Technische Universität Müchen
Ludwig-Maximillians-Universtät Müchen

The role of high nucleon density on neutrino production in exploding Supernovae

Sara Maggio

Thesis submitted
within the Elite Master Course
Theoretical and Mathematical Physics

Supervisor: Prof. Dr. Andreas Weiler
Second Referee: Dr. Pietro Baratella

28 November 2022

Declaration

I hereby certify that the thesis I am submitting is entirely my own original work except where otherwise indicated. I am aware of the University's regulations concerning plagiarism, including those regulations concerning disciplinary actions that may result from plagiarism. Any use of the works of any other author, in any form, is properly acknowledged at their point of use.

Sara Maggio

Acknowledgements

This Master's has been hard. From the very beginning, November 2020, when I moved from Rome to Munich in the high point of the pandemic, to the very last with the PhD application processes and the various end points. Luckily I have never been alone and I have always had many people that supported me.

I would like to thank my group T75, in particular Michael for his everyday help, for listening to me every time I was unsure about something and for the happy moments when we understood something together and Pietro for the useful discussions, for the support and for helping me entering the world of scattering amplitudes.

Many thanks are for Matthias, the person that has been with me since my very first day in Munich. He is not only my reference point when talking about physics, but he has always supported me especially during the various crisis in the Master's thesis year. Without him everything would have been way less happier.

Thanks to the people in my office, Julian was always there to both have a chat about physics and to help me going through the academic world. Mar and Asmaa are very nice and caring persons, I am very happy to have them in my daily life.

I want to thank Arianna, with whom I shared the period of my thesis, frustrations, but also bouldering and nice moments. Getting to know her in Munich was a fortunate coincidence.

Thanks to Giacomo. Talking to him has always been helpful and funny, he has always supported me, from when we met.

Thanks to Sheila and Leila, my favourite sisters. They made Munich my home from the very beginning.

Thanks to my friends Pablo, José and François, bouldering with them has been my favourite activity for more than 6 months now.

Thanks to my Master's and TMP friends, in particular Juanma, Beatrice, Michelangelo, Damiano, Angelica, Fotis and Marcin, I have learnt so much from them.

Back in Italy, my mother and my father followed all my adventures and decisions, to me it is really important to have their support. Thanks to my sister, her stories bring me always back in the familiar roman atmosphere.

Many thanks go to Davide, he is the first person with whom I really started discussing about physics and very soon became one of the most important people for me. I truly wish for him all the best.

Thanks to le Puellæ, they were my biggest motivation to survive Labs.

Thanks to Giorgia for her support throughout a lifetime, to Claudia, Elisa and Giulia for all our gossips and nice reunions.

Thanks to my relatives among which my grandfathers, grandmother, my uncles and my aunts. They are the ones that give me the most unconditioned support.

Finally, I would like to thank the German Academic Exchange Service (DAAD) for supporting my studies under the funding programme Study Scholarships for Graduates of All Disciplines, as well as the Elite Network of Bavaria.

Thesis advisor: Prof. Dr. Andreas Weiler *Sara Maggio*

The role of high nucleon density on neutrino production in exploding Supernovae

ABSTRACT

Chiral perturbation theory is an effective field theory for QCD, which allows us to describe baryons and mesons in the non-relativistic limit. Gauging its $SU(2)_L \times SU(2)_R$ group, we can add fields and external currents in the theory such as axions and electroweak gauge bosons. Density effects can be systematically included by modifying the nucleon propagator.

In this thesis, we systematically build the chiral Lagrangian describing the interaction of neutrinos with hadronic matter. We then use this construction to study the neutrino production processes in high nucleon density environments such as Supernovae and Neutron stars where neutral currents are affected by the high density of nucleons. We calculate the density dependence of this process at one loop order.

Contents

Chapter 1

Motivation

In this work we are looking at neutrinos in an environment characterized by an high density of protons and neutrons (nucleons). In this way we can study objects present in the universe such as Supernovae and Neutron stars.
Supernovae (SNe) are luminous optical transients, which, despite being rare events in a single galaxy, are nowadays discovered at a rate of ten per day. They have became a leading tool to probe the cosmic expansion, indeed SNe 1a provided the proof that the cosmic expansion is accelerated by what is now called dark energy. SNe can be used also as primary distance indicators: we can measure the value of the Hubble-Lemaître constant, H_0, with an error of only 1% [1]. Moreover, core-collapse supernova explosions seem anthropically needed as they are thought to be one among the few sources responsible for production of the intermediate-mass nuclei e.g. O, Fe, Ne, Na, mg, Al, that are the ones necessary for human life [2].
A very important role inside a SN is played by neutrinos, indeed a quantitatively accurate prediction of the fluxes and spectra of SN neutrinos is required for a theoretical understanding of many fascinating phenomena such as the explosion mechanism, the cause of neutron stars natal kicks, various aspects of SN nucleosynthesis and the interpretation of the neutrino signal with or without the assumption of neutrino masses and mixing [3]. Consequently, neutrinos are fundamental in the description of the SN, it is fundamental and interesting to study their scattering with the nucleons inside the SN. In doing so, we need to include the high density effect. In Figure 1.1 we show the densities that are reached during a core-collapse, we immediately realize that they need to be considered to have a consistent analysis of the processes inside a SN. In particular, neutrinos interact in two ways: with charged and neutral currents. In this work we focus on neutral current interactions because they are more affected by high density effects with respect to charged current interactions [3].
The dominant processes are extensively discussed in [3], here I want to briefly recall them. The elastic scattering $\nu N \to N\nu$ is the dominant SN opacity source for both μ and τ neutrinos. The dominant energy- and number-changing process is the bremsstrahlung $\nu\bar{\nu}NN \leftrightarrow NN$ and, to be more precise, for only the energy exchange, the related process $\nu NN \leftrightarrow NN\nu$ is even more effective by about a factor of 10.
Another reason to systematically study neutrino-nucleons interactions is that successful explosions are hard to get without other further assumptions [5]. The authors also

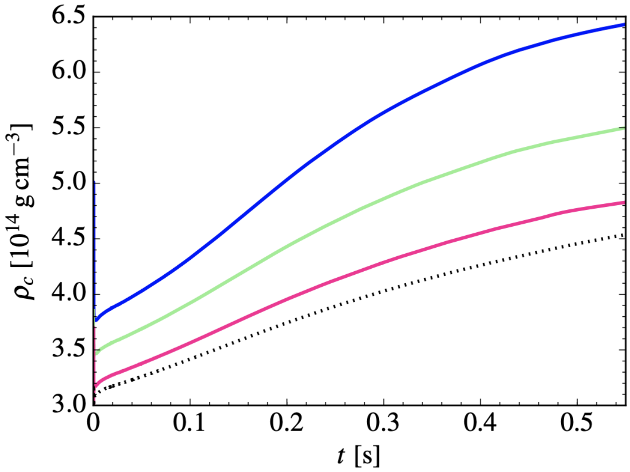


Figure 1.1: Plot of the densities during the core-collapse of a SN [4]. We trust our effective field theory up to around 1 nuclear saturation density, $n_0 \approx 3 \cdot 10^{14} g/cm^3$.

underline the importance of the study of neutral-current scattering since it is this one that gives relevant contributions for the neutrino-energy deposition behind the stalled shock wave.

These are the main reasons why we started this project: in this work we are systematically studying high density corrections using heavy baryon chiral perturbation theory and a modified nucleon propagator to account for the high density of nucleons. There are other approaches to tackle similar questions using the method of random phase approximation [6], where they study particle-hole excitations through a dynamic form factor.

However, it is worth mentioning that SNe are not the only environment interesting to study with this formalism. There is much experimental data on low-momentum nucleons and pions, particles that are well described in this formalism.

Chapter 2

Supernova explosion

Here we briefly summarize the mechanism that leads to the Supernova explosion and how the neutrinos play a fundamental role. We follow [2], so for more interested readers that paper is a nice reference. Before starting, is worth mentioning that the standard theoretical picture of core-collapse supernova explosions has not been established yet, but what we have now is consistent with observation, for example of neutrinos from SN1987A.

Everything starts from the collapse. The dynamics of stars is controlled by a balance between gravity and the energy released in nuclear fusion reactions, that are responsible for the formation of the heavier elements starting from lighter ones. The first stage is the conversion of the hydrogen in the core into helium. As soon as it can no longer react, gravity is not balanced anymore and so the core starts contracting. This causes a rise in the temperature of the core up to when the helium fusion begins. Now, the mass of the star plays a role: for stars with mass $m < 8m_{\odot}$ the fusion processes ends with the creation of an electron degenerate carbon core, a White Dwarf is born together with a planetary nebula. For massive stars with $m > 8m_{\odot}$, once helium is exhausted, we have a further raise in the temperature so that carbon burning begins. This is followed by nuclear fusions of neon, oxygen and silicon. Since the temperature raises, each heavier element burns faster than the previous one and neutrinos can carry away more energy, while nuclear reactions releases less energy. We start seeing the role of neutrinos since the very first stage. After the silicon, an iron core in the innermost part of the star is formed. This stops the fusion chain because iron is the most stable element. So we have now the structure of the supernova made by shells of successively lighter elements burning around an iron core, its radius R_{Fe} will be important in the following.

Since the iron core is the innermost shell, it is the electron degeneracy pressure that balances gravity. From this we can compute its maximum mass that is indeed determined by the Chandrasekhar limit, $m_{\mathrm{Ch}} \approx 1.4m_{\odot} \sim Y_e^2 m_{\mathrm{pl}}^3/m_n^2$, where $Y_e \sim 1/2$ is the fraction of electrons per nucleon. But the mass of the iron core is not fixed: as the silicon shell, that is the one surrounding the iron core, continues to burn, the iron mass slowly increases approaching the Chandrasekhar limit above which the electron degeneracy pressure is not enough anymore to balance gravity and the iron core starts collapsing.

Second step: the deleptonization. Here neutrinos come into play, indeed weak interactions contribute to the processes inside the star, and they become very important when the mass of the inert iron is $m \sim M_{\rm Ch}$, value that gives an high enough sub-nuclear density, $\rho \sim 10^{-6} m_n^4$, such that electrons and protons are converted into neutrons and neutrinos. Many elements participate in the inverse β decay reaction but the main contribution is given by $e + {}^{56}\text{Fe} \to {}^{56}\text{Mn} + \nu_e$. There is a difference between electrons and neutrinos: the former supports the core with their degeneracy pressure, while the latter freely escapes carrying energy. Subsequently, the core is de-leptonized, in a time scale $\tau_{weak} \sim 1/\sigma_{weak} n_n \propto v^4$ that is faster than the free-falling time-scale of the gravitational collapse $\tau_{grav} \sim \frac{m_{\rm Pl}}{m_n^2}$, in terms of numbers we indeed have $\tau_{weak} \approx 1/10^3 s \ll \tau_{grav} \approx 0.1 s$. This has an impact on $M_{\rm Ch}$ that suddenly decreases and consequently also the iron core start collapsing. To be more precise, we will have two regions, the innermost part with mass M_{ic} and radius R_{ic}, that is the one that start collapsing and an outer core. A way to define the boundary between these two regions is using the velocity of the collapse. In the inner core, this is proportional to the radial distance, and so we can define the boundary R_{ic} as the point where the inward radial velocity equals the sound speed of the fluid, which, on the contrary, lowers the radial distance.

Third step: rebounce. The inner core continues its collapse till when it reaches nuclear density: at that point, so for distances below 1 fm, the nuclear force is repulsive and so it can balance gravity. R_{ic} can be estimated from the third root of the number of nucleons inside the inner core multiplied by 1 fm obtaining $R_{ic} \approx 20$ km. We have again to do a splitting due to the mass, indeed if we keep the parametric dependence of R_{ic}, we can notice that is similar to the Schwarzschild radius. We need to keep all the coefficients to see that $m > 40 m_\odot$ for the neutron star to collapse into a black hole. However, we are interested in the other case, so for $m < 40 m_\odot$. Here we have R_{ic} decreasing, so, since the sound velocity of the fluid decreases with the radial distance, the sound velocity is bigger than the infalling one and so the inner core bounces. This generates an outward-going shock wave that starts propagating. The first layer that it encounters is the outer more stable part of the iron core, so it starts losing energy due to both iron dissociation and neutrino production via electron capture, so the shock wave stalls at $R_{\rm shock} \sim 100$ km.

Fourth step: neutrino trapping. Let us now focus on what happens to neutrinos during SN collapse dynamics. Matter density keeps increasing so neutrinos do not manage to escape: they get momentarily trapped up to a radius R_ν, of what we can call the neutrino-sphere, that is a bit larger than $R_{\rm ic}$. They play the important role of releasing the gravitational energy produced during the collapse. Two factors need to be taken into account, the time scale for the transmission inside the trapping volume and from the neutrino-sphere surface. Diffusion is what controls the flow of neutrinos inside the volume. We need to estimate the mean free path

$$l_\nu \sim \frac{1}{n_n \sigma_{weak}} \sim \frac{v^4}{m_n^3 T_{\rm ic}^2}, \tag{2.0.1}$$

that, after a precise calculation, results to be $l_\nu < R_{\rm ic}$ that means that neutrino are really trapped. Then we can compute the time scale as a random walk process, so we can estimate the number of steps needed to cover $R_{\rm ic}$ as $N_\nu \sim R_{\rm ic}^2/l_\nu^2$ obtaining for diffusion time

$$\tau_{\text{volume}} \sim \max(N_\nu, 1) l_\nu. \tag{2.0.2}$$

With this, we actually took into account only the diffusion of neutrinos inside the inner core, while before we pointed out that the radius of the neutrino sphere is bigger than the one of the inner core, so we need to analyze what happens there. Is it then possible to estimate R_ν as $n_n(R_\nu)\sigma_{weak} R_\nu \sim 1$, from this compute the power emitted in neutrinos as $L_\nu \sim R_\nu^2 T_\nu^4$ and eventually the cooling time of the surface, that is the time in which neutrinos go beyond the trapping sphere, as

$$\tau_{\text{surface}} = \frac{E_{tot}}{L_\nu} \sim \frac{m_{\text{Pl}}^{3/2}}{v^2 m_n^{1/2}}. \tag{2.0.3}$$

Fifth step: heating. We need now to study what happens with the energy released together with the neutrinos outside the neutrino sphere. The power we just computed heats nucleons immediately outside the neutrino sphere with a rate

$$Q_\nu \sim \sigma_{weak} \frac{L_\nu}{4\pi R_\nu^2}. \tag{2.0.4}$$

This power is split among all the interactions that happen: we have that the outgoing neutrinos heat free nucleons through weak reactions as $n + \nu_e \to e^- + p$ and $p + \bar{\nu}_e \to e^+ + n$. But the nucleons cool down capturing electrons and positrons. The relevant heating is then given by the difference between the two rates and it turns out that there is a radius, R_{gain}, above which the heating dominates. With numerical computations we get $R_{\text{gain}} \approx 3R_\nu$ and that $R_\nu < R_{\text{gain}} < R_{\text{shock}}$ allowing neutrinos to push the shock wave that realises the, delayed, explosion. In Figure 2.1 the important radii are shown.

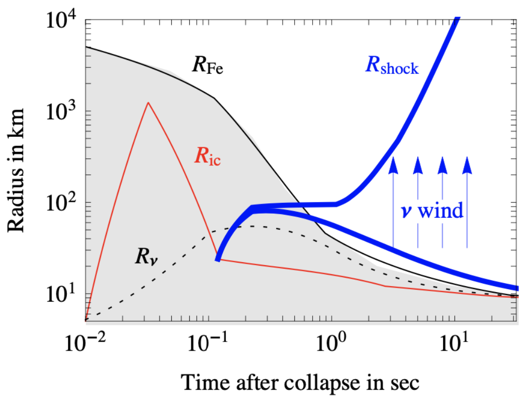


Figure 2.1: Figure from [2]. Relevant radii are shown to visualize the dynamics of the explosion.

This brief summary of a SN explosion does not want to be fully consistent, indeed not everything is discussed, but it aims to give an idea of how relevant are neutrinos. This emphasizes the motivation behind studying the scattering of neutrinos and nucleons accounting for density corrections. Is it worth mentioning that this paper notices that simulations of successful explosion require asphericity, magnetic fields, and so on.

Chapter 3

Goldstone bosons in Chiral Perturbation Theory

This section here is a review to set the basis of our investigation. In Chapter 5, we will generalize ChPT to include external fields and later we will take the heavy baryon limit. Here we will mostly follow Prof. Weiler's notes for QFT 2.

The relevant energy in our discussion is $E < v_{\text{Higgs}}$. This allows us to write the most general low energy effective Lagrangian, even for our strongly interacting theory, low energy QCD. The degrees of freedom won't be quarks and gluons anymore but the Goldstone bosons of a spontaneous breaking of an approximate flavor symmetry. In this case our symmetry is only approximately broken because quarks acquire mass due to the Higgs mechanism.

3.1 Non-linear parametrization

To make the GB-field explicit we can use the so-called non-linear parametrization

$$\Phi(x) = (v + \sigma(x))e^{i\tau^a \pi^a / v}, \tag{3.1.1}$$

were the light degrees of freedom will be $\Sigma(x) = e^{i\tau^a \pi^a / v}$. To keep the Lagrangian invariant the fields have to transform as

$$\begin{aligned} &\sigma \to \sigma, \\ &\Sigma(\pi) \to D_R \Sigma(\pi) D_L^\dagger. \end{aligned} \tag{3.1.2}$$

Plugging Σ inside the Lagrangian we notice that GBs appear only with derivatives so all interactions of GBs are proportional to their momentum and consequently become weak at low energies.

Since we are at low energies we can integrate out the $\sigma(x)$ field. One of the advantages of using the non-linear parametrization is that, since $\sigma(x)$ doesn't transform, the effective field theory retains the $SU(2)_L \otimes SU(2)_R$ symmetry.

3.2 Goldstone boson interactions

We can start looking at an EFT for GBs alone since they are the lightest particles in the spectrum, this means that their energy $E \ll$ mass of other particles. We want to construct the theory using the underlying, spontaneously broken, global symmetry. Each GB correspond to a dof, so we can parametrize the GB excitations in the following way

$$\phi(x) = \Sigma(x)\phi_0 = e^{i\hat{T}^{\hat{A}}\pi^{\hat{A}}(x)/v}\phi_0, \tag{3.2.1}$$

where $\hat{T}^{\hat{A}}$ are the broken generators and ϕ_0 is the vacuum expectation value. Now, we need to study the behaviour under symmetry transformations. Under a global transformation $g \in G$:

$$\phi(x) \to g\phi(x), \tag{3.2.2}$$

thus

$$\Sigma(x) \to g\Sigma(x), \tag{3.2.3}$$

which has not the same form of 3.2.1 because $g\Sigma(x)$ is a generic element of G and so can not be expressed as an exponential of a broken generator only. However, we can write it as

$$g\Sigma(\pi) = \Sigma(\pi')h, \tag{3.2.4}$$

where $h \in H$, form which will be helpful later.
The two matrices $g\Sigma(\pi)$ and $\Sigma(\pi')$ describe the same field configuration which differs by a transformation that leaves the vacuum invariant. So the Goldstone dynamics happens only in the vacuum manifold. Moreover the transformation h is non-trivial because the Goldstone manifold G/H is curved.
Equation (3.2.4) tells us that the transformation of $\Sigma(\pi)$ reads

$$\Sigma(\pi) \to g\Sigma(\pi)h^{-1}(g, \Sigma(\pi)). \tag{3.2.5}$$

We now want to study this transformation. We can always write for any group element $g \in G$

$$g = e^{i\alpha^A T^A} = e^{f_{\hat{a}}[\alpha]\hat{T}^{\hat{a}}} e^{if_i[\alpha]T^i}, \tag{3.2.6}$$

where $\hat{T}^{\hat{a}}$ generate the broken part and T^i the unbroken one. For infinitesimal transformation $|\alpha^A| \ll 1$ this equals

$$1 + i(\alpha^{\hat{a}}\hat{T}^{\hat{a}} + \alpha^i T^i) = (1 + if_{\hat{a}}[\alpha]\hat{T}^{\hat{e}_f})(1 + if_i[\alpha]T^i), \tag{3.2.7}$$

which gives

$$\begin{aligned} \alpha^{\hat{a}} &= f_{\hat{a}}[\alpha] + \mathcal{O}(a^2) \\ \alpha^i &= f_i[\alpha] + \mathcal{O}(a^2). \end{aligned} \tag{3.2.8}$$

We can now transform the GB field (3.2.1)

$$g\phi = g\Sigma(\pi)\phi_0 = \Sigma(\pi')h\phi_0 = \Sigma(\pi')\phi_0, \tag{3.2.9}$$

where we notice that the transformation leaves ϕ in the same form but transforms π

$$\pi \to \pi'. \tag{3.2.10}$$

3.3 General G/H coset

Let us study the G/H coset.
We can decompose the generators $\{T^A\}$ into T^i with $i = 1, ..., \mathrm{dim}H$ generators of the unbroken group H and $\hat{T}^{\hat{a}}$ generators of the broken G/H with $\hat{a} = 1, ..., \mathrm{dim}G - \mathrm{dim}H$

$$\{T^A\} = \{T^i, \hat{T}^{\hat{a}}\}, \tag{3.3.1}$$

this means

$$\begin{aligned} T^i \phi_0 &= 0 \\ T^{\hat{a}} \phi_0 &\neq 0. \end{aligned} \tag{3.3.2}$$

We can write the algebra for the generators starting from

$$[T^A, T^B] = i f^{AB}{}_C T^C. \tag{3.3.3}$$

We have three possible classes of commutators

$$\begin{aligned} [T^i, T^j] &= i f^{ij}{}_k T^k + i f^{ij}{}_{\hat{a}} \hat{T}^{\hat{a}} \\ [T^i, \hat{T}^{\hat{b}}] &= i f^{i\hat{b}}{}_{\hat{c}} \hat{T}^{\hat{c}} + i f^{i\hat{b}}{}_j T^j \\ [\hat{T}^{\hat{a}}, \hat{T}^{\hat{b}}] &= i f^{\hat{a}\hat{b}}{}_c T^c + i f^{\hat{a}\hat{b}}{}_{\hat{c}} \hat{T}^{\hat{c}}, \end{aligned} \tag{3.3.4}$$

not all these terms contribute, it is possible to further simplify. The first equation in (3.3.4) is the commutator of generator of the subgroup H, that has a closed algebra, so

$$[T^i, T^j] = i f^{ij}{}_k T^k \qquad \text{with} \quad i f^{ij}{}_k = (T^i_{adj})^j_k \quad \text{and} \quad f^{ij}{}_{\hat{a}} = 0. \tag{3.3.5}$$

This affects the second equation in (3.3.4) as well, because due to antisymmetry of $f^{AB}{}_C$ also $f^{i\hat{b}}{}_j = 0$, so we get

$$[T^i, \hat{T}^{\hat{b}}] = i f^{i\hat{b}}{}_{\hat{c}} \hat{T}^{\hat{c}}, \qquad \text{with} \quad i f^{i\hat{b}}{}_{\hat{c}} = (T^i_\pi)^{\hat{b}}_{\hat{c}} \quad \text{and} \quad f^{i\hat{b}}{}_j = 0, \tag{3.3.6}$$

where we defined $i f^{i\hat{b}}{}_{\hat{c}} = (T^i_\pi)^{\hat{b}}_{\hat{c}}$, that can be identified as the unbroken generators in the adjoint representation acting on G/H generators.
As regarding the third relation in (3.3.4), we know that for symmetric coset, with an algebra automorphism $\hat{T} \to -\hat{T}$, $f^{\hat{a}\hat{b}}{}_{\hat{c}} = 0$.
Finally, from the standard normalization

$$\mathrm{Tr}\,(T^a T^b) = \frac{1}{2} \delta^{ab}, \tag{3.3.7}$$

we get

$$\begin{aligned} \mathrm{Tr}\,(T^i T^j) &= \frac{1}{2} \delta^{ij}, \\ \mathrm{Tr}\,(T^i \hat{T}^{\hat{a}}) &= 0, \\ \mathrm{Tr}\,(\hat{T}^{\hat{a}} \hat{T}^{\hat{b}}) &= \frac{1}{2} \delta^{\hat{a}\hat{b}}. \end{aligned} \tag{3.3.8}$$

3.4 Transformation properties under G/H and H

Before we studied the symmetry transformation of the GB field under a group element $g \in G$. Having now discussed the generators of H and of G/H, we can look at the transformations under the respective group elements.
Let us start with the unbroken group H. Taking a generic group element $h = e^{iT^i\alpha^i}$, we get

$$\begin{aligned} \phi &\to h\phi, \\ \Sigma(\pi)\phi_0 &\to h\Sigma(\pi)\phi_0 \\ &= h\Sigma(\pi)h^{-1}h\phi_0 \\ &= \Sigma(h\pi h^{-1})\phi_0, \end{aligned} \tag{3.4.1}$$

since

$$\begin{aligned} h\Sigma(\pi)h^{-1} &= h(e^{i\pi^{\hat{a}}\hat{T}^{\hat{a}}})h^{-1} \\ &= h\left(\mathbb{I} + i\pi^{\hat{a}}\hat{T}^{\hat{a}} + \frac{(i\pi^{\hat{a}}\hat{T}^{\hat{a}})^2}{2} + ...\right)h^{-1} \\ &= \left(\mathbb{I} + ih\pi^{\hat{a}}\hat{T}^{\hat{a}}h^{-1} + \frac{(ih\pi^{\hat{a}}\hat{T}^{\hat{a}}h^{-1}ih\pi^{\hat{b}}\hat{T}^{\hat{b}}h^{-1})}{2} + ...\right) \\ &= e^{ih\pi^{\hat{a}}\hat{T}^{\hat{a}}h^{-1}} \\ &= \Sigma(h\pi h). \end{aligned} \tag{3.4.2}$$

This can be further simplified using the BCH-formula

$$e^X Y e^{-X} = \sum_{m=0}^{\infty} \frac{1}{m!}[X, Y]_m, \tag{3.4.3}$$

with $[X, Y]_m = [X, [X, Y]]_{m-1}$ and $[X, Y]_0 = Y$. To understand how this works we can write the first commutators, from (3.3.6) we get

$$[T^i, \hat{T}^{\hat{a}}]_1 = (T^i_\pi)^{\hat{a}}_{\hat{c}} T^{\hat{c}} \tag{3.4.4}$$

and so

$$\begin{aligned} [T^i, \hat{T}^{\hat{a}}]_2 &= [T^i, [T^i, \hat{T}^{\hat{a}}]] \\ &= (T^i_\pi)^{\hat{a}}_{\hat{c}}[T^i, \hat{T}^{\hat{c}}] \\ &= (T^i_\pi)^{\hat{a}}_{\hat{c}}(T^i_\pi)^{\hat{c}}_{\hat{d}}\hat{T}^{\hat{d}}. \end{aligned} \tag{3.4.5}$$

We can now apply all of it to our expression

$$\begin{aligned} h\pi^{\hat{a}}\hat{T}^{\hat{a}}h^{-1} &= e^{i\alpha^i T^i}\pi^{\hat{a}}\hat{T}^{\hat{a}}e^{-i\alpha^j T^j} \\ &= \sum_{n=0}^{\infty}\frac{(i\alpha^i T^i_\pi)^n}{n!}\pi^{\hat{a}}\hat{T}^{\hat{a}} \\ &= e^{i\alpha^i T^i_\pi}\pi^{\hat{a}}\hat{T}^{\hat{a}}, \end{aligned} \tag{3.4.6}$$

so, summarizing we have that

$$\begin{aligned} \phi &\to h\phi, \\ \Sigma(\pi)\phi_0 &\to \Sigma(e^{i\alpha^i T^i_\pi}\pi)\phi_0, \end{aligned} \tag{3.4.7}$$

which shows that the transformation of the GB field under the subgroup H is linear. The representation R_π of the linear transformation under H can be found according to

$$\mathrm{adj}(G) = \mathrm{adj}(H) \oplus R_\pi. \tag{3.4.8}$$

Let's now study the transformation under G/H, the broken generators,

$$\begin{aligned} \phi &\to e^{i\alpha^{\hat{a}}\hat{T}^{\hat{a}}}\phi, \\ \Sigma(\pi)\phi_0 &\to e^{i\alpha^{\hat{a}}\hat{T}^{\hat{a}}} e^{i\pi(x)^{\hat{b}}\hat{T}^{\hat{b}}/v}\phi_0 \\ &= e^{i\pi'(x)^{\hat{b}}\hat{T}^{\hat{b}}/v}\phi_0. \end{aligned} \tag{3.4.9}$$

To get a physical meaning we look at an infinitesimal transformation $|\alpha| \ll 1$,

$$(\mathbb{I} + i\pi^{\hat{a}}T^{\hat{a}}/v + ...) \to \left(\mathbb{I} + i\left(\frac{\pi^{\hat{b}}}{v} + \alpha^{\hat{b}}\right)\hat{T}^{\hat{b}} + ...\right), \tag{3.4.10}$$

where, focusing on the linear order, we read off the shift symmetry

$$\frac{\pi^{\hat{b}}}{v} \to \frac{\pi^{\hat{b}}}{v} + \alpha^{\hat{b}}, \tag{3.4.11}$$

while the higher order terms are a function of the symmetry parameter $\alpha^{\hat{a}}$.

3.5 Maurer-Cartan form and the invariant Lagrangian

Now we understood how the transformations work. We can now build the effective Lagrangian. To do so we notice that there are objects which have simpler transformation properties, we will call them $d^{\hat{a}}_\mu(\pi) \in R_\pi$ and $e^i_\mu(\pi) \in H$. They are defined using the maurer-Cartan-form decomposing on G algebra generators

$$i\Sigma(\pi)^{-1}\partial_\mu\Sigma(\pi) \equiv d^{\hat{a}}_\mu(\pi)\hat{T}^{\hat{a}} + e^i_\mu(\pi)T^i. \tag{3.5.1}$$

We want to study this object's transformation properties under an element of the global group $g \in G$, recalling (3.2.5) we get

$$\begin{aligned} i\Sigma^{-1}\partial_\mu\Sigma &\to ih\Sigma(\pi)^{-1}g^{-1}\partial_\mu(g\Sigma(x)h^{-1}(x)) \\ &= h(i\Sigma^{-1}\partial_\mu\Sigma)h^{-1} + ih\partial_\mu h^{-1}. \end{aligned} \tag{3.5.2}$$

We can already see that the inhomogeneous term purely contains element of the unbroken group, this can be translated into the transformation of the d and e symbol

$$\begin{aligned} d_\mu^{\hat{a}}(\pi)\hat{T}^{\hat{a}} &\to h[\pi,g] d_\mu^{\hat{a}}\hat{T}^{\hat{a}} h^{-1}[\pi,g], \\ e_\mu^i(\pi)T^i &\to h[\pi,g](e_\mu^i T^i + i\partial_\mu)h^{-1}[\pi,g]. \end{aligned} \tag{3.5.3}$$

Recalling equation (3.4.6) we can write the transformation under g for the d symbol as

$$d_\mu^{\hat{a}} \to d_\mu'^{\hat{a}} = (e^{i\beta^i T_\pi^i})_{\hat{b}}^{\hat{a}} d_\mu^{\hat{b}}, \tag{3.5.4}$$

that is the same transformation that the GB field has but under the unbroken group H. From this we understand why is useful to introduce this symbol, because it transform linearly even under the global group G.
Let us now study the e symbol. Since there is an inhomogeneous term in the transformation, it looks the same as a gauge transformation for a non-abelian symmetry under the H group, so, we can use it to construct covariant derivative to make the Lagrangian invariant under the group G. The upshot is that we can obtain all G-invariant operators of the GB-EFT Lagrangian using d_μ, e_μ and applying derivatives ∂_μ. The kinetic term for the GB field is indeed hidden in d: this can be seen expanding the Maurer-Cartan form at linear order in $\hat{T}^{\hat{a}}$

$$i\Sigma^{-1}\partial_\mu\Sigma \simeq -\frac{1}{v}\partial_\mu\pi^{\hat{a}}T^{\hat{a}}, \tag{3.5.5}$$

so

$$d_\mu^{\hat{a}} \simeq -\frac{1}{v}\partial_\mu\pi^{\hat{a}} + \mathcal{O}\left(\frac{\partial\pi}{v}\frac{\pi^2}{v^2}\right). \tag{3.5.6}$$

We can finally write the leading term as

$$\mathcal{L}^{(2)} = \sum_i \frac{f_i^2}{2} d_{\mu,\hat{a}_i} d^{\mu,\hat{a}_i}, \tag{3.5.7}$$

with $\{f_i\}$ are the vevs of the irreps because R_π can be a reducible representation under H.
Now let's look at the e symbol. As we notices before, this can be used to define a covariant derivative

$$\nabla_\mu \equiv \partial_\mu + ie_\mu \tag{3.5.8}$$

and a field strength

$$e_{\mu\nu} = \partial_\mu e_\nu - \partial_\nu e_\mu + i[e_\mu, e_\nu], \tag{3.5.9}$$

such that

$$e_{\mu\nu} \to h(\pi,g) e_{\mu\nu} h^{-1}(\pi,g). \tag{3.5.10}$$

We have now all the terms to construct the invariant effective chiral Lagrangian: d_μ, $e_{\mu\nu}$ and the covariant derivative ∇_μ acting on them.
Another important point is the coupling to matter field. We can use the covariant derivative just defined to get

$$(\nabla_\mu\psi)_r = \partial_\mu\psi_r - ie_{\mu,i}(T^i)_r^s\psi_s, \tag{3.5.11}$$

where the matter field under $g \in G$ transform as

$$\psi_r \to \psi_r^{(g)} = h[\pi, g]_r^s \psi_s. \tag{3.5.12}$$

The Lagrangian coupling Goldstone bosons to fermionic matter consequently reads

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \nabla_\mu \psi - m_\psi \bar{\psi}\psi \tag{3.5.13}$$

3.6 Spontaneous breaking of chiral symmetry

Here we briefly discuss what happens in QCD to understand what a symmetry breaking means and to connect this discussion to our investigation.
The global group is $SU(3)$ that makes QCD a non-abelian gauge theory with quarks in fundamental representation. Is empirically shown that quarks form bound states that use color-singlets, the color charge is confined in neutral composite objects.
As mentioned before, quarks do have a mass, in particular u, d, s are light quarks $m_f \leq \Lambda_{QCD}$ and c, b, t are heavy quarks $m_f \geq \Lambda_{QCD}$. Since we are interested in the low energy spectrum we can ignore the heavy quarks in the following analysis. If we now set $m_f = 0$ we can rotate q_{L_f} and q_{R_f} separately, since they are not connected by the mass term, by unitary matrices $L, R \in U(3)$ in flavor space:

$$\begin{aligned} q_{L_f} &\to L_{ff'} q_{L_{f'}}, \\ q_{R_f} &\to R_{ff'} q_{R_{f'}}, \end{aligned} \tag{3.6.1}$$

and so, for $m_f = 0$ the Lagrangian exhibit a global flavour symmetry

$$U(3)_L \otimes U(3)_R = U(1)_B \otimes U(1)_A \otimes SU(3)_V \otimes SU(3)_A, \tag{3.6.2}$$

where

$$\begin{aligned} U(1)_B: &\quad q_f \to e^{i\beta} q_f, \\ U(1)_A: &\quad q_f \to e^{i\alpha\gamma_5} q_f, \\ SU(3)_V: &\quad q_f \to e^{i\beta^a T^a} q_f, \\ SU(3)_A: &\quad q_f \to e^{i\alpha^a T^a \gamma_5} q_f. \end{aligned} \tag{3.6.3}$$

Now, once we include the mass, $U(1)$ stays unbroken and it is an accidental global symmetry, the baryon number. $U(1)_A$ is anomalous, is not a symmetry of the quantum theory. $SU(3)_V$ is an exact symmetry for $m_f \to 0$ or $m_f = m\mathbb{I}$, the quark masses different values explicitly break $SU(3)_V$ but the subgroup, that we are interested in, (u, d) is less strongly broken since $m_u, m_d \ll m_s$. As regarding $SU(2)_A$ (the subgroup of $SU(3)_A$ that we are considering), the spectrum has 3 excitations: $\pi^0, \pi^\pm$, they are the lightest particles, with negative parity and zero spin, the GBs, corresponding to a broken axial symmetry.

3.7 Effective chiral Lagrangian

We can now study the effective chiral Lagrangian. Above we said that pions come from the breaking of $SU(2)_A$ but to get this breaking we can make a dynamical assumption saying that the effect of the QCD force is to form quark-antiquark condensates

$$\langle\Omega|\bar{q}_L^i q_R^j + h.c.|\Omega\rangle \neq 0, \qquad i = u, d, c, s, b, t. \tag{3.7.1}$$

We can restrict to $q = (u, d)$ and study the symmetry of this new vacuum. If we now transform

$$\begin{aligned} q_L \to& L q_L, \qquad L \in SU(2)_L, \\ q_R \to& R q_R, \qquad R \in SU(2)_R, \end{aligned} \tag{3.7.2}$$

the condensate stays invariant only for $L = R$. Moreover the QCD vacuum does not have a flavour

$$\langle\Omega|\bar{q}_L^i q_R^j + \bar{q}_R^j q_L^i|\Omega\rangle = v_{QCD}^3 \delta^{ij}, \tag{3.7.3}$$

so we can parametrize the GB as

$$\langle\phi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v_{QCD}, \tag{3.7.4}$$

where $\phi \to L\phi R^\dagger$. Indeed in this way we get, as expected, the transformation under the unbroken part of the group with $L = R$

$$\langle\phi\rangle \to L\langle\phi\rangle L^\dagger = \langle\phi\rangle L L^\dagger = \langle\phi\rangle \tag{3.7.5}$$

and so the broken generators $L = R^\dagger$ parametrize the GB fields

$$\Sigma(x) = L(x)\langle\phi\rangle L(x), {}^{1}. \tag{3.7.6}$$

With this parametrization we can write

$$\Sigma(x) = e^{\frac{2i}{f_\pi}\pi(x)}, \qquad \pi(x) = \tau^a \pi^a(x), \quad a = 1, 2, 3, \quad \tau^a : \text{Pauli matrices} \tag{3.7.7}$$

and so also the transformation can be written as

$$\Sigma \to \Sigma' = L\Sigma R^\dagger, \tag{3.7.8}$$

with $L = e^{\frac{i}{2}\omega_L}$ and $R = e^{\frac{i}{2}\omega_R}$, where $\omega_L = \omega_L^a \lambda^a$ and $\omega_R = \omega_R^a \lambda^a$. We can study separately what happens under the unbroken and broken group: under the former (H)

$$\delta\pi = \frac{i}{2}[\omega_V, \pi(x)] + ..., \qquad \omega_L = \omega_R \equiv \omega_V \tag{3.7.9}$$

[1]Here $L(x) = R^\dagger(x)$ depends on x because is the generator of the broken symmetry. Being no more a symmetry of the vacuum can be used to parametrize the Goldstone field. This does not mean that we are gauging the global symmetry, we are not introducing any gauge field.

and under the latter (G/H)

$$\delta\pi = \frac{f_\pi}{2}\omega_A - \frac{1}{v}[\pi,[\pi,\omega_A]] + ..., \qquad \omega_L = -\omega_R \equiv \omega_A. \tag{3.7.10}$$

Again we see that we can only write terms with derivatives of the GB due to the shift symmetry. The unique term with least derivatives is[2]

$$\begin{aligned}\mathcal{L} =& \frac{f_\pi^2}{16}\,\mathrm{Tr}\,[\partial_\mu\Sigma\partial^\mu\Sigma^{-1}] \\ =& \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a + ... \quad .\end{aligned} \tag{3.7.11}$$

Explicit chiral symmetry breaking

The shift symmetry (3.7.10) forbids mass terms for pions, since would be not invariant. Here I want to mention how we can include a mass term.
Reintroducing quark masses we add

$$\mathcal{L}_m = \bar{q}_L^i M_{ij} q_R^j + \text{h.c.}, \qquad M_{ij} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{3.7.12}$$

To keep the Lagrangian invariant we can make the mass matrix transform, m_{ij} can indeed be viewed as a "spurion" of flavour symmetry breaking. We can postulate

$$M \to LMR^\dagger, \tag{3.7.13}$$

so that the Lagrangian stays formally invariant.
We need to include this in the chiral effective Lagrangian and the only term that can be written is

$$\mathcal{L}_m = \frac{v_{QCD}^3}{8}\,\mathrm{Tr}\,(M^\dagger\Sigma + \Sigma^\dagger M) \tag{3.7.14}$$

since $i\,\mathrm{Tr}\,(M^\dagger\Sigma - \Sigma^\dagger M)$ is not invariant under parity since π is a pseudoscalar. Explicitly in $SU(2)$ up to quadratic order we have

$$\Sigma = e^{\frac{2i}{f_\pi}\pi^a\tau^a} = \mathbb{I} + \frac{2i}{f_\pi}\pi^a\tau^a - \frac{4}{2f_\pi^2}\pi^a\tau^a\pi^b\tau^b + ..., \tag{3.7.15}$$

which gives

$$\begin{aligned}\mathrm{Tr}\,(M^\dagger\Sigma + \Sigma^\dagger M) =& \mathrm{Tr}\,(M(\Sigma + \Sigma^\dagger)) \\ =& -\frac{4}{f_\pi^2}\,\mathrm{Tr}\,(M\tau^a\tau^b)\pi^a\pi^b + ... \\ =& -\frac{2}{f_\pi^2}\,\mathrm{Tr}\,(M\{\tau^a,\tau^b\})\pi^a\pi^b + ... \\ =& -2\frac{2}{f_\pi^2}\,\mathrm{Tr}\,(M)\pi^a\pi^b + ... \quad .\end{aligned} \tag{3.7.16}$$

[2]Later when the Z boson will be included, the partial derivative will be substituted by a covariant derivative.

