left\langle \psi^m_{-1/2}, \phi^m_{-1/2} \right\rangle = 0.$$

[a]The upper index denoting the ${L^{\mathcal{M}}}_0$-eigenvalue and the lower denoting the ghost number.

5 The BRST Complex of the Bosonic String

We here present the BRST complex of the bosonic String. This will clarify why we still may consider $H^{-1/2}(\mathcal{F})$ as our space of physical states yet still apply the blueprint of the generic BRST complex. They key lies in that this space is canonically isomorphic (as vector spaces) to the physical space of the cohomology of a subcomplex of the full BRST complex, a subcomplex of which the generic BRST blueprint is applicable. The conclusion is that $H^{-1/2}(\mathcal{F})$ indeed identifies the correct vector states, in spite of it having the incorrect inner product (Remark 4.3.5). This does not contradict the conventional treatment, it agrees with [3], although here the connection made is based more on mathematics rather than more on physics.

In the first half we construct the Semi-infinite cohomology of the Virasoro algebra, $C^m_\infty(\mathbb{V};\mathcal{F}^\mathcal{M})$, and show that this indeed is a BRST complex. From this we then construct to sub-BRST complexes, $C_\infty(\mathbb{V},\mathbb{C}c;\mathcal{F}^\mathcal{M})$ and $C_\infty(\mathbb{V},\mathbb{V}_0;\mathcal{F}^\mathcal{M})$. $C_\infty(\mathbb{V},\mathbb{C}c;\mathcal{F}^\mathcal{M})$ is identified as the heuristic BRST complex of bosonic String as presented in section 4.3. From a physical perspective $C_\infty(\mathbb{V},\mathbb{V}_0;\mathcal{F}^\mathcal{M})$ will however turn out to have the more desirable hermitian structure. Since the physical subspaces of the respective sub-BRST complexes furthermore will turn out to be canonically isomorphic, $C_\infty(\mathbb{V},\mathbb{C}c;\mathcal{F}^\mathcal{M})$ will be identified as the BRST complex of the bosonic String. We then proceed to proving the No-ghost theorem for this complex, utilizing the methods of section 4, which indeed are applicable to this complex.

The following contents is highly inspired by [10] and by chapter 6 in [11] .

5.1 Representing the Virasoro algebra on the semi-infinite forms

5.1.1 Semi-infinite forms

The Virasoro algebra is a graded Lie algebra, $\mathbb{V} = \bigoplus_{n\in\mathbb{Z}}\mathbb{V}_n$ where $\mathbb{V}_0 = \mathrm{Span}_\mathbb{C}\{c, L_0\}$ and $\mathbb{V}_m = \mathrm{Span}_\mathbb{C}\{L_m\}$ for $m \in \mathbb{Z}\backslash\{0\}$. Where $\{L_m\}_{m\in\mathbb{Z}} \cup \{c\}$ is the canonical basis of $\mathbb{V}$ satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}, \tag{108}$$

$$[L_m, c] = 0. \tag{109}$$

Define $\mathbb{V}_\pm := \bigoplus_{\pm n>0}\mathbb{V}_n$. For every $m \in \mathbb{Z}$, let $\mathbb{V}'_m$ denote the dual space of $\mathbb{V}_m$. Set $\mathbb{V}' := \bigoplus_{n\in\mathbb{Z}}\mathbb{V}'_n$, known as the **restricted dual**.[44] Let $\{L'_m\}_{m\in\mathbb{Z}} \cup \{c'\}$

[44] It differs from the 'normal' dual because it only consists of finite linear combinations of elements of the $\mathbb{V}'_m$'s.

denote the canonically dual basis of $\{L_m\}_{m\in\mathbb{Z}} \cup \{c\}$; i.e. for every $m, n \in \mathbb{Z}$,

$$\begin{aligned} L'_m(L_n) &= \delta_{m-n}, \\ L'_m(c) &= c'(L_m) = 0, \\ c'(c) &= 1, \end{aligned}$$

Set $\mathbb{V}'_\pm := \bigoplus_{\pm n>0} \mathbb{V}'_n$.

The space of **semi-infinite forms**, $\bigwedge_\infty \mathbb{V}'$, is spanned by the formal monomials

$$\omega = L'_{i_1} \wedge L'_{i_2} \wedge \cdots , \tag{110}$$

where $\wedge$ denotes the exterior product, $i_1 > i_2 > \cdots$ and such that:

$$\exists K_\omega \in \mathbb{N} \text{ such that } i_{k+1} = i_k - 1 \text{ for all } k > K_\omega,$$

i.e. only finitely many basis elements of $\mathbb{V}'_-$ are missing, and where it is furthermore understood that c' as well can be included in the sequence of dual vectors of the semi-infinite form.[45]

We define two operations on $\bigwedge_\infty \mathbb{V}'$. For every $x \in \mathbb{V}$ and $x' \in \mathbb{V}'$, let

$$\iota(x) L'_{i_1} \wedge L'_{i_2} \wedge \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} L'_{i_k}(x) L'_{i_1} \wedge \cdots \wedge \widehat{L'_{i_k}} \wedge \cdots \tag{111}$$

$$\epsilon(x') L'_{i_1} \wedge L'_{i_2} \wedge \cdots = x' \wedge L'_{i_1} \wedge L'_{i_2} \wedge \cdots , \tag{112}$$

where $\widehat{L'_{i_k}}$ means that L'_{i_k} is removed. The sum in (111) is actually finite, since there exists some K for which $L'_{i_k}(x) = 0$ for all $k > K$.

Proposition 5.1.1. For any $x, y \in \mathbb{V}$ and $x', y' \in \mathbb{V}'$[a],

$$\{\iota(x), \epsilon(y')\} = y'(x) \tag{113}$$

$$\{\iota(x), \iota(y)\} = \{\epsilon(x'), \epsilon(y')\} = 0. \tag{114}$$

[a]Compare with the bc-ghost anti-commutator relations.

[45]I.e. (with slightly abusive notation) we might have $L'_{i_k} = c'$ for some k.

Proof.

$$\begin{aligned}
&\epsilon(y')\iota(x)L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\
&=\sum_{k=1}^{\infty}(-1)^{k+1}L'_{i_k}(x)\epsilon(y')L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_k}}\wedge\cdots\\
&=\sum_{k=1}^{\infty}(-1)^{k+1}L'_{i_k}(x)y'\wedge L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_k}}\wedge\cdots
\end{aligned}$$

and

$$\begin{aligned}
&\iota(x)\epsilon(y')L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\
&=\iota(x)y'\wedge L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\
&=y'(x)L'_{i_1}\wedge L'_{i_2}\wedge\cdots+\sum_{k=1}^{\infty}(-1)^{k+2}L'_{i_k}(x)y'\wedge L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_k}}\wedge\cdots .
\end{aligned}$$

Hence (113) follows.

$$\begin{aligned}
\epsilon(x')\epsilon(y')L'_{i_1}\wedge L'_{i_2}\wedge\cdots&=x'\wedge y'\wedge L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\
&=-y'\wedge x'\wedge L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\
&=-\epsilon(x')\epsilon(y')L'_{i_1}\wedge L'_{i_2}\wedge\cdots .
\end{aligned}$$

For the last statement, since ι is linear in its argument and since it is trivially zero if its argument is not part of the semi-infinite form, we need only check the case $\{\iota(L_{i_k}),\iota(L_{i_l})\}$, and for symmetry reasons we may assume $k<l$. So, since

$$\begin{aligned}
&\iota(L_{i_k})\iota(L_{i_l})L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\
&=(-1)^{l+1}\iota(L_{i_k})L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots\\
&=(-1)^{l+1}(-1)^{k+1}L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_k}}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots
\end{aligned}$$

and

$$\begin{aligned}
&\iota(L_{i_l})\iota(L_{i_k})L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\
&=(-1)^{k+1}\iota(L_{i_l})L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_k}}\wedge\cdots\\
&=(-1)^{l}(-1)^{k+1}L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_k}}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots ,
\end{aligned}$$

the sought result follows. □

5.1.2 Representation

We now define a representation, ρ, of $\mathbb{V}$ on $\bigwedge_\infty \mathbb{V}'$. For any $m \neq 0$ and $x \in \mathbb{V}_m$ we define

$$\rho(x)L'_{i_1} \wedge L'_{i_2} \wedge \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} L'_{i_1} \wedge \cdots \wedge \mathrm{ad}'x \cdot L'_{i_k} \wedge \cdots , \tag{115}$$

where ad' denotes the coadjoint representation. By the definition of the coadjoint representation,[46]

$$\begin{aligned}
\left(\mathrm{ad}' L_m \cdot L'_{i_k}\right)(L_n) &:= -L'_{i_k}(\mathrm{ad}\ L_m \cdot L_n) \\
&= -L'_{i_k}([L_m, L_n]) \\
&= -(m-n)L'_{i_k}(L_{m+n}) \\
&= -(m-n)\delta_{i_k - m - n}) \\
&= (i_k - 2m)L'_{i_k - m}(L_n),
\end{aligned} \tag{116}$$

where in the third step the central term vanishes by the definition the canonical dual vectors. Since also, by the definition of the semi-infinite forms, we can find a $K \in \mathbb{N}$ such that

$$L'_{i_k - m} = L'_{i_{k+1}-(m-1)} = \cdots = L'_{i_{k+m}}$$

for every $k > K$, the sum in (115) is indeed finite, since the semi-infinite form contains two $L'_{i_{k+m}}$.

From (116) we furthermore see that the definition (115) would be divergent if generalized to include even $x \in \mathbb{C}L_0$. But before we deal with this, we have the following result:

Proposition 5.1.2. For any $x, y \in \mathbb{V}$ and $y' \in \mathbb{V}'$,

$$[\rho(x), \iota(y)] = \iota(\mathrm{ad}x \cdot y) \tag{117}$$
$$[\rho(x), \epsilon(y')] = \epsilon(\mathrm{ad}'x \cdot y'). \tag{118}$$

Proof. By linearity of the argument of each operator we need only prove (117 - 118) on the basis vectors. We have

$$\rho(L_m)L'_{i_1} \wedge \cdots = \sum_{k=1}^{\infty} (i_k - 2m) L'_{i_1} \wedge \cdots \wedge L'_{i_k - m} \wedge \cdots ,$$

[46] ad' is in a sense the adjoint of ad with respect to the product $L'_m(L_n)$.

So

$$
\begin{aligned}
&\iota(L_n)\rho(L_m)L'_{i_1}\wedge\cdots\\
&=\sum_{k=1}^{\infty}(i_k-2m)\iota(L_n)L'_{i_1}\wedge\cdots\wedge L'_{i_k-m}\wedge\cdots\\
&=\sum_{k=1}^{\infty}\sum_{l=1}^{k-1}(i_k-2m)(-1)^{l+1}\delta_{n-i_l}L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots\wedge L'_{i_k-m}\wedge\cdots\\
&+\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}(i_k-2m)(-1)^{l+1}\delta_{n-i_l}L'_{i_1}\wedge\cdots\wedge L'_{i_k-m}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots\\
&+\sum_{k=1}^{\infty}(i_k-2m)(-1)^{k+1}\delta_{n+m-i_k}L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_k-m}}\wedge\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho(L_m)\iota(L_n)L'_{i_1}\wedge\cdots\\
&=\sum_{l=1}^{\infty}(-1)^{l+1}\delta_{n-i_l}\rho(L_m)L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots\\
&=\sum_{l=1}^{\infty}\sum_{k=1}^{l-1}(i_k-2m)(-1)^{l+1}\delta_{n-i_l}L'_{i_1}\wedge\cdots\wedge L'_{i_k-m}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots\\
&+\sum_{l=1}^{\infty}\sum_{k=l+1}^{\infty}(i_k-2m)(-1)^{l+1}\delta_{n-i_l}L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots\wedge L'_{i_k-m}\wedge\cdots\\
&=\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}(i_k-2m)(-1)^{l+1}\delta_{n-i_l}L'_{i_1}\wedge\cdots\wedge L'_{i_k-m}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots\\
&+\sum_{k=1}^{\infty}\sum_{l=1}^{k-1}(i_k-2m)(-1)^{l+1}\delta_{n-i_l}L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_l}}\wedge\cdots\wedge L'_{i_k-m}\wedge\cdots
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\iota(L_n),\rho(L_m)]L'_{i_1}\wedge\cdots&=\sum_{k=1}^{\infty}(i_k-2m)(-1)^{k+1}\delta_{n+m-i_k}L'_{i_1}\wedge\cdots\wedge\widehat{L'_{i_k-m}}\wedge\cdots\\
&=\sum_{k=1}^{\infty}(i_k-2m)\delta_{n+m-i_k}\iota(L_{i_k})L'_{i_1}\wedge\cdots\\
&=(n-m)\iota(L_{m+n})L'_{i_1}\wedge\cdots.
\end{aligned}
$$

Lastly,

$$\begin{aligned}\rho(L_m)\epsilon(L'_n)L'_{i_1}\wedge\cdots &= \rho(L_m)L'_n\wedge L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\ &= (n-2m)L'_{n-m}\wedge L'_{i_1}\wedge L'_{i_2}\wedge\cdots\\ &+\sum_{k=1}^{\infty}(i_k-2m)L'_n\wedge L'_{i_1}\wedge\cdots\wedge L'_{i_k-m}\wedge\cdots\\ &= (n-2m)\epsilon(L'_{n-m})L'_{i_1}\wedge\cdots+\epsilon(L_n)\rho(L_m)L'_{i_1}\wedge\cdots.\end{aligned}$$

Hence the proof is finished. □

We summarize for later referral:

$$[\rho(L_m),\iota(L_n)] = (m-n)\iota(L_{m+n}) \tag{119}$$

$$[\rho(L_m),\epsilon(L'_n)] = (n-2m)\epsilon(L'_{n-m}). \tag{120}$$

We move on towards defining $\rho(x)$ for any $x\in\mathbb{V}_0$. We do this by first defining it on a 'vacuum' of $\bigwedge_\infty\mathbb{V}'$, and then extending it linearly to all of $\bigwedge_\infty\mathbb{V}'$ under reinforcement of relations (117 - 118). A **vacuum** on $\bigwedge_\infty\mathbb{V}'$ is a vector ω_0 such that, $\forall x\in\mathbb{V}_m$ and $\forall y\in\mathbb{V}_{-m}$ with $m\neq 0$,

$$[\rho(x),\rho(y)]\omega_0 = f(x,y)\omega_0, \tag{121}$$

for some complex-valued antisymmetric bilinear function f. A typical example of a such is any[47]

$$\omega_0 = L'_{i_0}\wedge L'_{i_0-1}\wedge\cdots. \tag{122}$$

We will henceforth concern ourselves with such vacua only.[48] Moving on, for a fixed vacuum ω_0 and a fixed $\beta\in\mathbb{V}'_0$, we define

$$\rho(x)\omega_0 := \beta(x)\omega_0, \tag{123}$$

for any $x\in\mathbb{V}_0$. In fact, the resulting operators ρ may be written [10]

$$\rho(x) = \sum_{k\in\mathbb{Z}} :\epsilon(\mathrm{ad}'x\cdot L'_k)\iota(L_k): +\beta(x), \tag{124}$$

where the normal ordering, with respect to a vacuum ω_0 of the form (122), is

$$:\iota(L_k)\epsilon(L'_l) := \begin{cases}\iota(L_k)\epsilon(L'_l), & k\leq i_0\\ -\epsilon(L'_l)\iota(L_k), & k>i_0\end{cases}. \tag{125}$$

[47] That this indeed satisfies (121) will be shown in Theorem 5.1.

[48] Where it is here understood that c' is not in the vacuum. For our purposes this poses no restriction.

The form of ρ becomes more familiar[49] by looking at how it written in terms of the canonical basis elements of $\mathbb{V}$,[50]

$$\rho(L_m) = \sum_{k\in\mathbb{Z}} (k-m) : \epsilon(L'_k)\iota(L_{k+m}) : +\beta(L_m). \tag{126}$$

The following theorem shows that ρ defines a representation on $\bigwedge_\infty \mathbb{V}'$. The vacuum vectors ω_0 of the from (122) may have the physical interpretation of the vacuum of the Dirac sea. So our restrictions to such vacua only is not at odds with conventional physics. From this interpretation it should not matter which i_0 we choose, where we set our 'zero level', and it neither does mathematically, as the proof of Theorem 5.1 (below) for an arbitrary i_0 shows. We need just choose the β in (123) accordingly in order to get the canonical form of a representation of the Virasoro algebra. So not having to deal with tedious and (for our purposes) unnecessary notation, we with out loss of generality set $i_0 = -1$ for the remainder of this thesis.[51]

Theorem 5.1. ρ yields a representation of $\mathbb{V}$ on $\bigwedge_\infty \mathbb{V}'$ for which

$$\rho(c) = -26I,$$

i.e.

$$[\rho(L_m), \rho(L_n)] = (m-n)\rho(L_{m+n}) + \frac{-26}{12} m\left(m^2-1\right)\delta_{m+n}, \tag{127}$$

when in addition $\beta(L_0) = 1$.

Proof. By application of (119 - 120),

$$\begin{aligned}
&[\rho(L_m), \epsilon(L'_k)\iota(L_{k+n})] \\
&= [\rho(L_m), \epsilon(L'_k)]\iota(L_{k+n}) + \epsilon(L'_k)[\rho(L_m), \iota(L_{k+n})] \\
&= (k-2m)\epsilon(L'_{k-m})\iota(L_{k+n}) + (m-n-k)\epsilon(L'_k)\iota(L_{k+m+n})
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&[\rho(L_m), \iota(L_{k+n})\epsilon(L'_k)] \\
&= [\rho(L_m), \iota(L_{k+n})]\epsilon(L'_k) + \iota(L_{k+n})[\rho(L_m), \epsilon(L'_k)] \\
&= (m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) + (k-2m)\iota(L_{k+n})\epsilon(L'_{k-m}).
\end{aligned}$$

[49] Compare with the Virasoro operators on the ghost Fock space.
[50] We will make the full identification with the bc-ghosts later on.
[51] It will later become apparent that this choice of vacuum agrees with the conventional one considered in textbook treatments of the bosonic String.

Applying these, we get

$$
\begin{aligned}
&[\rho(L_m), \rho(L_n)] \\
&= \sum_{k\in\mathbb{Z}} (k-n)[\rho(L_m), : \epsilon(L'_k)\iota(L_{k+n}) :] \\
&= \sum_{k>-1-n} (k-n)[\rho(L_m), \epsilon(L'_k)\iota(L_{k+n})] \\
&- \sum_{k\leq -1-n} (k-n)[\rho(L_m), \iota(L_{k+n})\epsilon(L'_k)] \\
&= \sum_{k>-1-n} (k-n)\left((k-2m)\epsilon(L'_{k-m})\iota(L_{k+n}) + (m-n-k)\epsilon(L'_k)\iota(L_{k+m+n})\right) \\
&- \sum_{k\leq -1-n} (k-n)\left((m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) + (k-2m)\iota(L_{k+n})\epsilon(L'_{k-m})\right) \\
&= \sum_{k>-1-n} (k-n)(k-2m)\epsilon(L'_{k-m})\iota(L_{k+n}) - \sum_{k\leq -1-n} (k-n)(k-2m)\iota(L_{k+n})\epsilon(L'_{k-m}) \\
&- \sum_{k\leq -1-n} (k-n)(m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) + \sum_{k>-1-n} (k-n)(m-n-k)\epsilon(L'_k)\iota(L_{k+m+n}).
\end{aligned}
$$

By a suitable change of summation index,

$$
\begin{aligned}
&[\rho(L_m), \rho(L_n)] \\
&= \sum_{k>-1-n-m} (k+m-n)(k-m)\epsilon(L'_k)\iota(L_{k+m+n}) \\
&- \sum_{k\leq -1-n-m} (k+m-n)(k-m)\iota(L_{k+m+n})\epsilon(L'_k) \\
&- \sum_{k\leq -1-n} (k-n)(m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) + \sum_{k>-1-n} (k-n)(m-n-k)\epsilon(L'_k)\iota(L_{k+m+n}) \\
&= \sum_{k\in\mathbb{Z}} (k+m-n)(k-m) : \epsilon(L'_k)\iota(L_{k+m+n}) : \\
&- \sum_{k\leq -1-n} (k-n)(m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) + \sum_{k>-1-n} (k-n)(m-n-k)\epsilon(L'_k)\iota(L_{k+m+n}).
\end{aligned}
\tag{128}
$$

We deal with the last two sums: For $m \geq 0$,

$$
\begin{aligned}
\sum_{k>-1-n} &= \sum_{k>-1-n-m} - \sum_{-1-n\geq k>-1-n-m}, \\
\sum_{k\leq -1-n} &= \sum_{k\leq -1-n-m} + \sum_{-1-n\geq k>-1-n-m},
\end{aligned}
$$

and, for $m < 0$,

$$\sum_{k>-1-n} = \sum_{k>-1-n-m} + \sum_{-1-n-m\geq k>-1-n},$$
$$\sum_{k\leq -1-n} = \sum_{k\leq -1-n-m} - \sum_{-1-n-m\geq k>-1-n}.$$

So for the $m \geq 0$ case,

$$
\begin{aligned}
&- \sum_{k\leq -1-n} (k-n)(m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) + \sum_{k>-1-n} (k-n)(m-n-k)\epsilon(L'_k)\iota(L_{k+m+n}) \\
&= -\left(\sum_{k\leq -1-n-m} + \sum_{-1-n\geq k>-1-n-m} \right) (k-n)(m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) \\
&+ \left(\sum_{k>-1-n-m} - \sum_{-1-n\geq k>-1-n-m} \right) (k-n)(m-n-k)\epsilon(L'_k)\iota(L_{k+m+n}) \\
&= \sum_{k\in\mathbb{Z}} (k-n)(m-n-k) :\epsilon(L'_k)\iota(L_{k+m+n}): - \delta_{m+n} \sum_{-1-n\geq k>-1-n-m} (k-n)(m-n-k) \\
&= \sum_{k\in\mathbb{Z}} (k-n)(m-n-k) :\epsilon(L'_k)\iota(L_{k+m+n}): - \delta_{m+n} \sum_{k=0}^{m-1} (k+m)(2m-k) \\
&= \sum_{k\in\mathbb{Z}} (k-n)(m-n-k) :\epsilon(L'_k)\iota(L_{k+m+n}): - \delta_{m+n} \sum_{k=1}^{m} (k-1+m)(2m-k+1) \\
&= \sum_{k\in\mathbb{Z}} (k-n)(m-n-k) :\epsilon(L'_k)\iota(L_{k+m+n}): - \delta_{m+n} \frac{26}{12} m \left(m^2 - \frac{1}{13} \right),
\end{aligned}
$$

where we have: in the third step we have simply used that $n = -m$, in the fourth made a suitable change of summation index, and in the last evaluated the sum.

For the $m > 0$ case we similarly get

$$- \sum_{k \le -1-n} (k-n)(m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) + \sum_{k > -1-n} (k-n)(m-n-k)\epsilon(L'_k)\iota(L_{k+m+n})$$

$$= -\left(\sum_{k \le -1-n-m} - \sum_{-1-n-m \ge k > -1-n} \right) (k-n)(m-n-k)\iota(L_{k+m+n})\epsilon(L'_k)$$

$$+ \left(\sum_{k > -1-n-m} + \sum_{-1-n-m \ge k > -1-n} \right) (k-n)(m-n-k)\epsilon(L'_k)\iota(L_{k+m+n})$$

$$= \sum_{k \in \mathbb{Z}} (k-n)(m-n-k) : \epsilon(L'_k)\iota(L_{k+m+n}) : + \delta_{m+n} \sum_{-1-n-m \ge k > -1-n} (k-n)(m-n-k)$$

$$= \sum_{k \in \mathbb{Z}} (k-n)(m-n-k) : \epsilon(L'_k)\iota(L_{k+m+n}) : + \delta_{m+n} \sum_{-1-n-m \ge k > -1-n} (k-n)(m-n-k)$$

$$= \sum_{k \in \mathbb{Z}} (k-n)(m-n-k) : \epsilon(L'_k)\iota(L_{k+m+n}) : + \delta_{m+n} \sum_{k=0}^{-m} (k-1+(-m))(2(-m)-k+1)$$

$$= \sum_{k \in \mathbb{Z}} (k-n)(m-n-k) : \epsilon(L'_k)\iota(L_{k+m+n}) : + \delta_{m+n} \frac{26}{12}(-m)\left((-m)^2 - \frac{1}{13}\right)$$

$$= \sum_{k \in \mathbb{Z}} (k-n)(m-n-k) : \epsilon(L'_k)\iota(L_{k+m+n}) : - \delta_{m+n} \frac{26}{12} m \left(m^2 - \frac{1}{13}\right).$$

To summarize, for any $m \in \mathbb{Z}$,

$$- \sum_{k \le -1-n} (k-n)(m-n-k)\iota(L_{k+m+n})\epsilon(L'_k) + \sum_{k > -1-n} (k-n)(m-n-k)\epsilon(L'_k)\iota(L_{k+m+n})$$

$$= \sum_{k \in \mathbb{Z}} (k-n)(m-n-k) : \epsilon(L'_k)\iota(L_{k+m+n}) : - \delta_{m+n} \frac{26}{12} m \left(m^2 - \frac{1}{13}\right).$$

We use this in (128) so that we get

$$
\begin{aligned}
&[\rho(L_m), \rho(L_n)] \\
&= \sum_{k\in\mathbb{Z}} (k+m-n)(k-m) : \epsilon(L'_k)\iota(L_{k+m+n}) : \\
&+ \sum_{k\in\mathbb{Z}} (k-n)(m-n-k) : \epsilon(L'_k)\iota(L_{k+m+n}) : + \delta_{m+n} \frac{-26}{12} m \left(m^2 - \frac{1}{13}\right) \\
&= (m-n) \sum_{k\in\mathbb{Z}} (k-m-n)(k-m) : \epsilon(L'_k)\iota(L_{k+m+n}) : \\
&+ \delta_{m+n} \frac{-26}{12} m \left(m^2 - \frac{1}{13}\right) \\
&= (m-n)\rho(L_m) - (m-n)\beta(L_{m+n}) \\
&+ \delta_{m+n} \frac{-26}{12} m \left(m^2 - \frac{1}{13}\right) \\
&= (m-n)\rho(L_m) - (m-n)\delta_{m+n}\beta(L_{m+n}) \\
&+ \delta_{m+n} \frac{-26}{12} m \left(m^2 - \frac{1}{13}\right) \\
&= (m-n)\rho(L_m) - 2m\delta_{m+n}\beta(L_0) \\
&+ \delta_{m+n} \frac{-26}{12} m \left(m^2 - \frac{1}{13}\right) \\
&= (m-n)\rho(L_m) + \delta_{m+n} \frac{-26}{12} m \left(m^2 - \frac{1-12\beta(L_0)}{13}\right)
\end{aligned}
$$

where in the third step $(m-n)\beta(L_{m+n})$ was added and subtracted so that (124) could be applied, noticing that by construction

$$\delta_{m+n}\beta(L_{m+n}) = \beta(L_{m+n}).$$

So by choosing $\beta(L_0) = -1$, the theorem has been proved.