From here we can read off the mass term (equal) for all the 3 pions (for simplicity we rename $v_{QCD} = v$)

$$\mathcal{L}_m \supset -\frac{v^3}{2f_\pi^2}\,\mathrm{Tr}\,M\pi^a\pi^a + ..., \tag{3.7.17}$$

so

$$m_\pi^2 = (m_u + m_d)\frac{v^3}{f_\pi^2}. \tag{3.7.18}$$

Higher orders

In writing the Lagrangian we only kept the leading order with the least number of derivatives

$$\mathcal{L} = \frac{f_\pi^2}{16}\,\mathrm{Tr}\,(\partial_\mu\Sigma)\partial^\mu\Sigma^\dagger + m_0^2\frac{f_\pi^2}{8}\,\mathrm{Tr}\,\Sigma^\dagger M + \mathrm{h.c.}, \qquad m_\pi^2 \equiv (m_u + m_d)m_0^2. \tag{3.7.19}$$

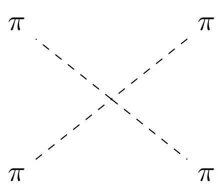
Higher orders are suppressed

$$\begin{aligned} &\mathrm{Tr}\,(\partial^\mu\partial_\nu\Sigma^\dagger\partial_\mu\partial_\nu\Sigma) \to \frac{p^2}{\Lambda^2}, \\ &\mathrm{Tr}\,(\partial^\nu\Sigma^\dagger\partial_\nu\Sigma\Sigma^\dagger m_0^2 m) \to \frac{m_0^2 M}{\Lambda^2}, \end{aligned} \tag{3.7.20}$$

with Λ the cut-off scale, and this can be generalized for more fields as well

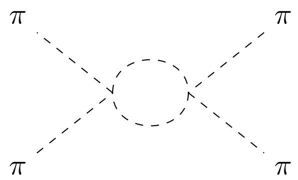
$$\begin{aligned} &\text{extra} \quad \partial_\mu\partial^\mu \to \frac{p^2}{\Lambda^2}, \\ &\text{extra} \quad m_0^2 M \to \frac{m_0^2 M}{\Lambda^2}, \end{aligned} \tag{3.7.21}$$

so that the expansion is valid for $p^2, m_0^2 M \ll \Lambda^2$. We now want to understand the value of the cut-off scale, and to do that one way is to compare the tree level scattering to the loop level asking that the second contributes less than the first one.
We can take the four pions interaction and evaluate it in momentum space



$$\frac{f_\pi^2}{16}\,\mathrm{Tr}\,(\partial^\mu\Sigma\partial_\mu\Sigma^{-1}) \to \frac{p^2}{f_\pi^2}, \tag{3.7.22}$$

where p is the external pion momentum, and then at 1 loop order



$$\sim \frac{1}{(4\pi^2)} \int \frac{d^4k}{k^4} \frac{k^4}{f_\pi^4} \sim \frac{\Lambda^4}{(4\pi)^2 f_\pi^4} \to p^2 \frac{\Lambda^2}{(4\pi)^2 f^4}, \tag{3.7.23}$$

because we need at least a two derivatives interaction to not violate the shift symmetry. We require for the cut-off

$$\frac{p^2}{f_\pi^2} > p^2 \frac{\Lambda^2}{(4\pi)^2 f_\pi^4}, \tag{3.7.24}$$

that gives

$$\Lambda < 4\pi f_\pi \qquad \to \qquad \Lambda \simeq (1-2)\text{GeV}. \tag{3.7.25}$$

Chapter 4

Electroweak symmetry breaking, Z_μ boson and neutrino production

Here we will briefly discuss the theoretical framework of the Z_μ boson and its decay, we will follow [7].
In the previous section we discussed Goldstone Bosons and Symmetry breaking in the context of global symmetry. We here want to briefly mention what happens when it is a local symmetry to be broken.
The electroweak part of the Standard model Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + (D_\mu H)^\dagger (D_\mu H) + m^2 H^\dagger H - \lambda (H^\dagger H)^2, \tag{4.0.1}$$

where the gauge group is $SU(2) \times U(1)_Y$, B_μ is the $U(1)_Y$ hypercharge gauge boson with $B_{\mu\nu}$ its field strength, W^a_μ the $SU(2)$ gauge bosons, $W^a_{\mu\nu}$ their field strength, H is the Higgs multiplet and D_μ is the appropriate covariant derivative, such that

$$D_\mu H = \partial H - igW^a_\mu \tau^a H - \frac{1}{2} ig' B_\mu H. \tag{4.0.2}$$

The gauge group gets broken: $SU(2) \times U(1)_Y \to U(1)_{EM}$ by the vacuum expectation value of the Higgs field.
The vev is found as the minimum of the Higgs potential $V(H) = -m^2|H|^2 + \lambda|H|^4$. Expanding H around its vev it is a way to study the behaviour at low energy. Without loss of generality we can take the vev to be real and in the lower component of the Higgs multiplet, the H expansion reads

$$H = \exp\left(2i\frac{\pi^a \tau^a}{v}\right)\left(\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{h}{\sqrt{2}}, \end{pmatrix}\right), \tag{4.0.3}$$

where $v = \frac{m}{\sqrt{\lambda}}$ is the vev and $\tau^a = \frac{1}{2}\sigma^a$ are the canonically normalized $SU(2)$ generators. π^a and h are the fluctuations aroud the vev but we are now interested in what happens when H assumes its expectation value, we can anyways avoid π^a using unitary gauge where we can set $\pi^a = 0$.
We want to rewrite the Lagrangian when the Higgs assumes its vev, so we need to rewrite

its covariant derivative (4.0.2). Plugging in and computing the matrix product we get

$$|D_\mu H|^2 = g^2 \frac{v^2}{8} \left((W^1_\mu)^2 + (W^2_\mu)^2 + \left(\frac{g'}{g} B_\mu - W^3_\mu \right)^2 \right), \tag{4.0.4}$$

since they are squared field terms, in the Lagrangian they will be the mass terms: we started with massless gauge bosons but when the Higgs field acquires a vev they get massive.
These terms are not diagonalized. To understand how to proceed we notice that the kinetic terms are already canonically normalized, so we need only to rotate the fields, in particular only the ones that are not perfect squares, so B_μ and W^3_μ. To do so we define

$$\begin{aligned} Z_\mu &\equiv \cos\theta_W W^3_\mu - \sin\theta_W B_\mu, \\ A_\mu &\equiv \sin\theta_W W^3_\mu + \cos\theta_W B_\mu, \end{aligned} \tag{4.0.5}$$

and inverting we get

$$\begin{aligned} B_\mu &= \cos\theta_W A_\mu - \sin\theta_W Z_\mu, \\ W^3_\mu &= \sin\theta_W A_\mu + \cos\theta_W Z_\mu, \end{aligned} \tag{4.0.6}$$

with

$$\tan\theta_W = \frac{g'}{g}. \tag{4.0.7}$$

This is the choice that diagonalizes the mass matrix. We can now focus on the Z_μ boson only, the part of the Lagrangian where it is described is

$$\mathcal{Z} = -\frac{1}{4} Z^2_{\mu\nu} + \frac{1}{2} m^2_Z Z^\mu Z_\mu, \tag{4.0.8}$$

with $m_Z = \frac{1}{2\cos\theta_W} gv$ and $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$.
Since the gauge group is a non-abelian gauge group the gauge bosons transform in the adjoint representation and their interaction is given by the commutators. In particular, together with the Z_μ boson we defined the photon field A_μ and its coupling as the coefficient of the commutator representing its coupling to the W^a_μ:

$$g[A_\mu, W^a_\nu \tau^a] = g \sin\theta_W W^3_\mu W^a_\nu [\tau^3, \tau^a], \tag{4.0.9}$$

giving

$$e = g \sin\theta_W = g' \cos\theta_W. \tag{4.0.10}$$

Now that we understood how the Z_μ boson appears after the Higgs acquires a vev, we can discuss its coupling with the fermion sector of the SM, in particular with neutrinos. Till now we only looked at the free part of the Lagrangian, here we need to include interactions. In this work we are focusing on the neutral current interactions, so here I will only report the neutral gauge boson interactions. The coupling to the Z_μ boson reads

$$\mathcal{L} = \frac{e}{\sin\theta_W} Z_\mu J_\mu, \tag{4.0.11}$$

with

$$J_\mu^Z = \frac{1}{\cos\theta_W}(J_\mu^3 - \sin^2\theta_W J_\mu^{EM}). \tag{4.0.12}$$

At the best of our knowledge $SU(2)_{\text{weak}}$ acts only on left-handed states, so the currents are

$$\begin{aligned} J_\mu^3 &= \sum_i \bar{\psi}_i^L \gamma^\mu T^3 \psi_i^L, \\ J_\mu^{Em} &= \sum_i Q_i(\bar{\psi}_i^L \gamma^\mu T^3 \psi_i^L + \bar{\psi}_i^R \gamma^\mu T^3 \psi_i^R). \end{aligned} \tag{4.0.13}$$

Neutrinos are neutral particles so $J_{EM} = 0$, so the only relevant interaction is given by J_μ^3.
In this way we understand how the Z_μ couples with neutrinos, giving a link between ChPT with external fields and neutrinos.
With this we end our review on the basis topics that are needed to understand our work. From the next chapter on we are showing our original results.

Chapter 5

Construction of the HBChPT Lagrangian with Z_μ

In this section we construct the Lagrangian describing the interaction between nucleons N and Z_μ boson. To do so we will follow some references later specified but we will also add some new contributions peculiar to our research.

5.1 QCD Lagrangian

The QCD Lagrangian

$$\mathcal{L}_{QCD,0} = -\frac{1}{4} G^{\mu\nu} G_{\mu\nu} + i\bar{q}\not{D}q, \tag{5.1.1}$$

is invariant under

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A. \tag{5.1.2}$$

The axial subgroup $U(1)_A$ is broken spontaneously by the chiral condensate as well as by anomaly effects at Λ_{QCD}. The $SU(N_f)_L \times SU(N_f)_R$ symmetry is spontaneously broken by the chiral condensate $\langle \bar{q}_R q_L \rangle$ which is invariant only under the vectorial subgroup

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A \to SU(N_f)_V \times U(1)_V. \tag{5.1.3}$$

This results in new low energy degrees of freedom, the Goldstone bosons, which acquire mass due to the additional explicit symmetry breaking due to the quark mass matrix. The number of the GB depends on N_f, for $N_f = 3$ we should get 9, 8 from $SU(3)_A$ and 1 from $U(1)_A$, but since the axial anomaly leads to large explicit breaking, the associated Goldstone boson η' it is a non-perturbative effect and can not be calculated by an effective Lagrangian at energies below Λ_{QCD}.

5.2 Chiral symmetry and CCWZ

We want to be able to construct the QCD Lagrangian together with Z_μ boson. To do this we will make use of the QCD global symmetry.

We start following the review [8], by parametrizing the vacuum with an element of the broken symmetry. From now on we will work with 2 quarks u, d, so $N_f = 2$, extending to three quarks would be a possible next step.
The Nambu GB can be parametrized using configurations that are related to a vacuum by a local transformation, after dropping the redundant ones, the ones that leave the vacuum invariant, as

$$U(\pi(x)) = e^{i\frac{\pi^a(x)\tau^a}{f_\pi}}. \tag{5.2.1}$$

Under chiral symmetry this transforms as $U' = R'UL'^\dagger$, with $R' \in U(2)_R$ and $L' \in U(2)_L$. For the following we also denote $R \in SU(2)_R$ and $L \in SU(2)_L$, their relation is $R' = e^{-i(\Theta_V + \Theta_A)}R$ and $L' = e^{-i(\Theta_V - \Theta_A)}L$.
To write the Lagrangian we use the CCWZ construction, which exploits the maurer-Cartan form of U

$$iU^{-1}\partial_\mu U, \tag{5.2.2}$$

and its derivatives to build terms invariant under the symmetry group.
Before proceeding we need to add to the bare QCD Lagrangian, the external fields we are interested in. First of all we need to include the mass term, that explicitly breaks the symmetry. To do so we need scalar and pseudoscalar external fields which transform as the field U, we then define

$$\chi = 2B(s - ip), \tag{5.2.3}$$

where s is the scalar and p the pseudoscalar field. B is a constant related to the explicit chiral symmetry breaking $B(m_u + m_d) = m_\pi^2$, where in our case we have $s - ip = \Updownarrow_q$ so that we get that the field transformation is $\chi' = R\chi L^\dagger$, the same as U, so it can be used together with U to write invarint terms for the Lagrangian.
Now we need to implement the external vector and axial vector fields, which will be called respectively v_μ and a_μ. It is more convenient to make manifest their isospin structure, they are indeed either isovector, $v_\mu^{(IV)}$ and $a_\mu^{(IV)}$, or isoscalar fields $v_\mu^{(IS)}$ and $a_\mu^{(IS)}$, where the isovector are proportional to τ^3 in the isospin space and the isoscalar to the identity matrix. The isovector gauge fields $v_\mu^{(IV)}$, $a_\mu^{(IV)}$ belongs to the groups $SU(2)_V$ and $SU(2)_A$, whilst the isoscalar $v_\mu^{(IS)}$ and $a_\mu^{(IS)}$ to $U(1)_V$ and $U(1)_A$.
Our target is to include the Z boson. To do so we notice that the electroweak gauge group $SU(2)_L \times U(1)_Y$ could be thought to be a subgroup of the gauged version of the $SU(2)_L \times SU(2)_R \times U(1)_V$ flavour group in the limit of zero mass. So, what we can do is to gauge the full flavor group and only later to identify the electroweak subgroup [9].
We can then define matrix-valued gauge fields $l_\mu(x)$ and $r_\mu(x)$ transforming as

$$\begin{aligned} l_\mu &\to L l_\mu L^\dagger + iL\partial_\mu L^\dagger, \\ r_\mu &\to R r_\mu R^\dagger + iR\partial_\mu R^\dagger, \end{aligned} \tag{5.2.4}$$

with $L(x)$ and $R(x)$ 2×2 unitary matrices corresponding to a general $SU(2)_L \times SU(2)_R \times U(1)_V$ gauge transformation; we restrict the $U(1)$ part of the transformation to the vector subgroup by requiring $\det L = \det R$ and $\mathrm{Tr}\, l_\mu = \mathrm{Tr}\, r_\mu$. We can extend the transformation for the U field as well and make it local

$$U \to LUR^\dagger. \tag{5.2.5}$$

Now we focus on the pion part of the Lagrangian. To now make it gauge invariant we just replace ordinary derivatives with appropriate covariant derivatives $\partial_\mu \to D_\mu$. To determine the form for each field, we require it to transform in the same way as the field itself, e.g. $D_\mu U \to L(D_\mu U)R^\dagger$, finding

$$\begin{aligned} D_\mu U &= \partial_\mu U - il_\mu U + iUr_\mu, \\ D_\mu U^\dagger &= \partial_\mu U^\dagger + iU^\dagger l_\mu - ir_\mu U^\dagger. \end{aligned} \tag{5.2.6}$$

Before moving on, is important to clarify what happens to the GBs (pions) after gauging the group, because there is also another symmetry breaking, namely the electroweak trough the Higgs mechanism. So, one question that may arise is why do the electroweak $SU(2)_L \times U(1)$ gauge bosons not eat the pions emerging from the symmetry breaking due to the chiral condensate. This is because we have two very different scales, where the electroweak one is way bigger than the chiral condensate. So at first we have the massless electroweak gauge bosons, with 2 degrees of freedom, after the breaking due to the Higgs mechanism, each of them eats one Goldstone boson and they become massive, so with 3 degrees of freedom. Since we have 3 gauge fields, the Higgs loses 3 degrees of freedom and it is left with 4-3=1 degree of freedom. When the, under our identification, same symmetry gets broken again, due to the quark condensate, the gauge fields are already massive and so they can't eat any GB more. However what will happen is that they will eat a combination of the Goldstone bosons generated by the two symmetry breakings. Since the energy scales are so far apart approximately this is as if there would just eat the Goldstone bosons from the symmetry breaking at a higher energy scale and the pions can be viewed as the Goldstone bosons from symmetry breaking due to the quark condensate, they are not eaten. However the masses of the electroweak gauge bosons gain an additional (though tiny) mass correction from the symmetry breaking at the energy scale of the quark condensate.

Having this in mind, we can write all the invariant combinations of U, χ and covariant derivatives of them, finding at leading order

$$\mathcal{L}_\pi^{(2)} = \frac{1}{4} f_\pi^2 \{ \mathrm{Tr}[\nabla_\mu U^\dagger \nabla^\mu U + \chi^\dagger U + \chi U^\dagger] \} \tag{5.2.7}$$

So far we didn't include baryons. To do this we introduce an isospin doublet given by 2 Dirac spinors, the proton and the neutron, as

$$\mathcal{N} = (p, n)^T. \tag{5.2.8}$$

There are many ways to properly describe the transformation of this isospin doublet. In this contest usually people proceed as in [10], defining $u(x)$

$$u^2(x) = U(x) \tag{5.2.9}$$

and so

$$u = \sqrt{U} = e^{i\frac{\pi^a \tau^a}{2f_\pi}}. \tag{5.2.10}$$

We now need to find the transformation property for u, this is induced by U since $U'(x) = RU(x)L^\dagger = u'^2$ and can be written as the following

$$Ru = u'K. \tag{5.2.11}$$

Solving for K it gives

$$K = \sqrt{LU^\dagger R^\dagger}R\sqrt{U} = \sqrt{RUL^\dagger}L\sqrt{U^\dagger} \tag{5.2.12}$$

and, as explained in [10] the field K is the correct one to parametrize the transformation of $\mathcal{N}$

$$\mathcal{N}' = K(L,R,U)\mathcal{N}. \tag{5.2.13}$$

Until now we only studied the transformation behaviour under $SU(2)_V \times SU(2)_A$ and we now need to go to the full chiral symmetry $SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$. To do so we just need to substitute in the previous equations R' and L' to R and L. We can immediately notice from (5.2.12) that $\mathcal{N}$ doesn't transform under $U(1)_A$, whilst it does under $U(1)_V$, picking up a phase $\mathcal{N} \to e^{-i\Theta'_V}\mathcal{N}$. It is now fundamental to stress that $\Theta'_V \neq \Theta_V$ in the UV theory, however, they can be matched, finding that $\Theta'_V = 3\Theta_V$. Θ'_V represents the baryon number and Θ_V the quark number, that explains the factor of 3 difference. This will be relevant in defining properly the external isoscalar vector gauge field.
Having clarified the transformation properties, we can now include the nucleon doublet in the Lagrangian; here as well we need to define an appropriate covariant derivative

$$D_\mu\mathcal{N} = \partial_\mu\mathcal{N} + \Gamma_\mu\mathcal{N}, \tag{5.2.14}$$

such that

$$D_\mu\mathcal{N} \to K(L,R,U)D_\mu\mathcal{N}, \tag{5.2.15}$$

so we need

$$D'_\mu = KD_\mu K^\dagger, \quad \Gamma'_\mu = K\Gamma_\mu K^\dagger + K\partial_\mu K^\dagger. \tag{5.2.16}$$

5.3 Construction

We now have the basis to set up the chiral Lagrangian. We still need to include the external fields that in our case will describe the Z_μ boson.
Let us briefly summarize the procedure. We start by rewriting the UV Lagrangian containing the Z_μ boson in order to make explicit what are our vector and axial fields. After that we can construct the invariant building block of the Lagrangian using P and C symmetry as well. We can then employ the equation of motion at leading order to eliminate some terms up to higher order corrections. There is still one simplification

left, we can apply the heavy baryon limit, since nucleons inside a SN are slowly moving, that leads us to integrate out the heavy part of the nucleon field.

We start by writing the UV Lagrangian after Electroweak symmetry breaking. We need to consider only the part with the Z_μ boson; from [7] we have

$$\mathcal{L} = -\frac{e}{\sin\theta_W \cos\theta_W} Z_\mu \left[\overline{q}_i^L \gamma^\mu \frac{\tau^3}{2} q_i^L - \sin\theta_W^2 \left(\frac{2}{3}\overline{u}\gamma^\mu u - \frac{1}{3}\overline{d}\gamma^\mu d \right) \right], \tag{5.3.1}$$

where $\frac{2}{3}$ and -$\frac{1}{3}$ are the electromagnetic charges of respectively u and d. We will use the convention $e = g\sin\theta_W$.
We want to rewrite this as the QCD Lagrangian with external fields

$$\mathcal{L} = \mathcal{L}_{QCD,0} + \bar{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \bar{q}(s - i\gamma_5 p)q, \qquad q = \begin{pmatrix} u \\ d \end{pmatrix}, \tag{5.3.2}$$

where

$$\mathcal{L}_{QCD,0} = -\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + i\bar{q}\not{D}q. \tag{5.3.3}$$

To do so we first split the external fields in the following way

$$-\bar{q}(s - i\gamma_5 p)q + (\bar{q}\gamma^\mu(v_\mu + \frac{1}{3}v_\mu^s)q)_{q=(u,d)^T} + (\bar{q}\gamma^\mu(a_\mu + a_\mu^s)\gamma_5 q)_{q=(u,d)^T}, \tag{5.3.4}$$

and we rewrite $\overline{q}_i^L\gamma^\mu q_i^L\frac{\tau^3}{2}$ as

$$\begin{aligned} \overline{q}\frac{\mathbb{I}+\gamma_5}{2}\gamma^\mu\frac{\mathbb{I}-\gamma_5}{2}q\frac{\tau^3}{2} &= \overline{q}\left[\gamma^\mu - \gamma^\mu\gamma_5 + \gamma_5\gamma^\mu - \gamma_5\gamma^\mu\gamma_5\right]q\frac{\tau^3}{8} = \\ &= \overline{q}\gamma^\mu\frac{\tau^3}{4}q - \overline{q}\gamma^\mu\gamma_5\frac{\tau^3}{4}q. \end{aligned} \tag{5.3.5}$$

Now we need to collect the u and d in the vector $q = (u,d)^T$:

$$\begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} = \frac{1}{6}\mathbb{I} + \frac{1}{2}\tau^3. \tag{5.3.6}$$

So we get

$$\begin{aligned} \mathcal{L} &= -\frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\left[\overline{q}\gamma^\mu\frac{\tau^3}{4}q - \overline{q}\gamma^\mu\gamma_5\frac{\tau^3}{4}q - \sin\theta_W^2\overline{q}\gamma^\mu\left(\frac{\mathbb{I}}{6} + \frac{\tau^3}{2}\right)q\right] \\ \mathcal{L} &= -\frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\overline{q}\left[\gamma^\mu\left(\frac{\tau^3}{4} - \sin\theta_W^2\frac{\tau^3}{2} - \sin\theta_W^2\frac{\mathbb{I}}{6}\right) - \gamma^\mu\frac{\tau^3}{4}\gamma_5\right]q, \end{aligned} \tag{5.3.7}$$

from where we can read off

$$\begin{aligned} v_\mu &= -\frac{g}{\cos\theta_W}Z_\mu\frac{\tau^3}{4} + eZ_\mu\tan\theta_W\frac{\tau^3}{2}, \\ a_\mu &= \frac{g}{\cos\theta_W}Z_\mu\frac{\tau^3}{4}, \\ v_\mu^s &= \frac{e}{2}Z_\mu\tan\theta_W, \end{aligned} \tag{5.3.8}$$

and define

$$\chi = 2B(s - ip), \qquad (s - ip) = \Updownarrow_q, \tag{5.3.9}$$

and so we can construct

$$\begin{aligned} r_\mu &= eZ_\mu \tan\theta_W \frac{\tau^3}{2}, \\ l_\mu &= -\frac{g}{2\cos\theta_W} Z_\mu \tau^3 + eZ_\mu \tan\theta_W \frac{\tau^3}{2}, \\ r^s_\mu &= l^s_\mu = v^s_\mu = \frac{e}{2} Z_\mu \tan\theta_W, \end{aligned} \tag{5.3.10}$$

where

$$v_\mu = \frac{1}{2}(r_\mu + l_\mu), \quad a_\mu = \frac{1}{2}(r_\mu - l_\mu). \tag{5.3.11}$$

It is important to clarify why in equation (5.3.4) we write $\mathcal{L} \supset \frac{1}{3} v^s_\mu$ and what are the requirements to be met by all the external fields under local $SU(2)_L \times SU(2)_R \times U(1)_V$. As discussed in [11], we can rewrite the external field part of the Lagrangian as

$$\begin{aligned} \bar{q}\gamma^\mu(v_\mu + \frac{1}{3}v(s)_\mu + \gamma_5 a_\mu)q =& \frac{1}{2}\bar{q}\gamma^\mu[r_\mu + l_\mu + \frac{2}{3}v^{(s)}_\mu + \gamma_5(r_\mu - l_\mu)] \\ =& \bar{q}_R\gamma^\mu \left(r_\mu + \frac{1}{3}v^{(s)}_\mu\right) q_R + \bar{q}_L\gamma^\mu \left(l_\mu + \frac{1}{3}v^{(s)}_\mu\right) q_L, \end{aligned} \tag{5.3.12}$$

and similarly

$$\bar{q}(s - i\gamma_5 p)q = \bar{q}_L(s - ip)q_R + \bar{q}_R(s + ip)q_L. \tag{5.3.13}$$

This means that we can rewrite (5.3.2) as

$$\begin{aligned} \mathcal{L} =& \mathcal{L}^0_{\text{QCD}} + \bar{q}_L\gamma^\mu \left(l_\mu + \frac{1}{3}v^{(s)}_\mu\right) q_L + \bar{q}_R\gamma^\mu \left(r_\mu + \frac{1}{3}v^{(s)}_\mu\right) q_R \\ &- \bar{q}_R(s + ip)q_L - \bar{q}_L(s - ip)q_R. \end{aligned} \tag{5.3.14}$$

With this form is easier to see that the Lagrangian remains invariant under the local transformations

$$\begin{aligned} q_R &\to \exp\left(-i\frac{\Theta(x)_V}{3}\right) R(x) q_R, \\ q_L &\to \exp\left(-i\frac{\Theta(x)_V}{3}\right) L(x) q_L, \end{aligned} \tag{5.3.15}$$

provided that the fields transform as

$$\begin{aligned} r_\mu &\to R r_\mu R^\dagger + iR\partial R^\dagger, \\ l_\mu &\to L l_\mu L^\dagger + iL\partial L^\dagger, \\ v^{(s)}_\mu &\to v^{(s)}_\mu - \partial_\mu \Theta_V, \\ s + ip &\to R(s + ip)L^\dagger, \\ s - ip &\to L(s - ip)R^\dagger. \end{aligned}$$

From the quark transformation we can understand why we introduce a factor of 1/3 in front of $v_\mu^{(s)}$: in our chiral perturbation theory Lagrangian we want to describe the interaction between nucleons that contain 3 quarks each.
We can now proceed with the construction of the other building blocks of the chiral Lagrangian

$$\begin{aligned}
\Gamma_\mu &= \frac{1}{2}[u^\dagger, \partial_\mu u] - \frac{i}{2}u^\dagger r_\mu u - \frac{i}{2}u l_\mu u^\dagger = \\
&= \frac{1}{2}[u^\dagger, \partial_\mu u] - \frac{i}{2}u^\dagger e Z_\mu \tan\theta_W \frac{\tau^3}{2}u + \frac{i}{2}u\frac{g}{2\cos\theta_W}Z_\mu \tau^3 u^\dagger - \frac{i}{2}u e Z_\mu \tan\theta_W \frac{\tau^3}{2}u^\dagger = \\
&= \frac{1}{2}[u^\dagger, \partial_\mu u] - \frac{i}{4}e\tan\theta_W Z_\mu u^\dagger \tau^3 u + \frac{i}{4}Z_\mu\left(\frac{g}{\cos\theta_W} - e\tan\theta_W\right)u\tau^3 u^\dagger, \\
\hat{\Gamma}_\mu &= -\frac{i}{2}u^\dagger r_\mu^s u - \frac{i}{2}u l_\mu^s u^\dagger = -i v_\mu^s = -i\frac{e}{2}Z_\mu \tan\theta_W, \\
u_\mu &= i\{u^\dagger, \partial_\mu u\} + u^\dagger r_\mu u - u l_\mu u^\dagger \\
&= i\{u^\dagger, \partial_\mu u\} + \frac{e}{2}\tan\theta_W Z_\mu u^\dagger \tau^3 u + \frac{Z_\mu}{2}\left(\frac{g}{\cos\theta_W} - e\tan\theta_W\right)u\tau^3 u^\dagger, \\
\hat{u}_\mu &= u^\dagger r_\mu^s u - u l_\mu^s u^\dagger = 0, \\
F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] = e\frac{\tau^3}{2}\tan\theta_W(\partial^\mu Z^\nu - \partial^\nu Z^\mu), \\
F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu] = \left(-\frac{g}{2\cos\theta_W} + e\frac{\tan\theta_W}{2}\right)\tau^3(\partial^\mu Z^\nu - \partial^\nu Z^\mu), \\
F_R^{s\mu\nu} &= F_L^{s\mu\nu} = \partial^\mu v^{s\nu} - \partial^\nu v^{s\mu} = \frac{e}{2}\tan\theta_W(\partial^\mu Z^\nu - \partial^\nu Z^\mu).
\end{aligned} \tag{5.3.16}$$

So now the covariant derivative reads: $D_\mu = \partial_\mu + \Gamma_\mu + \hat{\Gamma}_\mu$.
We can also associate a covariant derivative to U, which reads $\nabla_\mu = \partial_\mu U - i r_\mu U - i U l_\mu$, such that $\nabla_\mu U$ transforms the same way as U.
From the fields derived above we can define

$$\begin{aligned}
\chi_\pm &= u^\dagger \chi u^\dagger \pm u\chi^\dagger u, \\
F_{\mu\nu}^\pm &= u^\dagger F_{\mu\nu}^R u \pm u F_{\mu\nu}^L u^\dagger.
\end{aligned} \tag{5.3.17}$$

5.4 EOM eliminations

In this paragraph we briefly mention how to simplify our ChPT Lagrangian employing the equation of motion at leading order for the nucleon field. Together with the equation of motion there are various identities that reduce the minimal set of independent terms, this is extensively treated in [12]. Going back to the EOM elimination, the equation of motion at leading order let us make the following substitution

$$\not{D}\mathcal{N} \to -im\mathcal{N}, \tag{5.4.1}$$

that helps reducing the number of terms. However this is not the only replacement that we can make, Indeed we need to pay attention to other terms not explicitly containing $\not{D}\mathcal{N}$ but can be brought to a for which include that term. Keeping all into account many terms result to be redundant at leading order. This is extensively explained in [12], where the following set of operators has been found to be sufficient to generate the complete chiral Lagrangian up to fourth order:

$$\begin{aligned}
&\mathbb{I};\\
&\gamma_5\gamma_\mu, D_\mu;\\
&g_{\mu\nu}, \sigma_{\mu\nu}, \gamma_5\gamma_\mu D_\nu, D_{\mu\nu};\\
&g_{\mu\nu}\gamma_5\gamma_\rho, g_{\mu\nu}D_\rho, \sigma_{\mu\nu}D_\rho, \epsilon^{\lambda}_{\mu\nu\rho}D_\lambda, \gamma_5\gamma_\mu D_{\nu\rho}, D_{\mu\nu\rho};\\
&g_{\mu\nu}g_{\rho\tau}, \epsilon_{\mu\nu\rho\tau}, g_{\mu\nu}\sigma_{\rho\tau}, g_{\mu\nu}\gamma_5\gamma_\rho D_\tau, g_{\mu\nu}D_{\rho\tau}, \sigma_{\mu\nu}D_{\rho\tau}, \epsilon^{\lambda}_{\mu\nu\rho}D_{\lambda\tau}, \gamma_5\gamma_\mu D_{\nu\rho\tau}, D_{\mu\nu\rho\tau}.
\end{aligned} \tag{5.4.2}$$

It is worth mentioning that we can derive equation (5.4.1) also as a result of field transformations provided that the S-matrix elements are left untouched.

5.5 Discrete symmetries: Parity P and Charge Conjugation C

QCD Lagrangian is both P and C invariant, so has to be its HBChPT projection. Consequently, we can construct the terms requiring those invariance, to do so we follow [12]. However it is worth noticing that, even if electroweak interactions maximally break P and C, this does not modify our construction: we proceed keeping all the fields and building blocks general, e.g. r_μ, l_μ, and so on. We construct the terms consistently with P and C invariance and only after we break it substituting our values for the fields. We will end up with a Lagrangian invariant under CP only as it should be considering electroweak interactions.
To properly write the terms we have to make them hermitian, so every term in the relativistic Lagrangian should be written as

$$\bar{\mathcal{N}}A^{\mu\nu\cdots}\Theta_{\mu\nu\ldots}\mathcal{N} + \text{h.c.}, \tag{5.5.1}$$

where $A_{\mu\nu\ldots}$ is a product of pion and (or) external fields and their covariant derivatives, while $\Theta_{\mu\nu\ldots}$ is a product of a Clifford algebra element $\Gamma_{\mu\nu\ldots}$ and a totally symmetrized product of n covariant derivatives acting on the nucleon fields $D^n_{\alpha\beta\ldots\omega}$, so it has the following form

$$\Theta_{\mu\nu\ldots\alpha\beta\ldots} = \Gamma_{\mu\nu\ldots}D^n_{\alpha\beta\ldots}, \tag{5.5.2}$$

with $D^n_{\alpha\beta\ldots\omega} = \{D_\alpha, \{D_\beta, \{\ldots, D_\omega\}\}\}$. The relevant fields here are

$$u_\mu, \quad \tilde{\chi}_+, \quad \langle\tilde{\chi}_+\rangle, \quad F^+_{\mu\nu}, \quad \langle F^+_{\mu\nu}\rangle \tag{5.5.3}$$

and combinations of them and together with the covariant derivative. The Clifford algebra elements are expanded in the standard basis $(\mathbb{I}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu})$ and we include

there as well the metric and the Levi-Civita tensors. There are some restrictions on the structure of $\Theta_{\mu\nu\ldots}$ that are discussed in the appendix of [12]. As regarding $A^{\mu\nu\cdots}$, it is made by all possible products of the pion, external fields and their covariant derivatives, but every index increases the chiral order by one, so since our analysis goes up to $\mathcal{O}(p^2)$, we need a limited number of terms. To get simple transformation properties under charge and hermitian conjugation we write everything in terms of commutators and anticommutators, that for our case translates into decomposing each term into an isoscalar and isovector part.
Having said that we can start studying the transformation properties. We define the constant h_A, h_Γ, c_A and c_Γ

$$\begin{aligned} A^\dagger = (-1)^{h_A} A, \quad A^c = (-1)^{c_A} A^\top, \\ \Gamma^\dagger = (-1)^{h_\Gamma}\gamma^0\Gamma\gamma^0 \quad \Gamma^c = (-1)^{c_\Gamma}\Gamma^\top, \end{aligned} \tag{5.5.4}$$

where h_A, c_A, h_Γ and c_Γ depend on the specific choice of the field. So any term in $\mathcal{L}_{\pi N}$ takes the following form

$$\bar{\mathcal{N}} A^{\mu\nu\ldots\alpha\beta\ldots}\Gamma_{\mu\nu\ldots}D^n_{\alpha\beta\ldots}\mathcal{N} + (-1)^{h_A+h_\Gamma}\bar{N}\vec{D}^n_{\alpha\beta\ldots}\Gamma_{\mu\nu\ldots}A^{\mu\nu\ldots\alpha\beta}\mathcal{N}. \tag{5.5.5}$$

Our target here is to derive the Lagrangian up to $O(p^2)$, so we need expressions with u_μ up to quadratic order. Let's give a look on how the building block transform under C and P symmetry. To understand how it works it is better to start at the level of the pion field. C sends π^+ into π^- and viceversa that it is exactly $u \to u^T$, whilst the pion field gets a - sign when transforming under P, since it is a pseudoscalar field, that corresponds to $u \to u^\dagger$. From this we can easily derive the transformation properties of u_μ.
To avoid confusions, as already mentioned, it is better to separate isoscalar from isovector component of the fields, we can do this manually by defining for a generic field X in our chiral Lagrangian

$$\tilde{X} = X - \frac{1}{2}\langle X\rangle, \tag{5.5.6}$$

where $\langle\ldots\rangle$ stands for the trace in flavour space. Consequently we can write the Lagrangian in terms of the isovector $\tilde{X}$ and the isoscalar $\langle X\rangle$ component of each field. For example the anticommutator can be rewritten in terms of an isoscalar piece

$$\{u_\mu, u_\nu\} = \langle u_\mu u_\nu\rangle\mathbb{I}. \tag{5.5.7}$$

We now need to study the specific transformations of the building blocks in the Lagrangian. Let us briefly recap the transformations of the external fields. To compute it we need to ask for the invariance under P and C symmetry of the Lagrangian in equation (5.3.4). Doing so we get

$$\begin{aligned} &v_\mu \xrightarrow{\text{P}} v_\mu, \quad v^{(s)}_\mu \xrightarrow{\text{P}} v^{(s)}_\mu, \quad a_\mu \xrightarrow{\text{P}} -a_\mu, \quad a^{(s)}_\mu \xrightarrow{\text{P}} -a^{(s)}_\mu, \quad s \xrightarrow{\text{P}} s, \quad p \xrightarrow{\text{P}} -p, \\ &v_\mu \xrightarrow{\text{C}} -v^\top_\mu, \quad v^{(s)}_\mu \xrightarrow{\text{C}} -v^{(s)T}_\mu, \quad a_\mu \xrightarrow{\text{C}} a^\top_\mu, \quad a^{(s)}_\mu \xrightarrow{\text{C}} a^{(s)T}_\mu, \quad s \xrightarrow{\text{C}} s^\top, \quad p \xrightarrow{\text{C}} p^\top. \end{aligned} \tag{5.5.8}$$

The nucleon $\mathcal{N}$ is a fermion so it transforms as follows

$$\begin{aligned} &\mathcal{N} \xrightarrow{C} C\bar{\mathcal{N}}^\top, \quad \bar{\mathcal{N}} \xrightarrow{C} \mathcal{N}^\top C, \quad C = i\gamma^2\gamma^0, \\ &\mathcal{N} \xrightarrow{P} i\gamma^0\mathcal{N}, \quad \bar{\mathcal{N}} \xrightarrow{P} -i\bar{\mathcal{N}}\gamma^0. \end{aligned} \tag{5.5.9}$$

We now can compute the transformations of Clifford algebra elements and fields. They are listed in table 5.1 and 5.2. In particular we report only plus or minus signs that follow from the transformation, because they are the important information that we need to keep in mind when constructing the terms in the Lagrangian. Indeed, we could also have some parity matrices with open indices: those will not play a role since they will be contracted. In equations we get

$$\begin{aligned} \bar{\mathcal{N}}\Theta_{\mu\nu\ldots}\mathcal{N} &\xrightarrow{P} (\pm 1)P_\mu^\alpha P_\nu^\beta \ldots \bar{\mathcal{N}}\Theta_{\alpha\beta\ldots}\mathcal{N}, \\ \bar{\mathcal{N}}\Theta_{\mu\nu\ldots}\mathcal{N} &\xrightarrow{C} (\pm 1)(\bar{\mathcal{N}}\Theta_{\mu\nu\ldots}\mathcal{N})^\top = (\pm 1)\bar{\mathcal{N}}\Theta_{\mu\nu\ldots}\mathcal{N}, \\ A^{\mu\nu\ldots} &\xrightarrow{P} (\pm 1)P^\mu_\alpha P^\nu_\beta A^{\alpha\beta\ldots} \\ A^{\mu\nu\ldots} &\xrightarrow{C} (\pm 1)(A^{\mu\nu\ldots})^\top, \end{aligned} \tag{5.5.10}$$

with $P^\mu_\nu = diag(1,-1,-1,-1)$. As we mentioned before, we don't need to keep track

$\Theta_{\mu\nu\ldots}$	$\mathbb{I}$	γ_5	γ_μ	$\gamma_\mu\gamma_5$	$\sigma_{\mu\nu}$	$g_{\mu\nu}$	$\epsilon_{\lambda\mu\nu\rho}$	$D_\mu\mathcal{N}$	$\gamma_5\gamma_\mu D_\nu\mathcal{N}$	$D_{\mu\nu}\mathcal{N}$
Chiral dimension	0	1	0	0	0	0	0	0	0	0
P	+	-	+	-	+	+	-	+	-	+
C	+	+	-	+	-	+	+	-	-	+

Table 5.1: Transformation properties and chiral dimensions of Clifford algebra elements and derivatives of the nucleon field.