□

Remark 5.1.1. So in Theorem 5.1 we have chosen $\beta = -26c' - L'_0$.

Remark 5.1.2. By the identification

$$\begin{aligned} b_m &\leftrightarrow \iota(L_m), \\ c_n &\leftrightarrow \epsilon(L'_{-n}); \end{aligned} \tag{129}$$

the similarities of the bc-ghost picture and the semi-infinite forms of the Virasoro algebra are already shining through. We are furthermore justified in denoting $\rho(L_m) = L^{\mathcal{G}}{}_m$, and will hence do so. However, since the operators $\epsilon(c')$ and $\iota(c)$ have no bc-ghost counter part, the full connection can not yet be fully made. This will have to wait until section 5.2.1.

For future convenience we construct a basis of $\bigwedge_\infty \mathbb{V}'$[52],

$$\{G^B_{C,C_0}\omega_0\}_{B,C\in 1_\infty,\ C_0=0,1}, \tag{130}$$

where

$$G^B_{C,C_0} := \left(\prod_{m\in\mathbb{N}} \iota(L_{-m})^{B_m}\right) \epsilon(c')^{C_0} \left(\prod_{n\in\mathbb{N}} \epsilon(L'_{n-1})^{C_n}\right).$$

it is clear from the definition of semi-infinite forms that this indeed is a basis.

5.1.3 Hermitian form

Moving on, we want to equip $\bigwedge_\infty \mathbb{V}'$ with an inner product. This inner product furthermore needs to facilitate certain involutive properties, namly those of the bc-ghosts. We follow the blueprint in [10] for the construction of a such.

We first define a map σ. Set

$$\begin{aligned} \sigma(L_m) &= L_{-m} \\ \sigma(c) &= c, \end{aligned}$$

and extend it anti-linearly to all of $\mathbb{V}$. It follows that σ defines a Lie algebra anti-automorphism, since

$$\begin{aligned} [\sigma(L_m), \sigma(L_n)] &= -(m-n)L_{-m-n} \\ &= \sigma([L_n, L_m]). \end{aligned}$$

It is furthermore clearly its own inverse. Hence it defines an involution on $\mathbb{V}$. We 'lift' σ to an anti-linear map on $\mathbb{V}'$, for which we will use the same symbol. We define it by imposing

$$\sigma(y')(\sigma(x)) = \overline{y'(x)}.$$

[52] In analogy with (76)

So that

$$\sigma(L'_m) = L'_{-m}$$
$$\sigma(c') = c'.$$

In fact, since

$$\begin{aligned}\sigma^2(y')(\sigma(x)) &= \overline{\sigma(y')(x)}\\ &= \overline{\sigma(y')(\sigma^2(x))}\\ &= y'(\sigma(x))\end{aligned}$$

and

$$\begin{aligned}\sigma([L'_m, L'_n])(\sigma(x)) &= \overline{([L_m, L_n])'(x)}\\ &= (m-n)\overline{L'_{m+n}(x)} + A(m,n)\overline{c'(x)}\\ &= (m-n)\overline{\sigma(L'_{m+n})(\sigma(x))} + A(m,n)\overline{\sigma(c')(\sigma(x))}\\ &= (m-n)\overline{L'_{-m-n}(\sigma(x))} + A(m,n)\overline{c'(\sigma(x))}\\ &= [L'_{-n}, L'_{-m}](\overline{\sigma}(x))\\ &= [\sigma(L'_n), \sigma(L'_m)](\sigma(x)),\end{aligned}$$

where $A(m,n)$ denotes the anomaly in the Virasoro algebra; it follows that σ defines an involution on both $\mathbb{V}$ and $\mathbb{V}'$. Based on Remark 5.1.2, the *bc*-ghost mode identification, and involutive properties *bc*-ghost modes (74); The corresponding involutions we here want to impose are:

$$\iota(x)^* = \iota(\sigma(x)) \tag{131}$$
$$\epsilon(x')^* = \epsilon(\sigma(x')). \tag{132}$$

We move on to constructing the inner product, $\langle\cdot,\cdot\rangle$, on $\bigwedge_\infty \mathbb{V}'$. Consider the semi-infinite form

$$\omega_c = ic' \wedge L'_0 \wedge L'_{-1} \wedge \cdots . \tag{133}$$

Set $\langle \omega_0, \omega_c\rangle = 1$ and extend to all of $\bigwedge_\infty \mathbb{V}'$ by enforcing the properties (131 - 132).[53] A straightforward, although tedious, calculation shows that

$$\left\langle G^B_{C,C_0}\omega_0, G^{\widetilde{B}}_{\widetilde{C},\widetilde{C}_0}\omega_0\right\rangle = \delta_{1-C_0-\widetilde{C}_0}\delta_{1-C_1-\widetilde{C}_1}(-1)^{N_B N_{\widetilde{B}}}\delta_{B-T\widetilde{C}}\delta_{\widetilde{B}-TC}. \tag{134}$$

Hence the inner product between any two semi-infinite forms becomes calculable.

[53] By the same reasoning as in Remark 4.3.1, setting $\langle\omega_0, \omega_o\rangle = 1$ leads to a contradiction.

In fact, we may have chosen any two vectors

$$\begin{aligned}\omega_I &= a \cdot L'_{i_1} \wedge L'_{i_2} \wedge \cdots \\ \omega_J &= b \cdot c' \wedge L'_{j_1} \wedge L'_{j_2} \wedge \cdots ,\end{aligned}$$

such that $\{c'\} \cup \{L'_{i_k}\}_{k\in\mathbb{N}} \cup \{L'_{-j_l}\}_{l\in\mathbb{N}}$ is a basis of $\mathbb{V}'$ and $a, b \in \mathbb{C}$, to construct our inner product. Two different choices would only differ by a complex multiplicative factor. This follows from the fact that for each choice,

$$\cdots \wedge L'_{-j_2} \wedge L'_{-j_1} \wedge c' \wedge L'_{i_1} \wedge L'_{i_2} \wedge \cdot$$

defines a volume form, and different such differ by a constant multiple only. In particular this means that

$$\overline{\langle \omega_I, \omega_J \rangle} = z \langle \omega_J, \omega_I \rangle ,$$

for some $z \in \mathbb{C}$. Iterating, we even get

$$\langle \omega_I, \omega_J \rangle = \overline{z \langle \omega_J, \omega_I \rangle} = |z|^2 \langle \omega_I, \omega_J \rangle .$$

Hence, $z = e^{i\theta}$. So by picking $w = e^{i\theta/2}$ and considering $w \langle \cdot, \cdot \rangle$ in place of $\langle \cdot, \cdot \rangle$, we get an hermitian product. This is source of the factor i in (133).

Remark 5.1.3. We make a similar remark here as the one made in Remark 4.3.1. If we were to make a similar construction for two vectors ω_I and ω_J for which $\{c'\} \cup \{L'_{i_k}\}_{k\in\mathbb{N}} \cup \{L'_{-j_l}\}_{l\in\mathbb{N}}$ does not constitute a basis of $\mathbb{V}'$, we get a contradiction. The contradiction is derived by first noting that

$$\begin{aligned}\iota(L_M)\omega_I &= 0 \\ \iota(L_{-M})\omega_J &= 0,\end{aligned}$$

for some M in this case. Hence

$$\begin{aligned}\langle \omega_I, \omega_J \rangle &= \langle \iota(L_M)\epsilon(L_M)\omega_I, \omega_J \rangle \\ &= \langle \epsilon(L_M)\omega_I, \iota(L_{-M})\omega_J \rangle = 0.\end{aligned}$$

So we cannot set $\langle \omega_I, \omega_J \rangle \neq 0$, and thus not $\langle \omega_0, \omega_0 \rangle = 1$.

Hence, when considering this full complex we cannot ignore any element in $\{c'\} \cup \{L'_{i_k}\}_{k\in\mathbb{N}} \cup \{L'_{-j_l}\}_{l\in\mathbb{N}}$, and in particular not c'. Since c' does not have a counter part in the heuristic theory section 4.3, this 'full' complex is really not the complex we are looking for.

We note that under this involution

$$L^{\mathcal{G}*}{}_{m} = L^{\mathcal{G}}{}_{-m}, \tag{135}$$

as we seek.

5.1.4 Gradings

We now introduce two gradings on $\bigwedge_\infty \mathbb{V}'$, Deg and deg. We define **Deg** by first fixing $\mathrm{Deg}\ \omega_0 \in \mathbb{R}$ and then setting

$$\mathrm{Deg}\ \epsilon(x') = 1, \qquad \mathrm{Deg}\ \iota(x) = -1.$$

Deg on $\bigwedge_\infty \mathbb{V}'$ then takes the form of[54]

$$\mathrm{Deg}\ \hat{G}^{B}_{C,C_0}\omega_0 = \sum_{m\in\mathbb{N}} (B_m - C_m) + C_0 + \mathrm{Deg}\ \omega_0.$$

As seen from (124) $\mathrm{Deg}\ \rho(x) = 0$, so the ghost number is invariant under the action of $\mathbb{V}$. Hence the subspaces of a fixed Deg,

$$\bigwedge\nolimits^{m}_{\infty}\mathbb{V}' := \left\{\omega \in \bigwedge\nolimits_{\infty}\mathbb{V}' : \mathrm{Deg}\ \omega = m\right\}.$$

form subrepresentations of $\mathbb{V}$.

The second grading gives each $\bigwedge^m_\infty \mathbb{V}'$ the structure of a graded $\mathbb{V}$-module. We define **deg** by first fixing $\deg\ \omega_0 \in \mathbb{R}$ and then setting

$$\deg\ \epsilon(x') = m \quad \text{and} \quad \deg\ \iota(x) = -m,$$

for every $x' \in \mathbb{V}'_m$ and $x \in \mathbb{V}_m$. deg then takes the form on $\bigwedge_\infty \mathbb{V}'$ as[55]

$$\deg\ \hat{G}^{B}_{C,C_0}\omega_0 = \sum_{m\in\mathbb{N}} m\,(C_{m+1} + B_m) + \deg\ \omega_0.$$

Define

$$\bigwedge\nolimits^{m;n}_{\infty}\mathbb{V}' := \left\{\omega \in \bigwedge\nolimits^{m}_{\infty}\mathbb{V}' : \deg\ \omega = n\right\}.$$

It is clear that

$$\rho(\mathbb{V}_k) : \bigwedge\nolimits^{m;n}_{\infty}\mathbb{V}' \to \bigwedge\nolimits^{m;n+k}_{\infty}\mathbb{V}'$$

Hence

$$\bigwedge\nolimits^{m}_{\infty}\mathbb{V}' := \bigoplus_{n} \bigwedge\nolimits^{m;n}_{\infty}\mathbb{V}',$$

is a graded $\mathbb{V}$-module. It is clear that each $\bigwedge^{m;n}_\infty \mathbb{V}'$ is finite dimensional, since

[54] Which indeed is related to the ghost number, up to the C_0-term.
[55] Which indeed is the $L^{\mathcal{G}}{}_0$-level number.

for each fixed Deg and deg there are only finitely many basis vectors satisfying it. It is even clear that for each fixed Deg there is a degree, $|\deg|$, large enough so that $\bigwedge_{\infty}^{m;n}\mathbb{V}' = 0$ for all $\pm n > |\deg|$.[56] We notice that it hence makes sense to consider their respective q-character and q-signature[57].

5.2 The Semi-infinite cohomology of the Virasoro algebra

We define a differential Q on $\mathcal{F}^{\mathcal{M}} \otimes \bigwedge_{\infty}\mathbb{V}'$ according to the blueprint laid out in [10], which in our case becomes

$$\begin{aligned} Q = \sum_{m\in\mathbb{Z}} {L^{\mathcal{M}}}_m \epsilon(L'_m) - \frac{1}{2}\sum_{m,n\in\mathbb{Z}} (m-n) : \iota(L_{m+n})\epsilon(L'_m)\epsilon(L'_n) : \\ - \epsilon(L'_0) + (D-26)\epsilon(c'), \end{aligned} \tag{136}$$

The first sum is indeed well-defined because: ${L^{\mathcal{M}}}_m v = 0$ for m large enough, for any given $v \in \mathcal{F}^{\mathcal{M}}$; and $\epsilon(L'_m)\omega$ is zero for m small enough, for any semi-infinite form ω. The second sum is finite by virtue of the normal ordering (annihilation operators to the right).

By the identification (129) we see that Q is equal to the BRST charge Q, as defined in (83), all but for the last term $(D-26)\epsilon(c')$. However, as mentioned before, it is well-known that Q is nilpotent if and only if $D = 26$ (Corollary 5.1.1). So $D = 26$ is forced upon us. So the last term vanishes for $D = 26$. So the Q's are indeed identical as BRST operators.

As is easily seen, Deg $Q = 1$. We hence have a differential complex

$$\cdots \to C_{\infty}^{m}(\mathbb{V};\mathcal{F}^{\mathcal{M}}) \underset{Q}{\to} C_{\infty}^{m+1}(\mathbb{V};\mathcal{F}^{\mathcal{M}}) \to \cdots,$$

where $C_{\infty}^{m}(\mathbb{V};\mathcal{F}^{\mathcal{M}}) := \mathcal{F}^{\mathcal{M}} \otimes \bigwedge_{\infty}^{m}\mathbb{V}'$. This is the **Semi-infinite cohomology** of the Virasoro algebra with coefficients in $\mathcal{F}^{\mathcal{M}}$. We will denote this semi-infinite cohomology by $H_{\infty}(\mathbb{V};\mathcal{F}^{\mathcal{M}})$.

$\mathcal{F}^{\mathcal{M}} \otimes \bigwedge_{\infty}$ is a graded module by defining[58]

$$\deg(v \otimes \omega) := \deg(v) + \deg(\omega), \tag{137}$$

for $v \otimes \omega \in \mathcal{F}^{\mathcal{M}} \otimes \bigwedge_{\infty}$.[59] Furthermore, deg $Q = 0$ hence it preserves the deg-structure. For every n we hence get a subcohomology

$$\cdots \to C_{\infty}^{m;n}(\mathbb{V};\mathcal{F}^{\mathcal{M}}) \underset{Q}{\to} C_{\infty}^{m+1;n}(\mathbb{V};\mathcal{F}^{\mathcal{M}}) \to \cdots,$$

[56] The argumentation for this is the same as as the discussion following Remark 4.3.2.
[57] See Definition 3.1.
[58] Which indeed is the ${L^{\mathcal{T}}}_0$-level number, i.e. the energy.
[59] $\deg(v)$ denotes the grading on $\mathcal{F}^{\mathcal{M}}$ defined by (21).

and therefore

$$H^m_\infty(\mathbb{V}, \mathcal{F}^{\mathcal{M}}) = \bigoplus_n H^{m;n}_\infty(\mathbb{V}, \mathcal{F}^{\mathcal{M}}), \tag{138}$$

where each $\dim H^{m;n}_\infty(\mathbb{V}, \mathcal{F}^{\mathcal{M}}) < \infty$.

$\mathcal{F}^{\mathcal{M}}$ is a hermitian module, i.e it is equipped with a non-degenerate hermitian form such that

$$L^{\mathcal{M}}{}^*_m = L^{\mathcal{M}}{}_{-m}.$$

We thus have a hermitian form on the whole complex $\mathcal{F}^{\mathcal{M}} \otimes \bigwedge_\infty \mathbb{V}'$. Consequently

$$Q^* = Q.$$

$(C_\infty(\mathbb{V}, \mathcal{F}^{\mathcal{M}}), Q)$ defines a BRST complex. But does the decomposition theorem hold? In order for us here to be able to copy the proof of the decomposition theorem, we need the operator corresponding to the J-operator on our complex. We define J by generalizing the construction of the J for the BRST complex of the bosonic String, (95), as presented in section 4.3. We define J here as (ignoring the $\mathcal{F}^{\mathcal{M}}$-part)

$$\begin{aligned} &J\epsilon(x') = \iota(x)J, \qquad J\iota(x) = \epsilon(x')J, \\ &J\omega_0 = \omega_c, \end{aligned} \tag{139}$$

x' denoting the dual element of $x \in \mathbb{V}$. Duplicating the proof of Theorem 4.3, we get here the analogous result, an unitary self-adjoint operator J and a positive definite inner product $\langle \cdot, \cdot \rangle_J$.[60] Moreover, in order for the decomposition to hold we need $\operatorname{Deg} \omega_0 = -\operatorname{Deg} \omega_c$, so that J indeed reverses the ghost number. Hence,

$$\begin{aligned} \operatorname{Deg} \omega_0 &\stackrel{!}{=} -\operatorname{Deg} \epsilon(c')\epsilon(L_0')\omega_0 = -\left(1 + 1 + \operatorname{Deg} \omega_0\right) \\ &\Longleftrightarrow \\ \operatorname{Deg} \omega_0 &\stackrel{!}{=} -1. \end{aligned} \tag{140}$$

Hence for the choice $\operatorname{Deg} \omega_0 = -1$, the decomposition theorem indeed holds.

However, this complex is not the BRST complex of the bosonic String, it contains c' (the dual of the central element), which is not in our heuristic theory (section 4.3). Hence this BRST complex is not quite the one we are looking for. The BRST complex of the bosonic String is a relative subcomplex of this complex. We construct it in the next section.

[60] We notice that here $\langle \omega_0, \omega_0 \rangle_J = 1$.

5.2.1 The relative cohomology

We begin by presenting a result which justifies our construction of the relative subcomplex. We denote the representation of $\mathbb{V}$ on $\mathcal{F}^{\mathcal{M}}$ by π. Then $\theta := \pi + \rho$ defines a representation on $\mathcal{F}^{\mathcal{M}} \otimes \bigwedge_\infty \mathbb{V}'$.

Proposition 5.2.1.

$$\{Q, \iota(x)\} = \theta(x) \tag{141}$$
$$\{Q, \epsilon(y')\} = \epsilon(dy') \tag{142}$$
$$[Q, {L^{\mathcal{T}}}_n] = \frac{26-D}{12} n(n^2-1)c_n, \tag{143}$$

for any $n \in \mathbb{Z}$, $x \in \mathbb{V}$ and $y' \in \mathbb{V}'$.[a] In particular,

$$[Q, \theta(x)] = 0 \tag{144}$$

for all $x \in \mathbb{V}$ if and only if $D = 26$.

[a] Where dy' denotes the normal Lie algebra differential of y' [13], or just take (145) as a definition of notation.

Proof. By linearity we need only prove the theorem for $\theta(L_n) = {L^{\mathcal{T}}}_n$ and $\iota(L_n)$. For notational convenience we adopt the bc-notation (129). Since

$$\begin{aligned}
&\sum_{m,k} (m-k)\{: b_{k+m} c_{-m} c_{-k} :, b_n\} \\
&= \sum_m \sum_{m+k \leq -1} (m-k)\{b_{k+m} c_{-m} c_{-k}, b_n\} \\
&+ \sum_m \sum_{m+k > -1} (m-k)\{c_{-m} c_{-k} b_{k+m}, b_n\} \\
&= \sum_m \sum_{m+k \leq -1} (m-k)\left(b_{k+m} c_{-m} \delta_{n-k} - b_{k+m} c_{-k} \delta_{n-m}\right) \\
&- \sum_m \sum_{m+k > -1} (m-k)\left(c_{-m} b_{k+m} \delta_{n-k} - c_{-k} b_{k+m} \delta_{n-m}\right) \\
&= \sum_m \sum_k (m-k)\left(: b_{k+m} c_{-m} : \delta_{n-k} - : b_{k+m} c_{-k} : \delta_{n-m}\right) \\
&= \sum_m (m-n)\left(: b_{n+m} c_{-m} : + : b_{m+n} c_{-m} :\right) \\
&= -2\left({L^{\mathcal{G}}}_n + \delta_n\right),
\end{aligned}$$

$$\sum_m \{{L^{\mathcal{M}}}_m c_{-m}, b_n\} = {L^{\mathcal{M}}}_n$$

and $\{c_0, b_n\} = \delta_n$, from which (141) follows.

We move on to proving (142). Since

$$
\begin{aligned}
&\sum_{m,k}(m-k)\{:b_{k+m}c_{-m}c_{-k}:, c_n\} \\
&= \sum_m \sum_{m+k\le -1} (m-k)\{b_{k+m}c_{-m}c_{-k}, c_n\} \\
&+ \sum_m \sum_{m+k>-1} (m-k)\{c_{-m}c_{-k}b_{k+m}, c_n\} \\
&= \sum_m \sum_{m+k\le -1} (m-k)\delta_{k+m+n}c_{-m}c_{-k} \\
&+ \sum_m \sum_{m+k>-1} (m-k)c_{-m}c_{-k}\delta_{k+m+n} \\
&= \sum_m (2m+n)c_{-m}c_{m+n}.
\end{aligned}
$$

Since the Lie derivative [13] of L'_{-n} is

$$
dL'_{-n} = -\frac{1}{2}\sum_m (2m+n)L'_m \wedge L'_{-m-n}, \tag{145}
$$

and extending the action of ϵ to even include $\wedge\mathbb{V}'$, i.e.

$$
\epsilon(L'_{i_1}\wedge\cdots\wedge L'_{i_M})\omega = L'_{i_1}\wedge\cdots\wedge L'_{i_M}\wedge\omega,
$$

(142) follows.

In order to calculate $[Q, L^{\mathcal{G}}{}_n]$ it is convenient to first calculate

$$
\begin{aligned}
[b_{k+m}c_{-m}c_{-k}, L^{\mathcal{G}}{}_n] &= (m+k-n)b_{m+n+k}c_{-m}c_{-k} \\
&+ (2n-m)b_{m+k}c_{n-m}c_{-k} \\
&+ (2n-k)b_{m+k}c_{-m}c_{n-k}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_m \sum_{k+m\le -1} (m-k)[b_{k+m}c_{-m}c_{-k}, L^{\mathcal{G}}{}_n] \\
&= \sum_m \sum_{m+k\le -1} (m-k)(m+k-n)b_{m+n+k}c_{-m}c_{-k} \\
&+ \sum_m \sum_{m+k\le -1} (m-k)(2n-m)b_{m+k}c_{n-m}c_{-k} \\
&+ \sum_m \sum_{m+k\le -1} (m-k)(2n-k)b_{m+k}c_{-m}c_{n-k},
\end{aligned}
$$

which, after suitable changes of summation indices, gives

$$
\begin{aligned}
&\sum_m \sum_{k+m\le -1} (m-k)[b_{k+m}c_{-m}c_{-k}, L^{\mathcal{G}}{}_n] \\
&= \sum_m \sum_{k\le -1-m} (m-k)(m+k-n)b_{m+n+k}c_{-m}c_{-k} \\
&+ \sum_m \sum_{k\le -1-m-n} (m+n-k)(n-m)b_{m+n+k}c_{-m}c_{-k} \\
&+ \sum_m \sum_{k\le -1-m-n} (m-k-n)(n-k)b_{m+n+k}c_{-m}c_{-k} \\
&= \sum_m \left(\sum_{k\le -1-m} - \sum_{k\le -1-m-n} \right) (m-k)(m+k-n)b_{m+n+k}c_{-m}c_{-k}.
\end{aligned}
$$

We consider the **case** $n \geq 0$, the other case is analogous. Hence

$$
\begin{aligned}
&\sum_m \sum_{k+m\le -1} (m-k)[b_{k+m}c_{-m}c_{-k}, L^{\mathcal{G}}{}_n] \\
&= \sum_m \left(\sum_{-1-m-n<k\le -1-m} \right) (m-k)(m+k-n)b_{m+n+k}c_{-m}c_{-k}.
\end{aligned}
$$

We similarly get

$$
\begin{aligned}
&\sum_m \sum_{k+m>-1} (m-k)[c_{-m}c_{-k}b_{k+m}, L^{\mathcal{G}}{}_n] \\
&= -\sum_m \sum_{-1-m-n<k\le -1-m} (m-k)(m+k-n)c_{-m}c_{-k}b_{m+n+k} \\
&= -\sum_m \sum_{-1-m-n<k\le -1-m} (m-k)(m+k-n)b_{m+n+k}c_{-m}c_{-k} \\
&+ 2\sum_{k=1}^{n}(n+k-1)(k-1-2n)c_n,
\end{aligned}
$$

where we have commuted through the anti-ghost in the last step. So these results yield

$$
\begin{aligned}
&-\frac{1}{2}\sum_{m,k}(m-k)[:b_{k+m}c_{-m}c_{-k}:, L^{\mathcal{G}}{}_n] \\
&= -\sum_{m}\sum_{k+m>-1}(m-k)[c_{-m}c_{-k}b_{k+m}, L^{\mathcal{G}}{}_n]
\end{aligned}
$$

$$
-\sum_{m}\sum_{k+m\leq -1}(m-k)[b_{k+m}c_{-m}c_{-k}, L^{\mathcal{G}}{}_n]
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n}(n+k-1)(2n-k+1)c_n \\
&= \frac{26}{12}n(n^2-\frac{1}{13})c_n.
\end{aligned}
$$

We move on to the other terms of $[Q, L^{\mathcal{G}}{}_n]$. We have:

$$
-[c_0, L^{\mathcal{G}}{}_n] = -2nc_n = -\frac{26}{12}\frac{12}{13}nc_n
$$

and

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}} L^{\mathcal{M}}{}_m[c_{-m}, L^{\mathcal{G}}{}_n] &= \sum_{m\in\mathbb{Z}}(n-m)L^{\mathcal{M}}{}_{m+n}c_{-m} \\
&= \sum_{m\in\mathbb{Z}}\left([L^{\mathcal{M}}{}_n, L^{\mathcal{M}}{}_m] - \frac{D}{12}\delta_{m+n}n(n^2-1)\right)c_{-m} \\
&= \sum_{m\in\mathbb{Z}}[L^{\mathcal{M}}{}_n, L^{\mathcal{M}}{}_m]c_{-m} - \frac{D}{12}n(n^2-1)c_n
\end{aligned}
$$

Hence

$$
[Q, L^{\mathcal{G}}{}_n] = \sum_{m\in\mathbb{Z}}[L^{\mathcal{M}}{}_n, L^{\mathcal{M}}{}_m]c_{-m} + \frac{26-D}{12}n(n^2-1)c_n.
$$

Since also

$$
[Q, L^{\mathcal{M}}{}_n] = \sum_{m\in\mathbb{Z}}[L^{\mathcal{M}}{}_m, L^{\mathcal{M}}{}_n]c_{-m},
$$

adding all these together yields

$$
[Q, L^{\mathcal{T}}{}_n] = \frac{26-D}{12}n(n^2-1)c_n,
$$

which finishes the proof. $\square$

This Proposition allows us the prove the previously referenced result:

Corollary 5.1.1. Q is nilpotent if and only if $D = 26$.