$A_{\mu\nu\ldots}$	u_μ	$[u_\mu, u_\nu]$	$\langle u_\mu, u_\nu\rangle$	$\tilde{\chi}_+$	$\langle\tilde{\chi}_+\rangle$	$F^+_{\mu\nu}$	$\langle F^+_{\mu\nu}\rangle$
Chiral dimension	1	2	2	2	2	2	2
P	-	+	+	+	+	+	+
C	+	-	+	+	+	-	-

Table 5.2: Transformation properties and chiral dimension of the building blocks.

of P^μ_ν's in the tables, due to Lorentz invariance. Is also important to explain what do we mean by writing the transformation of gamma matrices and other objects that are not fields. They do not transform under P and C, however the nucleons do. Consequently their transformation is induced by the fields according to (5.5.9) and (5.5.10). In the tables the chiral dimension of the fields is displayed as well; it will be important in constructing the Lagrangian up to a fixed order, in our case $\mathcal{O}(p^2)$. It can be computed from the number of derivatives acting on all the fields except on the nucleon and from the pion mass insertions. γ_5 represents an exception: we assign to it 1 because the

projection to HBChPT does not give a chiral leading order term, indeed γ_5 first appears together with chiral order one terms.
Let us show one explicit example for each symmetry

$$\begin{aligned}\bar{\mathcal{N}}\gamma_\mu\gamma_5\mathcal{N} &\xrightarrow{\text{P}} -i\bar{\mathcal{N}}\gamma^0\gamma_\mu\gamma_5 i\gamma^0\mathcal{N}\\ &= \bar{\mathcal{N}}\gamma^0\gamma_\mu\gamma_5\gamma^0\mathcal{N}\\ &= -\bar{\mathcal{N}}\gamma^0\gamma_\mu\gamma^0\gamma_5\mathcal{N}\\ &= -\bar{\mathcal{N}}\gamma_\mu^\dagger\gamma_5\mathcal{N}\\ &= -P_\mu^\alpha\bar{\mathcal{N}}\gamma_\alpha\gamma_5\mathcal{N}\end{aligned} \tag{5.5.11}$$

so, it gets a $(-)$ sign. Under C it transform as

$$\begin{aligned}\bar{\mathcal{N}}\gamma_\mu\gamma_5\mathcal{N} &\xrightarrow{\text{C}} \mathcal{N}^\top C\gamma_\mu\gamma_5 C\bar{\mathcal{N}}^\top\\ &= -\mathcal{N}^\top C\gamma_\mu\gamma_5 C^{-1}\bar{\mathcal{N}}^\top\\ &= -\mathcal{N}^\top C\gamma_\mu C^{-1}C\gamma_5 C^{-1}\bar{\mathcal{N}}^\top\\ &= \mathcal{N}^\top(\gamma_\mu)^\top(\gamma_5)^\top\bar{\mathcal{N}}^\top\\ &= -(\bar{\mathcal{N}}\gamma_5\gamma_\mu\bar{\mathcal{N}})^\top\\ &= (\bar{\mathcal{N}}\gamma_\mu\gamma_5\bar{\mathcal{N}})^\top\\ &= \bar{\mathcal{N}}\gamma_\mu\gamma_5\bar{\mathcal{N}}.\end{aligned} \tag{5.5.12}$$

In this way we can work out all the transformations, some of them are explicitly computed in [12] and [13].
We now have all the ingredients to construct the ChPT Lagrangian. To do so, we need to take, from the terms in table 5.1 and 5.2, the combinations that are invariant under C and P symmetry and to keep into account the EOM elimination. Additionally requiring that the Lagrangian is invariant under chiral transformation and that all the operators are hermitian (this requires adding factors of i and adding the *h.c.* if necessary), we find for the most general form of the chiral Lagrangian up to $\mathcal{O}(p^2)$

$$\begin{aligned}\mathcal{L}_{\pi N} &= \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)}\\ \mathcal{L}_{\pi N}^{(1)} &= \bar{\mathcal{N}}(i\not{D} - m + \frac{g_A}{2}\not{u}\gamma_5)\mathcal{N}\\ \mathcal{L}_{\pi N}^{(2)} &= \sum_{i=1}^{7} c_i\bar{\mathcal{N}}\mathcal{O}_i^{(2)}\mathcal{N},\end{aligned} \tag{5.5.13}$$

where $\mathcal{O}_i^{(2)}$ are given in Table 5.3.

5.6 HBChPT formulation

Now that we have all the ingredients to properly write our Lagrangian in a relativistic manner, the next step is to study its Heavy Baryon limit. To do so, we closely follow section 5.5 of [11]. The idea is to split the four momentum of the nucleon k^μ into a large piece mv^μ and a residual smaller piece p^μ that represents the external momenta transferred by pions or external field as the Z_μ boson in our case. This can be written as

$$k^\mu = mv^\mu + p^\mu, \quad \text{with} \quad v \cdot p \ll m, \tag{5.6.1}$$

where v_μ is the nucleon four velocity, that in the rest frame is $v_\mu = (1, \vec{0})$ and $v^2 = 1$ has to hold. This fits well a SN environment, since nucleons are almost fixed.
With this we can expand any Lagrangian $\mathcal{L}^{(1)}_{\pi N}$ in powers of

$$\frac{p}{m} \quad \text{and} \quad \frac{p}{4\pi f_\pi}, \tag{5.6.2}$$

since the two scales are almost the same size. Consequently the nucleon field $\mathcal{N}$ splits into two fields, an heavy one with momentum v^μ and a light one with momentum p^μ that will be the interacting one.
One question that may arise is how can we use HBChPT with a Z_μ boson. Using this formalism to describe axions is very straightforward because axions are almost massless so they can be produced as external states from the scattering of nucleons with small momentum p^μ. On the contrary Z_μ bosons can have a small Cartesian momentum $\vec{p}$ but the zeroth component will always be not negligible. So to properly include them we need to treat them as off-shell particles, only as mediators, so that the relation $q^2 = m_Z^2$ is not satisfied. Indeed in this work they are used to describe the interactions among nucleons and neutrinos and they will never be studied as external particle: as soon as they are produced they will decay into neutrino-antineutrino that indeed are more similar to axions as they have a very small -here zero- mass, so they can be produced from very low energetic nucleons.

Expansion of the ChPT Lagrangian

Here we derive the Heavy Baryon limit for our Chiral Lagrangian, so starting from

$$\mathcal{L}^{(1)}_{\pi N} = \bar{\mathcal{N}} \left(i\gamma_\mu D^\mu - m + \frac{g_A}{2}\gamma^\mu \gamma_5 u_\mu \right), \tag{5.6.3}$$

we want to arrive to its HBChPT formulation

$$\hat{\mathcal{L}}^{(1)}_{\pi N} = \bar{N}\{iv \cdot D + g_A S \cdot u\} N \tag{5.6.4}$$

where the ˆ stands for the HB projection and N and S will be defined in the following.
Using the previously defined four momentum v^μ, with the properties $v^2 = 1$ and $v^0 \geq 1$ we can define the projectors

$$P_{v\pm} \equiv \frac{1 \pm \not{v}}{2}, \tag{5.6.5}$$

which can be used to split the nucleon isospin doublet into two pieces

$$N_v \equiv e^{imv \cdot x} P_{v+} \mathcal{N}, \quad H_v \equiv e^{imv \cdot x} P_{v-} \mathcal{N}, \tag{5.6.6}$$

such that

$$\mathcal{N}(x) = e^{-imv\cdot x}[N_v(x) + H_v(x)]. \tag{5.6.7}$$

Now, we can write the EOM from (5.6.3) using the projected fields to get

$$\begin{aligned} 0 =& \left(i\not{D} - m + \frac{g_A}{2}\not{u}\gamma_5\right) e^{-imv\cdot x}(N_v + H_v) \\ =& e^{-imv\cdot x}\left(m\not{v} + i\not{D} - m + \frac{g_A}{2}\not{u}\gamma_5\right)(N_v + H_v), \end{aligned} \tag{5.6.8}$$

which can be simplified using

$$\not{v}N_v = N_v, \quad \not{v}H_v = -H_v, \tag{5.6.9}$$

to get

$$\left(i\not{D} + \frac{g_A}{2}\not{u}\gamma_5\right) N_v + \left(i\not{D} - 2m + \frac{g_A}{2}\not{u}\gamma_5\right) H_v = 0. \tag{5.6.10}$$

We can now use $P_{v\pm}$ to project this equation onto the two subspaces. To do so we need the following identities [11]

$$\begin{aligned} &P_{v+}\not{A}P_{v+} = v\cdot AP_{v+}, \\ &P_{v+}\not{A}P_{v-} = \not{A}_\perp P_{v-} = P_{v+}\not{A}_\perp, \\ &P_{v-}\not{A}P_{v-} = \not{A}_\perp P_{v-} = -v\cdot AP_{v-}, \\ &P_{v-}\not{A}P_{v+} = \not{A}_\perp P_{v+} = P_{v-}\not{A}_\perp, \\ &P_{v+}\not{B}\gamma_5 P_{v+} = \not{B}_\perp\gamma_5 P_{v+}, \\ &P_{v+}\not{B}\gamma_5 P_{v-} = v\cdot B\gamma_5 P_{v-} = v\cdot BP_{v+}\gamma_5, \\ &P_{v-}\not{B}\gamma_5 P_{v-} = \not{B}_\perp\gamma_5 P_{v-}, \\ &P_{v-}\not{B}\gamma_5 P_{v+} = -v\cdot B\gamma_5 P_{v+} = -v\cdot BP_{v-}\gamma_5, \end{aligned} \tag{5.6.11}$$

where

$$A^\mu_\perp = A^\mu - v\cdot Av^\mu, \quad v\cdot A_\perp = 0. \tag{5.6.12}$$

We can now multiply the EOM for P_{v+} and P_{v-} and use the other projector operator inside N_v and H_v to get the two projected equations

$$\begin{aligned} &\left(iv\cdot D + \frac{g_A}{2}\not{u}_\perp\gamma_5\right) N_v + \left(i\not{D}_\perp + \frac{g_A}{2}v\cdot u\gamma_5\right) H_v = 0, \\ &\left(i\not{D}_\perp - \frac{g_A}{2}v\cdot u\gamma_5\right) N_v + \left(-iv\cdot D - 2m + \frac{g_A}{2}\not{u}_\perp\gamma_5\right) H_v = 0. \end{aligned} \tag{5.6.13}$$

We can now solve the second equation for H_v

$$\left(iv\cdot D + 2m - \frac{g_A}{2}\not{u}_\perp\gamma_5\right)^{-1}\left(i\not{D}_\perp - \frac{g_A}{2}v\cdot u\gamma_5\right) N_v = H_v, \tag{5.6.14}$$

where we can already see that H_v is at least suppressed by one more power of $\frac{1}{m}$ with respect to N_v. We can plug this back into the first projected EOM in order to have an equation only dependent on N_v

$$\begin{aligned}&\left(iv\cdot D+\frac{g_A}{2}\not{u}_\perp\gamma_5\right)N_v+\left(i\not{D}_\perp+\frac{g_A}{2}v\cdot u\gamma_5\right)\times\\&\times\left(iv\cdot D+2m-\frac{g_A}{2}\not{u}_\perp\gamma_5\right)^{-1}\left(i\not{D}_\perp-\frac{g_A}{2}v\cdot u\gamma_5\right)N_v=0.\end{aligned} \tag{5.6.15}$$

From this, we can trace back the form of Lagrangian from which this EOM could come from

$$\begin{aligned}\mathcal{L}_{\text{eff}}=&\bar{N}_v\left(iv\cdot D+\frac{g_A}{2}\not{u}_\perp\gamma_5\right)N_v+\bar{N}_v\left(i\not{D}_\perp+\frac{g_A}{2}v\cdot u\gamma_5\right)\times\\&\times\left(iv\cdot D+2m-\frac{g_A}{2}\not{u}_\perp\gamma_5\right)^{-1}\left(i\not{D}_\perp-\frac{g_A}{2}v\cdot u\gamma_5\right)N_v,\end{aligned} \tag{5.6.16}$$

where we see that the second term is suppressed by at least $\frac{1}{m}$ and does not contribute to the leading order terms. This shows that we can expand the Lagrangian in a series in $\frac{1}{m}$:

$$\mathcal{L}_{\text{eff}}=\bar{N}_v\left(iv\cdot D+\frac{g_A}{2}\not{u}_\perp\gamma_5\right)N_v+\sum_{n=1}^{\infty}\frac{1}{(2m)^n}\mathcal{L}_{\text{eff},n}. \tag{5.6.17}$$

We are integrating out the heavy part of the spinor. This can be thought as we are effectively integrating out the anti-particle: then, it can not appear in a loop calculation and so all pure nucleon vacuum loops in HBChPT are zero. This is however not true for finite density loops.
To get to the final expression we need to define the operator that will appear in HBChPT, that is S_v^μ,

$$S_v^\mu=\frac{i}{2}\gamma_5\sigma^{\mu\nu}v_\nu=-\frac{1}{2}\gamma_5(\gamma^\mu\not{v}-v^\mu),\quad \sigma^{\mu\nu}\equiv\frac{i}{2}[\gamma^\mu,\gamma^\nu]. \tag{5.6.18}$$

The relations

$$\begin{aligned}\bar{N}_v\gamma^\mu N_v&=v^\mu\bar{N}_vN_v,\\\bar{N}_v\gamma^\mu\gamma_5N_v&=2\bar{N}_vS_v^\mu N_v\end{aligned} \tag{5.6.19}$$

finally allow us to write the leading order Lagrangian, dropping the index to simplify the notation, as

$$\hat{\mathcal{L}}_{\pi N}^{(1)}=\bar{N}\{iv\cdot D+g_AS\cdot u\}N. \tag{5.6.20}$$

For the derivation of $\hat{\mathcal{L}}_{\pi N}^{(2)}$ refer to [11]. We get

$$\hat{\mathcal{L}}_{\pi N}^{(2)}=\frac{1}{2m}\bar{N}((v\cdot D)^2-D^2-ig_A\{S\cdot D,v\cdot u\})N+\sum_{i=1}^{7}\hat{c}_i\bar{N}\hat{\mathcal{O}}_i^{(2)}N, \tag{5.6.21}$$

where the operators $\hat{\mathcal{O}}_i^{(2)}$ are collected in table 5.3 as done in [12] and discussed in the next subsection. They come from projecting the $\mathcal{O}_i^{(2)}$ part of the relativistic ChPT

Lagrangian.
From now on, unless otherwise stated, we choose

$$v^{\mu} = (1,0,0,0)^{\top}. \tag{5.6.22}$$

Using this we can show that $S^{\mu} = (0, \frac{\vec{\sigma}}{2})$:

$$\begin{aligned} \frac{i}{2}\gamma_5\sigma_{\mu\nu}v^{\nu} &= \frac{i}{2}\gamma_5\sigma_{\mu 0} \\ &= -\frac{1}{4}\gamma_5(\gamma_\mu\gamma_0 - \gamma_0\gamma_\mu) \\ &= \begin{cases} 0 & \text{for } \mu\text{=0} \\ -\frac{1}{4}\gamma_5 \cdot 2\gamma_k\gamma_0 = -\frac{1}{2}\gamma_5\gamma_k\gamma_0 & \text{for } \mu\text{=}k\text{=1,2,3} \end{cases} \end{aligned} \tag{5.6.23}$$

and

$$\gamma_5\gamma_k\gamma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\sigma_k \\ -\sigma_k & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \tag{5.6.24}$$

so we get

$$S^{\mu} = \left(0, \frac{\vec{\sigma}}{2}\right). \tag{5.6.25}$$

5.7 Construction of the relevant HBChPT Lagrangian

We can now write the relevant parts of the Heavy Baryon Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\pi} + \mathcal{L}_{NN} + \mathcal{L}_{NNN} + \mathcal{L}_{\pi N} + \mathcal{L}_{\pi NN}, \tag{5.7.1}$$

where we can expand each term accordingly to his chiral order:

$$\begin{aligned} \mathcal{L}_{\pi} &= \mathcal{L}_{\pi}^{(2)} + \mathcal{L}_{\pi}^{(4)} + ..., \\ \mathcal{L}_{NN} &= \mathcal{L}_{NN}^{(0)} + \mathcal{L}_{NN}^{(2)} + ..., \\ \mathcal{L}_{NNN} &= \mathcal{L}_{NNN}^{(0)} + \mathcal{L}_{NNN}^{(2)} + ..., \\ \mathcal{L}_{\pi N} &= \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + ..., \\ \mathcal{L}_{\pi NN} &= \mathcal{L}_{\pi NN}^{(1)} + \mathcal{L}_{\pi NN}^{(2)} + ..., \end{aligned} \tag{5.7.2}$$

explicitly they are given by

$$\begin{aligned}
\mathcal{L}_{\pi}^{(2)} &= \frac{1}{4} f_{\pi}^{2} \{\mathrm{Tr}[\nabla_{\mu} U^{\dagger} \nabla^{\mu} U + \chi^{\dagger} U + \chi U^{\dagger}]\} \\
\mathcal{L}_{NN}^{(0)} &= -\frac{1}{2} C_S (\bar{N}N)(\bar{N}N) + 2C_T (\bar{N}SN)\cdot(\bar{N}SN) \\
\mathcal{L}_{NNN}^{(0)} &= -\frac{1}{2}\frac{c_E}{f_{\pi}^{4}\Lambda_{\chi}} (\bar{N}N)(\bar{N}\tau N)\cdot(\bar{N}\tau N) \\
\mathcal{L}_{\pi N}^{(1)} &= \bar{N}(iv\cdot D + g_A S\cdot u)N \\
\mathcal{L}_{\pi N}^{(2)} &= -\frac{1}{2m}\bar{N}\left(D^2 + ig_A\{S\cdot D, v\cdot u\}\right)N + \sum_{i=1}^{7}\hat{c}_i \bar{N}\hat{O}_i^{(2)} N \\
\mathcal{L}_{\pi NN}^{(1)} &= \frac{c_D}{2f_{\pi}^{2}\Lambda_{\chi}} (\bar{N}N)(\bar{N}S_{\mu}u^{\mu}N)
\end{aligned} \tag{5.7.3}$$

where the terms are constructed requiring Lorentz, parity and charge invariance. Transformations properties of operators and fields are collected respectively in table 5.1 and 5.2. $\hat{O}_i^{(2)}$ can be found in the table 5.3.

i	$O_i^{(2)}$	$\hat{O}_i^{(2)}$	$2m(\hat{c}_i - c_i)$
1	$\langle\chi_+\rangle$	$\langle\chi_+\rangle$	0
2	$-\frac{1}{8m^2}\langle u_\mu u_\nu\rangle D^{\mu\nu}$+h.c.	$\frac{1}{2}\langle(v\cdot u)^2\rangle$	$-\frac{1}{4}g_A^2$
3	$\frac{1}{2}\langle u\cdot u\rangle$	$\frac{1}{2}\langle u\cdot u\rangle$	0
4	$\frac{i}{4}[u_\mu, u_\nu]\sigma^{\mu\nu}$	$\frac{1}{2}[S^\mu, S^\nu][u_\mu, u_\nu]$	$\frac{1}{2}$
5	$\tilde{\chi}_+$	$\tilde{\chi}_+$	0
6	$\frac{1}{8m}\tilde{F}^+_{\mu\nu}\sigma^{\mu\nu}$	$-\frac{i}{4m}[S^\mu, S^\nu]\tilde{F}^+_{\mu\nu}$	$2m$
7	$\frac{1}{8m}\langle F^+_{\mu\nu}\rangle\sigma^{\mu\nu}$	$-\frac{i}{4m}[S^\mu, S^\nu]\langle F^+_{\mu\nu}\rangle$	0

Table 5.3: Independent operators with chiral dimension equal to 2 in ChPT and HBChPT Lagrangian. The $1/m$ corrections are listed as well.

$\mathcal{L}_{\pi N}^{(2)}$ written above is simplified, indeed the full expression would be

$$\mathcal{L}_{\pi N}^{(2)} = \frac{1}{2m}\bar{N}\left((v\cdot D)^2 - D^2 - ig_A\{S\cdot D, v\cdot u\}\right)N + \sum_{i=1}^{7}\hat{c}_i\bar{N}\hat{O}_i^{(2)}N, \tag{5.7.4}$$

but we can use a field redefinition or the equation of motions to simplify the first term, here we can define [11]

$$N = \left[1 + \frac{iv\cdot D}{4m} - \frac{g_A S\cdot u}{4m}\right]\tilde{N}. \tag{5.7.5}$$

Inserting this into the lowest order Lagrangian (5.7.3) we get

$$\bar{\tilde{N}}(iv\cdot D + g_A S\cdot u)\tilde{N} - \frac{1}{2m}\bar{\tilde{N}}(v\cdot D)^2\tilde{N} - \frac{g_A^2}{2m}\bar{\tilde{N}}S\cdot uS\cdot u\tilde{N} + \text{total derivative} + O(\frac{1}{m^2}). \tag{5.7.6}$$

So we see that the second term here derived cancels the same term in $\mathcal{L}^{(2)}_{\pi N}$ and the last term can be rewritten in terms of independent ones already present in the Lagrangian

$$S \cdot u S \cdot u = \frac{1}{4}[(v \cdot u)^2 - u \cdot u] + \frac{1}{2} i \epsilon^{\mu\nu\rho\sigma} v_\rho S_\sigma u_\mu u_\nu. \tag{5.7.7}$$

These fields should be added to the operators already present. This, as already said, does not change the number of independent operators. However does change the LECs. Anyways this is a $\mathcal{O}(\frac{1}{m})$ suppressed, so we can neglect it.
To write the nucleon contact terms in the Lagrangian, we kept only the independent ones and we used the EOM for the light field to write the leading order. Possible building blocks can be found in table 5.4. As explained in [14] and as we will show in the following, terms that contain two nucleon bilinears can be constructed without isospin matrices. They can be considered as possible part of monomials only for at least three bilinears where Fierz reshuffling is not enough to eliminate those terms.
Let us start with the construction. We can already make the first simplification, only all the two nucleon bilinears with maximum one spacetime and one isospin index are considered. Only bilinears because, both for spin and isospin matrices it is possible to apply Fierz identity

$$\begin{aligned} \tau^a_{ij} \tau^a_{kl} &= 2\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}, \\ \sigma^i_{ab}\sigma^i_{cd} &= 2\delta_{ad}\delta_{cb} - \delta_{ab}\delta_{cd}, \end{aligned} \tag{5.7.8}$$

which reduces possible non bilinear terms to bilinear. As regarding the isospin space we get similar simplifications using the commutator and anticommutator

$$\{\tau^a, \tau^b\} = 2\delta^{ab}, \qquad [\tau^a, \tau^b] = 2i\epsilon^{abc}\tau^c, \tag{5.7.9}$$

while terms where ϵ^{abc} has two open indices have to be contracted with two indices from the other bilinear. But these two indices can just come from a δ_{ab} which gives zero or from a $\tau_a\tau_b$ or another epsilon tensor. $\tau_a\tau_b$ are reduced to either δ_{ab} or ϵ_{abc}, but also the contraction of two epsilon tensors is reduced to product of Kronecker delta. The same is true for $\epsilon^{\mu\nu\rho\sigma}$ with more than one open index, which has to be contracted and again gives redundant terms: with more than one v_μ goes to zero, more than one S_μ is reduced using the identities below, with another epsilon tensor gives Kronecker delta.
Only one spacetime index because terms are built using S^μ, v^μ, $\epsilon^{\mu\nu\rho\sigma}$ and since

$$\begin{aligned} S^\mu S^\nu &= \frac{1}{2}\{S^\mu, S^\nu\} + \frac{1}{2}[S^\mu, S^\nu], \\ \{S^\mu, S^\nu\} &= \frac{1}{2}(v^\mu v^\nu - g^{\mu\nu}), \\ [S^\mu, S^\nu] &= i\epsilon^{\mu\nu\rho\sigma} v_\rho S_\sigma, \\ v^2 &= 1, \\ S^2 &= -\frac{3}{4}, \\ v \cdot S &= 0, \end{aligned} \tag{5.7.10}$$

we see that contractions of these vectors do add only redundant terms in the Lagrangian. In Table 5.4 we show the relevant monomials and their transformations.
Terms as:

$$\begin{aligned}(\bar{N}(v\cdot D)N)(\bar{N}N),\\ (\bar{N}(v\cdot D)S_\mu N)(\bar{N}S^\mu N),\end{aligned} \tag{5.7.11}$$

can be eliminated using EOM. Others as

$$\begin{aligned}(\bar{N}S^\mu N)(\bar{N}D_\mu N),\\ (\bar{N}(S\cdot D)N)(\bar{N}N),\\ (\bar{N}v^\mu N)(\bar{N}u_\mu N),\end{aligned} \tag{5.7.12}$$

are neither parity neither charge invariant. We could ask ourselves what about terms with more derivatives. As explained in [14], they are part of $\mathcal{L}^{(2)}_{NN}$. Using this table we

Operator	Parity (P)	Charge (C)
$\bar{N}N$	+	+
$\bar{N}S^\mu N$	-	+
$\bar{N}D^\mu N$	+	+
$\bar{N}v_\mu N$	+	+
$\bar{N}u_\mu N$	-	+
$\bar{N}S^\mu u_\mu N$	+	+
$\bar{N}v^\mu D_\mu N$	+	+
$\bar{N}S_\nu u_\rho N\epsilon^{\mu\nu\rho\sigma}v_\sigma$	-	-

Table 5.4: Transformation properties of the monomials for the four nucleons contact term in HBChPT.

can construct terms with 2 nucleons bilinear either pure nucleons or also with a pion insertion.
We still need to explain how to get the trilinear term in $\mathcal{L}_{NNN}$. We will not derive it here but we refer to [15] and [16].

5.8 Chiral power counting

We need a power counting to understand the correct relevance of the diagrams for our scattering process. The power counting in effective field theories, in particular in ChPT, was pioneered by Weinberg [17], [18], [19], [20], while for a more modern view we refer to [11], [21], [22]. The idea is to expand our EFT in powers of momentum over mass scale. In particular, in ChPT this corresponds to $(\frac{p}{\Lambda})^\nu$ where $\Lambda \approx 700$ MeV is the QCD scale and p is the transferred small momentum. A Feynman diagram will in general be written in terms of the following elements:

$$\delta^{(4)}(p)^C \int (d^4q)^L \frac{1}{(q^2)^{I_p}} \frac{1}{q_0^{I_n}} \Pi_i (q^{d_i})^{V_i}, \tag{5.8.1}$$

where L is the number of loops, I_n (I_p) is the number of internal nucleon (meson) lines, d_i is the number of derivatives or pion mass insertions in the i^{th} vertex appearing V_i times in the diagram, C is the number of disconnected diagrams. From this we can compute the power ν that here we call $\tilde{\nu}$

$$\begin{aligned} \tilde{\nu} &= 4 - 4C + 4L - 2I_p - I_n + \sum_i V_i d_i \\ &= 4 + 4(L - C) - 2I_p - I_n + \sum_i V_i d_i. \end{aligned} \tag{5.8.2}$$

To simplify this we can use the identities

$$\begin{aligned} \sum_i V_i n_i &= 2I_n + E_n, \\ \sum_i V_i p_i &= 2I_p + E_p, \end{aligned} \tag{5.8.3}$$

and a topological one

$$L - C = I_p + I_n - \sum_i V_i, \tag{5.8.4}$$

where E_n (E_p) is the number of external nucleon (meson) lines in the diagram, n_i (p_i) is the number of nucleons (mesons) lines attached to each vertex with index i. With the two previous identities we can rewrite the topological one as

$$2(L - C) = -E_n - E_p + \sum_i V_i(n_i + p_i - 2). \tag{5.8.5}$$

Plugging this back in the expression for $\tilde{\nu}$ we get

$$\begin{aligned} \tilde{\nu} &= 4 + 2(L - C) - E_n - E_p - 2I_p - I_n + \sum_i V_i(d_i + n_i + p_i - 2) \\ &= 4 + 2(L - C) - \frac{E_n}{2} + \sum_i V_i(d_i + \frac{n_i}{2} - 2). \end{aligned} \tag{5.8.6}$$

From this we can immediately see that this power counting introduces problems when we add external nucleons that are just spectators and do not interact, because we have E_n in the expression[1]. The reason for this is the different normalization of the 2/3 particle

[1]Moreover, as pointed out by Weinberg in [19], expression (5.8.6) has its leading contribution for tree level diagrams when $L = 0$ and when $(d_i + \frac{n_i}{2} - 2) = 0$ since any chiral invariant interaction must have either no nucleon fields and at least two pions derivatives or two nucleon fields and one derivative. This would rule out the existence of bound nuclear states that indeed arise from graphs with arbitrary number of loops. The issue here is that this higher order terms are afflicted with infrared divergences, that arise from the non-relativistic expansion. He discuss about it with the help of old-fashioned perturbation theory, we will no longer mention it since it is not relevant to our problem.

states:

$$\begin{aligned} &2N : \langle \vec{p}_1 \vec{p}_2 | \vec{p}_1' \vec{p}_2' \rangle = \delta^{(3)}(\vec{p}_1' - \vec{p}_1)\delta^{(3)}(\vec{p}_2' - \vec{p}_2), \\ &3N : \langle \vec{p}_1 \vec{p}_2 \vec{p}_3 | \vec{p}_1' \vec{p}_2' \vec{p}_3' \rangle = \delta^{(3)}(\vec{p}_1' - \vec{p}_1)\delta^{(3)}(\vec{p}_2' - \vec{p}_2)\delta^{(3)}(\vec{p}_3' - \vec{p}_3). \end{aligned} \tag{5.8.7}$$

As explained in [22] this can be solved by assigning a chiral dimension to the transition operator, where we effectively consider only the interacting particles, rather than to the matrix element in the N-nucleon space, resulting in the addition of $\frac{3}{2}E_n - 6$ to $\tilde{\nu}$ that now becomes ν. This results in the leading order Feynman diagram to be given by $\nu = 0$ independently of the number of the external non interacting nucleons. Thus we get for the power counting in ChPT

$$\nu = -2 + E_n + 2(L - C) + \sum_i V_i \Delta_i, \qquad \Delta_i = d_i + \frac{1}{2} n_i - 2. \tag{5.8.8}$$

Power counting in HBChPT

Here we have to be careful because we are expanding our Lagrangian simultaneously in terms of $(\frac{p}{\Lambda_\chi})^\nu$ and $\frac{1}{m}$ with $m \approx 1$ GeV, the nucleon mass. The two scales, although similar, do not have the same value, so the difference has to be taken into account as discussed in [19], [23] and more recently in [24], [25], [26], [15]. We have two possibilities, either $\frac{p}{m}$ is one order higher

$$\frac{p}{m} \sim \frac{p^2}{\Lambda_\chi^2}, \tag{5.8.9}$$

or the same order

$$\frac{p}{m} \sim \frac{p}{\Lambda_\chi}. \tag{5.8.10}$$

Numerically, the terms $\frac{p}{m}$ are suppressed by a factor of 0.7, so we can actually decide. We will use the latter counting.

Last thing here, we want to explicitly write the *chiral* mass dimensions of fields and operators that will appear in the following, as done in [27], [13] (here Ψ represents the nucleon field)

$$\begin{aligned} &\mathcal{O}(1) : m, \Psi, \bar{\Psi}, D_\mu \Psi, \bar{\Psi}\Psi, U(x), \bar{\Psi}\gamma_\mu \Psi, \bar{\Psi}\gamma^\mu \gamma_5 \Psi, \bar{\Psi}\sigma^{\mu\nu}\Psi, \bar{\Psi}\sigma^{\mu\nu}\gamma_5 \Psi, \\ &\mathcal{O}(p) : (i\not{D} - m)\Psi, \bar{\Psi}\gamma_5 \Psi, \partial_\mu U(x), u_\mu(x), l_\mu(x), r_\mu(x), \\ &\mathcal{O}(p^2) : s(x), p(x), \chi_\pm, F_\pm^{\mu\nu}. \end{aligned} \tag{5.8.11}$$

the reason why $(i\not{D} - m)\Psi = \mathcal{O}(p)$ is given in [28]. The nucleon field has chiral dimension zero as a consequence of the manipulations done to go from $\tilde{\nu}$ to ν above.

5.9 Renormalization

In this work we are interested in computing the density loop corrections to the vertex NNZ_μ, to later use it for various types of neutral current interactions. However when computing the integrals with the corrected nucleon propagator not only integrals with a natural cut-off will appear but also divergent ones; these will be called vacuum corrections as they arise form the zero density part of the nucleon propagator. To properly take them into account, they need to be renormalized. The renormalization of the Pion-Nucleon interaction has been studied by Ecker in [29] at order $\mathcal{O}(p^3)$ and extended at $\mathcal{O}(p^4)$ in [30]. To renormalize our diagrams we will use the higher dimensional operators there analyzed and their β functions and we use dimensional regularization. As we will see from the results of the computations of the diagrams, loop contributions set in at $\mathcal{O}(p^3)$ only, so neither g_A nor the other LECs $c_1, ..., c_7$ of $\mathcal{O}(p^2)$ need to be renormalized. We can introduce the counterterm Lagrangian

$$\begin{aligned}\mathcal{L}^{\text{ct}}_{\pi N} &= \mathcal{L}^{(3)\text{ct}}_{\pi N}(x) + \mathcal{L}^{(4)\text{ct}}_{\pi N}(x)\\ &= \frac{1}{(4\pi f_\pi^2)^2}\sum_i b_i \bar{N}(x)\tilde{O}_i^{(3)}(x)N(x) + \frac{1}{(4\pi f_\pi^2)^2}\sum_i d_i \bar{N}(x)\tilde{O}_i^{(4)}(x)N(x)\end{aligned} \tag{5.9.1}$$

where b_i and d_i are dimensionless coupling constants. We can decompose them into a finite and divergent part

$$\begin{aligned}b_i &= b_i^r(\mu) + (4\pi)^2\beta_i\Lambda(\mu),\\ d_i &= d_i^r(\mu) + (4\pi)^2\delta_i\Lambda(\mu).\end{aligned} \tag{5.9.2}$$

The β_i are functions of g_A constructed such that they cancel the divergences up to $\mathcal{O}(p^3)$ and the complete list of operators and the respective β_i can be found in [29]. The δ_i depend both on g_A and the $\mathcal{O}(p^2)$ LECs and the complete list of $\mathcal{O}(p^4)$ operators and their δ_i can be found in [30]. The finite parts b_i^r and d_i^r are measurable quantities satisfying

$$\begin{aligned}\mu\frac{d}{d\mu}b_i^r(\mu) &= -\beta_i,\\ \mu\frac{d}{d\mu}d_i^r(\mu) &= -\delta_i,\end{aligned} \tag{5.9.3}$$

that implies

$$\begin{aligned}b_i^r(\mu) &= b_i^r(\mu_0) - \beta_i \ln\frac{\mu}{\mu_0},\\ d_i^r(\mu) &= d_i^r(\mu_0) - \delta_i \ln\frac{\mu}{\mu_0}.\end{aligned} \tag{5.9.4}$$

Before concluding this section is worth noticing that the tables presented in the papers list a complete set of operators for the renormalization of Green functions for off-shell baryons. As long as one is interested in on-shell nucleons only the number of operators gets reduced using the equation of motion

$$iv \cdot \nabla N = -g_A S \cdot uN. \tag{5.9.5}$$

Chapter 6

Feynman Rules

In this section we derive the Feynman rules that will be necessary to compute the corrections for the vertex NNZ_μ. We will display all the computations here, so if the reader is interested only in the results, please refer to the last lines of each subsection.

6.1 Formalism

Here we explain the notation, let us use as an example a 2-to-2 nucleons scattering. For the initial and final state we write

$$\begin{aligned} |i\rangle &= \bar{N}_k(p)\bar{N}_l(-p)|0\rangle \equiv |N_k(p)N_l(-p)\rangle \\ |f\rangle &= \bar{N}_j(p+q)\bar{N}_i(-p-q)|0\rangle \equiv |\bar{N}_j(p+q)\bar{N}_i(-p-q)\rangle, \end{aligned} \tag{6.1.1}$$

here the nucleon carries an index that represents both spin and isospin. We need as well

$$\langle f| = |f\rangle^\dagger = \langle 0|(\bar{N}_j(p+q)\bar{N}_i(-p-q))^\dagger = \langle 0|N_i(-p-q)N_j(p+q) \equiv \langle N_j(p+q)N_i(-p-q)|. \tag{6.1.2}$$

Additionally, when we go to the Fourier space we use $\partial_\mu \to ip_\mu$ with p_μ outgoing.