Proof. We have, for any n,

$$[Q^2, b_n] = [Q, \{Q, b_n\}].$$

By first applying relation (141) and then relation (143), we hence get

$$[Q^2, b_n] = [Q, L^{\mathcal{T}}{}_n] = \frac{26 - D}{12} n(n^2 - 1)c_n.$$

From which it becomes clear; if Q is nilpotent, then $D = 26$.

For the other way around, we argue as follows: Q^2 has ghost number 2. Meaning that Q^2 is of the form,

$$Q^2 = \sum_{k,l} A_{k,l} c_k c_l,$$

for some $A_{k,l}$'s being operators of ghost number zero. But for such,

$$\begin{aligned}
[A_{k,l} c_k c_l, b_n] &= [A_{k,l}, b_n] c_k c_l + A_{k,l} [c_k c_l, b_n] \\
&= [A_{k,l}, b_n] c_k c_l + A_{k,l} \left(c_k \{c_l, b_n\} - \{c_k, b_n\} c_l \right) \\
&= [A_{k,l}, b_n] c_k c_l + A_{k,l} \left(\delta_{n+l} c_k - \delta_{n+k} c_l \right),
\end{aligned}$$

implying that

$$\begin{aligned}
\sum_{k,l} [A_{kl} c_k c_l, b_n] &= \sum_{k,l} [A_{kl}, b_n] c_k c_l + \sum_k A_{k,-n} c_k - \sum_l A_{-n,l} c_l && (146) \\
&= \sum_{k,l} [A_{kl}, b_n] c_k c_l + \sum_k \left(A_{k,-n} - A_{-n,k} \right) c_k. && (147)
\end{aligned}$$

Now suppose that $D = 26$. Then $[Q^2, b_n] = 0$ and of the form (147). So that, in particular, the last sum in (147) must vanish. But that happens only if the $A_{k,l}$'s are symmetric under exchange of indices. This means that

$$Q^2 = \sum_{k,l} A_{k,l} c_k c_l = 0.$$

Thus finishing the proof. $\square$

For any subalgebra $\mathbb{H} \subseteq \mathbb{V}_0$, we define the **semi-infinite forms relative to** $\mathbb{H}$ as

$$C_\infty(\mathbb{V}, \mathbb{H}; \mathcal{F}^{\mathcal{M}}) := \{\omega \in C_\infty(\mathbb{V}; \mathcal{F}^{\mathcal{M}}) : \iota(x)\omega = \theta(x)\omega = 0, \ \forall x \in \mathbb{H}\}.$$

By Proposition 5.2.1,

$$
\begin{aligned}
Q\iota(x)\omega &= \theta(x)\omega + \iota(x)Q\omega,\\
Q\theta(x)\omega &= \theta(x)Q\omega,
\end{aligned}
$$

hence Q stabalizes $C_\infty(\mathbb{V},\mathbb{H};\mathcal{F}^{\mathcal{M}})$. So it makes sense to talk about the subcohomology, the **relative semi-infinite cohomology**, denoted by $H_\infty(\mathbb{V},\mathbb{H};\mathbb{M})$. The relative subcohomologies which interests us are those corresponding to $\mathbb{H}=\mathbb{V}_0$ and $\mathbb{H}=\mathbb{C}c$ respectively. Where the latter corresponds to the differential complex of the bosonic String as presented in section 4.3. However, the former will be the one in which we will find the physical state space.[61]

> **Remark 5.2.1.** Remember,
>
> $$c^{\mathcal{T}} = (D-26)I.$$
>
> Hence, if $D\neq 26$, then both subcomplexes are trivial. Meaning that the physics would be trivial. Notice furthermore, because of (143), that any (sub-)cohomology is a subrepresentation of the Virasoro algebra if and only if $D=26$. These are yet other reasons for requiring 26 dimensions for the bosonic String.

Our goal is the No-ghost theorem. Our road towards this goes through the decomposition theorem. Hence we need to define a hermitian structure and a J-operator on each of these relative subcomplexes. These subcomplexes clearly inherits the hermitian structure already defined for the full structure. However, the construction of this hermitian form involves the terms we wish to remove[62], and thus is non-desirable. Furthermore, in order for the proof of the decomposition theorem to carry over to $C_\infty(\mathbb{V},\mathbb{H};\mathcal{F}^{\mathcal{M}})$ we need the J to stabilize $C_\infty(\mathbb{V},\mathbb{H};\mathcal{F}^{\mathcal{M}})$. But

$$\iota(x)J\omega = J\epsilon(x')\omega$$

and J is injective, so $J\omega,\omega\in C_\infty(\mathbb{V},\mathbb{H};\mathcal{F}^{\mathcal{M}})$ only if $\epsilon(x')\omega=0$ for all $x\in\mathbb{H}$. But for our subcomplexes this means

$$0=\epsilon(c')\iota(c)\omega = \{\epsilon(c'),\iota(c)\}\omega=\omega,$$

so only the zero vector is stabilized. So we need modifications.

We proceed as follows: Since we are considering subcomplexes, we can con-

[61] See Remark 4.3.4.

[62] c' in the $\mathbb{H}=\mathbb{C}c$-case and c', L_0' in the $\mathbb{H}=\mathbb{V}_0$-case

struct respective hermitian products by switching ω_c for

$$\iota(c)\omega_c = i\epsilon(L_0')\omega_0$$

in the $\mathbb{H} = \mathbb{C}c$ case, and for

$$\iota(L_0)\iota(c)\omega_c = i\omega_0$$

in the $\mathbb{H} = \mathbb{V}_0$ case.[63] We abbreviate both cases by

$$i[\epsilon(L_0')]\omega_0.$$

J then gets modified to

$$\begin{aligned} &J\epsilon(x') = \iota(x)J, \qquad && J\iota(x) = \epsilon(x')J, \\ &J\omega_0 = i[\epsilon(L_0')]\omega_0, \end{aligned} \tag{148}$$

for every $x \in$ span $\{L_m\}_{m\in\mathbb{Z}\setminus\{0\}}\,[\cup\{L_0\}]$. The proof that this indeed defines a J-operator follows the blueprint set by the proof of Theorem 4.3. But as remarked before in the case of the full complex, (140), in order for J to reverse the ghost number,

$$\begin{aligned} &\text{Deg } \omega_0 \overset{!}{=} -\text{Deg } [\epsilon(L_0')]\omega_0 = -\left([1] + \text{Deg } \omega_0\right) \\ &\qquad\Longleftrightarrow \\ &\text{Deg } \omega_0 \overset{!}{=} -[1]/2. \end{aligned} \tag{149}$$

So in Deg $\omega_0 = 0$ for the $\mathbb{H} = \mathbb{V}_0$-case and Deg $\omega_0 = -1/2$ for the $\mathbb{H} = \mathbb{C}c$-case.[64] With all this now set, we get that the decomposition theorem holds also for these relative subcomplexes.

Remark 5.2.2. From this it furthermore follows that Deg of $C_\infty(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$ agrees with the ghost number (the eignevalues of $N_{\mathcal{G}}$) from section 4.3. Deg of $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ in fact agrees with the eigenvalues of $N_{\mathcal{G}} - \frac{1}{2}(c_0b_0 - b_0c_0)$. Since we will be mostly concerned with the latter, we will denote it by $N_{\mathcal{G}}$ while the other switches notation to ${N_{\mathcal{G}}}^c$

Regarding the respective hermitian structures, we may in the $\mathbb{H} = \mathbb{V}_0$-case set

$$\langle \omega_0, \omega_0 \rangle = 1 \tag{150}$$

[63] Which is okay here since $\{L_m\}_{m\in\mathbb{Z}\setminus\{0\}}\,[\cup\{L_0\}]$ are bases in these respective cass. See [10].

[64] This is the reason for the half-integer numbering of the bosonic String BRST complex of section 4.3.

and extending as was done in section 5.1.3, since we here avoid the potential caveat of contradiction presented in Remark 5.1.3. Similarly, for the $\mathbb{H} = \mathbb{C}c$-case we may set

$$\langle \omega_0, \epsilon(L'_0)\omega_0 \rangle = 1, \tag{151}$$

noticing that this agrees with the inner product of the textbook ghost Fock space of the bosonic String from section 4.3.1.[65]

5.3 The No-ghost theorem

As mentioned, our interest in $C_\infty(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$ comes from it being the BRST complex of the bosonic string as seen in section 4.3. But why are we then interested in the subcomplex $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$? As mentioned in 4.3, the physical space of the bosonic String is $H_\infty^{-1/2}(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$ indeed, in the sense that is contains the correct vector states. It does however come with the wrong inner product[66]. However, in the identification of $H_\infty^{-1/2}(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$ as our physical space in 4.3 we were even led to conclude that the physical vectors are part of $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, as seen in Remark 4.3.4. As we will show in Theorem 5.2, their connection goes even further than that. Theorem 5.2 will justify us in identifying the zeroth order cohomology space of $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ as our physical space.

But before we present Theorem 5.2 we make some preliminary observations about the relative subcomplex $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$.[67] Consider the BRST operator (83), it can be decomposed as

$$Q = \mathcal{Q} + c_0 L^{\mathcal{T}}{}_0 - T b_0, \tag{152}$$

where

$$T := \sum_{m \in \mathbb{N}} m c_{-m} c_m, \tag{153}$$

isolating the parts containing the bc-ghost zero modes, i.e $\mathcal{Q}$ does not contain b_0 nor c_0. We notice that $\mathcal{Q}$, $L^{\mathcal{T}}{}_0$ and T all, but $\mathcal{Q}$ with it self, (anti)commute amongst each other. For $\mathcal{Q}$, since Q is nilpotent, we get

$$\mathcal{Q}^2 = T L^{\mathcal{T}}{}_0.$$

Hence we see that $\mathcal{Q}$ is nilpotent on Ker $L^{\mathcal{T}}{}_0$. So it is in particular nilpotent on $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$. Since also $Q = \mathcal{Q}$ on $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, we may identify $\mathcal{Q}$ as the BRST operator on $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$. On $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ we have thus

[65] Which we furthermore showed to have to give a trivial physical theory in Remark 4.3.5.
[66] Remark 4.3.5
[67] These observations are highly inspired by [15].

identified,

$$H_\infty(\mathcal{Q}) = H_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}}),$$

where $H_\infty(\mathcal{Q})$ denotes the $\mathcal{Q}$-cohomology on $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$.

Taking the J-adjoint of Q, yields

$$Q^\dagger = \mathcal{Q}^\dagger + b_0 {L^{\mathcal{T}}}_0 - T^\dagger c_0. \tag{154}$$

Again, $\mathcal{Q}^\dagger$, ${L^{\mathcal{T}}}_0$ and T all, but $\mathcal{Q}^\dagger$ with it self, (anti)commute amongst each other and

$$\left(\mathcal{Q}^\dagger\right)^2 = T^\dagger {L^{\mathcal{T}}}_0.$$

So $\mathcal{Q}^\dagger$ too is nilpotent on $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$.

The following lemma will be essential in proving Theorem 5.2.

Lemma 5.3.1. There exists a canonical injection

$$M : H_\infty^{g+\frac{1}{2}}(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}}) \to H_\infty^g(\mathcal{Q}) \oplus H_\infty^{g+1}(\mathcal{Q}) \tag{155}$$

for any $g \in \mathbb{Z}$.

Before we prove this, we make the following convenient remark.

Remark 5.3.1. In the proofs of Lemma 5.3.1 and Theorem 5.2 we will rely heavily in the decomposition theorem. It lets us uniquely identify any element of a BRST cohomology with an element $\mathrm{Ker}\, Q \cap \mathrm{Ker}\, Q^\dagger$, known as a **$Q$-harmonic** element. We will hence implicitly work with the Q-harmonic representative of a cohomology equivalence class.

Proof. Let $\mathcal{F}^{\mathcal{G}}$ denote the subspace of $\bigwedge_\infty \mathbb{V}'$ in which the L_0''s and c''s are removed, i.e. $\mathcal{F}^{\mathcal{G}}$ is spanned by semi-infinite forms of the form

$$L'_{i_1} \wedge L'_{i_2} \wedge \cdots, \tag{156}$$

where $L'_{i_k} \neq L'_0, c'$ for each k. We equip $\mathcal{F}^{\mathcal{G}}$ with the hermitian form of $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$.[68] Take any vector $\Psi \in C_\infty(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$. Since ${N_{\mathcal{G}}}^c$ commutes with ${L^{\mathcal{T}}}_0$ we may assume without loss of generality that Ψ is an eigenvectors of both ${N_{\mathcal{G}}}^c$ and ${L^{\mathcal{T}}}_0$.[69] Now, $\{b_0, Q\} = {L^{\mathcal{T}}}_0$. Hence, if $Q\Psi = 0$ and Ψ has

[68] The construction of the inner product on $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ is really a construction of an inner product on $\mathcal{F}^{\mathcal{G}}$. $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ is just the zero ${L^{\mathcal{T}}}_0$-level subspace of $\mathcal{F}^{\mathcal{G}}$.

[69] For instance we may assume without loss of generality that is a basis vector of the canonical basis, since these are are simultaneous eigenvectors of ${N_{\mathcal{G}}}^c$ and ${L^{\mathcal{T}}}_0$.

non-zero ${L^{\mathcal{T}}}_0$-eigenvalue, denoted L, then

$$\Psi = L^{-1}{L^{\mathcal{T}}}_0\Psi = L^{-1}Qb_0\Psi.$$

In which case, Ψ is BRST exact. So, since we in the end are only interested in elements of the cohomology, within which such Ψ's are zero, we may without loss of generality restrict to $\Psi \in \text{Ker } {L^{\mathcal{T}}}_0$.[70] Now, every such Ψ is of the form

$$\begin{aligned}\Psi &= \{b_0, c_0\}\Psi \\ &= \underbrace{b_0c_0\Psi}_{=:\psi^0} + c_0\underbrace{b_0\Psi}_{=:\psi^c},\end{aligned}$$

where hence both $\psi^0, \psi^c \in \mathcal{F} := \mathcal{F}^{\mathcal{M}} \otimes \mathcal{F}^{\mathcal{G}}$. We suppose furthermore that Ψ has ${N_{\mathcal{G}}}^c$-ghost number $g + \frac{1}{2}$. Then ψ^0 has $N_{\mathcal{G}}$-ghost number $g+1$ and ψ^c has $N_{\mathcal{G}}$-ghost number g.

We have

$$\begin{aligned}Q\Psi &= \mathcal{Q}\psi^0 + \mathcal{Q}c_0\psi^c - T\psi^c \\ &= \mathcal{Q}\psi^0 + c_0\mathcal{Q}\psi^c - T\psi^c.\end{aligned}$$

Hence: $Q\Psi = 0$ if and only if[71]

$$\begin{aligned}\mathcal{Q}\psi^0 - T\psi^c &= 0 \\ \mathcal{Q}\psi^c &= 0.\end{aligned} \tag{157}$$

We also have

$$\begin{aligned}Q^\dagger\Psi &= \mathcal{Q}^\dagger\Psi + {L^{\mathcal{T}}}_0 b_0\Psi - c_0T^\dagger\Psi \\ &= \mathcal{Q}^\dagger\Psi + b_0{L^{\mathcal{T}}}_0\Psi - T^\dagger c_0\Psi \\ &= \mathcal{Q}^\dagger\Psi - c_0T^\dagger\Psi \\ &= \mathcal{Q}^\dagger\psi^0 - c_0\mathcal{Q}^\dagger\psi^c - c_0T^\dagger\psi^0.\end{aligned}$$

Hence $Q^\dagger\Psi = 0$ if and only if

$$\begin{aligned}\mathcal{Q}^\dagger\psi^c - T^\dagger\psi^0 &= 0 \\ \mathcal{Q}^\dagger\psi^0 &= 0.\end{aligned} \tag{158}$$

We wish to define a map

$$M : H_\infty^{g+\frac{1}{2}}(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}}) \to H_\infty^{g+1}(\mathcal{Q}) \oplus H_\infty^{g}(\mathcal{Q}). \tag{159}$$

[70] Notice further that this is one of the requirements on the vectors in $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$.

[71] Since $c_0\mathcal{Q}\psi^c$ contains a c_0 term, while the others do not, so they are linearly independent.

Hence Ψ is taken as Q-harmonic, and thus satisfies (157) and (158). Hence $[\psi^0]^\dagger \in H^{g+1}(\mathcal{Q}^\dagger)$ and $[\psi^c] \in H^g(\mathcal{Q})$. Now, we define M as

$$M : \Psi \in H_\infty^{g+\frac{1}{2}}(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}}) \mapsto \left([\psi^0]^\dagger, [\psi^c]\right) \in H^{g+1}(\mathcal{Q}^\dagger) \oplus H^g(\mathcal{Q}).$$

We need to show that M is injective. By definition, $\Psi \in \text{Ker } M$ if and only if $[\psi^0]^\dagger = [\psi^c] = 0$, i.e. $\psi^0 \in \text{Im } \mathcal{Q}^\dagger$ and $\psi^c \in \text{Im } \mathcal{Q}$. Hence conditions (157) and (158) read respectively

$$\begin{aligned} \mathcal{Q}\psi^0 - T\psi^c &= 0, \\ \mathcal{Q}^\dagger\psi^c - T^\dagger\psi^0 &= 0. \end{aligned} \tag{160}$$

So in order for M to be injective we need to show that this system of equations only has solution $\psi^0 = \psi^c = 0$. With this in mind, we define the operator

$$D := \begin{pmatrix} -T^\dagger & \mathcal{Q}^\dagger \\ \mathcal{Q} & -T \end{pmatrix}, \tag{161}$$

and think of it as a linear map on $\text{Im } \mathcal{Q}^\dagger \oplus \text{Im } \mathcal{Q}$. D even is an endomorphism on $\text{Im } \mathcal{Q}^\dagger \oplus \text{Im } \mathcal{Q}$, since

$$\mathcal{Q}\psi^0 - T\psi^c \in \text{Im } \mathcal{Q}$$

and

$$\mathcal{Q}^\dagger\psi^c - T^\dagger\psi^0 \in \text{Im } \mathcal{Q}^\dagger$$

due to

$$[\mathcal{Q}, T] = [\mathcal{Q}^\dagger, T^\dagger] = 0$$

and since $\psi^0 \in \text{Im } \mathcal{Q}^\dagger$ and $\psi^c \in \text{Im } \mathcal{Q}$. (160) may hence be equivalently written as

$$0 = D \begin{pmatrix} \psi^0 \\ \psi^c \end{pmatrix}. \tag{162}$$

We may consider the adjoint operator

$$D^\dagger = \begin{pmatrix} -T & \mathcal{Q}^\dagger \\ \mathcal{Q} & -T^\dagger \end{pmatrix}, \tag{163}$$

which also is an endomorphism on $\text{Im } \mathcal{Q}^\dagger \oplus \text{Im } \mathcal{Q}$. We note here we are considering Ψ's in the kernel of ${L^{\mathcal{T}}}_0$[72] at ghost number $g+1/2$, we are hence considering M as linear a map from a finite dimensional vector space[73], and hence its image is finite dimensional. That is D and $D^\dagger$ are viewed as endomorphism on finite

[72] ${L^{\mathcal{T}}}_0$-eigenvalue 0

[73] See discussion following Remark 4.3.2

dimensional vector spaces. Hence the following result from linear algebra holds,

$$\dim \text{ Ker } D = \dim \text{ Ker } D^\dagger = \dim \text{ Ker } DD^\dagger. \tag{164}$$

So it suffices to show that Ker $DD^\dagger$ is trivial to show that M is injective.

We have

$$DD^\dagger = \begin{pmatrix} \mathcal{Q}^\dagger\mathcal{Q} + T^\dagger T & 0 \\ 0 & \mathcal{Q}\mathcal{Q}^\dagger + TT^\dagger \end{pmatrix}, \tag{165}$$

which follows from a straight forward calculation utilizing that $[\mathcal{Q}, T] = [\mathcal{Q}^\dagger, T^\dagger] = 0$. From which it follows that $DD^\dagger$ is non-negative. Now,

$$\left\langle \psi^0 \oplus \psi^c, DD^\dagger\psi^0 \oplus \psi^c \right\rangle_J = \|T\psi^0\|_J^2 + \|\mathcal{Q}\psi^0\|_J^2 + \|T^\dagger\psi^c\|_J^2 + \|\mathcal{Q}^\dagger\psi^c\|_J^2, \tag{166}$$

which hence only vanishes if all terms vanish separately.[74] In particular this means that

$$\psi^0 \in \text{Ker } \mathcal{Q} \cap \text{Im } \mathcal{Q}^\dagger \quad \text{and} \quad \psi^c \in \text{Ker } \mathcal{Q}^\dagger \cap \text{Im } \mathcal{Q}.$$

Since Ker $\mathcal{Q} \perp$ Im $\mathcal{Q}^\dagger$ and Ker $\mathcal{Q}^\dagger \perp$ Im $\mathcal{Q}$, as is easily shown[75], ψ^0 and ψ^c would have to be perpendicular to them selves. Hence we must have $\psi^0 = \psi^c = 0$. So that the kernel of $DD^\dagger$ is trivial, and hence so is Ker D. So M is injective.

Lastly, by the decomposition theorem $H^{g+1}(\mathcal{Q}^\dagger) \simeq H^{g+1}(\mathcal{Q})$, hence M induces the desired injection. □

Theorem 5.2. If the vanishing theorem holds for $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^\mathcal{M})$[a], then

$$H_\infty^{\pm 1/2}(\mathbb{V}, \mathbb{C}c; \mathcal{F}^\mathcal{M}) \simeq H^0(\mathcal{Q}). \tag{167}$$

[a]which it does for $p \neq 0$.

Proof. The theorem is proved if we prove that the injection, M, from Lemma 5.3.1 also is a surjection. That is, for any $\left([\psi^0], [\psi^c]\right) \in H^1(\mathcal{Q}^\dagger) \oplus H^0(\mathcal{Q})$[76] we need to find a Q-harmonic element

$$\tilde{\Psi} = \tilde{\psi}^0 + c_0\tilde{\psi}^c$$

such that

$$[\tilde{\psi}^0] = [\psi^0], \qquad [\tilde{\psi}^c] = [\psi^c].$$

[74]Remember, $\langle\cdot,\cdot\rangle_J$ was constructed to be positive definite.
[75]Follows from, $\left\langle \psi\cdot, \mathcal{Q}^\dagger\phi \right\rangle_J = \langle \mathcal{Q}\psi\cdot, \phi\rangle_J$.
[76]Where it is important to note that the supindex in ψ^0 does not denote its ghost number.

First, because of the vanishing theorem, $[\psi^0] = 0$. Hence we chose $\tilde{\psi}^0 = 0$.

Second, the vanishing theorem also implies that any ϕ^c such that $[\psi^c] = [\psi^c + \mathcal{Q}\phi^c]$ must be in Im $\mathcal{Q}$, since ϕ^c has $N_{\mathcal{G}}$-ghost number -1. Thus $[\psi^c] = \{\psi^c\}$. In turn meaning that ψ^c must be $\mathcal{Q}$-harmonic, since there is only one element in the equivalence class.[77] Hence M is a surjection if and only if

$$\tilde{\Psi} = c_0\psi^c$$

is Q-harmonic, i.e.

$$T\psi^c = 0, \tag{168}$$

which is the relations (157 - 158) for $\tilde{\Psi}$. Since T raises the the ghost number by 2, the vanishing theorem implies that $T\psi^c \in$ Im $\mathcal{Q}$, i.e. $T\psi^c = \mathcal{Q}\phi$ for some $\phi \in \mathcal{F}_1$. But then the vanishing theorem gives $\phi \in$ Im $\mathcal{Q}$. Hence (168) indeed holds. So M defines a surjection. $\square$

Result 5.1. Theorem 5.2 lets us identify the physical vectors of the $C_\infty(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$-complex with vectors in $H^0_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$. The $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$-complex has the advantage that we may apply the methods from the generic BRST case. So we set

$$\mathcal{H}_{\text{phys}} := H^0_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}}). \tag{169}$$

Notice that $\mathcal{H}_{\text{phys}}$ with its hermitian form[a] possesses all the correct involutive properties of our theory. It further more avoids the caveat of Remark 4.3.5, since zero ghost number vectors indeed may have non-zero norm. Also, the vacuum ω_0 is by construction normalized.

[a]Constructed in the paragraph following Remark 5.2.2.

The No-ghost theorem of the bosonic string hence says that $H^0_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ contains no negative probabilities.

Let $\mathcal{F}^{\mathcal{G}}$ denote the same space as in the proof of Theorem 5.2. Then $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ correspond to the $L^{\mathcal{T}}{}_0$-level 0 vectors of $\mathcal{F} = \mathcal{F}^{\mathcal{M}} \otimes \mathcal{F}^{\mathcal{G}}$. So assuming the the vanishing theorem holds for $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, in the notation of Result 4.2, the No-ghost theorem for the bosonic String holds if and only if

$$\text{Ch } \mathcal{F}(0) = \text{Sgn } \mathcal{F}(0). \tag{170}$$

Hence,

[77]We may by the decompsition theorem always find such a representative.

Result 5.2. The No-ghost theorem of the bosonic String holds if[a]

$$\sum_\lambda q^\lambda \mathrm{Ch}\mathcal{F}(\lambda) = \sum_\lambda q^\lambda \mathrm{Sgn}\mathcal{F}(\lambda), \tag{171}$$

i.e. if[b]

$$\mathrm{Tr}_\mathcal{F}(-1)^{N_\mathcal{G}} q^{L^\mathcal{T}{}_0} = \mathrm{Tr}_\mathcal{F} J q^{L^\mathcal{T}{}_0}. \tag{172}$$

[a]Notice that we here settle with a sufficient but not necessary condition.
[b]By Remark 4.2.3.

Remark 5.3.2. The reason why Result 5.2 is so useful is that it allows us to apply the multiplicative property of the character and the sign (i.e. the trace) over tensor products. For $\mathcal{F}$ such a decomposition is right at hand, and even a suitable basis of simultaneous eigenvectors of $N_\mathcal{G}$ and $L^\mathcal{T}{}_0$. For $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^\mathcal{M})$ it is not.