6.2 Nucleon propagator

In this work we want to systematically keep into account the dense nucleon environment. One way to do this is modifying the nucleon propagator. Here we only report the main steps, but the full exposition can be found in Appendix A.
The fermion propagator at finite density in HBChPT, here the nucleon, is given by [31]

$$iG(p) = P_+ iG_0(p) P_+ = \frac{i}{k^0 + i\epsilon} - 2\pi\delta(k^0)\theta(p^0)\theta(k_f - |\vec{k}|) + \mathcal{O}(\frac{1}{m}), \tag{6.2.1}$$

where $p^\mu = mv^\mu + k^\mu$, with k^μ the residual momentum. Let us analyze the two terms. The first, $\frac{i}{k^0+i\epsilon}$ is the zero density non-relativistic propagator, while the density effects are included in the second term and as we can see from $\theta(k_f - |\vec{k}|)$, accounts for the filled Fermi sea of nucleons. The propagator could also be written with the v^μ vector explicit, while here we used

$$v \cdot k = -\frac{k^2}{2m} = k^0 = E - m \ll m \tag{6.2.2}$$

with our choice for $v^\mu = (1, 0, 0, 0)$.
Here we show the derivation of the vacuum non-relativistic part, which comes from taking the $m \to \infty$ limit at the relativistic propagator

$$\begin{aligned}
\lim_{m\to\infty} \frac{i(\not{p} + m)}{p^2 - m^2} &= \lim_{m\to\infty} i\frac{m\not{v} + \not{k} + m}{(mv + k)^2 - m^2} \\
&= \lim_{m\to\infty} i\frac{m(1 + \not{v}) + \not{k}}{m^2 + 2mv \cdot k + k^2 - m^2} \\
&= \lim_{m\to\infty} i\frac{m(1 + \not{v}) + \not{k}}{2mv \cdot k + k^2} = i\frac{(1 + \not{v})}{2v \cdot k} = \frac{i}{v \cdot k} P_{v+},
\end{aligned} \tag{6.2.3}$$

where

$$P_{v+} = \frac{(1 + \not{v})}{2}. \tag{6.2.4}$$

We don't need to write P_{v+} because we have already integrated out the heavy part of the nucleon field.
Following again [31] we report here the next to leading order contribution

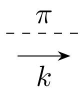
$$\frac{i}{2m}\left[1 - \frac{k^2}{(v \cdot k + i\epsilon)^2}\right]. \tag{6.2.5}$$

Using tools of QFT at finite density and temperature and taking the $T \to 0$ limit we get for the nucleon propagator at finite density

$$\begin{aligned}
iG(p, T) = \frac{i}{k^0 + i\epsilon} - 2\pi\Bigg[&\delta(k^0 - \frac{\vec{k}^2}{2m})\theta(p^0)\frac{1}{e^{\beta(\omega-\mu)} + 1} \\
&+ \delta(k^0 + 2m)\theta(-p^0)\frac{1}{e^{-\beta(\omega-\mu)} + 1}\Bigg] + \mathcal{O}\left(\frac{1}{m}\right),
\end{aligned} \tag{6.2.6}$$

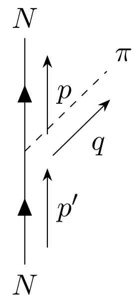
where $\omega \equiv p^0 \simeq m + k^0$.

6.3 Pion propagator

The pion propagator in ChPT is given by 

$$\frac{-i\delta^{ab}}{m_\pi^2 - q^2}. \tag{6.3.1}$$

6.4 $NN\pi$



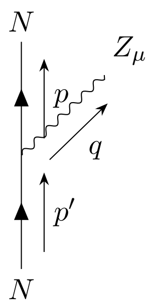
This vertex is computed from

$$\begin{aligned}\mathcal{L} \supset & g_A \bar{N} S^\mu u_\mu N \\ = & \frac{g_A}{2f_\pi} \bar{N} \vec{\sigma} \cdot \vec{\nabla} \pi^a \tau^a N.\end{aligned} \tag{6.4.1}$$

And the corresponding vertex rule is

$$\begin{aligned}i\langle N(p')\pi^a(q)|g_A \bar{N} S^\mu u_\mu N|N(p)\rangle = & -i\frac{1}{f_\pi}\langle N(p')\pi^a(q)|g_A \bar{N} S^\mu \tau^b (ip_{1\mu})\pi^b(p_1)N|N(p)\rangle \\ = & \frac{1}{f_\pi} g_A S^\mu q_\mu \tau^a \\ = & -\frac{1}{2f_\pi} g_A \vec{\sigma} \cdot \vec{q} \tau^a\end{aligned} \tag{6.4.2}$$

6.5 NNZ_μ vertex ($\Delta = 0$)



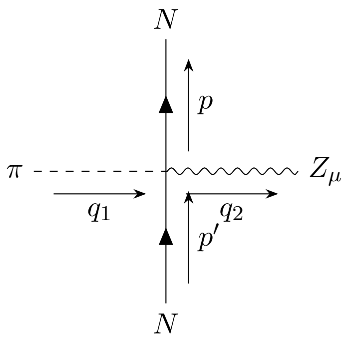
This vertex is given by the following part of the Lagrangian (5.7.1):

$$\begin{aligned}
\mathcal{L} &\supset g_A \bar{N}(iv^\mu D_\mu + S^\mu u_\mu)N \\
&\supset g_A \bar{N} S^\mu \left(\frac{e}{2}\tan\theta_W Z_\mu u^\dagger \tau^3 u + \frac{Z_\mu}{2}\frac{g}{\cos\theta_W}(1-\sin^2\theta_W)u\tau^3 u^\dagger\right) N + \\
&+ \bar{N} iv^\mu \left[+i\frac{g}{4\cos\theta_W}(1-2\sin^2\theta_W)Z_\mu \tau^3 - i\frac{e}{2}Z_\mu \tan\theta_W\right] N \\
&\supset \frac{g_A g}{2\cos\theta_W}\bar{N} S^\mu Z_\mu \tau^3 N - \bar{N} v^\mu \frac{g}{4\cos\theta_W}\left[(1-2\sin^2\theta_W)\tau^3 + e\tan\theta_W\right] Z_\mu N \\
&= -\frac{g_A g}{4\cos\theta_W}\bar{N}\vec{\sigma}\cdot\vec{Z}\tau^3 N - \bar{N} v^\mu \frac{g}{4\cos\theta_W}\left[(1-2\sin^2\theta_W)\tau^3 + e\tan\theta_W\right] Z_\mu N
\end{aligned} \tag{6.5.1}$$

Thus we find

$$\begin{aligned}
& i\langle N(p')Z_\mu(q)|\,(...)\,|N(p)\rangle \\
&= -i\frac{g_A g}{4\cos\theta_W}\vec{\sigma}\cdot\vec{\epsilon}\tau^3 - ig\frac{v\cdot\epsilon}{4\cos\theta_W}\left[(1-2\sin^2\theta_W)\tau^3 + e\tan\theta_W\right] \\
&= -i\frac{g}{4\cos\theta_W}\left[g_A(\vec{\sigma}\cdot\vec{\epsilon})\tau^3 + (v\cdot\epsilon)\left((1-2\sin^2\theta_W)\tau^3 + e\tan\theta_W\right)\right].
\end{aligned} \tag{6.5.2}$$

6.6 $NN\pi Z_\mu$ vertex ($\Delta = 0$)



Again from the same part of (5.7.1):

$$\mathcal{L} \supset g_A \bar{N}(iv^\mu D_\mu + S^\mu u_\mu)N. \tag{6.6.1}$$

Using

$$\tau^3\tau^a - \tau^a\tau^3 = [\tau^3, \tau^a] = 2i\epsilon^{3ab}\tau^b, \tag{6.6.2}$$

we can write

$$\begin{aligned}
u^\dagger r_\mu u &\supset r_\mu \frac{i\pi^a\tau^a}{2f_\pi} - \frac{i\pi^a\tau^a}{2f_\pi} r_\mu = \frac{i\pi^a}{f_\pi}(\hat{r}_\mu \tau^3) i\epsilon^{3ab}\tau^b, \\
-ul_\mu u^\dagger &\supset l_\mu \frac{i\pi^a\tau^a}{2f_\pi} - \frac{i\pi^a\tau^a}{2f_\pi} l_\mu = \frac{i\pi^a}{f_\pi}(\hat{l}_\mu \tau^3) i\epsilon^{3ab}\tau^b,
\end{aligned} \tag{6.6.3}$$

so

$$u^\dagger r_\mu u - u l_\mu u^\dagger = -\frac{2(v_\mu \tau^3)}{f_\pi}\pi^a \epsilon^{3ab}\tau^b = Z_\mu \frac{g}{2\cos\theta_W f_\pi}\left(1 - 2\sin^2\theta_W\right)\epsilon^{3ab}\pi^a\tau^b,$$
$$-\frac{i}{2}(u^\dagger r_\mu u + u l_\mu u^\dagger) = i\frac{2(a_\mu \tau^3)}{2f_\pi}\pi^a\epsilon^{3ab}\tau^b = iZ_\mu \frac{g}{4\cos\theta_W f_\pi}\epsilon^{3ab}\pi^a\tau^b. \tag{6.6.4}$$

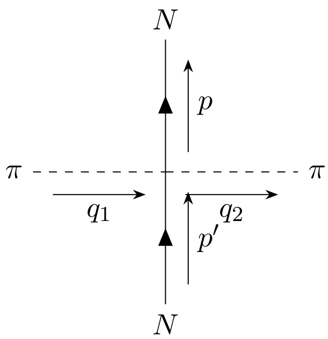
Now we can go back to the Lagrangian and we get

$$\begin{aligned}\mathcal{L} \supset & - g_A \bar{N}\vec{\sigma}\vec{Z}\left(\frac{g}{4\cos\theta_W f_\pi}\epsilon^{3ab}\pi^a\tau^b\left(1 - 2\sin^2\theta_W\right)\right)N \\ & - \bar{N}\left(\frac{g}{4\cos\theta f_\pi}\epsilon^{3ab}\pi^a\tau^b v\cdot Z\right)N.\end{aligned} \tag{6.6.5}$$

Thus

$$\begin{aligned} & i\langle N(p')Z_\mu(q_2)|(6.6.5)|\pi^a(q_1)N(p)\rangle \\ &= -i\frac{g}{4\cos\theta_W f_\pi}\epsilon^{3ab}\tau^b(v\cdot\epsilon) - i\frac{gg_A}{4\cos\theta_W f_\pi}\left(1-2\sin^2\theta_W\right)(\vec{\sigma}\cdot\vec{\epsilon})\epsilon^{3ab}\tau^b \\ &= -i\frac{g}{4\cos\theta_W f_\pi}\epsilon^{3ab}\tau^b\left[(v\cdot\epsilon) + g_A\left(1-2\sin^2\theta_W\right)(\vec{\sigma}\cdot\vec{\epsilon})\right].\end{aligned} \tag{6.6.6}$$

6.7 $NN\pi\pi$ vertex ($\Delta = 1$)



We need the $\mathcal{O}(p^2)$ part of the Lagrangian

$$\mathcal{L} \supset \hat{c}_1\bar{N}\langle\chi_+\rangle N + \hat{c}_3\bar{N}u^\mu u_\mu N + \hat{c}_4\bar{N}[S^\mu, S^\nu]u_\mu u_\nu N, \tag{6.7.1}$$

the $\hat{c}_2$ doesn't contribute, as can be seen from (6.2.2), it is additionally suppressed by

$\frac{1}{m}$ factor. We can split the computation in two terms

$$\begin{aligned}
&i\langle N(p')\pi^a(q_2)|(\hat{c}_1\bar{N}\langle\chi_+\rangle N + \hat{c}_3\bar{N}u^\mu u_\mu N)|\pi^b(q_1)N(p\rangle) \\
=&i\langle N(p')\pi^a(q_2)|\Bigg(-\frac{2\hat{c}_1 m_\pi^2}{f_\pi^2}\bar{N}(\pi^c\pi^d\delta^{cd})N \\
&\qquad - (i)^2\frac{\hat{c}_3}{f_\pi^2}\bar{N}(\tau^c p_1^\mu \pi^c(p_1)\tau^d p_{2\mu}\pi^d(p_2))N\Bigg)|\pi^b(q_1)N(p\rangle) \\
=&i\left(-\frac{4\hat{c}_1 m_\pi^2}{f_\pi^2}\delta^{ab} + \frac{\hat{c}_3}{f_\pi^2}(\tau^a\tau^b + \tau^b\tau^a)q_1^\mu q_{2\mu}\right) \\
=&\frac{i\delta^{ab}}{f_\pi^2}(-4\hat{c}_1 m_\pi^2 + 2\hat{c}_3 q_1^\mu q_{2\mu}) \\
=&-\frac{i\delta^{ab}}{f_\pi^2}(4\hat{c}_1 m_\pi^2 + 2\hat{c}_3\vec{q}_1\cdot\vec{q}_2),
\end{aligned} \tag{6.7.2}$$

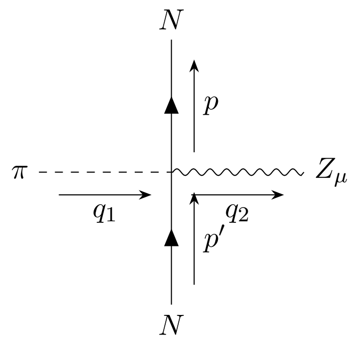
and the other term

$$\begin{aligned}
&i\langle N(p')\pi^a(q_2)|(\hat{c}_4\bar{N}[S^\mu, S^\nu]u_\mu u_\nu N)|\pi^b(q_1)N(p)\rangle \\
=&\frac{i}{f_\pi^2}\langle N(p')\pi^a(q_2)|\left(\hat{c}_4\bar{N}(-1)(i)^2\left[\frac{\sigma^i}{2},\frac{\sigma^j}{2}\right]p_1^i p_2^j \tau^c\pi^c(p_1)\tau^d\pi^d(p_2)N\right)|\pi^b(q_1)N(p)\rangle \\
=&\frac{i}{f_\pi^2}\left(\hat{c}_4\frac{1}{4}[\sigma^i,\sigma^j]q_1^i q_2^j(\tau^a\tau^b - \tau^b\tau^a)\right) \\
=&-\frac{1}{2f_\pi^2}(\hat{c}_4[\vec{\sigma}\cdot\vec{q}_1, \vec{\sigma}\cdot\vec{q}_2]\epsilon^{abc}\tau^c) \\
=&-\frac{i}{f_\pi^2}(\hat{c}_4\epsilon_{ijk}\sigma_k q_{1i}q_{2j}\epsilon^{abc}\tau^c) \\
=&-i\frac{\hat{c}_4}{f_\pi^2}(\vec{q}_1\cdot(\vec{q}_2\times\vec{\sigma}))\epsilon^{abc}\tau^c.
\end{aligned} \tag{6.7.3}$$

So the Feynman rule is

$$\begin{aligned}
&i\langle N(p')\pi^a(q_2)|(...)|\pi^b(q_1)N(p)\rangle \\
=&-\frac{i\delta^{ab}}{f_\pi^2}(4\hat{c}_1 m_\pi^2 + 2\hat{c}_3\vec{q}_1\cdot\vec{q}_2) - i\frac{\hat{c}_4}{f_\pi^2}(\vec{q}_1\cdot(\vec{q}_2\times\vec{\sigma}))\epsilon^{abc}\tau^c.
\end{aligned} \tag{6.7.4}$$

6.8 $NN\pi Z_\mu$ vertex ($\Delta = 1$)



This vertex can be read off the $O(p^2)$ part of the Lagrangian

$$\mathcal{L} \supset \frac{\hat{c}_3}{2}\bar{N}\langle u\cdot u\rangle N + \frac{\hat{c}_4}{2}[S^\mu, S^\nu][u_\mu, u_\nu] - i\frac{\hat{c}_6}{4m}[S^\mu, S^\nu]\tilde{F}^+_{\mu\nu}, \tag{6.8.1}$$

contributions from

$$\mathcal{L} \supset -i\frac{g_A}{2m}\bar{N}\{S\cdot D, v\cdot u\}N + \hat{c}_2\bar{N}\frac{1}{2}\langle (v\cdot u)^2\rangle N \tag{6.8.2}$$

are suppressed because, calling k the small momentum of the nucleon and choosing for $v^\mu = (1,0,0,0)$, then we get

$$v\cdot k = -\frac{\vec{k}^2}{2m} = k_0 = E - m << m. \tag{6.8.3}$$

For $-i\frac{g_A}{2m}\bar{N}\{S\cdot D, v\cdot u\}N$ is possible to see that the only way to get the right contribution is from taking the pion from u_μ and the Z_μ from D_μ, but $u_\mu \supset -\frac{\partial_\mu \pi^a\tau^a}{f_\pi}$, so it is suppressed.
Let us start with $\frac{\hat{c}_3}{2}\bar{N}\langle u\cdot u\rangle N$

$$\begin{aligned}\langle u_\mu u^\mu\rangle \supset &\Big\langle \left(-\frac{\partial_\mu\pi^a\tau^a}{f_\pi} + \frac{e}{2}\tan\theta_W Z_\mu\tau^3 + \frac{g}{2\cos\theta_W}Z_\mu\left(1-\sin^2\theta_W\right)\tau^3\right) \\ &\left(-\frac{\partial^\mu\pi^a\tau^a}{f_\pi} + \frac{e}{2}\tan\theta_W Z^\mu\tau^3 + \frac{g}{2\cos\theta_W}Z^\mu\left(1-\sin^2\theta_W\right)\tau^3\right)\Big\rangle \\ \supset &\Big\langle -\frac{\partial_\mu\pi^a\tau^a}{f_\pi}\frac{g}{2\cos\theta_W}\tau^3 Z^\mu - \frac{g}{2\cos\theta_W}\tau^3 Z^\mu\frac{\partial_\mu\pi^a\tau^a}{f_\pi}\Big\rangle,\end{aligned} \tag{6.8.4}$$

using

$$\langle\tau^a\tau^3\rangle + \langle\tau^3\tau^a\rangle = 2\langle\tau^a\tau^3\rangle = \langle\{\tau^a,\tau^3\}\rangle = 4\delta^{a3}, \tag{6.8.5}$$

we get for this term

$$\frac{\hat{c}_3}{2}\bar{N}\left(-\frac{2\delta^{a3}g}{f_\pi\cos\theta_W}\partial_\mu\pi^a Z^\mu\right)N. \tag{6.8.6}$$

Now for $\frac{\hat{c}_4}{2}[S^\mu, S^\nu][u_\mu, u_\nu]$, we use

$$[S^\mu, S^\nu] = \frac{1}{4}[\sigma_i, \sigma_j] \tag{6.8.7}$$

and

$$\begin{aligned}[u_\mu, u_\nu] =& \left(-\frac{\partial_\mu \pi^a \tau^a}{f_\pi} + \frac{e}{2}\tan\theta_W Z_\mu \tau^3 + \frac{g}{2\cos\theta_W} Z_\mu \left(1 - \sin^2\theta_W\right)\tau^3\right)(...)_\nu \\ & - (\mu \Longleftrightarrow \nu) \\ =& -\frac{g}{2f_\pi \cos\theta_W}[\tau^a\tau^3(\partial_\mu \pi^a Z_\nu) + \tau^3\tau^a(Z_\mu \partial_\nu \pi^a) \\ & - \tau^a\tau^3(Z_\mu\partial_\nu\pi^a) - \tau^3\tau^a(Z_\nu \partial_\mu \pi^a)] \\ =& -\frac{g}{f_\pi\cos\theta_W} i\epsilon^{3ab}\tau^b(Z_\mu\partial_\nu\pi^a - Z_\nu\partial_\mu\pi^a),\end{aligned} \tag{6.8.8}$$

where we used tricks already used before. So we get

$$-i\frac{\hat{c}_4 g}{8f_\pi\cos\theta_W}\bar{N}[\sigma_i,\sigma_j]\epsilon^{3ab}\tau^b(Z_i\partial_j\pi^a - Z_j\partial_i\pi^a). \tag{6.8.9}$$

Let us analyze now the last term $-\frac{i}{4m}\hat{c}_6[S^\mu, S^\nu]\tilde{F}^+_{\mu\nu}$,

$$\begin{aligned} F^+_{\mu\nu} =& \tilde{F}^+_{\mu\nu} = u^\dagger F^R_{\mu\nu} u + u F^L_{\mu\nu} u^\dagger \\ =& \left(\mathbb{I} - \frac{i\pi^a\tau^a}{2f_\pi}\right)\left(\frac{e}{2}\tan\theta_W\right)(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\tau^3\left(\mathbb{I} + \frac{i\pi^a\tau^a}{2f_\pi}\right) + \\ & + \left(\mathbb{I} + \frac{i\pi^a\tau^a}{2f_\pi}\right)\left(-\frac{g}{2\cos\theta_W} + \frac{e\tan\theta_W}{2}\right)(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\tau^3\left(\mathbb{I} - \frac{i\pi^a\tau^a}{2f_\pi}\right) \\ \supset& \left(\frac{e}{2}\tan\theta_W\right)(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\frac{i\pi^a}{2f_\pi}(\tau^3\tau^a - \tau^a\tau^3) + \\ & + \left(-\frac{g}{2\cos\theta_W} + \frac{e\tan\theta_W}{2}\right)(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\frac{i\pi^a}{2f_\pi}(-\tau^3\tau^a + \tau^a\tau^3) \\ =& -\frac{g}{2f_\pi\cos\theta_W}\epsilon^{3ab}\pi^a\tau^b(\partial_\mu Z_\nu - \partial_\nu Z_\mu),\end{aligned} \tag{6.8.10}$$

so we get

$$i\frac{\hat{c}_6 g}{32m\cos\theta_W f_\pi}\bar{N}[\sigma_i,\sigma_j]\epsilon^{3ab}\pi^a\tau^b(\partial_i Z_j - \partial_j Z_i)N. \tag{6.8.11}$$

We now want to check also what happens for $\frac{\hat{c}_2}{2}\langle(v\cdot u)^2\rangle$,

$$\begin{aligned}\langle u_\mu u_\nu\rangle \supset& -\frac{g}{2f_\pi\cos\theta_W}(\langle\tau^a\tau^3\rangle(\partial_\mu\pi^a Z_\nu) + \langle\tau^3\tau^a\rangle(Z_\mu\partial_\nu\pi^a)) \\ =& -\frac{g}{2f_\pi\cos\theta_W}(\langle\tau^a\tau^3\rangle(\partial_\mu\pi^a Z_\nu + Z_\mu\partial_\nu\pi^a)) \\ =& -\frac{g}{f_\pi\cos\theta_W}\delta^{a3}(\langle\tau^a\tau^3\rangle(\partial_\mu\pi^a Z_\nu + Z_\mu\partial_\nu\pi^a))\end{aligned} \tag{6.8.12}$$

doing similar computations as shown for the $\hat{c}_4$ term. So we get

$$\begin{aligned}
&-\frac{\hat{c}_2}{2}\frac{g}{f_\pi \cos\theta_W}\delta^{a3}\bar{N}(v\cdot\partial\pi^a v\cdot Z + v\cdot Z v\cdot\partial\pi^a)N \\
=&-\hat{c}_2\frac{g}{f_\pi \cos\theta_W}\delta^{a3}\bar{N}(v\cdot\partial\pi^a v\cdot Z)N.
\end{aligned} \tag{6.8.13}$$

We can finally compute the Feynman rules for them

$$\begin{aligned}
&i\langle N(p')Z_\mu(q_2)| - \hat{c}_2\frac{g}{f_\pi \cos\theta_W}\delta^{a3}\bar{N}(v\cdot\partial\pi^a v\cdot Z)N|\pi^a(q_1)N(p)\rangle \\
&= -\frac{\hat{c}_2 g}{f_\pi \cos\theta_W}(q_1\cdot v)\delta^{a3}v\cdot\epsilon,
\end{aligned} \tag{6.8.14}$$

$$\begin{aligned}
&i\langle N(p')Z_\mu(q_2)|\frac{\hat{c}_3}{2}\bar{N}\left(-\frac{2\delta^{a3}g}{f_\pi\cos\theta_W}\partial^\mu\pi^a Z_\mu\right)N|\pi^a(q_1)N(p)\rangle \\
&= -\frac{\hat{c}_3 g}{f_\pi \cos\theta_W}\delta^{a3}q_1\cdot\epsilon,
\end{aligned} \tag{6.8.15}$$

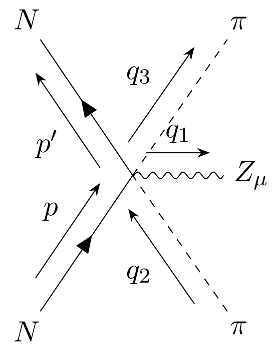
$$\begin{aligned}
&i\langle N(p')Z_\mu(q_2)| - i\frac{\hat{c}_4 g}{8f_\pi\cos\theta_W}\bar{N}[\sigma_i,\sigma_j]\epsilon^{3ab}\tau^b(Z_i\partial_j\pi^a - Z_j\partial_i\pi^a)|\pi^a(q_1)N(p)\rangle \\
=&\frac{\hat{c}_4 g}{8f_\pi\cos\theta_W}\epsilon^{3ab}\tau^b([\vec{\sigma}\cdot\vec{\epsilon},(-i)\vec{\sigma}\cdot\vec{q_1}] - [(-i)\vec{\sigma}\cdot\vec{q_1},\vec{\sigma}\cdot\vec{\epsilon}]) \\
=&-i\frac{\hat{c}_4 g}{4f_\pi\cos\theta_W}\epsilon^{3ab}\tau^b[\vec{\sigma}\cdot\vec{\epsilon},\vec{\sigma}\cdot\vec{q_1}] \\
=&\frac{\hat{c}_4 g}{2f_\pi\cos\theta_W}\epsilon^{3ab}\tau^b(\vec{q_1}\times\vec{\sigma})\cdot\vec{\epsilon},
\end{aligned} \tag{6.8.16}$$

$$\begin{aligned}
&i\langle N(p')Z_\mu(q_2)|i\frac{\hat{c}_6 g}{32m\cos\theta_W f_\pi}\bar{N}[\sigma_i,\sigma_j]\epsilon^{3ab}\pi^a\tau^b(\partial_i Z_j - \partial_j Z_i)N|\pi^a(q_1)N(p)\rangle \\
&\qquad = -i\frac{\hat{c}_6 g}{16m\cos\theta_W f_\pi}\epsilon^{3ab}\tau^b[\vec{q_2}\cdot\vec{\sigma},\vec{\sigma}\cdot\vec{\epsilon}] \\
&\qquad = -\frac{\hat{c}_6 g}{8m\cos\theta_W f_\pi}\epsilon^{3ab}\tau^b(\vec{q_2}\times\vec{\sigma})\cdot\vec{\epsilon},
\end{aligned} \tag{6.8.17}$$

where, as expected, the $\hat{c}_2$ term is suppressed. Summing we get

$$\begin{aligned}
&i\langle N(p')Z_\mu(q_2)|(...)|\pi^a(q_1)N(p)\rangle \\
=&-\frac{\hat{c}_3 g}{f_\pi\cos\theta_W}\delta^{a3}q_1\cdot\epsilon + \frac{\hat{c}_4 g}{2f_\pi\cos\theta_W}\epsilon^{3ab}\tau^b(\vec{q_1}\times\vec{\sigma})\cdot\vec{\epsilon} - \frac{\hat{c}_6 g}{8m\cos\theta_W f_\pi}\epsilon^{3ab}\tau^b(\vec{q_2}\times\vec{\sigma})\cdot\vec{\epsilon}.
\end{aligned} \tag{6.8.18}$$

6.9 $NN\pi\pi Z_\mu$ vertex $(\Delta = 0)$



We need to look at

$$\mathcal{L}^{(1)}_{\pi N} = \bar{N} i(v \cdot D)N + g_A \bar{N}(S^\mu u_\mu)N \tag{6.9.1}$$

For $g_A \bar{N} S^\mu u_\mu N$ we write

$$\begin{aligned} u_\mu \supset & \frac{e\tan\theta_W}{2} Z_\mu \left(\mathbb{I} - \frac{i\pi^a\tau^a}{2f_\pi} + \frac{1}{2}\frac{i\pi^a\tau^a}{2f_\pi}\frac{i\pi^b\tau^b}{2f_\pi}\right)\tau^3\left(\mathbb{I} + \frac{i\pi^a\tau^a}{2f_\pi} + \frac{1}{2}\frac{i\pi^a\tau^a}{2f_\pi}\frac{i\pi^b\tau^b}{2f_\pi}\right) \\ & + \frac{Z_\mu}{2}\left(\frac{g}{\cos\theta_W} - e\tan\theta_W\right)\left(\mathbb{I} + \frac{i\pi^a\tau^a}{2f_\pi} + \frac{1}{2}\frac{i\pi^a\tau^a}{2f_\pi}\frac{i\pi^b\tau^b}{2f_\pi}\right)\tau^3\left(\mathbb{I} - \frac{i\pi^a\tau^a}{2f_\pi} + \frac{1}{2}\frac{i\pi^a\tau^a}{2f_\pi}\frac{i\pi^b\tau^b}{2f_\pi}\right) \\ = & \left(Z_\mu \frac{g}{2\cos\theta_W}\right)\left[-\tau^3\frac{\pi^a\pi^b\tau^a\tau^b}{8f_\pi^2} - \frac{\pi^a\pi^b\tau^a\tau^b}{8f_\pi^2}\tau^3 - \frac{i\pi^a\tau^a}{2f_\pi}\tau^3\frac{i\pi^b\tau^b}{2f_\pi}\right], \end{aligned} \tag{6.9.2}$$

let us look at the square brackets. Notice that the first two terms are the same because since the τ matrices multiply the π, only the symmetric part survives, so

$$\begin{aligned} & \left[-\frac{1}{4f_\pi^2}\pi^a\pi^b\tau^3\tau^a\tau^b + \frac{1}{4f_\pi^2}\pi^a\pi^b\tau^a\tau^3\tau^b\right] \\ & = \frac{\pi^a\pi^b}{4f_\pi^2}(\tau^a\tau^3 - \tau^3\tau^a)\tau^b \\ & = (-2i\epsilon^{3ac}\tau^c\tau^b)\frac{\pi^a\pi^b}{4f_\pi^2} \\ & = (-2i\epsilon^{3ac}(\delta^{cb}\mathbb{I} + i\epsilon^{cbd}\tau^d))\frac{\pi^a\pi^b}{4f_\pi^2} \\ & = (-2i\epsilon^{3ab} + 2\tau^d(\delta^{3b}\delta^{ad} - \delta^{3d}\delta^{ab}))\frac{\pi^a\pi^b}{4f_\pi^2} \\ & = (\tau^a\delta^{3b} - \tau^3\delta^{ab})\frac{\pi^a\pi^b}{2f_\pi^2}. \end{aligned} \tag{6.9.3}$$

Then we get for the vertex

$$\frac{gg_A}{4\cos\theta_W f_\pi^2}\bar{N}\pi^a\pi^b(\tau^a\delta^{3b} - \tau^3\delta^{ab})S^\mu Z_\mu N \tag{6.9.4}$$

and the corresponding Feynman rule

$$\begin{aligned} i\langle N(p')Z_\mu(q)\pi^a\pi^b| &\frac{gg_A}{4\cos\theta_W f_\pi^2}\bar{N}\pi^a\pi^b(\tau^a\delta^{3b} - \tau^3\delta^{ab})S^\nu Z_\nu N|N(p)\rangle \\ &= i\frac{gg_A}{4\cos\theta_W f_\pi^2}S^\mu(\tau^b\delta^{3a} + \tau^a\delta^{3b} - 2\tau^3\delta^{ab}) \\ &= -i\frac{gg_A}{8\cos\theta_W f_\pi^2}\vec{\sigma}\cdot\vec{\epsilon}(\tau^b\delta^{3a} + \tau^a\delta^{3b} - 2\tau^3\delta^{ab}). \end{aligned} \tag{6.9.5}$$

Now, from $\bar{N}i(v\cdot D)N$ we get

$$u^\dagger\tau^3 u = u\tau^3 u^\dagger = (\tau^a\delta^{3b} - \tau^3\delta^{ab})\frac{\pi^a\pi^b}{2f_\pi^2}, \tag{6.9.6}$$

so that the covariant derivative

$$\begin{aligned} D_\mu \supset &- i\frac{e}{4}Z_\mu\tan\theta_W(\tau^a\delta^{3b} - \tau^3\delta^{ab})\frac{\pi^a\pi^b}{2f_\pi^2} \\ &+ \frac{i}{4}Z_\mu\left(\frac{g}{\cos\theta_W} - e\tan\theta_W\right)(\tau^a\delta^{3b} - \tau^3\delta^{ab})\frac{\pi^a\pi^b}{2f_\pi^2} \\ =& i(\tau^a\delta^{3b} - \tau^3\delta^{ab})\frac{\pi^a\pi^b}{4f_\pi^2}\left(\frac{g}{2\cos\theta_W} - e\tan\theta_W\right)Z_\mu, \end{aligned} \tag{6.9.7}$$

gives the following contribution

$$\bar{N}i(v\cdot D)N = \frac{1}{4f_\pi^2}\left(e\tan\theta_W - \frac{g}{2\cos\theta_W}\right)\bar{N}(v\cdot Z)(\tau^a\delta^{3b} - \tau^3\delta^{ab})\pi^a\pi^b N. \tag{6.9.8}$$

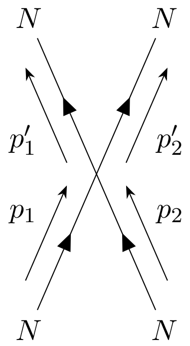
From this we can write the Feynman rule

$$\begin{aligned} &i\langle N(p')Z_\mu(q)\pi^a\pi^b|(...)|N(p)\rangle \\ &= \frac{i}{4f_\pi^2}\left(e\tan\theta_W - \frac{g}{2\cos\theta_W}(\tau^b\delta^{3a} + \tau^a\delta^{3b} - 2\tau^3\delta^{ab})(v\cdot\epsilon)\right). \end{aligned} \tag{6.9.9}$$

We would have another contribution coming from $\mathcal{L}_{\pi N}^{(2)} \supset -\frac{1}{2m}\bar{N}D^2N$, but this is $O(\frac{1}{m})$ suppressed.
So the relevant contribution is

$$\begin{aligned} &i\langle N(p')Z_\mu(q)\pi^a\pi^b|(...)|N(p)\rangle \\ =& - i\frac{gg_A}{8\cos\theta_W f_\pi^2}\vec{\sigma}\cdot\vec{\epsilon}(\tau^b\delta^{3a} + \tau^a\delta^{3b} - 2\tau^3\delta^{ab}) \\ &+ \frac{i}{4f_\pi^2}\left(e\tan\theta_W - \frac{g}{2\cos\theta_W}\right)(\tau^b\delta^{3a} + \tau^a\delta^{3b} - 2\tau^3\delta^{ab})v\cdot\epsilon \end{aligned} \tag{6.9.10}$$

6.10 $NNNN$ vertex ($\Delta = 0$)



The relevant part of the Lagrangian is

$$\begin{aligned}\mathcal{L} \supset &-\frac{1}{2}C_S(\bar{N}N)(\bar{N}N) + 2C_T(\bar{N}SN)\cdot(\bar{N}SN) \\ = &-\frac{1}{2}C_S(\bar{N}N)(\bar{N}N) - \frac{1}{2}C_T(\bar{N}\vec{\sigma}N)\cdot(\bar{N}\vec{\sigma}N).\end{aligned} \tag{6.10.1}$$

Let us split the computation of the Feynman rule in the two terms

$$\begin{aligned}&i\langle N(p'_1)N(p'_2)|(-\frac{1}{2}C_S(\bar{N}N)(\bar{N}N))|N(p_1)N(p_2)\rangle \\ =&-\frac{i}{2}C_S\langle 0|N_{2'}N_{1'}(\bar{N}N)(\bar{N}N)\bar{N}_1\bar{N}_2|0\rangle \\ =&-\frac{i}{2}C_S(\delta_{1'1}\delta_{2'2} + \delta_{2'2}\delta_{1'1} - \delta_{1'2}\delta_{2'1} - \delta_{2'1}\delta_{1'2}) \\ =&-iC_S(\delta_{1'1}\delta_{2'2} - \delta_{1'2}\delta_{2'1})\end{aligned} \tag{6.10.2}$$

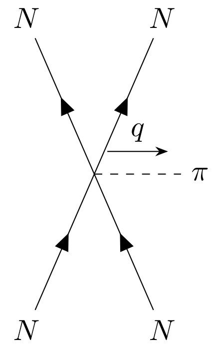
and

$$\begin{aligned}&i\langle N(p'_1)N(p'_2)|(-\frac{1}{2}C_T(\bar{N}\vec{\sigma}N)\cdot(\bar{N}\vec{\sigma}N))|N(p_1)N(p_2)\rangle \\ =&-\frac{i}{2}C_T\langle 0|N_{2'}N_{1'}(\bar{N}\vec{\sigma}N)\cdot(\bar{N}\vec{\sigma}N)\bar{N}_1\bar{N}_2|0\rangle \\ =&-\frac{i}{2}C_T(\sigma_{1'1}\sigma_{2'2} + \sigma_{2'2}\sigma_{1'1} - \sigma_{1'2}\sigma_{2'1} - \sigma_{2'1}\sigma_{1'2}) \\ =&-iC_T(\sigma_{1'1}\sigma_{2'2} - \sigma_{1'2}\sigma_{2'1}),\end{aligned} \tag{6.10.3}$$

so in total we get

$$= -i(C_S(\delta_{1'1}\delta_{2'2} - \delta_{1'2}\delta_{2'1}) + C_T(\sigma_{1'1}\sigma_{2'2} - \sigma_{1'2}\sigma_{2'1})). \tag{6.10.4}$$

6.11 $NNNN\pi$ vertex ($\Delta = 1$)



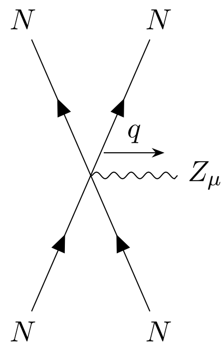
To compute the Feynman rule here we need

$$\mathcal{L}^{(1)}_{\pi NN} = \frac{c_D}{2f_\pi^2\Lambda_\chi}(\bar{N}N)(\bar{N}S_\mu u^\mu N), \tag{6.11.1}$$

so

$$\begin{aligned}
&i\langle N(p_1')N(p_2')\pi^a(q)|\left(\frac{c_D}{2f_\pi^2\Lambda_\chi}(\bar{N}N)(\bar{N}S^\mu u_\mu N)\right)|N(p_1)N(p_2)\rangle \\
=&-i\langle N(p_1')N(p_2')\pi^a(q)|\left(\frac{c_D}{2f_\pi^3\Lambda_\chi}(\bar{N}N)(\bar{N}S^\mu(i)q_{1\mu}\pi^b(q_1)\tau^b N)\right)|N(p_1)N(p_2)\rangle \\
=&\langle N(p_1')N(p_2')\pi^a(q)|\left(\frac{c_D}{2f_\pi^3\Lambda_\chi}(\bar{N}N)(\bar{N}S^\mu q_{1\mu}\pi^b(q_1)\tau^b N)\right)|N(p_1)N(p_2)\rangle \\
=&\frac{c_D}{2f_\pi^3\Lambda_\chi}[(S^\mu q_\mu\tau^a)_{1'1}\delta_{2'2} + (S^\mu q_\mu\tau^a)_{2'2}\delta_{1'1} - (S^\mu q_\mu\tau^a)_{2'1}\delta_{1'2} - (S^\mu q_\mu\tau^a)_{1'2}\delta_{2'1}] \\
=&-\frac{c_D}{4f_\pi^3\Lambda_\chi}[(\vec{\sigma}\cdot\vec{q}\tau^a)_{1'1}\delta_{2'2} + (\vec{\sigma}\cdot\vec{q}\tau^a)_{2'2}\delta_{1'1} - (\vec{\sigma}\cdot\vec{q}\tau^a)_{2'1}\delta_{1'2} - (\vec{\sigma}\cdot\vec{q}\tau^a)_{1'2}\delta_{2'1}].
\end{aligned} \tag{6.11.2}$$

6.12 $NNNNZ_\mu$ vertex ($\Delta = 1$)

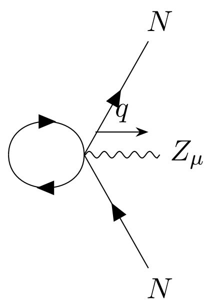


Here we need

$$\begin{aligned} \mathcal{L}^{(1)}_{\pi NN} =& \frac{c_D}{2f_\pi^2\Lambda_\chi}(\bar{N}N)(\bar{N}S_\mu u^\mu N) \\ \supset& \frac{c_D}{2f_\pi^2\Lambda_\chi}(\bar{N}N)\left(\bar{N}S_\mu \frac{g}{2\cos\theta}\tau^3 Z_\mu N\right) \\ =& -\frac{c_D g}{8f_\pi^2\Lambda_\chi\cos\theta_W}(\bar{N}N)(\bar{N}\tau^3\vec{\sigma}\cdot\vec{Z}N), \end{aligned} \tag{6.12.1}$$

so the vertex is

$$\begin{aligned} &i\langle N(p_1')N(p_2')Z_\mu(q)| - \frac{c_D g}{8f_\pi^2\Lambda_\chi\cos\theta_W}(\bar{N}N)(\bar{N}\tau^3\vec{\sigma}\cdot\vec{Z}N)|N(p_1)N(p_2)\rangle \\ &- i\frac{c_D g}{8f_\pi^2\Lambda_\chi\cos\theta_W}[(\tau^3\sigma_i)_{1'1}\delta_{2'2} + (\tau^3\sigma_i)_{2'2}\delta_{1'1} - (\tau^3\sigma_i)_{2'1}\delta_{1'2} - (\tau^3\sigma_i)_{1'2}\delta_{2'1}]. \end{aligned} \tag{6.12.2}$$

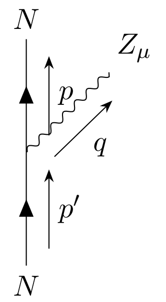


We can compute the loop vertex as well. Here we should have the vacuum and the density contributions. The vacuum one is zero because the nucleon propagator in HBChPT has only one pole, this means that we can choose the contour to avoid it, so that we are left with the density contribution alone

$$\begin{aligned} i\int_{|\vec{k}|\leq k_f}\frac{d^3\vec{k}}{(2\pi)^3}\langle N_i(p')Z_\mu(q)| - \frac{c_D g}{8f_\pi^2\Lambda_\chi\cos\theta_W}(\bar{N}N)(\bar{N}\tau^3\vec{\sigma}\cdot\vec{Z}N)|N_j(p)\rangle \\ = -i\frac{c_D g}{8f_\pi^2\Lambda_\chi\cos\theta_W}\left(-\delta_{ij}(\tau^3\sigma_k)_{ll}\frac{\rho}{4} - \frac{\rho}{4}\delta_{ll}(\tau^3\sigma_k)_{ij} + \frac{\rho}{4}(\tau^3\sigma_k)_{ij} + \frac{\rho}{4}(\tau^3\sigma_k)_{ij}\right) \\ = i\frac{c_D g\rho}{16f_\pi^2\Lambda_\chi\cos\theta_W}(\tau^3\sigma_k)_{ij} \end{aligned} \quad (6.12.3)$$

where the trace is both over the spin and the isospin space.

6.13 NNZ_μ vertex ($\Delta = 1$)



This vertex is not present in the axion Lagrangian, indeed comes from

$$\mathcal{L}_{\pi N}^{(2)} \supset \hat{c}_6\bar{N}\left(-\frac{i}{4m}[S^\mu, S^\nu]\tilde{F}_{\mu\nu}^+\right)N + \hat{c}_7\bar{N}\left(-\frac{i}{4m}[S^\mu, S^\nu]\langle F_{\mu\nu}^+\rangle\right) \quad (6.13.1)$$

in particular

$$\begin{aligned} \tilde{F}_{\mu\nu}^+ \supset& \frac{e}{2}\tan\theta_W(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\tau^3 + \left(-\frac{g}{2\cos\theta_W} + \frac{e}{2}\tan\theta_W\right)(\partial_\mu Z_\nu)\tau^3 \\ =& (\partial_\mu Z_\nu - \partial_\nu Z_\mu)\tau^3\left(e\tan\theta_W - \frac{g}{2\cos\theta_W}\right), \\ \langle F_{\mu\nu}^+\rangle =& e\tan\theta_W(\partial_\mu Z_\nu - \partial_\nu Z_\mu), \end{aligned} \quad (6.13.2)$$

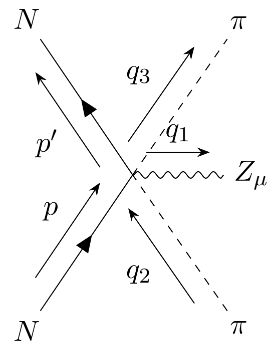
so,

$$\begin{aligned} \mathcal{L}_{\pi N}^{(2)} \supset& -i\frac{\hat{c}_6}{16m}\left(e\tan\theta_W - \frac{g}{2\cos\theta}\right)\bar{N}[\sigma_i, \sigma_j](\partial_i Z_j - \partial_j Z_i)\tau^3 N \\ & -i\frac{\hat{c}_7 e\tan\theta_W}{16m}\bar{N}[\sigma_i, \sigma_j](\partial_i Z_j - \partial_j Z_i)N. \end{aligned} \quad (6.13.3)$$

This gives the following Feynman rule

$$\begin{aligned}
&i\langle N(p')Z_\mu(q)|(...)|N(p)\rangle \\
=& \left(\frac{\hat{c}_6}{16m}\left(e\tan\theta_W - \frac{g}{2\cos\theta_W}\right)\tau^3 + \frac{\hat{c}_7 e\tan\theta_W}{16m}\right)([i\vec{\sigma}\cdot\vec{q},\vec{\sigma}\cdot\vec{\epsilon}] - [\vec{\sigma}\cdot\vec{\epsilon}, i\vec{\sigma}\cdot\vec{q}]) \\
&= 2i[\vec{\sigma}\cdot\vec{q},\vec{\sigma}\cdot\vec{\epsilon}]\left(\frac{\hat{c}_6}{16m}\left(e\tan\theta_W - \frac{g}{2\cos\theta_W}\right)\tau^3 + \frac{\hat{c}_7 e\tan\theta_W}{16m}\right) \\
&= \frac{(\vec{q}\times\vec{\sigma})\cdot\vec{\epsilon}}{4m}\left(\hat{c}_6\left(e\tan\theta_W - \frac{g}{2\cos\theta_W}\right)\tau^3 + \hat{c}_7 e\tan\theta_W\right).
\end{aligned} \tag{6.13.4}$$

6.14 $NN\pi\pi Z_\mu$ vertex ($\Delta = 1$)



With the same reasoning as is (3.3), immediately follows that $\hat{c}_2$ term is suppressed. Contributions come from

$$\mathcal{L} \supset \hat{c}_3\bar{N}\langle u\cdot u\rangle N + \frac{\hat{c}_4}{2}[S^\mu, S^\nu][u_\mu, u_\nu] - i\frac{\hat{c}_6}{4m}[S^\mu, S^\nu]\tilde{F}^+_{\mu\nu}. \tag{6.14.1}$$

We start from $\hat{c}_3$

$$\begin{aligned}
u_\mu &\supset -\frac{\tau^a\partial_\mu\pi^a}{f_\pi} - \frac{Z_\mu}{f_\pi}\left(-\frac{g}{2\cos\theta_W} + e\tan\theta_W\right)\epsilon^{3ab}\pi^a\tau^b \\
u_\mu u^\mu &\supset \frac{\tau^a\partial_\mu\pi^a}{f_\pi^2}Z^\mu\left(-\frac{g}{2\cos\theta_W} + e\tan\theta_W\right)\epsilon^{3cb}\pi^c\tau^b \\
&+ \frac{Z_\mu}{f_\pi^2}\left(-\frac{g}{2\cos\theta_W} + e\tan\theta_W\right)\epsilon^{3cb}\pi^c\tau^b\tau^a\partial_\mu\pi^a,
\end{aligned} \tag{6.14.2}$$

we need to analyze the isospin structure

$$\begin{aligned}
\epsilon^{3cb}\tau^a\tau^b =& \epsilon^{3cb}(\delta^{ab}\mathbb{I} + i\epsilon^{abd}\tau^d) = \epsilon^{3ca}\mathbb{I} + i\epsilon^{b3c}\epsilon^{bda}\tau^d \\
=& \epsilon^{3ca}\mathbb{I} + i(\delta^{3d}\delta^{ca} - \delta^{3a}\delta^{cd})\tau^d \\
=& \epsilon^{3ca}\mathbb{I} + i\delta^{ca}\tau^3 - i\delta^{3a}\tau^c \\
\epsilon^{3cb}\tau^b\tau^a =& \epsilon^{3ca}\mathbb{I} - i\delta^{ca}\tau^3 + i\delta^{3a}\tau^c,
\end{aligned} \tag{6.14.3}$$

tracing over isospin indices we get for both

$$\langle \epsilon^{3cb}\tau^a\tau^b\rangle = \langle \epsilon^{3cb}\tau^b\tau^a\rangle = 2\epsilon^{3ca}. \tag{6.14.4}$$

So we get

$$\frac{\hat{c}_3}{2}\bar{N}\langle u\cdot u\rangle N \supset \frac{2}{f_\pi^2}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)\epsilon^{3ca}\bar{N}Z\cdot\partial\pi^a\pi^c. \tag{6.14.5}$$

For $\hat{c}_4$ we proceed similarly, we indeed have the same isospin structure, but here the δ^{ab} does not simplifies:

$$\begin{aligned}
u_\mu u_\nu =& \frac{\tau^a\partial_\mu\pi^a}{f_\pi}\frac{Z_\nu}{f_\pi}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)\epsilon^{3cb}\pi^c\tau^b\\
&+\frac{Z_\mu}{f_\pi}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)\epsilon^{3cb}\pi^c\tau^b\frac{\tau^a\partial_\nu\pi^a}{f_\pi}\\
[u_\mu,u_\nu]\supset&\frac{\partial_\mu\pi^a Z_\nu}{f_\pi^2}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)2i(\delta^{ca}\tau^3-\delta^{3a}\tau^c)\pi^c\\
&-\frac{\partial_\nu\pi^a Z_\mu}{f_\pi^2}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)2i(\delta^{ca}\tau^3-\delta^{3a}\tau^c)\pi^c\\
&=\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)\frac{2i}{f_\pi^2}(\delta^{ca}\tau^3-\delta^{3a}\tau^c)(\partial_\mu\pi^a Z_\nu - Z_\mu\partial_\nu\pi^a)\pi^c,
\end{aligned} \tag{6.14.6}$$

so we get

$$i\frac{\hat{c}_4}{4f_\pi^2}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)\bar{N}[\sigma_i,\sigma_j](\partial_i\pi^a Z_j - Z_i\partial_j\pi^a)(\delta^{ca}\tau^3-\delta^{3a}\tau^c)\pi^c N. \tag{6.14.7}$$

For computing the vertex, we need to contract the pions obtaining

$$i(\vec{q}_2\cdot\vec{\epsilon})i(\delta^{ba}\tau^3-\delta^{3a}\tau^b)+i(\vec{q}_3\cdot\vec{\epsilon})i(\delta^{ab}\tau^3-\delta^{3b}\tau^a) \tag{6.14.8}$$

and the same for j and multiply with $[\sigma_i,\sigma_j]$,

$$\begin{aligned}
&-[\vec{\sigma}\cdot(\vec{q}_2(\delta^{ab}\tau^3-\delta^{3a}\tau^b)+\vec{q}_3(\delta^{ab}\tau^3-\delta^{3b}\tau^a)),\vec{\sigma}\cdot\vec{\epsilon}]\\
&+[\vec{\sigma}\cdot\vec{\epsilon},\vec{\sigma}\cdot(\vec{q}_2(\delta^{ab}\tau^3-\delta^{3a}\tau^b)+\vec{q}_3(\delta^{ab}\tau^3-\delta^{3b}\tau^a))]\\
=&2[\vec{\sigma}\cdot\vec{\epsilon},\vec{\sigma}\cdot(\vec{q}_2(\delta^{ab}\tau^3-\delta^{3a}\tau^b)+\vec{q}_3(\delta^{ab}\tau^3-\delta^{3b}\tau^a))].
\end{aligned} \tag{6.14.9}$$

For $\hat{c}_6$ we need

$$\begin{aligned}
\tilde{F}^+_{\mu\nu} &= u^\dagger F^R_{\mu\nu}u + uF^L_{\mu\nu}u^\dagger\\
&\supset\left(\mathbb{I}-\frac{i\pi^a\tau^a}{2f_\pi}+\frac{1}{2}\frac{i\pi^a\tau^a}{2f_\pi}\frac{i\pi^b\tau^b}{2f_\pi}\right)\left(\frac{e}{2}\tan\theta_W\right)(\partial_\mu Z_\nu-\partial_\nu Z_\mu)\tau^3\left(\mathbb{I}+\frac{i\pi^a\tau^a}{2f_\pi}+\frac{1}{2}\frac{i\pi^a\tau^a}{2f_\pi}\frac{i\pi^b\tau^b}{2f_\pi}\right)\\
&+\left(\mathbb{I}+\frac{i\pi^a\tau^a}{2f_\pi}+\frac{1}{2}\frac{i\pi^a\tau^a}{2f_\pi}\frac{i\pi^b\tau^b}{2f_\pi}\right)\left(-\frac{g}{2\cos\theta_W}+\frac{e\tan\theta_W}{2}\right)\\
&\quad(\partial_\mu Z_\nu-\partial_\nu Z_\mu)\tau^3\left(\mathbb{I}-\frac{i\pi^a\tau^a}{2f_\pi}+\frac{1}{2}\frac{i\pi^a\tau^a}{2f_\pi}\frac{i\pi^b\tau^b}{2f_\pi}\right).
\end{aligned} \tag{6.14.10}$$

With the same simplification as for $NN\pi\pi Z_\mu$ at $\Delta = 0$ we get

$$\bar{N}\left(i\frac{g\hat{c}_6}{32m\cos\theta_W}[\sigma_i,\sigma_j]\frac{\pi^a\pi^b}{2f_\pi^2}(\tau^a\delta^{3b}-\tau^3\delta^{ab})\left(1-2\sin_W^\theta\right)(\partial_i Z_j-\partial_j Z_i)\right)N \quad (6.14.11)$$

So we get the corresponding Feynman rule

$$\begin{aligned}
&i\langle N(p')Z_\mu(q_1)\pi^a(q_2)\pi^b(q_3)|(...)|N(p)\rangle\\
&=-\frac{g\hat{c}_3}{\cos\theta_W f_\pi^2}\left(1-2\sin^2\theta_W\right)((q_2-q_3)\cdot\epsilon)\epsilon^{3ab}\\
&+\frac{g\hat{c}_4}{2\cos\theta_W f_\pi^2}\left(1-2\sin^2\theta_W\right)((\delta^{ab}\tau^3-\delta^{3a}\tau^b)(\vec{q}_2\times\vec{\sigma})\cdot\vec{\epsilon}+(\delta^{ab}\tau^3-\delta^{3b}\tau^a)(\vec{q}_3\times\vec{\sigma})\cdot\vec{\epsilon})\\
&-\frac{g\hat{c}_6}{16\cos\theta_W m f_\pi^2}\left(1-2\sin^2\theta_W\right)(\tau^b\delta^{3a}+\tau^a\delta^{3b}-2\tau^3\delta^{ab})(\vec{q}_1\times\vec{\sigma})\cdot\vec{\epsilon}.
\end{aligned} \quad (6.14.12)$$

6.15 Why do we never include the Goldstone Bosons of the electroweak symmetry breaking?

Here we computed all the Feynman rules that we will later use to compute the matrix element squared for $NN\nu\nu$ process. We will use for the Z_μ boson the propagator [32]:

$$-\frac{i}{k^2-m_Z^2}\eta_{\mu\nu}, \quad (6.15.1)$$

this means that we choose the Feynman gauge, $\xi_Z = 1$, for our gauge field. The complete propagator indeed reads

$$-\frac{i}{k^2-m_Z^2}\left(\eta_{\mu\nu}-(1-\xi_Z)\frac{k^\mu k^\nu}{k^2-\xi_Z m_Z^2}\right). \quad (6.15.2)$$

With this choice the propagator of the Goldstone boson G^0 is not 0, and has to be taken into account: this Goldstone boson corresponds to the longitudinal degree of freedom that the massive gauge field has

$$\frac{i}{k^2-\xi_Z m_Z^2}=\frac{i}{k^2-m_Z^2}. \quad (6.15.3)$$

We need to understand where the GB is relevant. For the computation of the vertex correction, we never inserted a Z_μ boson propagator because it has the biggest mass compared to all the others and so its effect becomes negligible. But in the computation of the scattering amplitude we do need the Z_μ propagating from the nucleons to the neutrinos. Here we should consider the GB as well. From [32] we can see that the vertex $\nu\nu G^0$ is

$$\begin{aligned}&-\frac{2}{v}\delta_{f_1 f_2} m_{\nu_{f_1}}\gamma^5 - v\left(U_{g_2 f_2}U^*_{g_1 f_1}C^{\phi l_1}_{g_1 g_2}\not{p}_3 P_L - U_{g_1 f_1}U^*_{g_2 f_2}C^{\phi l_1}_{g_2 g_1}\not{p}_3 P_R\right)\\ &+ v\left(U_{g_2 f_2}U^*_{g_1 f_1}C^{\phi l_3}_{g_1 g_2}\not{p}_3 P_L - U_{g_1 f_1}U^*_{g_2 f_2}C^{\phi l_3}_{g_2 g_1}\not{p}_3 P_R\right).\end{aligned} \tag{6.15.4}$$

For our purpose this vertex is zero because we are considering the mass of the neutrino to be equal to zero and the C coefficients are 5 dimensional operators, so higher order compared to the others.

Chapter 7

Results

The aim of this section is to report what are the corrections that follow from inserting vacuum and density loops to NNZ_μ vertex.
At first we show that at tree level the HBChPT formulation is equivalent to the effective four-fermion interactions. Then we compute both the vacuum and the density vertex corrections and we notice that various new independent structures arise. We plot the density dependence of the relevant ones that will be used to compute the scattering matrix element squared. Finally, we report its density dependence, comparing it with the tree level result, showing what is the role of density in a $NN\nu\nu$ scattering process. We compute only 1PI diagrams up to chiral power counting $\nu = 3$.

7.1 Equivalence between HBChPT and four-fermion interaction

The interaction between neutrinos and nucleons can also be written using the effective four-fermion interactions Lagrangian [33]:

$$\mathcal{L}_{nc} = \frac{G_F}{\sqrt{2}} l_\mu j^\mu = \frac{G_F}{\sqrt{2}} \bar{\psi}_\nu \gamma_\mu (1-\gamma_5)\psi_\nu \bar{\psi}_N \gamma^\mu (C_V - C_A\gamma_5)\psi_N. \tag{7.1.1}$$

We can make a consistency check for our formalism: we can indeed compute $NN \to \nu\nu$ in both the effective theories.
Let us start with $\mathcal{L}_{nc}$ in 7.1.1. Projecting out the heavy part of the nucleon field and writing the neutrino field in terms only of the left component, we get

$$\mathcal{L}_{nc} = \sqrt{2} G_F (\bar{\nu}_L \gamma^\nu \nu_L) g_{\mu\nu} \bar{N}\gamma^\mu (C_V - C_A\gamma_5) N. \tag{7.1.2}$$

Focusing on the nucleon part, we can use the relations [11]

$$\begin{aligned} \bar{N}\gamma^\mu N &= v^\mu \bar{N} N, \\ \bar{N}\gamma^\mu\gamma_5 N &= 2\bar{N} S^\mu N, \end{aligned} \tag{7.1.3}$$

so that

$$\begin{aligned}
&\bar{N}(C_V\gamma^\mu - C_A\gamma^\mu\gamma_5)N \\
&= \bar{N}C_V\gamma^\mu N - C_A\bar{N}\gamma^\mu\gamma_5 N \\
&= C_V v^\mu \bar{N}N - 2C_A\bar{N}S^\mu N \\
&= C_V v^\mu \bar{N}N - 2C_A\bar{N}S^\mu N.
\end{aligned} \tag{7.1.4}$$

The Lagrangian now reads

$$\begin{aligned}
\mathcal{L}_{nc} &= \sqrt{2}G_F(\bar{\nu}_L\gamma^\nu\nu_L)g_{\mu\nu}(C_V v^\mu\bar{N}N - 2C_A\bar{N}S^\mu N) \\
&= \sqrt{2}G_F(\bar{\nu}_L\gamma_\mu\nu_L)(\bar{N}(C_V v^\mu - 2C_A S^\mu)N).
\end{aligned} \tag{7.1.5}$$

Now, we should get the same result also using the vertex NNZ_μ computed in the previous section.
We need the $Z_\mu\nu\nu$ vertex and the Z_μ boson propagator as well. We can read them off from the lepton sector of the standard model as in [9]

$$\mathcal{L}_{\text{int}} \supset \frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\frac{1}{2}\bar{\mathcal{N}}_L\gamma^\mu\mathcal{N}_L, \qquad \mathcal{N}_L := P_L\mathcal{N} = \begin{pmatrix}\nu_L \\ 0\end{pmatrix}, \tag{7.1.6}$$

while the propagator can be read off from the free part of the Lagrangian to be $g^{\mu\nu}/m_Z^2$.
We can then rewrite the Lagrangian as

$$\begin{aligned}
\mathcal{L}_{\text{int}} \supset &\frac{e}{2\sin\theta_W\cos\theta_W}Z_\mu\begin{pmatrix}0 & \bar{\nu}_L\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}\nu_L \\ 0\end{pmatrix} \\
= &\frac{e}{2\sin\theta_W\cos\theta_W}Z_\mu\bar{\nu}_L\gamma^\mu\nu_L,
\end{aligned} \tag{7.1.7}$$

so we get the following vertex

$$\begin{aligned}
i\langle\nu_L|(...)|Z_\nu\nu\rangle &= i\frac{e}{2\sin\theta_W\cos\theta_W}\gamma_\nu \\
&= i\frac{g}{2\cos\theta_W}\gamma_\nu.
\end{aligned} \tag{7.1.8}$$

We can now multiply this with the other vertex and the propagator to get the Amplitude for $NN \to \nu\nu$

$$\begin{aligned}
&(\bar{\nu}_L i\frac{g}{2\cos\theta_W}\gamma_\nu\nu_L)\frac{\eta^{\nu\mu}}{m_Z^2}\left(\bar{N}\left(-i\frac{g_A g}{4\cos\theta_W}\sigma_i\tau^3\right.\right. \\
&\qquad\qquad\left.\left. + i\frac{v^\mu}{2}\left[\left(-\frac{g}{2\cos\theta_W} + e\tan\theta_W\right)\tau^3 + e\tan\theta_W\right]\right)N\right) \\
&= \frac{g^2}{4\cos^2\theta_W m_Z^2}\left(\bar{\nu}_L\gamma_\nu\nu_L\right)\eta^{\mu\nu}\left(\bar{N}\left(\frac{g_A}{2}\sigma_i\tau^3 - \frac{v_\mu}{2}((2\sin^2\theta_W) - 1)\tau^3 + 2\sin^2\theta_W\right)N\right).
\end{aligned} \tag{7.1.9}$$

To compare (7.1.5) and (7.1.9) we observe that the $\bar{\nu}_L(...)\nu_L$ factor is the same, so we only need to look at the other factor. We need to simplify the $\bar{N}(...)N$ factor for the neutron and the proton, knowing that

$$C_A^n = -g_A/2, \quad C_V^n = -1/2, \quad C_A^p = g_A/2, \quad C_V^p = 1/2 - 2\sin^2\theta_W \tag{7.1.10}$$

and

$$\sqrt{2}G_F = \frac{g^2}{4\cos^2\theta_W m_Z^2}. \tag{7.1.11}$$

Starting from the proton, that we get from considering only the element $(1,1)$ of τ^3, we get

$$\begin{aligned} &\bar{N}\left(-g_A S^\nu - v^\nu(2\sin^2\theta_W - \frac{1}{2})\right) \\ &= \bar{N}(C_V v^\nu - 2C_A S^\nu)N \end{aligned} \tag{7.1.12}$$

that is the same as 7.1.5. For the neutron, element $(1,-1)$ we get

$$\begin{aligned} &\bar{N}\left(\frac{g_A}{2}\sigma^j - \frac{v^\nu}{2}\right)N \\ &= \bar{N}(C_V v^\nu - 2C_A S^\nu)N \end{aligned} \tag{7.1.13}$$

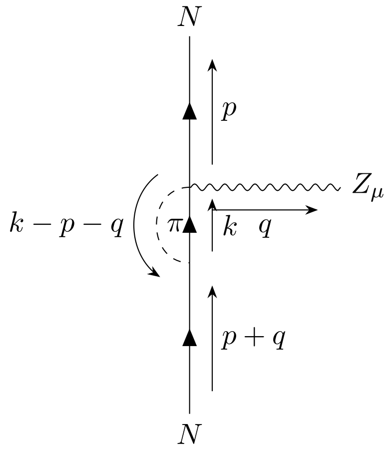
again, in agreement. As it will be discussed later, we can do the consistency check directly computing the matrix element squared for the scattering process at tree level in the two different ways. This has been done and we get the same result.

7.2 Finite density vertex corrections

Here we compute the corrections to the vertex NNZ_μ that derive from the density part of the propagator. The integrals used can be found in Appendix C. We treat separately the Feynman diagrams accordingly to the number of nucleons present in the loop and to the power counting ν. We contract the expressions with the Z_μ polarization vector ϵ^μ. This will not change the results because in this work we are always considering the Z_μ boson as an external particle since it is too massive and its propagator would be suppressed, but this is useful to understand what structures appear. The difference between $\nu = 2$ and $\nu = 3$ lies in what value of Δ have the vertices that we use.
In all the integrals here shown we don't write the fermion polarization, these will be considered directly in the computation of the matrix element squared.

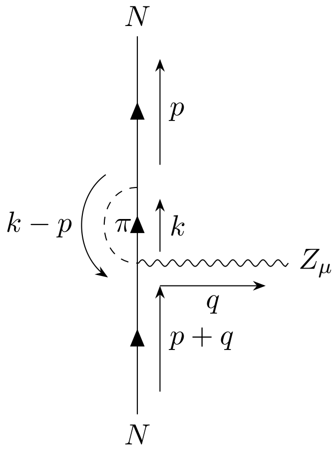
V_1 : NNZ_μ vertex with one nucleon in the loop (contributes at $\nu = 2$)

Here we have both $NN\pi Z_\mu$ and $NN\pi$ at $\Delta = 0$.



$$
\begin{aligned}
&= \int \frac{d^4k}{(2\pi)^4} \left(-i \frac{g}{4\cos\theta_W f_\pi} \epsilon^{3\alpha b} \tau^b (v\cdot\epsilon) + i \frac{g_A}{2f_\pi} \left(-\frac{g}{2\cos\theta_W} + e\tan\theta_W \right) (\vec{\sigma}\cdot\vec{\epsilon}) \epsilon^{3\alpha b}\tau^b \right) \\
&\quad (-2\pi\delta(k^0)\Theta(k_f - |\vec{k}|)) \left(\frac{i\delta^{a\alpha}}{(k-p-q)^2 - m_\pi^2 + i\epsilon} \right) \left(\frac{g_A}{2f_\pi} \vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})\tau^a \right) \\
&= -\int \frac{d^4k}{(2\pi)^4} \frac{2\pi\delta(k^0)\Theta(k_f - |\vec{k}|)(\vec{k}-\vec{p}-\vec{q})\cdot\vec{\sigma}}{(k-p-q)^2 - m_\pi^2 + i\epsilon} \frac{g_A}{4f_\pi^2} \epsilon^{3ab}\tau^b\tau^a \\
&\quad \left(\frac{g}{2\cos\theta_W}(v\cdot\epsilon) - g_A \left(-\frac{g}{2\cos\theta_W} + e\tan\theta_W \right) (\vec{\sigma}\cdot\vec{\epsilon}) \right) \\
&= \int_{|\vec{k}|\le k_f} \frac{d^3\vec{k}}{(2\pi)^3} \frac{(\vec{k}-\vec{p}-\vec{q})\cdot\vec{\sigma}}{\tilde{m}_\pi^2 + (\vec{k}-\vec{p}-\vec{q})^2} \frac{g_A}{4f_\pi^2} (-)2i\tau^3 \left(\frac{g}{2\cos\theta_W}(v\cdot\epsilon) - g_A \left(-\frac{g}{2\cos\theta_W} + e\tan\theta_W \right) (\vec{\sigma}\cdot\vec{\epsilon}) \right) \\
&= \int_{|\vec{k}|\le k_f} \frac{d^3\vec{k}}{(2\pi)^3} \frac{(\vec{k}+\vec{p}+\vec{q})\cdot\vec{\sigma}}{\tilde{m}_\pi^2 + (\vec{k}+\vec{p}+\vec{q})^2} \frac{g_A}{4f_\pi^2} 2i\tau^3 \left(\frac{g}{2\cos\theta_W}(v\cdot\epsilon) - g_A \left(-\frac{g}{2\cos\theta_W} + e\tan\theta_W \right) (\vec{\sigma}\cdot\vec{\epsilon}) \right) \\
&= i\frac{g_A}{2f_\pi^2} \left(\frac{g}{2\cos\theta_W}(v\cdot\epsilon) + g_A \left(\frac{g}{2\cos\theta_W} - e\tan\theta_W \right) (\vec{\sigma}\cdot\vec{\epsilon}) \right) \tau^3 \tilde{I}_3(\sigma, p+q) \\
&= (v\cdot\epsilon) i \frac{g_A g}{4f_\pi^2\cos\theta_W} \tau^3 \tilde{I}_3(\sigma, p+q) + (\vec{\sigma}\cdot\vec{\epsilon}) i \frac{g_A^2 g}{4f_\pi^2\cos\theta_W} (1 - 2\sin^2\theta_W) \tau^3 \tilde{I}_3(\sigma, p+q).
\end{aligned}
\tag{7.2.1}
$$

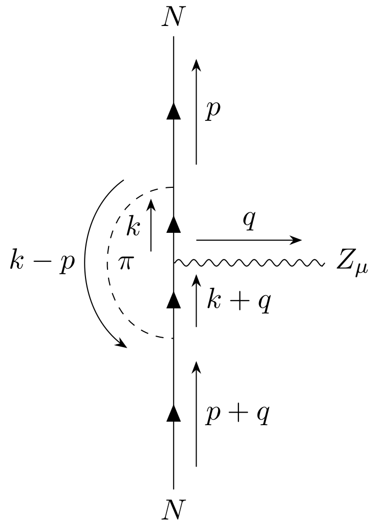
We have a similar diagrams but with vertices exchanged



$$
\begin{aligned}
&= \int \frac{d^4k}{(2\pi)^4}\left(-\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^\alpha\right)\left(\frac{i\delta^{a\alpha}}{(k-p)^2-m_\pi^2+i\epsilon}\right)(-2\pi\delta(k^0)\Theta(k_f-|\vec{k}|)) \\
&\qquad \left(-i\frac{g}{4\cos\theta_W f_\pi}\epsilon^{3ab}\tau^b(v\cdot\epsilon)+i\frac{g_A}{2f_\pi}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)(\vec{\sigma}\cdot\vec{\epsilon})\epsilon^{3ab}\tau^b\right) \\
&= \int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{\vec{\sigma}\cdot(\vec{k}+\vec{p})}{m_\pi^2+(\vec{k}+\vec{p})^2}\frac{g_A}{4f_\pi^2}2i\tau^3\left(\frac{g}{2\cos\theta_W}(v\cdot\epsilon)+g_A\left(\frac{g}{2\cos\theta_W}-e\tan\theta_W\right)(\vec{\sigma}\cdot\vec{\epsilon})\right) \\
&= i\frac{g_A}{2f_\pi^2}\tau^3 I_3(\sigma,p)\left(\frac{g}{2\cos\theta_W}(v\cdot\epsilon)+g_A\left(\frac{g}{2\cos\theta_W}-e\tan\theta_W\right)(\vec{\sigma}\cdot\vec{\epsilon})\right) \\
&= (v\cdot\epsilon)i\frac{g_A g}{4f_\pi^2\cos\theta_W}\tau^3 I_3(\sigma,p)+i\frac{g_A^2 g}{4f_\pi^2\cos\theta_W}(1-2\sin^2\theta_W)\tau^3 I_3(\sigma,p)(\vec{\sigma}\cdot\vec{\epsilon}).
\end{aligned}
\tag{7.2.2}
$$

V_2 : NNZ_μ vertex with two nucleons in the loop (contributes at $\nu = 2$)

Here we have both the vertices $NN\pi$ and NNZ_μ at $\Delta = 0$.



$$
\begin{aligned}
&= \int \frac{d^4k}{(2\pi)^4}\left(-\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^\alpha\right)\left(\frac{i}{v\cdot k+i\epsilon}-2\pi\delta(k^0)\Theta(k_f-|\vec{k}|)\right)\\
&\quad\left(-i\frac{g_A g}{4\cos\theta_W}\vec{\sigma}\cdot\vec{\epsilon}\tau^3+i\frac{(v\cdot\epsilon)}{2}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)\tau^3+e\tan\theta_W\right)\\
&\quad\left(\frac{i}{v\cdot(k+q)+i\epsilon}-2\pi\delta(k^0+q^0)\Theta(k_f-|\vec{k}+\vec{q}|)\left(\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^\alpha\right)\right)\left(\frac{-i\delta^{a\alpha}}{m_\pi^2-(k-p)^2}\right)\\
&= -\int \frac{d^4k}{(2\pi)^4}\frac{\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^a(...\vec{\sigma}\cdot\vec{\epsilon}\tau^3+...(v\cdot\epsilon)\epsilon_\mu\tau^3+...)(-)2\pi\delta(k^0+q^0)\Theta(k_f-|\vec{k}+\vec{q}|)\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^a}{k^0(m_\pi^2-(k-p)^2)}\frac{g_A^2}{4f_\pi^2}\\
&\quad-\int \frac{d^4k}{(2\pi)^4}\frac{\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^a(-)2\pi\delta(k^0)\Theta(k_f-|\vec{k}|)(...\vec{\sigma}\cdot\vec{\epsilon}\tau^3+...(v\cdot\epsilon)\epsilon_\mu\tau^3+...)\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^a}{(k^0+q^0)(m_\pi^2-(k-p)^2)}\frac{g_A^2}{4f_\pi^2}.
\end{aligned}
\tag{7.2.3}
$$

We notice that this integral is made by two very similar parts, we compute them separately. Moreover it is better to divide each part in two, considering the non trivial vertex structure. We call respectively 1) an 2) the first and second integral. Inside them, we call 1a) the part with $(\vec{\sigma}\cdot\vec{\epsilon})\tau^3$ and 1b) the rest. Same is done for 2).

1a)

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\frac{\sigma_j(k-p)_j\tau^a(-i)}{k^0(m_\pi^2-(k-p)^2)}\frac{g_Ag}{4\cos\theta_W}\vec{\sigma}\cdot\vec{\epsilon}\epsilon_i\tau^3(-)2\pi\delta(k^0+q^0)\Theta(k_f-|\vec{k}+\vec{q}|)\sigma_k(k-p)_k\tau^a(-)\frac{g_A^2}{4f_\pi^2}\\
&=i\frac{g_A^3g}{16\cos\theta_W f_\pi^2q^0}\tau^a\tau^3\tau^a\int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{\sigma_j\vec{\sigma}\cdot\vec{\epsilon}\sigma_k\epsilon_i(k-p-q)_j(k-p-q)_k}{\tilde{m}_\pi^2+(\vec{k}-\vec{p}-\vec{q})^2}\\
&=i\frac{g_A^3g}{16\cos\theta_W f_\pi^2q^0}\tau^a\tau^3\tau^a\int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{(k-p-q)_j(k-p-q)_k}{\tilde{m}_\pi^2+(\vec{k}-\vec{p}-\vec{q})^2}(\delta_{ji}\sigma_k-\delta_{jk}\vec{\sigma}\cdot\vec{\epsilon}+\delta_{ik}\sigma_j+i\epsilon_{jik})\epsilon_i\\
&=i\frac{g_A^3g}{16\cos\theta_W f_\pi^2q^0}\tau^a\tau^3\tau^a\int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{(2\vec{\epsilon}\cdot(\vec{k}+\vec{p}+\vec{q})(\vec{k}+\vec{p}+\vec{q})\cdot\vec{\sigma}-(\vec{k}+\vec{p}+\vec{q})\cdot(\vec{k}+\vec{p}+\vec{q})\vec{\sigma}\cdot\epsilon)}{\tilde{m}_\pi^2+(\vec{k}+\vec{p}+\vec{q})^2}\\
&=i\frac{g_A^3g}{16\cos\theta_W f_\pi^2q^0}\tau^a\tau^3\tau^a(2\tilde{I}_1(\sigma,p+q,\epsilon)-\tilde{I}_2(p+q)\vec{\sigma}\cdot\vec{\epsilon})\\
&=-i\frac{g_A^3g}{16\cos\theta_W f_\pi^2q^0}\tau^3(2\tilde{I}_1(\sigma,p+q,\epsilon)-\tilde{I}_2(p+q)\vec{\sigma}\cdot\vec{\epsilon}),
\end{aligned} \tag{7.2.4}$$

where we can use the same standard integrals and we use

$$\begin{aligned}
&\sigma_i\sigma_j\sigma_k=\delta_{ij}\sigma_k+i\epsilon_{ijl}\sigma_l\sigma_k=\delta_{ij}\sigma_k+i\epsilon_{ijl}(\delta_{lk}+i\epsilon_{lkm}\sigma_m)\\
&=\delta_{ij}\sigma_k+i\epsilon_{ijk}-\delta_{ik}\sigma_j+\delta_{jk}\sigma_i\\
&=\delta_{ij}\sigma_k-\delta_{ik}\sigma_j+\delta_{jk}\sigma_i+i\epsilon_{ijk}.
\end{aligned} \tag{7.2.5}$$

The same can be done for 2a) and we get

$$-i\frac{g_A^3g}{16\cos\theta_W f_\pi^2q^0}\tau^3(-2I_1(\sigma,p,\epsilon)+I_2(p)\vec{\sigma}\cdot\vec{\epsilon}). \tag{7.2.6}$$

For 1b)

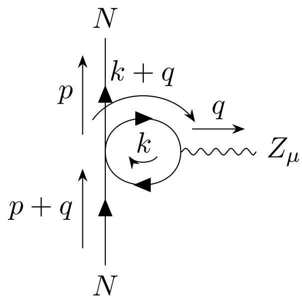
$$\begin{aligned}
&\int\frac{d^4k}{(2\pi)^4}\frac{\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^a c\cdot\epsilon(-)2\pi\delta(k^0+q^0)\Theta(k_f-|\vec{k}+\vec{q}|)\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^a}{k^0(m_\pi^2-(k-p)^2)}(-)\frac{g_A^2}{4f_\pi^2}\\
&=-\frac{g_A^2}{4f_\pi^2q^0}\tau^a c\cdot\epsilon\tau^a\int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{\vec{\sigma}\cdot(\vec{k}+\vec{p}+\vec{q})\vec{\sigma}\cdot(\vec{k}+\vec{p}+\vec{q})}{\tilde{m}_\pi^2+(\vec{k}+\vec{p}+\vec{q})^2}\\
&=-\frac{g_A^2}{4f_\pi^2q^0}\tau^a c\cdot\epsilon\tau^a\int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{(\vec{k}+\vec{p}+\vec{q})\cdot(\vec{k}+\vec{p}+\vec{q})}{\tilde{m}_\pi^2+(\vec{k}+\vec{p}+\vec{q})^2}\\
&=-\frac{g_A^2}{4f_\pi^2q^0}\tau^a c\cdot\epsilon\tau^a\tilde{I}_2(p+q)\\
&=-(v\cdot\epsilon)i\frac{g_A^2g}{16f_\pi^2q^0\cos\theta_W}((1-2\sin^2\theta)\tau^3+6\sin^2\theta)\tilde{I}_2(p+q),
\end{aligned} \tag{7.2.7}$$

where we called c^μ the part of the vertex proportional to v^μ.
Immediately we get for 2b)

$$\begin{aligned}
&-\frac{g_A^2}{4f_\pi^2 q^0}\tau^a c\cdot\epsilon\tau^a I_2(p)\\
&=(v\cdot\epsilon)i\frac{g_A^2 g}{16 f_\pi^2 q^0\cos\theta_W}((1-2\sin^2\theta)\tau^3+6\sin^2\theta)I_2(p).
\end{aligned}\tag{7.2.8}$$

NNZ_μ vertices with vanishing loops (contributes at $\nu=2$)

Here we have both $NNNN$ and NNZ_μ at $\Delta = 0$.