We show that (172) indeed holds: Since the trace is multiplicative over tensor products, we get

$$\mathrm{Tr}_\mathcal{F}(-1)^{N_\mathcal{G}} q^{L^\mathcal{T}{}_0} = \mathrm{Tr}_{\mathcal{F}^\mathcal{M}} q^{L^\mathcal{M}{}_0} \cdot \mathrm{Tr}_{\mathcal{F}^\mathcal{G}}(-1)^{N_\mathcal{G}} q^{L^\mathcal{G}{}_0}.$$

We have already calculate the matter factor, it is the q-character of $\mathcal{F}^\mathcal{M}$ (49), i.e

$$\mathrm{Tr}_{\mathcal{F}^\mathcal{M}} q^{L^\mathcal{M}{}_0} = q^{p^2/2} \prod_{n\in\mathbb{N}} (1-q^n)^{-26}. \tag{173}$$

For the ghost part we first notice that (130) with $C_0 = C_1 = 0$ provides an orthonormal[78] basis of $\mathcal{F}^\mathcal{G}$ simultaneous eigenvectors of $N_\mathcal{G}$ and $L^\mathcal{G}{}_0$. For ease of notation let (130) denote this basis, where, again, it is understood that $C_0 = C_1 = 0$. We calculate the q-character of the ghost part:

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^\mathcal{G}}\left(q^{L^\mathcal{G}{}_0}(-1)^{N_\mathcal{G}}\right) \\
&= \sum_{B,C} \left\langle G^B_C\omega_0, q^{L^\mathcal{G}{}_0}(-1)^{N_\mathcal{G}} G^B_C\omega_0 \right\rangle_J \\
&= \sum_{B,C} q^{\sum_{m\in\mathbb{N}} m(C_{m+1}+B_m)-1} (-1)^{\sum_{n\in\mathbb{N}}(C_{n+1}-B_n)} \left\langle G^B_C\omega_0, G^B_C\omega_0\right\rangle_J \\
&= \sum_{B,C} q^{\sum_{m\in\mathbb{N}} m(C_{m+1}+B_m)-1} (-1)^{\sum_{n\in\mathbb{N}}(C_{n+1}-B_n)}
\end{aligned}$$

where we have applied that $G^B_C\omega_0$ are eigenvectors of both $N_\mathcal{G}$ and $L^\mathcal{G}{}_0$, whose

[78]with respect to $\langle\cdot,\cdot\rangle_J$

respective eigenvalues are given by[79]

$$\sum_{n\in\mathbb{N}}(C_{n+1}-B_n), \tag{174}$$

and[80]

$$\sum_{m\in\mathbb{N}} m(C_{m+1}+B_m)-1; \tag{175}$$

and in the last step used that this basis is orthonormal. By some rearranging,

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^{\mathcal{G}}}\left(q^{L^{\mathcal{G}}{}_0}(-1)^{N_{\mathcal{G}}}\right)\\
&=q^{-1}\prod_{m\in\mathbb{N}}\sum_{C_{m+1},B_m=0,1}(-1)^{C_{m+1}-B_m}q^{m(C_{m+1}+B_m)}\\
&=q^{-1}\prod_{m\in\mathbb{N}}\sum_{C_{m+1}=0,1}\left((-1)^{C_{m+1}}q^{mC_{m+1}}+(-1)^{C_{m+1}-1}q^{m(C_{m+1}+1)}\right)\\
&=q^{-1}\prod_{m\in\mathbb{N}}\left(1-q^m-q^m+q^{2m}\right)\\
&=q^{-1}\prod_{m\in\mathbb{N}}\left(1-q^m\right)^2
\end{aligned}$$

Multiplying this with the matter q-character (173) hence gives

$$\mathrm{Tr}_{\mathcal{F}}(-1)^{N_{\mathcal{G}}}q^{L^{\mathcal{T}}{}_0}=q^{p^2/2-1}\prod_{n\in\mathbb{N}}(1-q^n)^{-24}. \tag{176}$$

We move on to $\mathrm{Tr}_{\mathcal{F}}Jq^{L^{\mathcal{T}}{}_0}$, the q-signature, for which we again can utilize that the trace is multiplicative over tensor products. We have already calculated the matter part. It is the q-signature of $\mathcal{F}^{\mathcal{M}}$, (50),

$$\mathrm{Tr}_{\mathcal{F}^{\mathcal{M}}}Jq^{L^{\mathcal{M}}{}_0}=q^{\frac{p^2}{2}}\prod_{n\in\mathbb{N}}(1+q^n)^{-1}(1-q^n)^{-25}. \tag{177}$$

For the ghost part we have

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^{\mathcal{G}}}\left(Jq^{L^{\mathcal{G}}{}_0}\right)\\
&=\sum_{B,C}\left\langle G^B_C\omega_0, Jq^{L^{\mathcal{G}}{}_0}G^B_C\omega_0\right\rangle_J\\
&=q^{-1}\prod_{m\in\mathbb{N}}\sum_{C_{m+1},B_m=0,1}q^{\sum_{n\in\mathbb{N}}n(C_{n+1}+B_n)}\left\langle G^B_C\omega_0, JG^B_C\omega_0\right\rangle_J\\
&=q^{-1}\prod_{m\in\mathbb{N}}\sum_{C_{m+1},B_m=0,1}q^{\sum_{n\in\mathbb{N}}n(C_{n+1}+B_n)}\left\langle G^B_C\omega_0, J^2G^B_C\omega_0\right\rangle,
\end{aligned}$$

[79]By Proposition 4.3.2, noting that where we in this complex have only the case $C_0=C_1=0$ and the term $-1/2$ is replaced by 0.

[80]By 94

where we again have made use of the form of the eigenvalues of $L^{\mathcal{G}}{}_0$, (175) and, in the last step, the definition of $\langle\cdot,\cdot\rangle_J$. In fact, since J is idempotent, the last factor is just the inner product of $\langle\cdot,\cdot\rangle$. The $\langle\cdot,\cdot\rangle$-product on $\mathcal{F}^{\mathcal{G}}$ takes the form (134) but with the C_0, C_1-dependence removed, i.e.

$$\left\langle G^B_C\omega_0, G^{\widetilde{B}}_{\widetilde{C}}\omega_0\right\rangle = (-1)^{N_B N_{\widetilde{B}}}\delta_{B-T\widetilde{C}}\delta_{\widetilde{B}-TC}.$$

Which in our case boils down to

$$\langle G^B_C\omega_0, G^B_C\omega_0\rangle = \prod_{n\in\mathbb{N}}(-1)^{B_n}\delta_{C_{n+1}-B_n}.$$

We apply this to evaluate the q-signature of $\mathcal{F}^{\mathcal{G}}$ and perform some rearranging,

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{F}^{\mathcal{G}}}\left(Jq^{L^{\mathcal{G}}{}_0}\right)\\
&= q^{-1}\prod_{m\in\mathbb{N}}\sum_{C_{m+1},B_m=0,1} q^{\sum_{n\in\mathbb{N}} n(C_{n+1}+B_n)}\prod_{n\in\mathbb{N}}(-1)^{B_n}\delta_{C_{n+1}-B_n}\\
&= q^{-1}\prod_{m\in\mathbb{N}}\sum_{C_{m+1},B_m=0,1} q^{m(C_{m+1}+B_m)}(-1)^{B_m}\delta_{C_{m+1}-B_m}\\
&= q^{-1}\prod_{m\in\mathbb{N}}\sum_{B_m=0,1} q^{2mB_m}(-1)^{B_m}\\
&= q^{-1}\prod_{m\in\mathbb{N}}\left(1-q^{2m}\right)\\
&= q^{-1}\prod_{m\in\mathbb{N}}\left(1+q^m\right)\left(1-q^m\right),
\end{aligned}$$

where in the third step we have performed sum over the Kronecker delta $\delta_{C_{m+1}-B_m}$. So multiplying this with the q-signature of the matter part (177) gives

$$\mathrm{Tr}_{\mathcal{F}}Jq^{L^{\mathcal{T}}{}_0} = q^{\frac{p^2}{2}-1}\prod_{n\in\mathbb{N}}(1-q^n)^{-24}. \tag{178}$$

All in all, comparing (176) to (178) we have shown that $\mathrm{Tr}_{\mathcal{F}}(-1)^{N_{\mathcal{G}}}q^{L^{\mathcal{T}}{}_0} = \mathrm{Tr}_{\mathcal{F}}Jq^{L^{\mathcal{T}}{}_0}$ indeed. Hence:

Result 5.3. If the vanishing theorem holds for $C_\infty(\mathbb{V},\mathbb{V}_0;\mathcal{F}^{\mathcal{M}})$, then the No-ghost theorem of the bosonic String indeed holds.

But it still remains to prove the vanishing theorem for $H_\infty(\mathbb{V},\mathbb{V}_0;\mathcal{F}^{\mathcal{M}})$.

5.3.1 Vanishing theorem for the relative subcomplex

We follows the lines of the corresponding meterial in [11] chapter 6.

We simplify the notation by setting

$$C^g := C^g_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$$

and

$$C := \bigoplus_g C^g.$$

We may bigrade each C^g by defining the grading,

$$C^g = \bigoplus_{c-b=m} C^{b,c}, \tag{179}$$

where $C^{b,c}$ is spanned by semi-infinite forms with b number of missing elements of $\mathbb{V}'_-$ and c number of missing elements from $\mathbb{V}'_+$.[81] We hence have a decomposition

$$C = \left(\mathcal{F}^{\mathcal{M}} \otimes \bigwedge \mathbb{V}'_+ \otimes \bigwedge\nolimits_\infty \mathbb{V}'_-\right)^{\mathbb{V}_0},$$

where the supindex '$^{\mathbb{V}_0}$' mean the $\mathbb{V}_0$-invariant vectors in the representation, i.e. the vectors in the kernel of ${L^{\mathcal{T}}}_0$, which indeed are those we are interested in.[82]

We construct a **filtration degree** on

$$\mathcal{F}^{\mathcal{M}} \otimes \bigwedge \mathbb{V}'_+ \otimes \bigwedge\nolimits_\infty \mathbb{V}'_-, \tag{180}$$

a grading which is equal to deg from before but also can be considered individually on the individual components of (180). Let

$$\text{Fdeg}\,(m \otimes \omega_+ \otimes \omega_-) := \deg\, m - \deg\, \omega_+ + \deg\, \omega_-,$$

where $m \in \mathcal{F}^{\mathcal{M}}$, $\omega_+ \in \bigwedge \mathbb{V}'_+$ and $\omega_- \in \bigwedge_\infty \mathbb{V}'_-$. deg stands for the degree as before, all but for the deg ω_+-term, for which

$$\deg \bigwedge_{k=1}^{N} L'_{i_k} := \sum_{k=1}^{N} i_k.$$

It follows that

$$\begin{aligned} \text{Fdeg}\, {L^{\mathcal{M}}}_n &= n \\ \text{Fdeg}\, b_n &= -|n| \\ \text{Fdeg}\, c_n &= |n| \end{aligned}$$

[81] This clearly is compatible with the ghost number grading of the complex.

[82] Strictly speaking, they should also be in the kernel of $c^{\mathcal{T}}$, but $c^{\mathcal{T}} = 0$ in our considerations ($D = 26$).

for any $n \neq 0$.[83] From which we get

$$\text{Fdeg }\left(L^{\mathcal{M}}{}_n c_{-n}\right) = n + |n|$$
$$\text{Fdeg }(: b_{m+n} c_{-m} c_{-n} :) = |m| + |n| - |m+n|.$$

Hence every term of Q has non-negative filtration degree.

Now define

$$F^p C := \{\omega \in C : \text{Fdeg } \omega \geq p\}.$$

Then clearly $F^{p+1}C \subseteq F^p C$ and $\cup_p F^p C = C$. Such a set of subspaces $FC := \{F^p C\}_p$ is generally known as a **filtration**. From the similarities of Fdeg and deg it follows that there are p_0 and p_1 such that

$$F^p C = \begin{cases} C & \text{if} \quad p \leq p_0 \\ 0 & \text{if} \quad p \geq p_1 \end{cases}, \tag{181}$$

a property known as **filtration**[84]. Furthermore, since Q has non-negative filtration degree, $QF^pC \subseteq F^pC$, an extra property for a filtrated differential complex making it a **filtered differential complex**. That is, each F^pC is a complex under Q.

For any filtration FC we may construct the **associated graded space**

$$\text{Gr } C := \bigoplus_p \text{Gr}^p C,$$

where

$$\text{Gr}^p C := \frac{F^p C}{F^{p+1} C}. \tag{182}$$

It follows that Gr C and C are isomorphic as vector spaces [11]. We notice that this construction is not only possible for C but for the cohomology of C, $H(C)$ as well, yielding its associated graded space Gr $H(C)$. For which we also have

$$\text{Gr } H(C) \simeq H(C). \tag{183}$$

For (FC, Q) a filtered differential complex (as in our case), Q induces a differential on Gr C, which preserves its grading, i.e.

$$Q(\text{Gr}^p C) \subseteq \text{Gr}^p C.$$

Hence in the grading of Gr C we get that Q has degree 0. We denote the

[83] The $n = 0$ case is not part of our relative subcomplex.
[84] Which is an adjective, not to be confused with a 'filtration', FC, which is a noun.

cohomology of the complex $(\mathrm{Gr}\ C, Q)$ by $H(\mathrm{Gr}\ C)$. In the complex $(\mathrm{Gr}\ C, Q)$ it is simpler to calculate the cohomology than in (C, Q) and (FC, Q). This since the parts of Q having positive filtration degree takes $F^pC \to F^{p+1}$, i.e. maps to zero in $\mathrm{Gr}\ C$.

In order for us to take advantage of this simpler nature of $(\mathrm{Gr}\ C, Q)$ we make use of the concept of spectral sequences. A **spectral sequence** is a sequence $\{(E_r, d_r)\}_{r=0}^{\infty}$ of differential complexes for which E_{r+1} is the cohomology of the preceding, i.e.

$$E_{r+1} = H(E_r) := \frac{\mathrm{Ker}\ d_r}{\mathrm{Im}\ d_r}. \tag{184}$$

The spectral sequence is said to **spectral sequence**[85] to E_∞ if there exists $R \in \mathbb{N}$ such that $E_r = E_{r+1} = E_\infty$ for all $r > R$. One writes $(E_r) \Rightarrow E_\infty$. For our purposes, i.e proving the vanishing theorem, the usefulness of spectral sequences stems from (184). This since if we have a spectral sequence converging to our considered cohomology and we at some stage of this spectral sequence can show that its cohomology vanishes, then we have effectively shown that our considered cohomology vanishes as well.