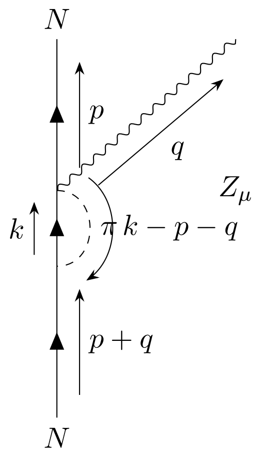


$$\begin{aligned}
&=\int\frac{d^4k}{(2\pi)^4}V_{1'12'2}\left(\frac{i}{k^0}-2\pi\delta(k^0)\Theta(k_f-|\vec{k}|)\right)_{12}\left(-i\frac{g_A g}{4\cos\theta_W}\sigma_i\tau^3+i\frac{v^\mu}{2}\left(\left(-\frac{g}{2\cos\theta_W}\right.\right.\right.\\
&\left.\left.\left.+e\tan\theta_W\right)\tau^3+e\tan\theta_W\right)\right)_{22'}\left(\frac{i}{k^0+q^0}-2\pi\delta(k^0+q^0)\Theta(k_f-|\vec{k}+\vec{q}|)\right)_{2'1'}\\
&\supset\int\frac{d^4k}{(2\pi)^4}V_{1'12'2}\left(-i\frac{g_A g}{4\cos\theta_W}\sigma_i\tau^3+i\frac{v^\mu}{2}\left(\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)\tau^3+e\tan\theta_W\right)\right)\\
&\quad\left(-\frac{i}{k^0}2\pi\delta(k^0+q^0)\Theta(k_f-|\vec{k}+\vec{q}|)-\frac{i}{k^0+q^0}2\pi\delta(k^0)\Theta(k_f-|\vec{k}|)\right)\\
&=\int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}(\ldots)\left(\frac{i}{q^0}\Theta(k_f-|\vec{k}+\vec{q}|)-\frac{i}{q^0}\Theta(k_f-|\vec{k}|)\right)\\
&=\int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}(\ldots)\left(\frac{i}{q^0}\Theta(k_f-|\vec{k}|)-\frac{i}{q^0}\Theta(k_f-|\vec{k}|)\right)=0,
\end{aligned}\tag{7.2.9}$$

where $V_{1'12'2}$ is the four nucleon vertex computed in paragraph 6.10.

V_3 : NNZ_μ vertex with one nucleon in the loop (contributes at $\nu = 3$)

Here and below we have $NN\pi Z_\mu$ at $\Delta = 1$ and $NN\pi$ at $\Delta = 0$.



We have three contributions coming from $\hat{c}_3$, $\hat{c}_4$, $\hat{c}_6$, it is easier to look at them separately, so we split the integral

$$\int \frac{d^4k}{(2\pi)^4}\left(\frac{2\hat{c}_3 g}{f_\pi \cos\theta_W}\delta^{\alpha 3}(k-p-q)^\mu \epsilon_\mu + i\frac{\hat{c}_4 g}{4f_\pi\cos\theta_W}\epsilon^{3\alpha b}\tau^b[\vec{\sigma}\cdot\vec{\epsilon}, \vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})]\right.$$
$$\left. - i\frac{\hat{c}_6 g}{16m\cos\theta_W f_\pi}\epsilon^{3\alpha b}\tau^b\,[\vec{q}\cdot\vec{\sigma}, \vec{\sigma}\cdot\vec{\epsilon}]\right)\left(\frac{i}{v\cdot k + i\epsilon} - 2\pi\delta(k^0)\Theta(k_f - |\vec{k}|)\right)$$
$$\left(\frac{-i\delta^{a\alpha}}{m_\pi^2 - (k-p-q)^2}\right)\left(\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})\tau^a\right) \tag{7.2.10}$$

in 1_3), 1_4), 1_6).
1_3)

$$\int \frac{d^4k}{(2\pi)^4}\frac{(k-p-q)^\mu\epsilon_\mu(-)2\pi\delta(k^0)\Theta(k_f-|\vec{k}|)\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})}{m_\pi^2-(k-p-q)^2}(-)i\frac{\hat{c}_3 g g_A}{f_\pi^2\cos\theta_W}\tau^3. \tag{7.2.11}$$

It is easier to split the integral for $\mu = 0$ and $\mu = i$,
for $\mu = 0$

$$\begin{aligned} & i\frac{\hat{c}_3 g g_A \tau^3}{f_\pi^2\cos\theta_W}(p^0+q^0)\epsilon_0\int_{|\vec{k}|\le k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{\vec{\sigma}\cdot(\vec{k}+\vec{p}+\vec{q})}{\tilde{m}_\pi^2+(k+p+q)^2} \\ &= i\frac{\hat{c}_3 g g_A \tau^3}{f_\pi^2\cos\theta_W}(p^0+q^0)\epsilon_0\tilde{I}_3(\sigma, p+q) \\ &= (v\cdot(p+q))(v\cdot\epsilon)i\frac{\hat{c}_3 g g_A\tau^3}{f_\pi^2\cos\theta_W}\tilde{I}_3(\sigma,p+q) \end{aligned} \tag{7.2.12}$$

and for $\mu = i$

$$\begin{aligned}&-i\frac{\hat{c}_3 g g_A \tau^3}{f_\pi^2 \cos\theta_W}\int_{|\vec{k}|\leq k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{(\vec{k}+\vec{p}+\vec{q})\cdot\vec{\epsilon}\vec{\sigma}\cdot(\vec{k}+\vec{p}+\vec{q})}{\tilde{m}_\pi^2+(\vec{k}+\vec{p}+\vec{q})^2}\\&=-i\frac{\hat{c}_3 g g_A \tau^3}{f_\pi^2 \cos\theta_W}\tilde{I}_1(\sigma,p+q,\epsilon).\end{aligned}\tag{7.2.13}$$

1_6)

$$\begin{aligned}&\left(i\frac{\hat{c}_6 g g_A}{16m\cos\theta_W f_\pi^2}[\vec{q}\cdot\vec{\sigma},\vec{\sigma}\cdot\vec{\epsilon}]\tau^3\right)\int\frac{d^4k}{(2\pi)^4}\frac{-2\pi\delta(k^0)\Theta(k_f-|\vec{k}|)\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})}{m_\pi^2-(k-p-q)^2}\\&=i\frac{\hat{c}_6 g g_A}{16m\cos\theta_W f_\pi^2}[\vec{q}\cdot\vec{\sigma},\vec{\sigma}\cdot\vec{\epsilon}]\tau^3\int_{|\vec{k}|\leq k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{\vec{\sigma}\cdot(\vec{k}+\vec{p}+\vec{q})}{\tilde{m}_\pi^2+(\vec{k}+\vec{p}+\vec{q})^2}\\&=i\frac{\hat{c}_6 g g_A}{16m\cos\theta_W f_\pi^2}[\vec{q}\cdot\vec{\sigma},\vec{\sigma}\cdot\vec{\epsilon}]\tau^3\tilde{I}_3(\sigma,p+q)\\&=-(\vec{q}\cdot(\vec{\epsilon}\times\vec{\sigma}))\frac{\hat{c}_6 g g_A}{8m\cos\theta_W f_\pi^2}\tau^3\tilde{I}_3(\sigma,p+q),\end{aligned}\tag{7.2.14}$$

where we have to pay attention to the order of the Pauli matrices since they do not commute.

1_4)

$$\begin{aligned}&\int\frac{d^4k}{(2\pi)^4}\frac{[\vec{\sigma}\cdot\vec{\epsilon},\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})](-)2\pi\delta(k^0)\Theta(k_f-|\vec{k}|)\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})}{m_\pi^2-(k-p-q)^2}(-)i\frac{\hat{c}_4 g g_A\tau^3}{4f_\pi^2\cos\theta_W}\\&=i\frac{\hat{c}_4 g g_A\tau^3}{4f_\pi^2\cos\theta_W}\int_{|\vec{k}|\leq k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{(\vec{\sigma}\cdot\vec{\epsilon}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})-\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})\vec{\sigma}\cdot\vec{\epsilon})\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})}{\tilde{m}_\pi^2+(\vec{k}-\vec{p}-\vec{q})^2}\\&=i\frac{\hat{c}_4 g g_A\tau^3}{4f_\pi^2\cos\theta_W}\int_{|\vec{k}|\leq k_f}\frac{d^3\vec{k}}{(2\pi)^3}\frac{\vec{\sigma}\cdot\vec{\epsilon}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})-\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})\vec{\sigma}\cdot\vec{\epsilon}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})}{\tilde{m}_\pi^2+(\vec{k}-\vec{p}-\vec{q})^2}\\&=i\frac{\hat{c}_4 g g_A\tau^3}{2f_\pi^2\cos\theta_W}((\vec{\sigma}\cdot\vec{\epsilon})\tilde{I}_2(p+q)-\tilde{I}_1(\sigma,p+q,\epsilon)),\end{aligned}\tag{7.2.15}$$

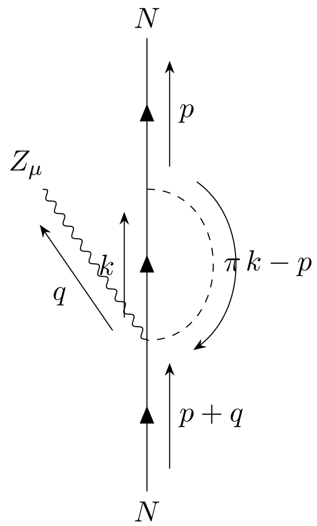
where we simplified the numerator in the following way

$$\begin{aligned}&\sigma_i\sigma_j\sigma_k\epsilon_i(k-p-q)_j(k-p-q)_k\\&=(\delta_{ij}\sigma_k-\delta_{ik}\sigma_j+\delta_{jk}\sigma_i+i\epsilon_{ijk})\epsilon_i(k-p-q)_j(k-p-q)_k\\&=\vec{\sigma}\cdot\vec{\epsilon}(\vec{k}-\vec{p}-\vec{q})\cdot(\vec{k}-\vec{p}-\vec{q}),\\&\sigma_j\sigma_i\sigma_k\epsilon_i(k-p-q)_j(k-p-q)_k\\&=(\delta_{ji}\sigma_k-\delta_{jk}\sigma_i+\delta_{ik}\sigma_j+i\epsilon_{jik})\epsilon_i(k-p-q)_j(k-p-q)_k\\&=(\vec{k}-\vec{p}-\vec{q})\cdot\vec{\epsilon}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})-(\vec{k}-\vec{p}-\vec{q})\cdot(\vec{k}-\vec{p}-\vec{q})\vec{\sigma}\cdot\vec{\epsilon}+(\vec{k}-\vec{p}-\vec{q})\cdot\vec{\epsilon}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})\end{aligned}\tag{7.2.16}$$

and subtracting them we get

$$2\vec{\sigma}\cdot\vec{\epsilon}(\vec{k}-\vec{p}-\vec{q})\cdot(\vec{k}-\vec{p}-\vec{q})-2(\vec{k}-\vec{p}-\vec{q})\cdot\vec{\epsilon}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q}), \tag{7.2.17}$$

which gives us the integrals above.



The integral 2), as for $\nu=2$ case, is very similar, we only have some vertices exchanged and small differences in the momentum. Doing the explicit computations we see that we only need to replace $p+q$ with p and $\tilde{m}_\pi^2$ with m_π^2. The results are

2_3)

$$\begin{aligned}&\mu=0 \quad (p\cdot v)(\epsilon\cdot v)i\frac{\hat{c}_3 g g_A\tau^3}{f_\pi^2\cos\theta_W}I_3(\sigma,p),\\&\mu=-i \quad -i\frac{\hat{c}_3 g g_A\tau^3}{f_\pi^2\cos\theta_W}I_1(\sigma,p,\epsilon),\end{aligned} \tag{7.2.18}$$

2_4)

$$i\frac{\hat{c}_4 g g_A\tau^3}{2f_\pi^2\cos\theta_W}((\vec{\sigma}\cdot\vec{\epsilon})I_2(p)-I_1(\sigma,p,\epsilon)), \tag{7.2.19}$$

2_6)

$$\begin{aligned}&i\frac{\hat{c}_6 g g_A}{16m\cos\theta_W f_\pi^2}[\vec{q}\cdot\vec{\sigma},\vec{\sigma}\cdot\vec{\epsilon}]\tau^3 I_3(\sigma,p)\\&=-(\vec{q}\cdot(\vec{\epsilon}\times\vec{\sigma}))\frac{\hat{c}_6 g g_A}{8m\cos\theta_W f_\pi^2}\tau^3 I_3(\sigma,p),\end{aligned} \tag{7.2.20}$$

V_4 **:** NNZ_μ **vertex with two nucleons in the loop (contributes at** $\nu = 3$**)**

We have here the corresponding diagram as for $\nu = 2$, so we refer to diagram 7.2 for the choice of momenta. The difference is that NNZ_μ vertex has $\Delta = 1$.

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4} \left(-\frac{g_A}{2f_\pi} \vec{\sigma} \cdot (\vec{k} - \vec{p}) \tau^\alpha \right) \left(\frac{i}{v \cdot k + i\epsilon} - 2\pi\delta(k^0)\Theta(k_f - |\vec{k}|) \right) \\
&\quad 2i[\vec{\sigma} \cdot \vec{q}, \vec{\sigma} \cdot \vec{\epsilon}] \left(\frac{\hat{c}_6}{16m} \left(e \tan\theta_W - \frac{g}{2\cos\theta_W} \right) \tau^3 + \frac{\hat{c}_7 e \tan\theta_W}{16m} \right) \\
&\quad \left(\frac{i}{v \cdot (k+q) + i\epsilon} - 2\pi\delta(k^0 + q^0)\Theta(k_f - |\vec{k} + \vec{q}|) \right) \left(\frac{g_A}{2f_\pi} \vec{\sigma} \cdot (\vec{k} - \vec{p}) \tau^a \right) \left(\frac{-i\delta^{a\alpha}}{m_\pi^2 - (k-p)^2} \right) \\
&= i \frac{g_A^2}{2f_\pi^2 q^0} \left(\int_{|\vec{k}| \le k_f} \frac{d^3\vec{k}}{(2\pi)^3} \frac{\vec{\sigma} \cdot (\vec{k} + \vec{p}) \tau^a [\vec{\sigma} \cdot \vec{q}, \vec{\sigma} \cdot \vec{\epsilon}] c_i \vec{\sigma} \cdot (\vec{k} + \vec{p}) \tau^a}{m_\pi^2 + (\vec{k} + \vec{p})^2} \right. \\
&\quad \left. - \int_{|\vec{k}| \le k_f} \frac{d^3\vec{k}}{(2\pi)^3} \frac{\vec{\sigma} \cdot (\vec{k} + \vec{p} + \vec{q}) \tau^a [\vec{\sigma} \cdot \vec{q}, \vec{\sigma} \cdot \vec{\epsilon}] c_i \vec{\sigma} \cdot (\vec{k} + \vec{p} + \vec{q}) \tau^a}{\tilde{m}_\pi^2 + (\vec{k} + \vec{p} + \vec{q})^2} \right).
\end{aligned} \tag{7.2.21}$$

Here as well we need to analyze the Pauli matrices structure, here we write the one for the first integral, the second one works the same way:

$$\begin{aligned}
&(\sigma_j \sigma_k \sigma_i \sigma_l - \sigma_j \sigma_i \sigma_k \sigma_l)(k-p)_j q_k (k-p)_l \epsilon_i \\
&= (\delta_{jk} \sigma_i \sigma_l - \delta_{ji} \sigma_k \sigma_l + \delta_{ki} \delta_j \delta_l + i\epsilon_{jki} \sigma_l - \delta_{ji} \sigma_k \sigma_l - \delta_{jk} \sigma_i \sigma_l + \delta_{ik} \sigma_j \sigma_l + i\epsilon_{jik} \sigma_l)(k-p)_j q_k (k-p)_l \epsilon_i \\
&= (2\delta_{jk} \sigma_i \sigma_l - 2\delta_{ji} \sigma_k \sigma_l + 2i\epsilon_{jki} \sigma_l)(k-p)_j q_k (k-p)_l \epsilon_i \\
&= (2\delta_{jk}(\delta_{il} + i\epsilon_{ilm} \sigma_m) - 2\delta_{ji}(\delta_{kl} + i\epsilon_{klm} \sigma_m) + 2i\epsilon_{jki} \sigma_l)(k-p)_j q_k (k-p)_l \epsilon_i \\
&= 2(\delta_{jk} \delta_{il} + i\delta_{jk} \epsilon_{ilm} \sigma_m - \delta_{ji} \delta_{kl} - \delta_{ji} \epsilon_{klm} \sigma_m + i\epsilon_{jki} \sigma_l)(k-p)_j q_k (k-p)_l \epsilon_i \\
&= 2(i(\vec{k} - \vec{p}) \cdot \vec{q} \epsilon_{ilm} \sigma_m (k-p)_l - i(\vec{k} - \vec{p}) \cdot \vec{\epsilon} \epsilon_{klm} \sigma_m q_k (k-p)_l + i\epsilon_{jki} (k-p)_j q_k \vec{\sigma} \cdot (\vec{k} - \vec{p})) \\
&= 2(i(\vec{k} + \vec{p}) \cdot \vec{q} \vec{\epsilon} \cdot ((\vec{k} + \vec{p}) \times \vec{\sigma}) - i(\vec{k} + \vec{p}) \cdot \vec{\epsilon} \vec{q} \cdot ((\vec{k} + \vec{p}) \times \vec{\sigma}) + i\vec{\epsilon} \cdot ((\vec{k} + \vec{p}) \times \vec{q}) \vec{\sigma} \cdot (\vec{k} + \vec{p})) \\
&= 2i((\vec{k} + \vec{p}) \cdot \vec{q} (\vec{k} + \vec{p}) \cdot (\vec{\sigma} \times \vec{\epsilon}) - (\vec{k} + \vec{p}) \cdot \vec{\epsilon} (\vec{k} + \vec{p}) \cdot (\vec{\sigma} \times \vec{q}) + (\vec{k} + \vec{p}) \cdot (\vec{q} \times \vec{\epsilon}) \vec{\sigma} \cdot (\vec{k} + \vec{p})).
\end{aligned} \tag{7.2.22}$$

We can now use the usual standard integrals because everything works the same. So, going back to the integrals we get

$$\begin{aligned}
&-\frac{g_A^2}{f_\pi^2 q^0} \tau^a \hat{c}_i \tau^a \left(\int_{|\vec{k}| \le k_f} \frac{d^3\vec{k}}{(2\pi)^3} \frac{(\vec{k} + \vec{p}) \cdot \vec{q} (\vec{k} + \vec{p}) \cdot (\vec{\sigma} \times \vec{\epsilon}) - (\vec{k} + \vec{p}) \cdot \vec{\epsilon} (\vec{k} + \vec{p}) \cdot (\vec{\sigma} \times \vec{q})}{m_\pi^2 + (\vec{k} + \vec{p})^2} \right. \\
&\qquad \left. + \frac{(\vec{k} + \vec{p}) \cdot (\vec{q} \times \vec{\epsilon}) \vec{\sigma} \cdot (\vec{k} + \vec{p})}{m_\pi^2 + (\vec{k} + \vec{p})^2} - (\vec{p} \to \vec{p} + \vec{q}, \quad m_\pi^2 \to \tilde{m}_\pi^2) \right)
\end{aligned} \tag{7.2.23}$$

$$= -\frac{g_A^2}{f_\pi^2 q^0}\tau^a \hat{c}_i \tau^a (I_1(|\vec{\sigma}\times\vec{\epsilon}|, p, q) - I_1(|\vec{\sigma}\times\vec{q}|, p, \epsilon) + I_1(\sigma, p, |\vec{q}\times\vec{\epsilon}|)$$
$$- \tilde{I}_1(|\vec{\sigma}\times\vec{\epsilon}|, p+q, q) + \tilde{I}_1(|\vec{\sigma}\times\vec{q}|, p+q, \epsilon) - \tilde{I}_1(\sigma, p+q, |\vec{q}\times\vec{\epsilon}|))$$
$$= -\frac{g_A^2 g}{32 m f_\pi^2 q^0 \cos\theta_W}(\hat{c}_6(1-2\sin^2\theta_W)\tau^3 + \hat{c}_7 6\sin^2\theta_W)(I_1(|\vec{\sigma}\times\vec{\epsilon}|, p, q) - I_1(|\vec{\sigma}\times\vec{q}|, p, \epsilon)$$
$$+ I_1(\sigma, p, |\vec{q}\times\vec{\epsilon}|) - \tilde{I}_1(|\vec{\sigma}\times\vec{\epsilon}|, p+q, q) + \tilde{I}_1(|\vec{\sigma}\times\vec{q}|, p+q, \epsilon) - \tilde{I}_1(\sigma, p+q, |\vec{q}\times\vec{\epsilon}|)). \tag{7.2.24}$$

V_5 : NNZ_μ pure nucleon loop vertex (contributes at $\nu = 3$)

For the computation and the diagram we refer to (6.12.3).

NNZ_μ vertices with vanishing loops (contributes at $\nu = 3$)

It is equal to 0 doing the same computations as for the same integral that contributes at $\nu = 2$, the different vertex NNZ_μ at $\Delta = 1$ does not affect the integration.

7.3 Zero density vertex corrections

These are the corrections that come from the zero density non relativistic part of the nucleon propagator. Since they do not have any density dependence, they can be considered as an off-set for our density plots. Nonetheless to have a consistent treatment we need to include them as well. A relevant difference with respect to the previous ones is that here the loop momentum can go to infinity and so the diagrams need to be renormalized. We will first compute the diagrams and we will after explain how to renormalize them. The integrals here used can be found in Appendix C.

V_{v1} : NNZ_μ vertex with one nucleon in the loop (contributes at $\nu = 2$)

We refer to 7.2 for the diagrams.

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\left(-i\frac{g}{4\cos\theta_W f_\pi}\epsilon^{3\alpha b}\tau^b(v^\mu\epsilon_\mu)+i\frac{g_A}{2f_\pi}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)(\vec{\sigma}\cdot\vec{\epsilon})\epsilon^{3\alpha b}\tau^b\right)\\
&\quad(\frac{i}{v\cdot k+i\epsilon})\left(\frac{i\delta^{a\alpha}}{(k-p-q)^2-m_\pi^2+i\epsilon}\right)\left(\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})\tau^a\right)\\
&=-\left(-i\frac{g}{4\cos\theta_W f_\pi}\epsilon^{3\alpha b}\tau^b(v^\mu\epsilon_\mu)+i\frac{g_A}{2f_\pi}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)(\vec{\sigma}\cdot\vec{\epsilon})\epsilon^{3\alpha b}\tau^b\right)\left(\delta^{a\alpha}\frac{g_A}{2f_\pi}\tau^a\right)\\
&\quad i\sigma_j\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{(k-p-q)_j}{(v\cdot k+i\epsilon)((k-p-q)^2-m_\pi^2+i\epsilon)}=\\
&=\frac{g_A}{2f_\pi}\left(i\frac{g}{4\cos\theta}(v^\mu\epsilon_\mu)-i\frac{g_A}{2f_\pi}\left(\frac{-g+2g\sin^2\theta_W}{2\cos\theta_W}(\vec{\sigma}\cdot\vec{\epsilon})\right)\right)(-2i\tau^3)\\
&\quad i\sigma_j\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{-k_j}{(v\cdot(p+q)-v\cdot k+i\epsilon)(k^2-m_\pi^2+i\epsilon)}\\
&=-\frac{g_A}{f_\pi}\left(i\frac{g}{4\cos\theta}(v^\mu\epsilon_\mu)-i\frac{g_A}{2f_\pi}\left(\frac{-g+2g\sin^2\theta_W}{2\cos\theta_W}(\vec{\sigma}\cdot\vec{\epsilon})\right)\right)\tau^3\sigma_j v_j J_1(v\cdot(p+q))=0.
\end{aligned}\tag{7.3.1}$$

This can be considered as a sample integral since the tricks used here will be used for the others as well. These are: simplifications of products of τ and σ matrices and writing the integrals with only the loop momentum in the numerator together with the denominator that needs to have the form such as it is possible to use the standard integrals defined in Appendix C. To simplify even more we remind that

$$S\cdot v=0. \tag{7.3.2}$$

For the exchanged one we have

$$\begin{aligned}
&\int\frac{d^4k}{(2\pi)^4}\left(-\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^\alpha\right)\left(\frac{i\delta^{a\alpha}}{(k-p)^2-m_\pi^2+i\epsilon}\right)\left(\frac{i}{v\cdot k+i\epsilon}\right)\\
&\quad\left(-i\frac{g}{4\cos\theta_W f_\pi}\epsilon^{3ab}\tau^b(v^\mu\epsilon_\mu)+i\frac{g_A}{2f_\pi}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)(\vec{\sigma}\cdot\vec{\epsilon})\epsilon^{3ab}\tau^b\right)\\
&=-\sigma_j\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{(p-k)_j}{((k-p)^2-m_\pi^2+i\epsilon)(v\cdot k+i\epsilon)}\\
&\quad i\left(-\frac{ig}{4\cos\theta_W f_\pi}(v^\mu\epsilon_\mu)+i\frac{g_Ag}{2f_\pi}\frac{2\sin^2\theta_W-1}{2\cos\theta_W}(\vec{\sigma}\cdot\vec{\epsilon})\right)\frac{g_A}{2f_\pi}2i\tau^3\\
&=-\sigma_j J_1(v\cdot p)\frac{ig_Ag}{4f_\pi^2\cos\theta_W}((v^\mu\epsilon_\mu)-g_A(2\sin^2\theta_W-1)(\vec{\sigma}\cdot\vec{\epsilon}))\tau^3=0,
\end{aligned}\tag{7.3.3}$$

so they do not contribute.

V_{v2}:NNZ_μ vertex with two nucleons in the loop (contributes at $\nu = 2$)

Here we refer to 7.2 for the diagram.

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\left(-\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^\alpha\right)\left(\frac{i}{v\cdot k+i\epsilon}\right)\\
&\quad\left(-i\frac{g_A g}{4\cos\theta_W}(\vec{\sigma}\cdot\vec{\epsilon})\tau^3+i\frac{(v\cdot\epsilon)}{2}\left(-\frac{g}{2\cos\theta_W}+e\tan\theta_W\right)\tau^3+e\tan\theta_W\right)\\
&\quad\left(\frac{i}{v\cdot(k+q)+i\epsilon}\right)\left(\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^\alpha\right)\left(\frac{i\delta^{a\alpha}}{(k-p)^2-m_\pi^2}\right)\\
=&-\int\frac{d^4k}{(2\pi)^4}\frac{\vec{\sigma}\cdot(\vec{k}-\vec{p})(\vec{\sigma}\cdot\vec{\epsilon})\vec{\sigma}\cdot(\vec{k}-\vec{p})}{(v\cdot k+i\epsilon)(v\cdot(k+q)+i\epsilon)(m_\pi^2-(k-p)^2)}\left(-\frac{gg_A^3}{4(2f_\pi^2)\cos\theta_W}\tau^a\tau^3\tau^a\right)\\
&+\int\frac{d^4k}{(2\pi)^4}\frac{\vec{\sigma}\cdot(\vec{k}-\vec{p})\vec{\sigma}\cdot(\vec{k}-\vec{p})}{(v\cdot k+i\epsilon)(v\cdot(k+q)+i\epsilon)(m_\pi^2-(k-p)^2)}\\
&\quad\left(\frac{gg_A^2}{(2f_\pi)^2}\frac{(v\cdot\epsilon)}{2}\left(\frac{2\sin^2\theta_W-1}{2\cos\theta_W}\tau^a\tau^3\tau^a+\frac{\sin^2\theta_W}{\cos\theta_W}\tau^a\tau^a\right)\right)\\
=&-\int\frac{d^4k}{(2\pi)^4}\frac{2\vec{\epsilon}\cdot(\vec{k}-\vec{p})\vec{\sigma}\cdot(\vec{k}-\vec{p})-(\vec{k}-\vec{p})\cdot(\vec{k}-\vec{p})\vec{\sigma}\cdot\vec{\epsilon}}{(v\cdot k+i\epsilon)(v\cdot(k+q)+i\epsilon)(m_\pi^2-(k-p)^2)}\left(\frac{gg_A^3}{16f_\pi^2\cos\theta_W}\tau^3\right)\\
&-\delta^{jk}\int\frac{d^4k}{(2\pi)^4}\frac{(k-p)^j(k-p)^k}{(v\cdot k+i\epsilon)(v\cdot(k+q)+i\epsilon)(m_\pi^2-(k-p)^2)}\\
&\quad\left(\frac{gg_A^2}{16f_\pi^2\cos\theta_W}(v\cdot\epsilon)\left(-(2\sin^2\theta_W-1)\tau^3+6\sin^2\theta_W\right)\right),
\end{aligned}\tag{7.3.4}$$

where the numerator of the first integral has been computed as the respective finite density integral. We proceed analyzing the sum separately. With some manipulations and the identity mentioned in Appendix C, the first becomes

$$\begin{aligned}
&-\left(\frac{gg_A^3}{16f_\pi^2\cos\theta_W}\tau^3\right)\left((2i\epsilon_j\sigma_k-\vec{\sigma}\cdot\vec{\epsilon}\delta_{jk})\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{k_jk_\mu}{(v\cdot p-v\cdot k+i\epsilon)(v\cdot(p+q)-v\cdot k+i\epsilon)(k^2-m_\pi^2)}\right)\\
&=-\frac{gg_A^3}{16f_\pi^2\cos\theta_W}\tau^3(2i\epsilon_j\sigma_k-i\vec{\sigma}\vec{\epsilon}\delta_{jk})\frac{1}{-v\cdot q}(g_{\mu\nu}[J_2(v\cdot q)-J_2(0)]+v_\mu v_\nu[J_3(v\cdot q)-J_3(0)])\\
&=i\frac{g_A^3g}{16f_\pi^2\cos\theta_W(v\cdot q)}\tau^3(\vec{\epsilon}\cdot\vec{\sigma})[J_2(v\cdot q)-J_2(0)],
\end{aligned}\tag{7.3.5}$$

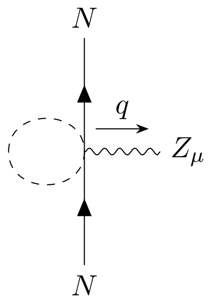
and the second one

$$\begin{aligned}
&-\delta_{jk} i \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \frac{k_j k_k}{(v \cdot p - v \cdot k + i\epsilon)(v \cdot (p+q) - v \cdot k + i\epsilon)(k^2 - m_\pi^2 + i\epsilon)} \\
&\left(\frac{g g_A^2}{16 f_\pi^2 \cos\theta_W} (v \cdot \epsilon) \left(-(2\sin^2\theta_W - 1)\tau^3 + 6\sin^2\theta_W \right) \right) \\
&= -\left(i \frac{3 g g_A^2}{16 f_\pi^2 \cos\theta_W} (v \cdot \epsilon) \left(-(2\sin^2\theta_W - 1)\tau^3 + 6\sin^2\theta_W \right) \right) [J_2(v \cdot q) - J_2(0)].
\end{aligned} \tag{7.3.6}$$

NNZ_μ vertices with vanishing loops (contributes at $\nu = 2$)

As in the density case 7.2, this integral gives no contribution since both the poles lie in negative imaginary half of the complex plane. This was expected since we integrated out the heavy part of the nucleon field.

NNZ_μ vertex with pion loop (contributes at $\nu = 2$)



This diagram contributes only as a vacuum corrections since nucleon propagators are not present here. This is a purely divergent contribution:

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4} \left(\frac{ig}{8\cos\theta_W f_\pi^2} (-g_A(\vec{\sigma} \cdot \vec{\epsilon}) + (2\sin^2\theta_W - 1)(v \cdot \epsilon))(\tau^b \delta^{3a} + \tau^a \delta^{3b} - 2\tau^3 \delta^{ab}) \right) \left(\frac{i\delta^{ab}}{k^2 - m_\pi^2 + i\epsilon} \right) \\
&= -\frac{ig}{4\cos\theta_W f_\pi^2} (-g_A(\vec{\sigma} \cdot \vec{\epsilon}) + (2\sin^2\theta_W - 1)(v \cdot \epsilon))\tau^3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_\pi^2 + i\epsilon} \\
&= -\frac{ig}{4\cos\theta_W f_\pi^2} \tau^3 (-g_A(\vec{\sigma} \cdot \vec{\epsilon}) + (2\sin^2\theta_W - 1)(v \cdot \epsilon))\Delta_\pi \\
&= -\frac{ig}{4\cos\theta_W f_\pi^2} \tau^3 (-g_A(\vec{\sigma} \cdot \vec{\epsilon}) + (2\sin^2\theta_W - 1)(v \cdot \epsilon)) m_\pi^2 \Lambda(\lambda),
\end{aligned} \tag{7.3.7}$$

using again Appendix C.

V_{v3} **:** NNZ_μ **vertex with one nucleon in the loop (contributes at** $\nu = 3$**)**

Here we proceed similarly as the $\nu = 2$ case, having in mind the diagrams 7.2. Some of the manipulation we did in the finite density case are needed here as well. In the first diagram

$$\int \frac{d^4k}{(2\pi)^4}\left(\frac{2\hat{c}_3 g}{f_\pi \cos\theta_W}\delta^{\alpha 3}(k-p-q)^\mu + i\frac{\hat{c}_4 g}{4f_\pi\cos\theta_W}\epsilon^{3\alpha b}\tau^b[\vec{\sigma}\cdot\vec{\epsilon},\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})]\right.$$
$$\left. - i\frac{\hat{c}_6 g}{16m\cos\theta_W f_\pi}\epsilon^{3\alpha b}\tau^b[\vec{q}\cdot\vec{\sigma},\vec{\sigma}\cdot\vec{\epsilon}]\right)\left(\frac{i}{v\cdot k + i\epsilon}\right) \tag{7.3.8}$$
$$\left(\frac{i\delta^{a\alpha}}{(k-p-q)^2 - m_\pi^2}\right)\left(\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})\tau^a\right)$$

we can be split the sum and analyze the three integrals separately
1)

$$\begin{aligned} &-\frac{\hat{c}_3 g g_A}{2f_\pi^2\cos\theta_W}\tau^3\sigma_j\epsilon^\mu i\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{(k-p-q)_\mu(k-p-q)_j}{(v\cdot k+i\epsilon)((k-p-q)^2-m_\pi^2)} \\ &= -\frac{i\hat{c}_3 g g_A}{2f_\pi^2\cos\theta_W}\tau^3(\vec{\sigma}\cdot\vec{\epsilon})J_2(v\cdot q) \qquad \text{in the nucleon rest frame,}\end{aligned} \tag{7.3.9}$$

2)

$$\begin{aligned} &-i\frac{\hat{c}_4 g g_A}{2f_\pi^2\cos\theta_W}\tau^3\epsilon_{ijk}\epsilon_i\sigma_k\sigma_l i\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{(k-p-q)_j(k-p-q)_l}{(v\cdot k+i\epsilon)((k-p-q)^2-m_\pi^2)} \\ &= i\frac{\hat{c}_4 g g_A}{f_\pi^2\cos\theta_W}\tau^3(\vec{\sigma}\cdot\vec{\epsilon})J_2(v\cdot q) \qquad \text{in the nucleon rest frame,}\end{aligned} \tag{7.3.10}$$

3)

$$i\frac{\hat{c}_6 g g_A}{8m\cos\theta_W f_\pi^2}\tau^3\vec{\sigma}\cdot(\vec{q}\times\vec{\epsilon})\sigma_j i\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{(k-p-q)_j}{(v\cdot k+i\epsilon)((k-p-q)^2-m_\pi^2)} = 0. \tag{7.3.11}$$

The other integral of this kind can be either computed or can be already solved comparing with the first, since for our purpose, the only different thing is the pion momentum that get shifted: $k-p-q \to k-p$. We immediately get
1)

$$-\frac{i\hat{c}_3 g g_A}{2f_\pi^2\cos\theta_W}\tau^3(\vec{\sigma}\cdot\vec{\epsilon})J_2(0) \tag{7.3.12}$$

2)

$$i\frac{\hat{c}_4 g g_A}{f_\pi^2\cos\theta_W}\tau^3\vec{\sigma}\cdot\vec{\epsilon}J_2(0), \tag{7.3.13}$$

in the nucleon rest frame and the third piece is zero here as well.

V_{v4} : NNZ_μ vertex with two nucleons in the loop (contributes at $\nu = 3$)

We have here the corresponding diagram as for $\nu = 2$ but with NNZ_μ vertex with $\Delta = 1$.

$$\int \frac{d^4k}{(2\pi)^4}\left(-\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^\alpha\right)\left(\frac{i}{v\cdot k+i\epsilon}\right)2i[\vec{\sigma}\cdot\vec{q},\vec{\sigma}\cdot\vec{\epsilon}]$$
$$\left(\frac{\hat{c}_6}{16m}g\frac{2\sin^2\theta_W-1}{2\cos\theta_W}\tau^3+\hat{c}_7 g\frac{\sin^2\theta_W}{16m\cos\theta_W}\right)\left(\frac{i}{v\cdot(k+q)+i\epsilon}\right)\left(\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p})\tau^a\right)\left(\frac{i\delta^{a\alpha}}{(k-p)^2-m_\pi^2}\right), \tag{7.3.14}$$

where we manipulate the Pauli matrices as in the density case and use the standard integrals to get

$$-\frac{g_A^2 g}{32f_\pi^2(v\cdot q)m\cos\theta_W}\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q})(-\hat{c}_6(2\sin^2\theta_W-1)\tau^3+6\hat{c}_7\sin^2\theta_W)[J_2(0)-J_2(v\cdot q)], \tag{7.3.15}$$

again in the nucleon reference frame.

NNZ_μ vertices with vanishing loops (contributes at $\nu = 3$)

As in the density and $\nu = 2$ case 7.2, this integral gives no contribution since both the poles lie in negative imaginary half of the complex plane.

NNZ_μ vertex with pion loop (contributes at $\nu = 3$)

As for $\nu = 2$ 7.3, this diagram contributes only as a vacuum corrections since nucleon propagators are not present here. This is a purely divergent contribution. Three LECs contribute to the vertex $NN\pi\pi Z_\mu$ at $\Delta = 1$ but only the $\hat{c}_6$ term doesn't cancel here, so we get

$$\begin{aligned}&\int \frac{d^4k}{(2\pi)^4}\frac{i\delta^{ab}}{k^2-m_\pi^2+i\epsilon}\left(i\frac{\hat{c}_6}{16mf_\pi^2}\frac{2\sin^2\theta_W-1}{2\cos\theta_W}\right)(\tau^b\delta^{3a}+\tau^a\delta^{3b}-2\tau^3\delta^{ab})[\vec{\sigma}\cdot\vec{q},\vec{\sigma}\cdot\vec{\epsilon}]\\&=\frac{\hat{c}_6 g}{8mf_\pi^2}\frac{2\sin^2\theta_W-1}{\cos\theta_W}\tau^3\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q})\Delta_\pi.\end{aligned} \tag{7.3.16}$$

Renormalization of zero density corrections

We here sum and regularize all the vacuum contributions, first at $\nu = 2$ and secondly at $\nu = 3$. To do so we use the functions defined in Appendix C. To simplify the notation we will write $\omega = v\cdot q$ and, as already done in the computation of the vacuum corrections,

we set the reference frame as the rest frame of the outgoing nucleon with momentum p such that $v \cdot p = 0$.
At $\nu = 2$ we get

$$
\begin{aligned}
&- i\frac{g}{48 f_\pi^2 \cos\theta_W \omega}\Bigg(\Big(3g_A^2(-(2\sin^2\theta_W - 1)\tau^3 + 6\sin^2\theta_W)(v\cdot\epsilon) - g_A^3\tau^3(\vec{\epsilon}\cdot\vec{\sigma})\Big) \\
&\qquad \Bigg((m_\pi^2 - \omega^2)\left(\frac{\omega}{8\pi^2} - \frac{1}{4\pi^2}\sqrt{m_\pi^2-\omega^2}\arccos\big(\frac{-\omega}{m_\pi}\big)\right) - \frac{m_\pi^3}{8\pi}\Bigg)\Bigg) \\
&-\Bigg(\Bigg(\frac{2ig}{3\cos\theta_W f_\pi^2} m_\pi^2 (v\cdot\epsilon)\left((\frac{3}{4} - \frac{9}{16}g_A^2)(-(2\sin^2\theta_W - 1)\right)\tau^3 - \frac{9}{16}g_A^2 2\sin^2\theta_W) \\
&- i\frac{g_A g m_\pi^2}{f_\pi^2\cos\theta_W}\tau^3(\vec{\epsilon}\cdot\vec{\sigma})\frac{4-g_A^2}{8} \\
&- \frac{ig\omega^2}{12 f_\pi^2\cos\theta_W}\Big(-3g_A^2(-(2\sin^2\theta_W - 1)\tau^3 + 6\sin^2\theta_W)(v\cdot\epsilon) + g_A^3\tau^3(\vec{\epsilon}\cdot\vec{\sigma})\Big)\Bigg)\Bigg)\Lambda(\lambda),
\end{aligned}
\tag{7.3.17}
$$

where we separated the finite piece, in the first 2 lines, from the divergent piece that can be recognized by $\Lambda(\lambda)$. Looking closer to it, as expected, this can be renormalized using only $\mathcal{O}_i(p^3)$ operators and β_i coefficients. In particular, following Table 1 in [29] we need the operators collected in Table 7.1, because at tree level they give contributions

$\tilde{O}_i(p^3)$	β_i
$b_9(\lambda)\bar{N}(S\cdot u\langle\chi_+\rangle)N$	$g_A(4-g_A^2)/8$
$b_{15}(\lambda)\bar{N}(i(v\cdot D)^3)N$	$-3g_A^2$
$b_{16}(\lambda)\bar{N}(v\cdot \overleftarrow{D} S\cdot u v\cdot D)N$	g_A^3
$b_{20}(\lambda)\bar{N}(\langle\chi_+\rangle iv\cdot D + \text{h.c})N$	$-9g_A^2/16$

Table 7.1: Operators at order $\mathcal{O}(p^3)$ and the respective β functions. Those are the ones needed to renormalize our contributions.

respectively proportional to

$$
\begin{aligned}
&b_9(\lambda)m_\pi^2(\vec{\sigma}\cdot\vec{\epsilon}), \\
&b_{15}(\lambda)\omega^2(v\cdot\epsilon), \\
&b_{16}(\lambda)\omega^2(\vec{\sigma}\cdot\vec{\epsilon}), \\
&b_{20}(\lambda)m_\pi^2(v\cdot\epsilon),
\end{aligned}
\tag{7.3.18}
$$

that are exactly the contributions that we want to renormalize. Looking carefully at the expression we can see that the all the 4 β_i functions appear in front of the correct tree level pieces. There is only the coefficient multiplying $(v\cdot\epsilon)$, coming from the pion loop, that seems not to be renormalized with these β_i. Our suspicious is that since in

a pion loop all the nucleons are on-shell, we can use the equation of motion to discard that piece and so we do not need to renormalize it. So, after we managed to remove all the infinities, it is now the turn of the finite pieces generated by the counterterms. The finite pieces that come from operators that can be eliminated by an appropriate field redefinition, e.g. proportional to $v \cdot D$, are zero, this removes the finite terms coming from $\tilde{O}_{15}$, $\tilde{O}_{16}$ and $\tilde{O}_{20}$. The one coming from O_9 is not zero, however, since it comes from an $\mathcal{O}(p^3)$ operator, it is negligible with respect to the other finite parts as it is suppressed by one more power of the nucleon mass.
Let us analyze now the $\nu = 3$ diagrams. Summing all together we get

$$
\begin{aligned}
&-\frac{igg_A}{f_\pi^2 \cos\theta_W}\tau^3(\vec{\sigma}\cdot\vec{\epsilon})(\hat{c}_3/2-\hat{c}_4)\left(\frac{1}{3}\left((m_\pi^2-\omega^2)\left(\frac{\omega}{8\pi^2}-\frac{1}{4\pi^2}\sqrt{m_\pi^2-\omega^2}\arccos(\frac{-\omega}{m_\pi})+\frac{m_\pi^3}{8\pi}\right)\right)\right)\\
&-\frac{g_A^2 g}{32 f_\pi^2 \omega m \cos\theta_W}\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q})(-\hat{c}_6(2\sin^2\theta_W-1)\tau^3+6\hat{c}_7\sin^2\theta_W)\\
&-\frac{1}{3}\left((m_\pi^2-\omega^2)\left(\frac{\omega}{8\pi^2}-\frac{1}{4\pi^2}\sqrt{m_\pi^2-\omega^2}\arccos(\frac{-\omega}{m_\pi})+\frac{m_\pi^3}{8\pi}\right)\right)\\
&-\left(-\frac{igg_A}{f_\pi^2\cos\theta_W}\tau^3(\vec{\sigma}\cdot\vec{\epsilon})(\hat{c}_3-2\hat{c}_4)m_\pi^2\omega+\frac{8igg_A}{3f_\pi^2\cos\theta_W}\tau^3(\vec{\sigma}\cdot\vec{\epsilon})(\hat{c}_3-2\hat{c}_4)\omega^3\right.\\
&-\frac{2(2\sin^2\theta_W-1)\hat{c}_6 g}{2mf_\pi^2\cos\theta_W}\tau^3\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q})m_\pi^2(\frac{1}{4}-\frac{g_A^2}{16})\\
&-\frac{3g_A^2 g}{32 f_\pi^2 2m}\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q})2\hat{c}_7 m_\pi^2\frac{2\sin^2\theta_W}{2\cos\theta_W}\\
&\left.-\frac{g_A^2 g}{2f_\pi^2 2m6\cos\theta_W}\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q})\omega^2\hat{c}_6(2\sin^2\theta_W-1)\tau^3-\frac{\sin^2\theta_W}{\cos\theta_W}\frac{(-2\hat{c}_4)g}{2m}\frac{3g_A^2}{g}4f_\pi^2 2m\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q})\omega^2\right)\Lambda(\lambda)\\
&+\mathcal{O}(d-4),
\end{aligned}
\tag{7.3.19}
$$

where the first three lines are the finite piece and the other lines need to be renormalized. To do so, here we need $\tilde{O}_i(p^4)$ operators and δ_i coefficients. In particular, using the same notation of in [30] in Table 1 we need the operators collected in Table 7.2, because at tree level they give contributions respectively proportional to

$$
\begin{aligned}
m_\pi^2\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q}),\\
m_\pi^2\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q}),\\
m_\pi^2\omega(\vec{\sigma}\cdot\vec{\epsilon}),\\
\omega^2\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q}),\\
\omega^2\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q}),\\
\omega^3(\vec{\sigma}\cdot\vec{\epsilon}),
\end{aligned}
\tag{7.3.20}
$$

that are exactly the ones that we want to renormalize. From this list above we can notice that 2 different operators give rise to the same tree level term, the difference lies

$\tilde{O}_i(p^4)$	δ_i
$d_{54}(\lambda)\bar{N}v_\rho\epsilon^{\rho\mu\nu\sigma}S_\sigma\tilde{F}^+_{\mu\nu}\langle\chi_+\rangle$N	$\hat{c}_6/2m\,(1/4-g_A^2/16)$
$d_{55}(\lambda)\bar{N}v_\rho\epsilon^{\rho\mu\nu\sigma}S_\sigma\langle\tilde{F}^+_{\mu\nu}\rangle\langle\chi_+\rangle$N	$3g_A^2\hat{c}_7/32m$
$d_{117}(\lambda)\bar{N}(\langle\chi_+\rangle S\cdot uiv\cdot D+\text{h.c.})N$	$g_A(\hat{c}_3-2\hat{c}_4)$
$d_{183}(\lambda)\bar{N}v_\rho\epsilon^{\rho\mu\nu\sigma}S_\sigma v\cdot\overleftarrow{D}\tilde{F}^+_{\mu\nu}v\cdot DN$	$g_A^2\hat{c}_6/4m$
$d_{186}(\lambda)\bar{N}v_\rho\epsilon^{\rho\mu\nu\sigma}S_\sigma v\cdot\overleftarrow{D}\langle F^+_{\mu\nu}\rangle v\cdot DN$	$-\hat{c}_4 3g_A^2/4m$
$d_{193}(\lambda)\bar{N}(S\cdot ui(v\cdot D)^3+\text{h.c.})N$	$8/3g_A(\hat{c}_3-2\hat{c}_4)$

Table 7.2: Operators at order $\mathcal{O}(p^4)$ and the respective β functions. Those are the ones needed to renormalize our contributions.

in the LECs that multiply the expressions. Here there is a complete agreement between the δ_i and the coefficients from of the sum of the diagrams, all the terms match. So we renormalize the $\nu=3$ part of the diagrams with $O(p^4)$ operators. As regarding the finite pieces that arise with the counterterms the same as for $\nu=2$ applies.

7.4 Mathematica implementation

In the course of this section we presented analytical result. The next important step is to graphically see their implementation, in order to understand the relevance of the density corrections. We will show our results for the two types of neutral currents: proton-proton and nucleon-nucleon scattering. We here present two main results. Everything has been done using Wolfram Mathematica.

Plot of the structures

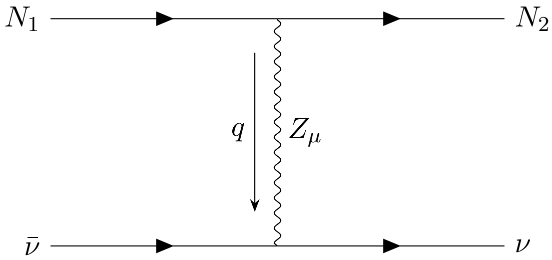
As soon as we compute loop corrections to NNZ_μ vertex we realize that new structures, with respect to the tree level case, appear. By saying structure we mean scalar or Cartesian products where Pauli matrices $\vec{\sigma}$, the reference vector v^μ and the Z_μ polarization ϵ_μ are contracted between each others and with the external momenta q^μ. These can not be plotted, but it is interesting to plot their coefficients and compare them with the tree level case. Two scenarios open here: either the structure is already present in the tree level vertex and so we see how it gets modified by density or the structure is not present at tree level and so we study the new ones and their coefficient, to understand how relevant they are, estimating the order of magnitude. We proceed doing an expansion in q^μ since the momentum of the Z_μ boson here is considered to be small. The plots will then refer to the case $q^\mu=0$. Let us recall the tree vertex; it is computed in section 6.5 and written in terms of structures reads

$$-i\frac{g_A g}{4\cos\theta_W}\tau^3(\vec{\epsilon}\cdot\vec{\sigma})-i\frac{g}{2\cos\theta_W}(v\cdot\epsilon)\left(\left(1-\sin^2\theta_W\right)\tau^3+2\sin^2\theta_W\right), \qquad (7.4.1)$$

where we see that there are only two structure appearing $\vec{\epsilon} \cdot \vec{\sigma}$ and $v \cdot \epsilon$.
Adding density effects we have now 5 different structures: $(\vec{p} \cdot \vec{\epsilon})(\vec{p} \cdot \vec{\sigma})$, $(\vec{p} \cdot (\vec{\epsilon} \times \vec{\sigma}))$, $(\vec{p} \cdot \vec{\epsilon})$, $(\vec{\epsilon} \cdot \vec{\sigma})$ and $(\vec{p} \cdot \vec{\sigma})(v \cdot \epsilon)$. We can immediately notice that there is no pure vectorial contribution as at tree level, instead we have $(\vec{p} \cdot \vec{\sigma})(v \cdot \epsilon)$. This is expected because the energy transfer in vector interactions vanishes as the momentum transfer goes to zero due to vector-current conservation [33]. However HBChPT allows us to keep track of mixed interactions and so terms like $(\vec{p} \cdot \vec{\sigma})(v \cdot \epsilon)$ can appear. This is relevant because this type of structure is what has not been considered in previous analysis [33], [34]. In Figure 7.1 and 7.2 we plot the five structures both for incoming protons and neutrons (the difference lies in an overall sign due to the isospin matrix), to understand which are the most relevant. As could be expected, the axial structure present at tree level is the most relevant one at loop level as well. However, looking at the plots we can immediately notice that the second most relevant structure is the one containing a vectorial part. This will later be important in the computation of the matrix element squared.

Computation of the matrix element squared

First step was computing the matrix element squared at tree level, using only the NNZ_μ of section (6.5) vertex, with no corrections.



There are two main reasons why we do this: the first is that computing a matrix element including only one vertex is way easier and let us understand how to code it in Mathematica, the second is that is actually of fundamental importance because can be compared with the matrix element squared that can be computed using four-fermion interactions, so it represents another proof of the reliability of the loop result.
Before proceeding, we list the numerical values used for the couplings in Tables 7.4 and 7.4.

Λ [meV]	f_π [meV]	g_A [meV]	g	$\cos\theta_W$	$\sin\theta_W$	e	m_π [meV]	m [meV]
700	130	1.2723	0.47212	0.88153	0.47212	1	136.16	939

$\hat{c}_3$ $[meV^{-1}]$	$\hat{c}_4$ $[meV^{-1}]$	$\hat{c}_6$	$\hat{c}_7$	c_D
(-5.34 ± 0.04) 10^{-3}	(2.84 ± 0.15) 10^{-3}	5.64 10^{-3}	-2.88$10^{-3}$	(-0.85 ± 2.15)

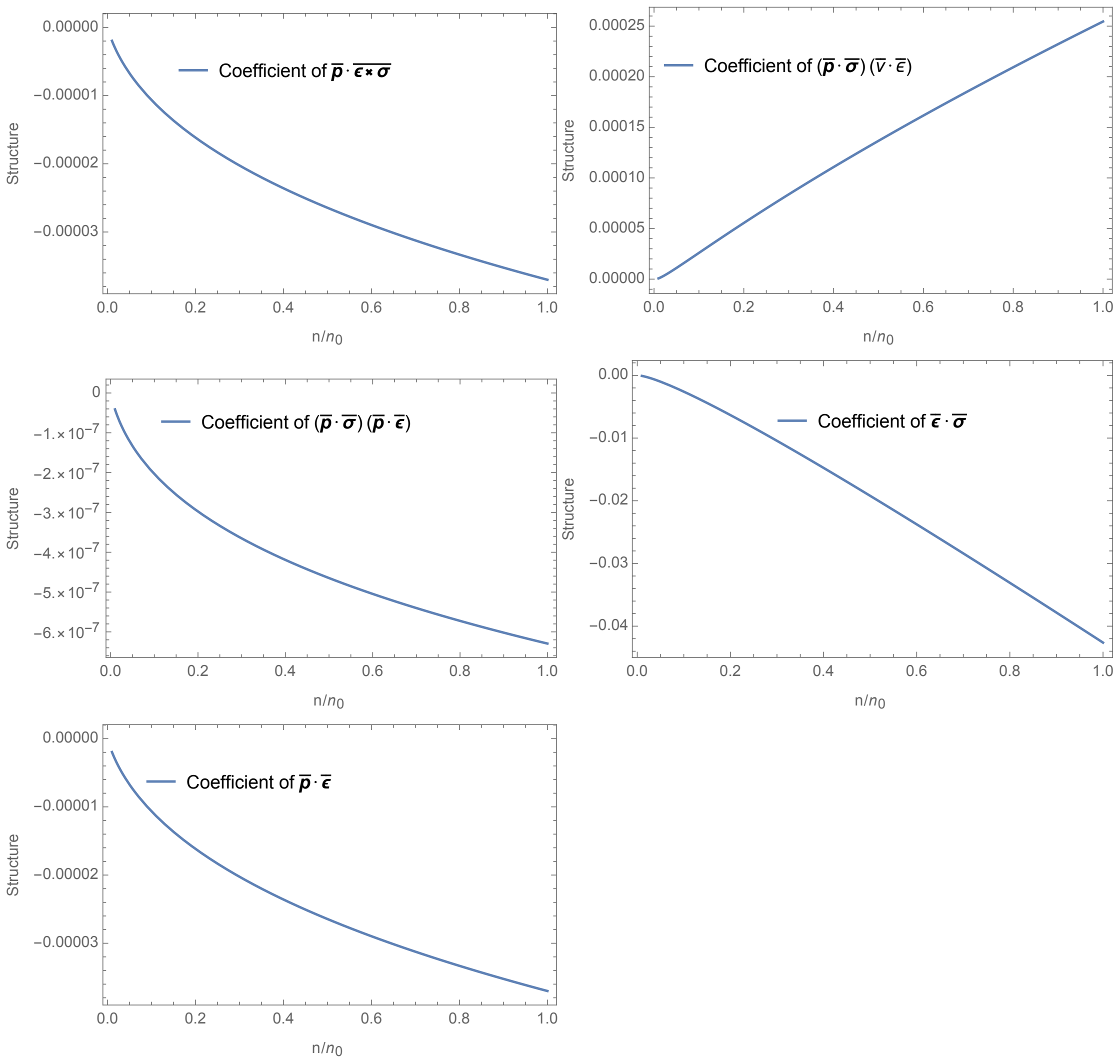


Figure 7.1: Plots of the coefficients of the structures appearing at $\mathcal{O}(q^0)$ with incoming protons. We show the value of the coefficient versus density expressed in terms of nuclear saturation density n_0.

All the $\hat{c}_i$ are taken from [35] and c_D from [36].
Back to the implementation, one difficulty that we encounter already at tree level is the matching between relativistic and non-relativistic formalism. HBChPT is a non-relativistic theory while the decay of the Z_μ boson is described using electroweak theory. This results in an expression where we both have Pauli matrices and Gamma matrices, structures that are not compatible. This is of course a problem that appears when multiplying these 2 different objects, computing traces and so on. Here, this has been solved writing all Pauli matrices, Cartesian scalar products and cross products in terms of 4-dimensional objects.

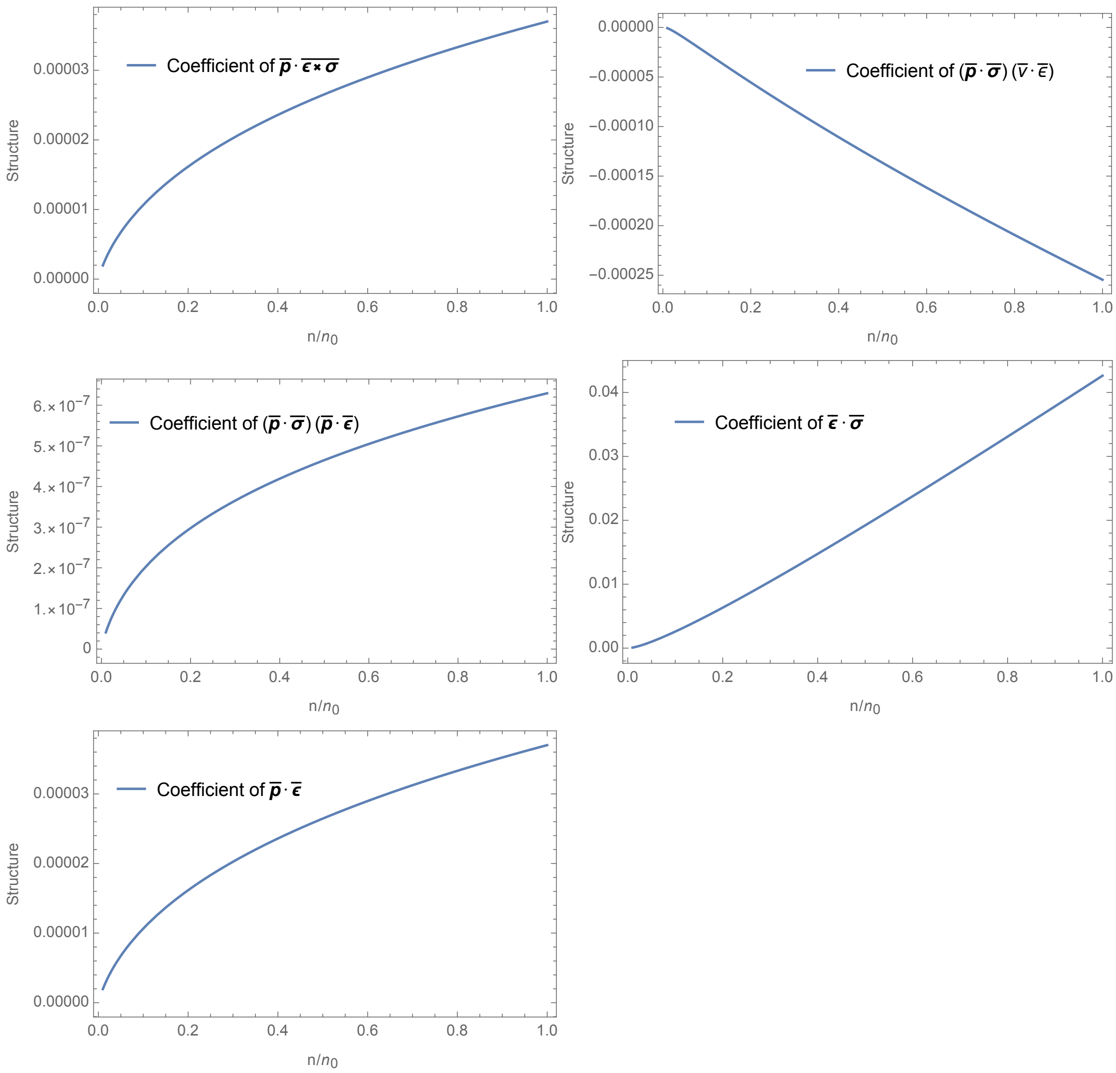


Figure 7.2: Plots of the coefficients of the structures appearing at $\mathcal{O}(q^0)$ with incoming neutrons. We show the value of the coefficient versus density expressed in terms of nuclear saturation density n_0.

Once these problems are solved, computing the tree level amplitude is almost straightforward. We need to be careful about the spinor sum for the non-relativistic case, here the following is true

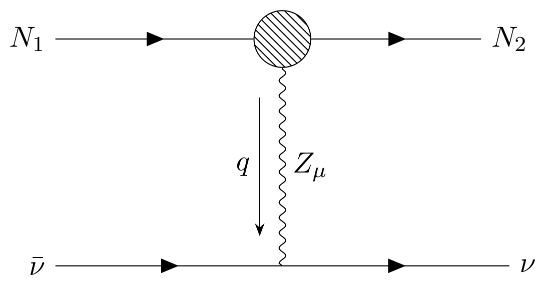
$$\begin{aligned} \sum_s u_s(p)\bar{u}_s(p) &= 2m \\ \bar{u}_s(p) &= u_s^\dagger(p). \end{aligned} \tag{7.4.2}$$

We get

$$\sum_s |M_{\text{spins}}|^2 = 32G_F^2(C_A^2(3-\cos\theta)+C_V^2(1+\cos\theta)), \tag{7.4.3}$$

in agreement with [33], where this is computed using the 4-fermion interactions. Let us stress the differences in the two approaches. When we compute the matrix element squared in HBChPT we need to implement the non-relativistic spinor sum 7.4.2. However we do not need to do any other simplification since the non relativistic limit is already included in the spinor sum and in HBChPT. While, when we do the computation using the four-fermions interaction we need to manually impose that the nucleons are non-relativistic e.g. saying that the scalar product between a nucleon and a neutrino momentum gives only the product of the two temporal components, that in the nucleon case it is its mass, or that the scalar product of 2 nucleons gives the mass of the nucleon squared, implying that the only angle that is relevant is the one between the two neutrinos.

Next step is to implement all the vertex corrections listed in the previous subsections.



A big difficulty here encountered is given by the fact that is computationally heavy, it requires a big amount of time, so also the debug process is slowed down. So we proceed with approximations. The computation, here presented, of the matrix element squared is given assuming $q \to 0$. This however is an assumption in agreement with all what has been said till now, since it's really at the heart of our implementation of HBChPT for our study. Here we will not report the analytic result but we plot the density dependence of the matrix element squared. First, we connect our results to [34] where they considered only the quenching of the axial coupling g_A. In Figures 7.3 and 7.4 we show the comparison between the matrix element squared computed in the two ways: taking the tree level one but substituting g_A with its quenched version and using HBChPT considering only pure axial contributions coming from the diagrams computed before.

We see that there is perfect agreement between the two methods. We now show what changes adding the vectorial contributions. In the literature this has been computed not in nucleon density environments but in high temperature ones in [37]. However, as pointed out in [36], there is an analogy between density and temperature corrections; the temperature T as well as the chemical potential μ, sets the effective energy of particle in a system, due to a coupling to a "heat bath". Moreover, we could account for temperature

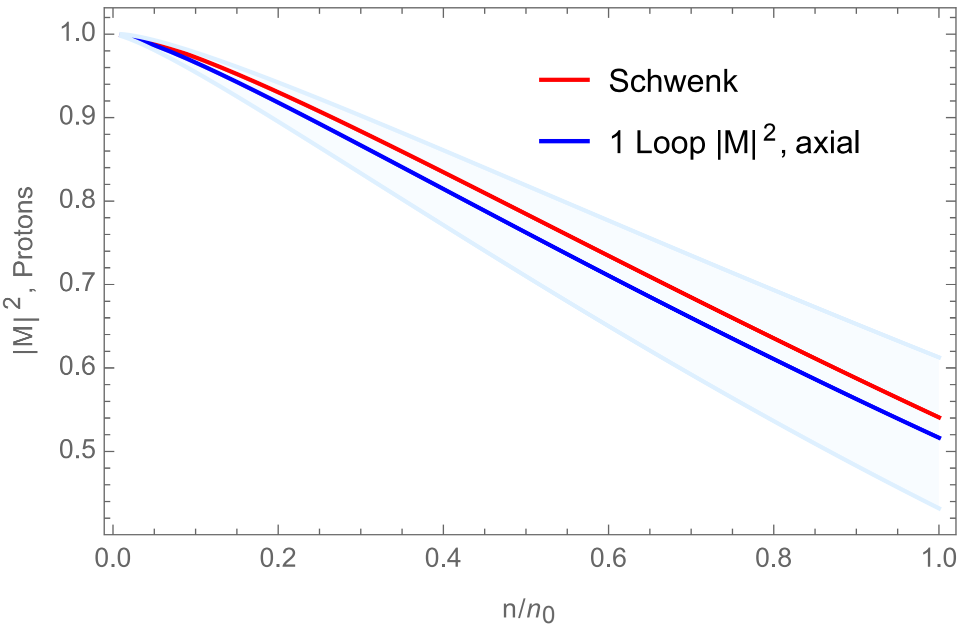


Figure 7.3: 1 Loop matrix element squared for incoming protons computed quenching g_A, in red, and considering only axial contributions from HBChPT in blue. The shaded region is due to the errors in the LECs.

corrections using a temperature corrected propagator that can be derived following [38] and [39].

Back to our results, we can see the effect of vectorial contributions in two ways. As recalled in [37], ultra-relativistic neutrinos preserve their helicity, thus back scattering can only occur through their axial current coupling to the nucleon spin. This is indeed what we get: our matrix element squared for back scattering doesn't change considering or not the vectorial contribution. In Figure 7.5 we show what we get in this case.

Finally, in Figure 7.6 we show the matrix element squared for forward scattering for both incoming protons and neutrons, compared to [34]. We get a very similar result compared to [37]; indeed the suppression of the axial response is partially compensated by the enhancement of the vector one.

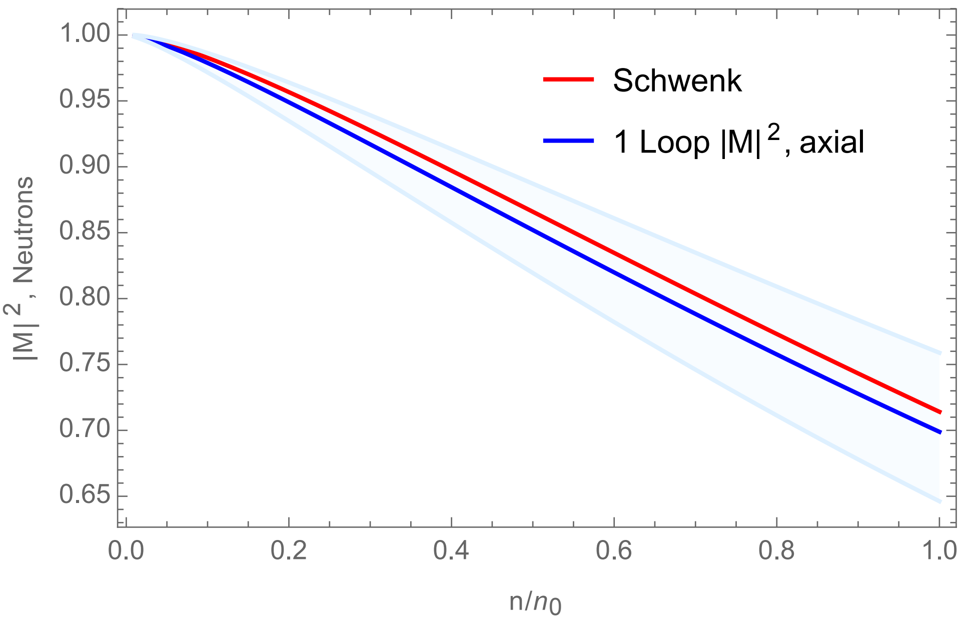


Figure 7.4: 1 Loop matrix element squared for incoming neutrons computed quenching g_A, in red, and considering only axial contributions from HBChPT in blue. The shaded region is due to the errors in the LECs.

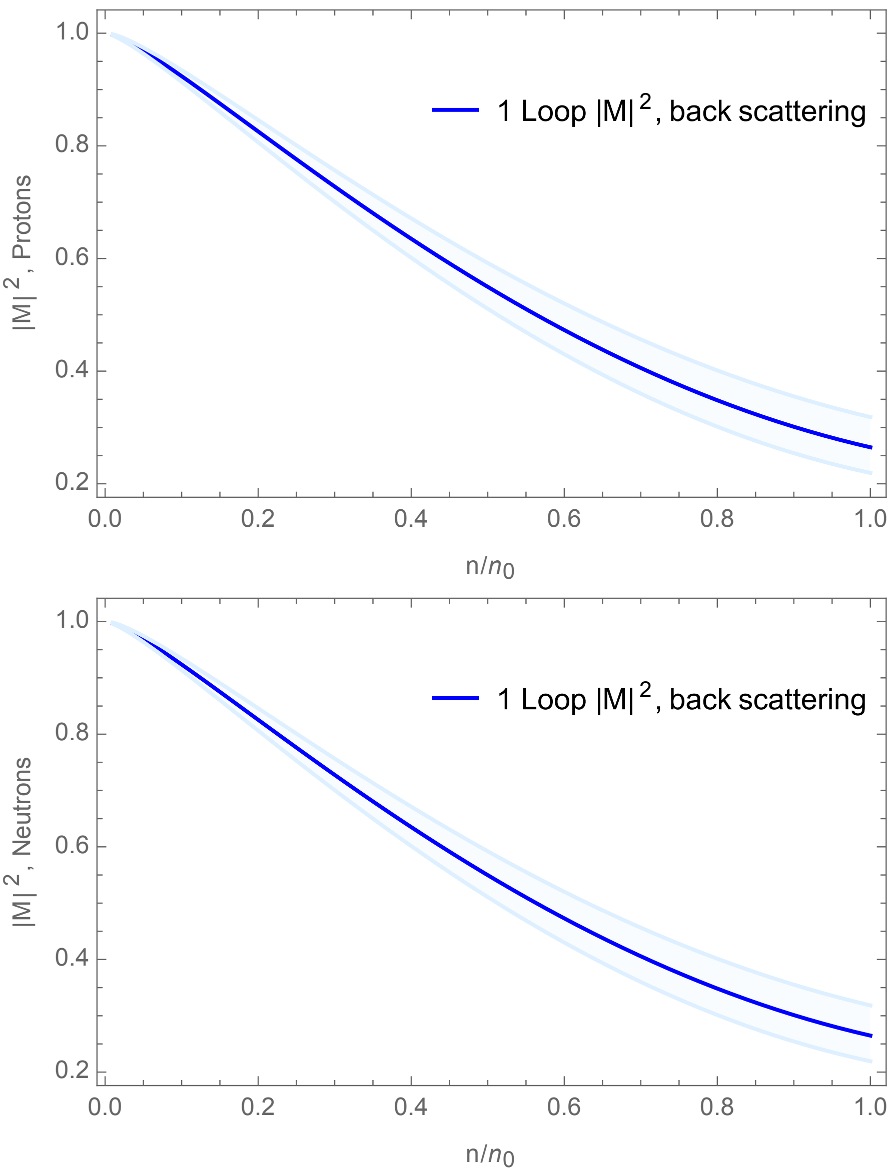


Figure 7.5: Plots of the matrix element squared for back neutrinos scattering. We show its value versus density expressed in terms of nuclear saturation density n_0.

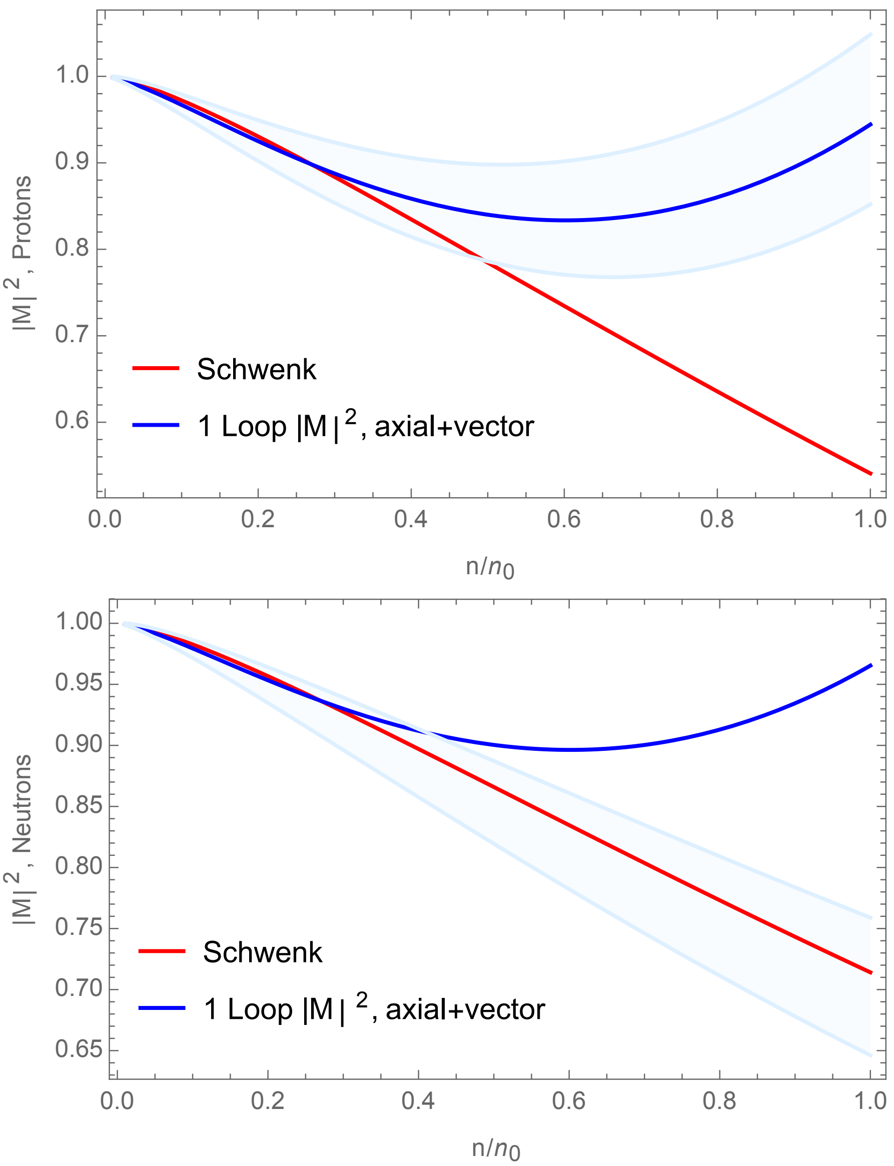


Figure 7.6: Plots of the matrix element squared for forward neutrinos scattering compared to what we get considering only the axial response. We show its value versus density expressed in terms of nuclear saturation density n_0.

Chapter 8

Axions in heavy baryon chiral perturbation theory

This section is here because this project is inserted in a bigger one where PhD students are investigating axions properties in environments characterized by high density of nucleons [40]. What is showed here is what has been done before starting the project on neutrinos. Here we compute zero density corrections to the nucleon-nucleon-axion vertex at $\nu = 3$. Other contributions are related to the understanding of some relevant points and to the redoing of some of their computations that, however, are not reported here.

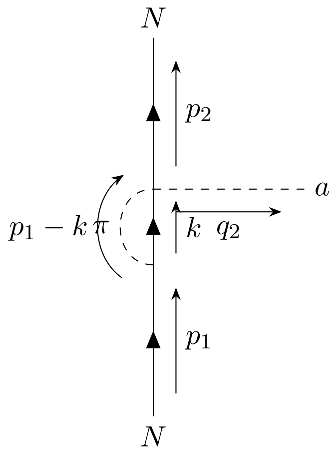
8.1 Brief theoretical overview

This section does not want to be fully self-consistent and exhaustive, for a complete analysis we refer to [40]. Here we will only underline the main differences and similarities that appear when studying neutrinos and axions interacting with nucleons.
For axions as well the most efficient production mechanism in a supernova core is the nucleon-nucleon bremsstrahlung process [41]. However, the axion-nucleon coupling does not depend only on one energy scale f_a as for the pion f_π, but is also model dependent. We would have a slight model dependence in the neutrino case as well since there are many models extending SM neutrinos, but in this work we only analyzed the SM ones, considering $m_\nu = 0$. One difference about their behaviour inside a SN is that neutrinos are trapped while QCD axions have a mean free-path bigger than the radius of a proto-neutron star. From this follows that it is interesting to study their production in a SN core since they can provide an additional source of energy loss or transfer with respect to the neutrino emission and so they do have a feedback on the dynamics of the star.
The study of the axion-nucleon coupling, in a bremsstrahlung process, provides constraints on the axion mass as well.
We want to point out that there is one main difference in the formalism: while the Z_μ boson can only be produced as an off-shell particle due to its large rest mass, the axion a can be produced as an on-shell particle, due to its almost zero mass, so we can have final states with an axion. However, processes with more than one axions are highly

suppressed due to the large value of f_a.