Result 5.4. By Theorem II.1.32 from [11], there exists for our filtered complex FC a spectral sequence $\{(E_r, d_r)\}$ of graded spaces

$$E_r = \bigoplus_p E_r^p$$

with

$$d_r : E_r^p \to E_r^{p+r}$$

and such that

$$\begin{aligned} E_0^p &= \mathrm{Gr}^p C \\ E_1^p &= H\left(\mathrm{Gr}^p C\right) \\ E_\infty^p &= \mathrm{Gr}^p H\left(C\right). \end{aligned}$$

That is, there exists a spectral sequence which spectral sequences to the cohomology of C[86], with

$$E_1 := \bigoplus_p E_1^p = H\left(\mathrm{Gr}\ C\right),$$

[85] An adjective.

[86] As mentioned before, (183) $H(C) \simeq \mathrm{Gr}\ H\left(C\right)$.

where we emphasize, $H\left(\mathrm{Gr}\ C\right)$ is the cohomology of the filtered complex $(\mathrm{Gr}\ C, Q)$.[87] So by Result 5.4, all we need to show in order to prove the vanishing cohomology theorem is that $E_1 = 0$. Since then

$$0 = \bigoplus_p E_\infty^p = \mathrm{Gr}\ H(C) \simeq H(C).$$

So we aim towards doing so.

As already mentioned, (182) tells us that the only non-zero components of the from the Q induced differential on Gr C, denoted Q_0, are those of zero filtration degree. Hence,

$$\begin{aligned} Q_0 = &\sum_{m\in\mathbb{N}} L^{\mathcal{M}}{}_{-m} c_m - \frac{1}{2}\sum_{m,n\in\mathbb{N}} (n-m) b_{-m-n} c_m c_n \\ &- \frac{1}{2}\sum_{m,n\in\mathbb{N}} (m-n) b_{m+n} c_{-m} c_{-n}, \end{aligned}$$

noticing that the normal ordering drops out. By definition

$$\mathbb{V}_\pm = \mathrm{Span}\ \{L_m : \pm m \in \mathbb{N}\}.$$

The first two terms may be identified with the differential of the semi-infinite cohomology $H(\mathbb{V}_-; \mathcal{F}^{\mathcal{M}})$, denoted Q_-, and the third with the differential in $\wedge\mathbb{V}'_+$ computing the ordinary Lie algebra cohomology of $\mathbb{V}_+$, denoted Q_+.[88] Hence E_1 is the cohomology of the complex $K^{\mathbb{V}_0}$ (under Q_0), where

$$K := \wedge\mathbb{V}'_+ \otimes C_\infty(\mathbb{V}_-; \mathcal{F}^{\mathcal{M}}).$$

Notice that K inherits the bigrading (179) from C, i.e.

$$K^{b,c} := \left(\wedge\mathbb{V}'_+\right)^c \otimes C_\infty^b(\mathbb{V}_-; \mathcal{F}^{\mathcal{M}}).$$

Now, by Proposition 5.2.1, $[Q_0, \mathbb{V}_0] = 0$, hence

$$H(K^{\mathbb{V}_0}) = H(K)^{\mathbb{V}_0}.$$

Applying the Künneth formula[89] we get, keeping track of the bigrading,[90]

$$E_1^{b,c} = \left(H^c(\mathbb{V}_+) \otimes H_\infty^b(\mathbb{V}_-; \mathcal{F}^{\mathcal{M}})\right)^{\mathbb{V}_0}.$$

[87] The ghost number grading is here left implicit, i.e. we are really considering the different E_1^g's of specific ghost numbers.
[88] $\mathbb{V}_\pm$ are both Lie subalgebras of $\mathbb{V}$.
[89] See (II.1.56) in [11]
[90] Here, the supindices now denotes the bigrading related to the ghost number grading, so not the filtration degree.

In [11][91] it is shown that

$$H_\infty^{b\neq 0}(\mathbb{V}_-;\mathcal{F}^{\mathcal{M}}) = 0. \tag{185}$$

In the proof of which $\mathcal{F}^{\mathcal{M}}$ being a Verma module becomes a necessity. Now, since $E_1^g = \oplus_{c-b=m} E_1^{b,c}$, c being non-negative, gives

$$E_1^{g<0} = 0.$$

So by the properties of spectral sequences, we must have

$$H_\infty^{g<0}(\mathbb{V},\mathbb{V}_0;\mathcal{F}^{\mathcal{M}}) = 0.$$

Hence, by the decomposition theorem, it holds for $g > 0$ as well. So we obtain the vanishing cohomology theorem:

Result 5.5. The vanishing theorem of the complex $C_\infty(\mathbb{V},\mathbb{V}_0;\mathcal{F}^{\mathcal{M}})$ hold, i.e.

$$H_\infty^{g\neq 0}(\mathbb{V},\mathbb{V}_0;\mathcal{F}^{\mathcal{M}}) = 0. \tag{186}$$

[91] In Appendix A.

6 Conclusion

We summarize the main results and conclusions.

In the textbook treatments of the BRST quantized bosonic String it (at least in the authors opinion) appears like $C_\infty(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$ should be identified with the Strings BRST complex. Given the involutive properties of that complex[92] the resulting hermitian form, $\langle \cdot, \cdot \rangle$, is uniquely defined up to complex multiplicative factor[93]. We constructed a J-operator, from which we constructed a the positive definite hermitian form $\langle \cdot, \cdot \rangle_J$. From which we got that the 1/2-ghost grading becomes a necessity[94] in agreement with the textbook treatment.[95] But we identify $H^{-1/2}(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$ as the physical space of this complex, and the inner product $\langle \cdot, \cdot \rangle$ of this complex is non-zero only if the vectors have opposite ghost number[96], hence the inner would be trivial on this physical space[97]. Usually one identifies zero ghost number cohomology as the physical space, for which every vector has opposite ghost number to itself, so the inner product does not have to be trivial. But the 1/2-grading is forced upon us, so we cannot even choose it differently as to circumvent this problem. We can in particular not even normalize the vacuum[98], i.e. set $\langle \omega_0, \omega_0 \rangle = 1$.

However, because of Theorem 5.2[99], the physical vectors of $C_\infty(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$ are identified with the physical vectors of $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, i.e.

$$H^{-1/2}(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}}) \simeq H^0(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}}).$$

Since the hermitian structure of $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ still possessed the correct involutive properties and, additionally, the vacuum is normalizable and the inner product is not a priori trivial on $H^0(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$[100]; we are led to identify $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ as the BRST complex of the bosonic String. This identification is also made in the standard litterature[101]. So it is not something new or unknown. It is however the authors opinion that the identification made possible by Theorem 5.2 makes the treatment less ad-hoc, the physical spaces being identical as vector spaces, but the integer-graded complex having the more physically desirable inner product

$C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ furthermore satisfies the properties classifying it as a generic BRST complex, in accordance with section 4. We are hence able to apply

[92] See (74).
[93] section 5.1.3
[94] See (149).
[95] We note further that our choice of J in constructing the positive defininte hermitian form is not unique. However, any such J would still have to map $\mathcal{F}_g$ to $\mathcal{F}_{-g}$ in order to induce a positive definite inner product (see (54). So (149) would still hold.
[96] By Remark 51.
[97] See Remark 4.3.5.
[98] See Remark 5.1.3.
[99] Which required the vanishing theorem (Result 5.5) for $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$.
[100] Since the vectors here have ghost number 0
[101] [3] pg. 135.

Result 4.2, the reformulation of the No-ghost theorem. This reformulation is useful since it allows us to prove the No-ghost theorem (in a relatively short calculation) by calculating traces over the whole complex[102], as opposed to needing to isolate a basis of $H^0(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$.

Both the identification of the BRST complex of the bosonic String as $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$ and the proof of its No-ghost theorem relied on the vanishing of its cohomology. The proof of which[103] required $\mathcal{F}^{\mathcal{M}}$ to be a Verma module, which by Result 3.1 meant $\mathcal{F}^{\mathcal{M}}$ having non-zero D-momentum. For the case of 0 D-momentum, the cohomology can however be explicitly calculated [10].

[102] See (172)

[103] Section 5.3.1

Nomenclature

$*$ Denotes the $\langle \cdot, \cdot \rangle$-adjoint, the one of the physical inner product, page 37

$\bigwedge_\infty \mathbb{V}'$ Space of semi-infinite forms of the Virasoro algebra, page 46

$\mathcal{Q}$ The equivalent operator to Q when considering the subcomplex $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, page 66

$\dagger$ Denotes the $\langle \cdot, \cdot \rangle_J$-adjoint, page 29

$\epsilon(x)$ The semi-infinite form counter part to the ghost, page 46

$\mathcal{F}$ Defined as $\mathcal{F}^{\mathcal{M}} \otimes \mathcal{F}^{\mathcal{G}}$, page 67

$\mathcal{F}_g$ Subspace of $\mathcal{F}$ of ghost number g, page 69

$\mathcal{F}^{\mathcal{G}}$ The subspace of $\bigwedge_\infty \mathbb{V}'$ in which the L_0''s and c''s are removed, equipped with the hermitian form of $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, page 67

$\mathcal{F}^{\mathcal{M}}$ The Fock space of the matter part of the quantized String, where the D-momentum subindex $\mathbf{p}$ has been omitted, page 12

$\iota(x)$ The semi-infinite form counter part to the anti-ghost, page 46

$\langle \cdot, \cdot \rangle_J$ Defined as $\langle \cdot, J(\cdot) \rangle$, page 28

Deg ω_0 For the String BRST complex case, page 65

Deg The ghost number grading on $\bigwedge_\infty \mathbb{V}'$, page 56

deg , page 57

Gr C Defined as $\bigoplus_p \mathrm{Gr}^p C$, page 74

$\mathrm{Gr}^p C$ Defined as $\frac{F^p C}{F^{p+1} C}$, page 74

$N_{\mathcal{G}}$ Ghost number operator for $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, page 65

${N_{\mathcal{G}}}^c$ Ghost number operator for $C_\infty(\mathbb{V}, \mathbb{C}c; \mathcal{F}^{\mathcal{M}})$, page 65

$\mathbb{V}_0 = \mathrm{Span}_{\mathbb{C}}\{c, L_0\}$, page 45

$\mathbb{V}_\pm$ Defined as Span $\{L_m : \pm m \in \mathbb{N}\}$, page 75

C Defined as $C_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, page 72

C^m Defined as $C^m_\infty(\mathbb{V}, \mathbb{V}_0; \mathcal{F}^{\mathcal{M}})$, page 72

$C^{b,c}$ Defined as the subspace of C^m spanned by semi-infinite forms with b number of missing elements of $\mathbb{V}'_-$ and c number of missing elements from $\mathbb{V}'_+$, page 72

$C_\infty(\mathbb{V},\mathbb{H};\mathcal{F}^{\mathcal{M}})$, page 64

c_n, b_m Ghost respectively anti-ghost, page 37

$C_\infty^m(\mathbb{V};\mathcal{F}^{\mathcal{M}})$ Defined as $\mathcal{F}^{\mathcal{M}} \otimes \bigwedge_\infty^m \mathbb{V}'$, page 58

$C_\infty^{m;n}(\mathbb{V};\mathcal{F}^{\mathcal{M}})$ The subspace of $C_\infty(\mathbb{V};\mathcal{F}^{\mathcal{M}})$ of fixed Deg $= m$ and deg $= n$ degrees , page 58

F^pC Subspace of C of filtration degree p, page 74

FC Defined as $\{F^pC\}_p$, page 74

$H(C)$ Short notation for the cohomology of C with respect to Q, page 74

$H_\infty^m(\mathbb{V},\mathcal{F}^{\mathcal{M}})$ The subspace of $H_\infty(\mathbb{V},\mathcal{F}^{\mathcal{M}})$ of Deg $= m$ degree., page 58

$H_\infty^{m;n}(\mathbb{V},\mathcal{F}^{\mathcal{M}})$ The subspace of $H_\infty^m(\mathbb{V},\mathcal{F}^{\mathcal{M}})$ of deg $= n$ degree., page 58

$H_\infty(\mathbb{V};\mathcal{F}^{\mathcal{M}})$ The cohomology of the complex $\big(C_\infty^m(\mathbb{V};\mathcal{F}^{\mathcal{M}}), Q\big)$, page 58

$H_\infty\,(\mathcal{Q})$ The $\mathcal{Q}$-cohomology on $C_\infty(\mathbb{V},\mathbb{V}_0;\mathcal{F}^{\mathcal{M}})$, page 66

J For the String BRST complex case, page 64

M The injection from Lemma 5.3.1, page 68

Q The BRST differential/operator, page 57

T Defined as $\sum_{m\in\mathbb{N}} c_{-m}c_m$, page 66

$\bigwedge_\infty^{m;n}\mathbb{V}'$ The subspace of $\bigwedge_\infty^m \mathbb{V}'$ of deg $= n$-vectors, page 57

$\bigwedge_\infty^m\mathbb{V}'$ The subspace of $\bigwedge_\infty \mathbb{V}'$ of Deg $= m$-vectors, page 56

Fdeg The filtration degree, page 73

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