8.2 $\nu = 3$ NNa vertex zero density contribution

Here we proceed the same way as for the Z_μ boson case 7.3. There is a slight difference in the functions used, they will be defined in the following.
As for the Z_μ, we have the same two diagrams that can be obtained just by exchanging the two vertices:

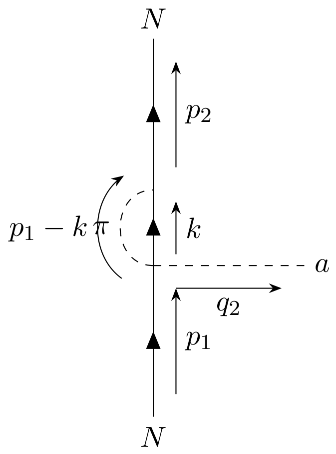


$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\overline{N(2)}\Bigg[&-2i\delta^{3a}\frac{c_{u-d}}{f_\pi f_a}\hat{c}_3(p_1-k)\cdot q_2+\frac{1}{2}\frac{c_{u-d}}{f_\pi f_a}\hat{c}_4[\vec{\sigma}\cdot(\vec{p}_1-\vec{k}),\vec{\sigma}\cdot\vec{q}_2](\epsilon_{a3b}\tau^b)\\ &-i\frac{\hat{c}_9c_{u+d}}{f_\pi f_a}\tau^a(p_1-k)\cdot q_2+4\hat{c}_5 i\frac{m_a^2f_a}{f_\pi^3}\tau^a\Bigg]\left[\frac{i\delta^{ac}}{(p_1-k)^2-m_\pi^2}\right]\left[\frac{i}{v\cdot k+i\epsilon}\right]\\ &\left[-\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{p}_1-\vec{k})\tau^c\right]N(1)=\end{aligned} \tag{8.2.1}$$

$$\begin{aligned}=\overline{N(2)}\Bigg[&2i\delta^{3a}\frac{c_{u-d}}{f_\pi f_a}\hat{c}_3 i\delta^{ac}i\frac{g_A}{2f_\pi}\tau^c q_{2\mu}(\vec{\sigma}\cdot\vec{\epsilon})(I_1)^{\mu i}-\frac{1}{2}\frac{c_{u-d}}{f_\pi f_a}\hat{c}_4 2i\epsilon_{ijk}\sigma_k q_{2j}\epsilon_{a3b}\tau^b i\delta^{ac}i\frac{g_A\sigma_l}{2f_\pi}\tau^c(I_1)^{il}\\ &+i\frac{\hat{c}_9c_{u+d}}{f_\pi f_a}\tau^a q_i i\delta^{ac}i\frac{g_A}{2f_\pi}\sigma_j\tau^c(I_1)^{ij}-4\hat{c}_5 i\frac{m_a^2f_a}{f_\pi^3}\tau^a i\delta_{ac}i\frac{g_A}{2f_\pi}(\vec{\sigma}\cdot\vec{\epsilon})\tau^c(I_2)^i\Bigg]N(1)=\end{aligned} \tag{8.2.2}$$

$$
\begin{aligned}
= \overline{N(2)}\Bigg[&-i\frac{c_{u-d}}{f_\pi^2 f_a}\hat{c}_3 g_A \tau^3 q_{2\mu}(\vec{\sigma}\cdot\vec{\epsilon})(I_1)^{\mu i} - i\frac{c_{u-d}}{f_\pi^2 f_a} g_A \hat{c}_4 q_{2j}\tau^3(\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl})\sigma_n (I_1^{il}) \\
&- i\frac{\hat{c}_9 c_{u+d}}{f_\pi^2 f_a}\frac{3}{2}\mathbb{I} g_A q_i \sigma_j (I_1)^{ij} + 2i\frac{\hat{c}_5}{f_\pi^4} m_a^2 f_a g_A 3\mathbb{I}(\vec{\sigma}\cdot\vec{\epsilon})(I_2)^i\Bigg]N(1),
\end{aligned}
\tag{8.2.3}
$$

and



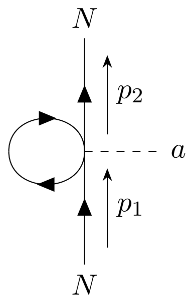
$$
\begin{aligned}
\int \frac{d^4k}{(2\pi)^4}\overline{N(2)}\left[-\frac{g_A}{2f_\pi}\vec{\sigma}\cdot(\vec{k}-\vec{p}_1)\tau^a\right]\left[\frac{i\delta^{ac}}{(p_1-k)^2-m_\pi^2}\right]\left[\frac{i}{v\cdot k+i\epsilon}\right]\Bigg[-2i\delta^{3c}\frac{c_{u-d}}{f_\pi f_a}\hat{c}_3(k-p_1)\cdot q_2 \\
+\frac{1}{2}\frac{c_{u-d}}{f_\pi f_a}\hat{c}_4[\vec{\sigma}\cdot(\vec{k}-\vec{p}_1),\vec{\sigma}\cdot\vec{q}_2](\epsilon_{c3b}\tau^b) - i\frac{\hat{c}_9 c_{u+d}}{f_\pi f_a}\tau^c(k-p_1)\cdot q_2 + 4\hat{c}_5 i\frac{m_a^2 f_a}{f_\pi^3}\tau^c\Bigg]N(1) =
\end{aligned}
\tag{8.2.4}
$$

$$
\begin{aligned}
\overline{N(2)}\frac{g_A}{2f_\pi}\Bigg[&-2i\tau^3\frac{c_{u-d}}{f_\pi f_a}\hat{c}_3(\vec{\sigma}\cdot\vec{\epsilon})q_j(I_1)^{ij} - 2i\tau^3\frac{c_{u-d}}{f_\pi f_a}\hat{c}_4 q_{2l}(\delta_{ji}\delta_{lk}-\delta_{jk}\delta_{li})\sigma_k(I_1)^{ij} \\
&- i\frac{\hat{c}_9 c_{u+d}}{f_\pi f_a}3\mathbb{I}(\vec{\sigma}\cdot\vec{\epsilon})q_{2\mu}(I_1)^{i\mu} - 4\hat{c}_5 i\frac{m_a^2 f_a}{f_\pi^3}3\mathbb{I}(\vec{\sigma}\cdot\vec{\epsilon})(I_2)^i\Bigg]N(1),
\end{aligned}
\tag{8.2.5}
$$

where we used:

$$
\begin{aligned}
&[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k, \\
&\epsilon_{a3b}\tau^b\tau^a = \epsilon_{a3b}i\epsilon^{bac}\tau^c = 2\delta_c^3\tau^c = 2\tau^3, \\
&\tau^a\tau^a = 3\mathbb{I}, \\
&\epsilon_{ijk}\sigma_k\sigma_l = \epsilon_{ijk}(\delta_{kl}\mathbb{I} + i\epsilon_{kln}\sigma_n) = \epsilon_{ikl}\mathbb{I} + i(\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl})\sigma_n,
\end{aligned}
\tag{8.2.6}
$$

and in the last line the $\epsilon_{ikl}\mathbb{I}$ drops when multiplied by $(I_1)^{il}$ because the integral is symmetric under index exchange.
The last integral is



$$-\int \frac{d^4k}{(2\pi)^4}\overline{N(2)}\left[\frac{i}{v\cdot k+i\epsilon}\right]\frac{1}{2f_\pi^2 f_a\Lambda_\chi}((c_D c_{u-d}\tau^3+\tilde{c}_D c_{u+d})\vec{\sigma}\cdot\vec{p}_a)_{ij}N(1)=0, \quad (8.2.7)$$

because has only one pole and so choosing the proper contour we don't get any residue.

$(I_1)^{\mu\nu}$ and $(I_2)^\mu$ are defined and computed in the Appendix C of [11], as follows:

$$\begin{aligned}(I_1)^{\mu\nu} &= i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{(p_1-k)^\mu(p_1-k)^\nu}{[(p_1-k)^2-m_\pi^2+i\epsilon](v\cdot k+i\epsilon)} \overset{k\,\to\,k\,+\,p_1}{=}\\ &= i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{k^\mu k^\nu}{(k^2-m_\pi^2+i\epsilon)(v\cdot k+v\cdot p_1+i\epsilon)}=\\ &= v^\mu v^\nu C_{20}(v\cdot p_1,m_\pi^2)+g^{\mu\nu}C_{21}(v\cdot p_1,m_\pi^2)=\\ &= v^\mu v^\nu\left[\frac{1}{3}(2m_\pi^2+(v\cdot p_1)^2)J_{\pi N}(0;v\cdot p_1)-\frac{1}{3}v\cdot p_1 I_\pi(0)+\frac{v\cdot p_1}{12\pi^2}\left(\frac{m_\pi^2}{2}-\frac{(v\cdot p_1)^2}{3}\right)\right]+\\ &\quad + g^{\mu\nu}\left[\frac{1}{3}(m_\pi^2-(v\cdot p_1)^2)J_{\pi N}(0;v\cdot p_1)+(v\cdot p_1)I_\pi(0)-\frac{v\cdot p_1}{12\pi^2}\left(\frac{m_\pi^2}{2}-\frac{(v\cdot p_1)^2}{3}\right)\right],\end{aligned} \quad (8.2.8)$$

and

$$\begin{aligned}(I_2)^\mu &= i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{(p_1-k)^\mu}{[(p_1-k)^2-m_\pi^2+i\epsilon](v\cdot k+i\epsilon)} \overset{k\,\to\,k\,+\,p_1}{=}\\ &= i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{k^\mu}{(k^2-m_\pi^2+i\epsilon)(v\cdot k+v\cdot p_1+i\epsilon)}=\\ &= v^\mu C_1(v\cdot p_1,m_\pi^2)=\\ &= v^\mu\left(I_\pi(0)-v\cdot p_1 J_{\pi N}(0;v\cdot p_1)\right),\end{aligned} \quad (8.2.9)$$

with:

$$J_{\pi N}(0; v\cdot p_1) = \frac{v\cdot p_1}{8\pi^2}\left[R + ln\left(\frac{m_\pi^2}{\mu^2}\right) - 1\right] + \\ + \frac{1}{8}\left\{\begin{array}{ll} 2\sqrt{(v\cdot p_1)^2 - m_\pi^2}\cosh^{-1}\left(\frac{v\cdot p_1}{m_\pi}\right) - 2\pi i\sqrt{(v\cdot p_1)^2 - m_\pi^2}, & \text{for } v\cdot p_1 > m_\pi \\ 2\sqrt{m_\pi^2 - (v\cdot p_1)^2}\cosh^{-1}\left(-\frac{v\cdot p_1}{m_\pi}\right), & \text{for } (v\cdot p_1)^2 < m_\pi^2 \\ -2\sqrt{(v\cdot p_1)^2 - m_\pi^2}\cosh^{-1}\left(-\frac{v\cdot p_1}{m_\pi}\right), & \text{for } v\cdot p_1 < -m_\pi \end{array}\right\}, \tag{8.2.10}$$

and

$$I_\pi(0) = \frac{m_\pi^2}{16\pi^2}\left[R + ln\left(\frac{m_\pi^2}{\mu^2}\right)\right] + O(n-4) \tag{8.2.11}$$

$$R = \frac{2}{n-4} - [\ln(4\pi) + \Gamma^{'}(1) + 1]. \tag{8.2.12}$$

Additionally there is no $\Delta = 1$ contribution for the vertex $NN\pi\pi a$ from $\{S\cdot D, v\cdot u\}$ and $\{S\cdot D, v\cdot \hat{u}\}$ in the Lagrangian because they are order $\frac{1}{m^2}$ suppressed.

Chapter 9

Conclusions and Outlook

In this thesis we analyzed nucleons-neutrinos scattering in environments characterized by the high density of nucleons, as Supernovae and Neutron Stars. We systematically constructed the Heavy Baryon Chiral Perturbation Theory Lagrangian and modified the nucleon propagator to properly take in account nucleons density. With our formalism we computed some Feynman rules that have been used to calculate the $NN \longrightarrow \nu\nu$ neutral scattering. We are in agreement with the previous results, which were taking into account only modification in the axial coupling, and we compute here the contribution of the vectorial coupling.

Possible outlook are:

- to compute the (differential) cross-section;
- to consider higher powers of the transfer momentum;
- to study the density dependence on the other neutral current scattering processes, e.g. $\nu\bar{\nu}NN \leftrightarrow NN$ and $\nu NN \leftrightarrow NN\nu$;
- to study the full $SU(3)_A$ symmetry breaking, as in [42] for axions;
- add other effects as weak magnetism; strange quarks and so on;
- study positivity constraints to give bounds on low-energy-couplings characterizing the axion-Lagrangian;
- add temperature corrections given that $T \neq 0$ during the core-collapse.

Appendix A

Nucleon propagator at finite density

In this appendix we derive the propagator at finite density and temperature using the tools of QFT at finite temperature among which the real-time formalism. We start adding a chemical potential to the theory so we use the grand canonical ensemble. Our main references are [38] and [39].

A.1 Real-time formalism

To properly describe systems with chemical potential and temperature we can start with the grand canonical ensemble, in particular the temperature average is defined as

$$\langle\langle \hat{A} \rangle\rangle \equiv \frac{\mathrm{Tr}\{e^{-\beta \hat{K}} \hat{A}\}}{\mathrm{Tr}\{e^{-\beta \hat{K}}\}} \equiv Z^{-1}\,\mathrm{Tr}\{e^{-\beta \hat{K}} \hat{A}\}, \qquad \hat{K} = \hat{H} - \mu' \hat{B}. \tag{A.1.1}$$

To simplify the notation we set $\mu' = 0$ in the first part of the discussion, we will extend it later. For simplicity we define $\alpha \equiv -\beta\mu$ and we assume $\mu > 0$. We can now define the generating functional for scalar field and manipulate it:

$$\begin{aligned} Z[j] \equiv & Z\langle\langle T_c e^{i\int_C d^4x j(x)\hat{\phi(x)}} \rangle\rangle \\ = & \mathrm{Tr}\{e^{-\beta \hat{H}} T_c e^{i\int_C d^4x j(x)\hat{\phi(x)}}\} \\ = & \sum_\phi \langle \phi(t_0)| e^{-\beta \hat{H}} T_c e^{i\int_C d^4x j(x)\hat{\phi(x)}} |\phi(t_0)\rangle \\ = & \sum_\phi \langle \phi(t_0 - i\beta)| T_c e^{i\int_C d^4x j(x)\hat{\phi(x)}} |\phi(t_0)\rangle. \end{aligned} \tag{A.1.2}$$

From the last formula we can notice that the contour C has to start at t_0 and end at $t_0 - i\beta$. As long as the propagator is not divergent at any point in the complex plane the path passes through, the result is independent on the path.

The propagator is given by

$$
\begin{aligned}
i\Delta^C(x-x') \equiv& \langle\langle T_C\hat{\phi}(x)\hat{\phi}(x')\rangle\rangle \\
=&\theta_C(t-t')\langle\langle\hat{\phi}(x)\hat{\phi}(x')\rangle\rangle + \theta_C(t'-t)\langle\langle\hat{\phi}(x)\hat{\phi}(x')\rangle\rangle \\
=&\theta_C(t-t')C^>(x-x') + \theta_C(t'-t)C_<(x-x'),
\end{aligned}
\tag{A.1.3}
$$

with

$$
\begin{aligned}
C^>(x-x') =& C^<(x'-x) = Z^{-1}\,\mathrm{Tr}\{e^{-\beta\hat{H}}\hat{\phi}(x)\hat{\phi}(x')\} \\
=&Z^{-1}\sum_{mn}\langle n|\hat{\phi}(\mathbf{x})|m\rangle\langle m|\hat{\phi}(\mathbf{x}')|n\rangle e^{-iE_m(t-t')}e^{iE_n(t-t'+i\beta)}.
\end{aligned}
\tag{A.1.4}
$$

Thus the propagator converges only if

$$
\begin{aligned}
-\beta < \mathrm{Im}(t-t') < 0, &\quad t \longleftarrow t', \\
0 < \mathrm{Im}(t-t') < \beta, &\quad t' \longleftarrow t,
\end{aligned}
\tag{A.1.5}
$$

where $\theta_C(t-t')$ means that t' is before t on the path C and $t \longleftarrow t'$ means that t follows t' on the contour. Is also possible, following [38], to extend this till the boundary. With this we can constraint the range in the complex t-plane where the path C can go.
We could choose as a contour a straight vertical line that only goes through the imaginary part that corresponds to the complex-time formalism, but here we will focus on the real-time formalism. We can then choose the other path. For simplicity it is better to split the path in three parts C_1, C_2 and C_3 and it is possible to show that the generating functionals for C_1 and C_2 can be factorized out from the one describing C_3

$$
\lim_{t_f\to+\infty}\lim_{t_0\to-\infty} Z[j] = Z_{12}Z_m[j]. \tag{A.1.6}
$$

However, this happens in a non-trivial way by rewriting (and therefore regularizing) the delta distribution appearing in the Fourier transform of the propagator in a way such that the spectral density stays a meromorphic function. This trick changes the convergence behaviour in such a way that the factorization can be proven. We will analyze the example of the fermion field propagator, that is the one that we need for our work. It is important to emphasize that the factorization of equation (A.1.6) implies also that all convolutions of propagator from path C_1 or C_2 with C_3 vanish identically in the infinite time limit (as in [38]). Thus we can ignore them and focus on the paths C_1 and C_2. Indeed the physical external states will have propagators just coming from path C_1 since their time arguments are purely real. Furthermore, since all vertices will be diagonal in C_1 and C_2, the only thing we really need for our calculations is the propagator from the path C_1, so whenever we are talking about "the propagator" we will now mean the propagator from the path C_1.
We start by writing down the partition function for a fermion field with an additional chemical potential as

$$
Z[\bar{\xi},\xi] = \mathcal{N}\int_{-\psi}^{\psi}\exp\,[i\int_c d^4x(\mathcal{L} + \mu\bar{\psi}\gamma^0\psi + \bar{\xi}\psi + \bar{\psi}\xi)], \tag{A.1.7}
$$

with $\mathcal{L}$ the fermion Lagrangian. To find the Feynman propagator we can split the Lagrangian into a free and interacting part

$$\begin{aligned} Z[\bar{\xi},\xi] = Z_0 \exp\left[i \int d^4 y \mathcal{L}_{int}\left[\frac{\delta}{i\delta\bar{\xi}(y)}, \frac{\delta}{-i\delta\bar{\xi}(y)} \right]\right] \\ \times \exp\left[-i \int_c d^4 x \int_c d^4 x' \xi(t,x) G_0^{(c)}(t-t', x-x')\xi(t',x') \right], \end{aligned} \tag{A.1.8}$$

where

$$[\gamma_\mu (i\partial^\mu + \mu n^\mu) - m] G_0^c(t,\tilde{x}) = \delta_c^{(4)}, \tag{A.1.9}$$

with $n^\mu = (1,0,0,0)^T$ and the propagator can be written as

$$iG_0^{(c)}(t,x) \equiv \langle\langle T_c \hat{\psi}(t,x)\hat{\bar{\psi}}(0)\rangle\rangle_0 = \theta_c(t)\langle\langle \hat{\psi}(t,x)\hat{\bar{\psi}}(0)\rangle\rangle_0 - \theta_c(-t)\langle\langle \hat{\bar{\psi}}(0)\hat{\psi}(t,x)\rangle\rangle_0; \tag{A.1.10}$$

we can additionally notice that the two terms are not independent since

$$\langle\langle \hat{\bar{\psi}}(0)\hat{\psi}(t,x)\rangle\rangle_0 = \langle\langle \hat{\psi}(t-i\beta,x)\hat{\bar{\psi}}(0)\rangle\rangle_0. \tag{A.1.11}$$

The next step is to find an explicit expression for $iG_0^{(c)}(t,x)$, to do so we need the solutions for ψ and $\bar{\psi}$

$$\begin{aligned} \psi(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} (a_p^s u_p^s e^{-ipx} + b_p^{s\dagger} v_p^s e^{ipx}) e^{i\mu t}, \\ \psi(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} (a_p^{s\dagger} \bar{u}_p^s e^{ipx} + b_p^{s} \bar{v}_p^s e^{-ipx}) e^{-i\mu t}. \end{aligned} \tag{A.1.12}$$

Since we will need to do the thermal average, we notice that

$$\begin{aligned} &\langle\langle a_p^s a_k^{s'\dagger} \rangle\rangle = (2\pi)^3 \delta_{ss'} \delta^{(3)}(\vec{p}-\vec{k}) \bar{n}_f(\omega,\alpha), \quad \bar{n}_f \equiv 1 - n_f(\omega,\alpha) \\ &\langle\langle b_p^{s\dagger} b_k^{s'} \rangle\rangle = (2\pi)^3 \delta_{ss'} \delta^{(3)}(\vec{p}-\vec{k}) n_f(\omega,\alpha), \end{aligned} \tag{A.1.13}$$

with

$$n_f(\omega,\alpha) = \frac{1}{e^{\beta\omega-\alpha}+1}, \quad \bar{n}_f(\omega,\alpha) = \frac{1}{e^{-\beta\omega+\alpha}+1}, \quad \omega = p^0. \tag{A.1.14}$$

We will now work out the explicit form of the propagator.

A.2 Particle part of the propagator

We start with the thermal average

$$
\begin{aligned}
\langle\langle\psi(x)\bar{\psi}(0)\rangle\rangle =&\sum_{s,s'}\int \frac{d^3p d^3k}{(2\pi)^6\sqrt{2\omega_p}\sqrt{2\omega_k}}\langle\langle a_p^s a_k^{s'\dagger}\rangle\rangle u_p^s \bar{u}_k^{s'} e^{-ipx}e^{i\mu t}\\
=&\sum_{s,s'}\int \frac{d^3p d^3k}{(2\pi)^6\sqrt{2\omega_p}\sqrt{2\omega_k}}(2\pi)^3\delta_{ss'}\delta^{(3)}(\vec{p}-\vec{k})\bar{n}_f(\omega,\alpha)u_p^s \bar{u}_k^{s'} e^{-ipx}e^{i\mu t}\\
=&\sum_{s}\int \frac{d^3p}{(2\pi)^3 2\omega_p}\bar{n}_f(\omega,\alpha)u_p^s \bar{u}_k^{s'} e^{-ipx}e^{i\mu t}\\
=&\int \frac{d^3p}{(2\pi)^3 2\omega_p}(\not{p}+m)\bar{n}_f(\omega,\alpha)e^{-ipx}e^{i\mu t}\\
=&\int \frac{d^4p}{(2\pi)^4}(2\pi)\theta(p^0)\delta(p^2-m^2)(\not{p}+m)\bar{n}_f(\omega,\alpha)e^{-ipx}e^{i\mu t}\\
=&e^{i\mu t}(i\not{\partial}+m)\int \frac{d^4p}{(2\pi)^4}(2\pi)\theta(p^0)\delta(p^2-m^2)\bar{n}_f(\omega,\alpha)e^{-ipx},
\end{aligned}
\tag{A.2.1}
$$

and similarly

$$
\langle\langle\bar{\psi}(0)\psi(x)\rangle\rangle = e^{i\mu t}(i\not{\partial}+m)\int \frac{d^4p}{(2\pi)^4}(2\pi)\theta(p^0)\delta(p^2-m^2)n_f(\omega,\alpha)e^{-ipx}. \tag{A.2.2}
$$

Plugging it back in the particle propagator

$$
\begin{aligned}
iG_0^{\text{particle}}(x) &= e^{i\mu t}(i\not{\partial}+m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}(2\pi)\theta(p^0)\delta(p^2-m^2)[\theta_c(t)\bar{n}_f(\omega,\alpha)-\theta_c(-t)n_f(\omega,\alpha)]\\
&= e^{i\mu t}(i\not{\partial}+m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}(2\pi)\theta(p^0)\delta(p^2-m^2)[\theta_c(t)-n_f(\omega,\alpha)].
\end{aligned}
\tag{A.2.3}
$$

Let's analyze the first term

$$
\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4}e^{-ipx}(2\pi)\theta(p^0)\delta(p^2-m^2)\theta_c(t)\\
=&\int \frac{d^4p}{(2\pi)^4}e^{-ipx}(2\pi)\frac{1}{2\omega_p}\delta(p^0-\omega_p)\theta_c(t)\\
=&\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{1}{2\omega_p}i\left[\frac{1}{p^0-\omega_p+i\epsilon}-\frac{1}{p^0-\omega_p-i\epsilon}\right]\theta_c(t)\\
=&\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{1}{2\omega_p}i\left[\frac{1}{p^0-\omega_p+i\epsilon}\right],
\end{aligned}
\tag{A.2.4}
$$

where, remembering that the integration contour has to be closed in the lower half plane, the θ function cancels the second fraction. We can now use the identity

$$\lim_{\epsilon\to 0}\frac{1}{\omega\pm i\epsilon}=\mathcal{P}\frac{1}{\omega}\mp i\pi\delta(\omega). \tag{A.2.5}$$

With this we can rewrite the particle propagator as

$$\begin{aligned} iG_0^{\text{particle}}(x) &= e^{i\mu t}(i\partial\!\!\!/+m)\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\left[\frac{1}{2\omega_p}\left(\frac{i}{p^0-\omega_p+i\epsilon}\right)-(2\pi)\theta(p^0)\delta(p^2-m^2)n_f(\omega,\alpha)\right]\\ &= e^{i\mu t}\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\left[\frac{1}{2\omega_p}\left(\frac{i(p\!\!\!/+m)}{p^0-\omega_p+i\epsilon}\right)-(2\pi)\theta(p^0)(p\!\!\!/+m)\delta(p^2-m^2)n_f(\omega,\alpha)\right], \end{aligned} \tag{A.2.6}$$

thus, Fourier transforming in momentum space, we get

$$iG_0^{\text{particle}}(p)=\frac{1}{2\omega_p}\left(\frac{i(p\!\!\!/+m)}{p^0-\omega_p+i\epsilon}\right)-(2\pi)\theta(p^0)(p\!\!\!/+m)\delta(p^2-m^2)n_f(\omega,\alpha). \tag{A.2.7}$$

A.3 Anti-particle part of the propagator

We do the same for the anti-particle part

$$\begin{aligned} \langle\langle\psi(x)\bar\psi(0)\rangle\rangle &= \sum_{s,s'}\int\frac{d^3pd^3k}{(2\pi)^6\sqrt{2\omega_p}\sqrt{2\omega_k}}\langle\langle b_p^{s\dagger}b_k^{s'}\rangle\rangle v_p^s\bar v_k^{s'}e^{ipx}e^{i\mu t}\\ &= \sum_{s,s'}\int\frac{d^3pd^3k}{(2\pi)^6\sqrt{2\omega_p}\sqrt{2\omega_k}}(2\pi)^3\delta_{ss'}\delta^{(3)}(\vec p-\vec k)n_f(\omega,-\alpha)v_p^s\bar v_k^{s'}e^{ipx}e^{i\mu t}\\ &= \sum_{s}\int\frac{d^3p}{(2\pi)^3 2\omega_p}n_f(\omega,-\alpha)v_p^s\bar v_k^{s'}e^{ipx}e^{i\mu t}\\ &= \int\frac{d^3p}{(2\pi)^3 2\omega_p}(p\!\!\!/-m)n_f(\omega,-\alpha)e^{ipx}e^{i\mu t}\\ &= \int\frac{d^4p}{(2\pi)^4}(2\pi)\theta(p^0)\delta(p^2-m^2)(p\!\!\!/-m)n_f(\omega,-\alpha)e^{ipx}e^{i\mu t}\\ &= -\int\frac{d^4p}{(2\pi)^4}(2\pi)\theta(-p^0)\delta(p^2-m^2)(p\!\!\!/+m)n_f(-\omega,-\alpha)e^{-ipx}e^{i\mu t}\\ &= -e^{i\mu t}(i\partial\!\!\!/+m)\int\frac{d^4p}{(2\pi)^4}(2\pi)\theta(-p^0)\delta(p^2-m^2)\bar n_f(\omega,\alpha)e^{-ipx} \end{aligned} \tag{A.3.1}$$

and similarly

$$\langle\langle\bar\psi(0)\psi(x)\rangle\rangle=-e^{i\mu t}(i\partial\!\!\!/+m)\int\frac{d^4p}{(2\pi)^4}(2\pi)\theta(-p^0)\delta(p^2-m^2)n_f(\omega,\alpha)e^{-ipx}. \tag{A.3.2}$$

Plugging it back in the anti-particle propagator

$$\begin{aligned} iG_0^{\text{anti-particle}}(x) &= -e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}(2\pi)\theta(-p^0)\delta(p^2-m^2)[\theta_c(t)\bar{n}_f(\omega,\alpha) - \theta_c(-t)n_f(\omega,\alpha)] \\ &= e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}(2\pi)\theta(-p^0)\delta(p^2-m^2)[\theta_c(-t) - \bar{n}_f(\omega,\alpha)]. \end{aligned} \tag{A.3.3}$$

Let's analyze the first term in the same way as before

$$\begin{aligned} &\int \frac{d^4p}{(2\pi)^4}e^{-ipx}(2\pi)\theta(-p^0)\delta(p^2-m^2)\theta_c(-t) \\ =&\int \frac{d^4p}{(2\pi)^4}e^{-ipx}(2\pi)\frac{1}{2\omega_p}\delta(p^0+\omega_p)\theta_c(-t) \\ =&\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{1}{2\omega_p}i\left[\frac{1}{p^0+\omega_p+i\epsilon} - \frac{1}{p^0+\omega_p-i\epsilon}\right]\theta_c(-t) \\ =&\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{1}{2\omega_p}i\left[-\frac{1}{p^0+\omega_p-i\epsilon}\right]. \end{aligned} \tag{A.3.4}$$

With this we can rewrite the particle propagator as

$$\begin{aligned} iG_0^{\text{anti-particle}}(x) &= e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\left[\frac{1}{2\omega_p}\left(-\frac{i}{p^0+\omega_p-i\epsilon}\right) - (2\pi)\theta(-p^0)\delta(p^2-m^2)\bar{n}_f(\omega,\alpha)\right] \\ &= e^{i\mu t}\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\left[-\frac{1}{2\omega_p}\left(\frac{i(p\!\!\!/+m)}{p^0+\omega_p-i\epsilon}\right) - (2\pi)\theta(-p^0)(p\!\!\!/+m)\delta(p^2-m^2)\bar{n}_f(\omega,\alpha)\right], \end{aligned} \tag{A.3.5}$$

thus, in momentum space

$$iG_0^{\text{ant-particle}}(p) = -\frac{1}{2\omega_p}\left(\frac{i(p\!\!\!/+m)}{p^0+\omega_p-i\epsilon}\right) - (2\pi)\theta(-p^0)(p\!\!\!/+m)\delta(p^2-m^2)\bar{n}_f(\omega,\alpha). \tag{A.3.6}$$

A.4 Full propagator at finite density

Combining the results from the last two sections we find:

$$\begin{aligned} \langle\langle\psi(x)\bar{\psi}(0)\rangle\rangle =& e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}(2\pi)[\theta(p^0)-\theta(-p^0)]\delta(p^2-m^2)\bar{n}_f(\omega,\alpha)e^{-ipx} \\ =& e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}\rho(p)\bar{n}_f(\omega,\alpha)e^{-ipx} \end{aligned} \tag{A.4.1}$$

and

$$
\begin{aligned}
\langle\langle\bar{\psi}(0)\psi(x)\rangle\rangle =& e^{i\mu t}(i\partial\!\!\!/ + m)\int\frac{d^4p}{(2\pi)^4}(2\pi)[\theta(p^0)-\theta(-p^0)]\delta(p^2-m^2)n_f(\omega,\alpha)e^{-ipx} \\
=& e^{i\mu t}(i\partial\!\!\!/ + m)\int\frac{d^4p}{(2\pi)^4}\rho(p)n_f(\omega,\alpha)e^{-ipx},
\end{aligned}
\tag{A.4.2}
$$

having defined

$$
\rho(p)\equiv(2\pi)[\theta(p^0)-\theta(-p^0)]\delta(p^2-m^2). \tag{A.4.3}
$$

We can finally write the fermion propagator at finite density

$$
\begin{aligned}
iG_0^{(c)}(t,x) &= e^{i\mu t}(i\partial\!\!\!/ + m)\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\rho(p)[\theta_c(t)\bar{n}_f(\omega,\alpha)-\theta_c(-t)n_f(\omega,\alpha)] \\
&= e^{i\mu t}(i\partial\!\!\!/ + m)\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\rho(p)[\theta_c(t)-n_f(\omega,\alpha)],
\end{aligned}
\tag{A.4.4}
$$

that can be further manipulated using the identity (A.2.5) to rewrite ρ since

$$
\begin{aligned}
\rho_0(k) &= \frac{i}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\epsilon}-\frac{1}{k_0-\omega_k-i\epsilon}-\frac{1}{k_0+\omega_k+i\epsilon}+\frac{1}{k_0+\omega_k-i\epsilon}\right] \\
&= \frac{\pi}{\omega_k}[\delta_\epsilon(k_0-\omega_k)-\delta_\epsilon(k_0+\omega_k)] \\
&= 2\pi\mathrm{sgn}(k_0)\delta_\epsilon(k^2-m^2) \\
&= i[\theta(k_0)-\theta(-k_0)][\Delta_{0F}(k)-\Delta_{0F}^\dagger(k)],
\end{aligned}
\tag{A.4.5}
$$

so we get for the first term

$$
\begin{aligned}
&e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\rho(p)\theta_c(t)\\
=&e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{i}{2\omega_p}\left[\frac{1}{p_0-\omega_p+i\epsilon}-\frac{1}{p_0-\omega_p-i\epsilon}-\frac{1}{p_0+\omega_p+i\epsilon}+\frac{1}{p_0+\omega_p-i\epsilon}\right]\theta_c(t)\\
=&e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{i}{2\omega_p}\left[\frac{1}{p_0-\omega_p+i\epsilon}-\frac{1}{p_0+\omega_p+i\epsilon}\right]\theta_c(t)\\
=&e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{i}{2\omega_p}\left[\frac{2\omega_p}{p_0^2+\omega_p^2+2p_0i\epsilon}\right]\theta_c(t)\\
=&e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{i}{2\omega_p}\left[\frac{2\omega_p}{p_0^2-\omega_p^2+2p_0i\epsilon}\right]\theta_c(t)\\
=&e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}i\left[\frac{\theta(p^0)}{p_0^2-\omega_p^2+2p_0i\epsilon}+\frac{\theta(-p^0)}{p_0^2-\omega_p^2+2p_0i\epsilon}\right]\\
=&e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}i\left[\frac{\theta(p^0)}{p_0^2-\omega_p^2+2p_0i\epsilon}+\frac{\theta(-p^0)}{p_0^2-\omega_p^2+2p_0i\epsilon}\right][n_f(\omega,\alpha)+\bar{n}_f(\omega,\alpha)]\\
=&e^{i\mu t}(i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}e^{-ipx}i[\theta(p^0)\Delta_{0F}(p)+\theta(-p^0)\Delta_{0F}^\dagger(p)][n_f(\omega,\alpha)+\bar{n}_f(\omega,\alpha)],
\end{aligned}
\tag{A.4.6}
$$

and its Fourier transform reads

$$
\boxed{G_0^{(c)}(p) = G_{0F}(p) + 2\pi i(p\!\!\!/ + m)\delta_\epsilon(p^2-m^2)N_f(\omega,\alpha)}, \tag{A.4.7}
$$

where we defined

$$
\begin{aligned}
&N_f(\omega,\alpha) \equiv \theta(p^0)n_f(\omega,\alpha) + \theta(-p^0)\bar{n}_f(\omega,\alpha),\\
&\Delta_{0F}(k) = \frac{1}{k^2-m^2+i\epsilon}, \quad \Delta_{0F}^\dagger(k) = \frac{1}{k^2-m^2-i\epsilon},\\
&G_{0F}(k) = \frac{(k\!\!\!/ + m)}{k^2-m^2+i\epsilon}, \quad G_{0F}^\dagger(k) = \frac{(k\!\!\!/ + m)}{k^2-m^2-i\epsilon}
\end{aligned}
\tag{A.4.8}
$$

and used that

$$
[\Delta_{0F}(k) - \Delta_{0F}^\dagger(k)] = -i(2\pi)\delta_\epsilon(k^2-m^2). \tag{A.4.9}
$$

In the case of Supernova matter we can take the $T \to 0$ limit that simplifies the expression to

$$
\boxed{G^0(p) = (p\!\!\!/ + m)\left[\frac{1}{p^2-m^2+i\epsilon} + \frac{i\pi}{\omega_p}\delta(p^0-\omega_p)\theta(p^0)\theta(k_F-|\vec{p}|)\right]}, \tag{A.4.10}
$$

where we can see that the first term is the usual vacuum propagator and the second term is what is called either density insertion or density propagator. We can further notice that in this limit the density part of the anti-particle propagator vanishes as we can see from comparing (A.1.14) and (A.3.6) with (A.4.10). Here we are still in a relativistic framework, we need to take another limit to go to HBChPT, that is what we are going to do next.

In HBChPT we split our fermion field into an heavy and light component, so we just need to do the projection in the computations done right above, or we can simply do the projection at the very end, let us now see why.
We recall equation (5.6.6)

$$N_v = e^{imv\cdot x} P_{v+}\mathcal{N}, \quad H_v = e^{imv\cdot x} P_v\mathcal{N} \tag{A.4.11}$$

with $\psi(x) \equiv \mathcal{N}(x)$ from now on and our choice for $v^\mu = (1,0,0,0)^\top$. We can the substitute in (A.1.10) our $\mathcal{N}$

$$iG_0^{(c)}(t,x) = \langle\langle T_c\hat{\mathcal{N}}(x)\hat{\bar{\mathcal{N}}}(0)\rangle\rangle_0 = \theta_c(t)\langle\langle\hat{\mathcal{N}}(x)\hat{\bar{\mathcal{N}}}(0)\rangle\rangle_0 - \theta_c(-t)\langle\langle\hat{\bar{\mathcal{N}}}(0)\hat{\mathcal{N}}(x)\rangle\rangle_0 \tag{A.4.12}$$

and project to get the heavy and light components propagators (we remove the hat to simplify the notation)

$$\begin{aligned} iG_0^{\text{light}}(x) &= \theta_c(t)\langle\langle N(x)\bar{N}(0)\rangle\rangle_0 - \theta_c(-t)\langle\langle\bar{N}(0)N(x)\rangle\rangle_0, \\ iG_0^{\text{heavy}}(x) &= \theta_c(t)\langle\langle H(x)\bar{H}(0)\rangle\rangle_0 - \theta_c(-t)\langle\langle\bar{H}(0)H(x)\rangle\rangle_0. \end{aligned} \tag{A.4.13}$$

Since we are mainly interested in the light component field we will proceed with it but the computations are analogous for the heavy component. We need to relate it to the relativistic fermion

$$\langle\langle N(x)\bar{N}(0)\rangle\rangle = \langle\langle e^{imt}P_+\mathcal{N}(x)\bar{\mathcal{N}}(0)P_+e^{-imt}\rangle\rangle = P_+\langle\langle\mathcal{N}(x)\bar{\mathcal{N}}(0)\rangle\rangle P_+ \tag{A.4.14}$$

and the last step is correct because the projectors act on the spinors and they commute with the thermal average. The same holds for

$$\langle\langle N(0)\bar{N}(x)\rangle\rangle = \langle\langle e^{imt}P_+\mathcal{N}(0)\bar{\mathcal{N}}(x)P_+e^{-imt}\rangle\rangle = P_+\langle\langle\mathcal{N}(0)\bar{\mathcal{N}}(x)\rangle\rangle P_+ \tag{A.4.15}$$

It is now straightforward to write the HBChPT propagators

$$\begin{aligned} iG_0^{\text{light}}(p) &= P_+iG_0(p)P_+ = \frac{i}{l^0+i\epsilon} - 2\pi\delta(l^0)\theta(p^0)\theta(k_f-|\vec{l}|) + \mathcal{O}(\frac{1}{m}), \\ iG_0^{\text{heavy}}(p) &= P_-iG_0(p)P_- = -\frac{i}{2m} + \pi\frac{l^0}{m}\delta(l^0)\theta(p^0)\theta(k_f-|\vec{l}|) + \mathcal{O}(\frac{1}{m^2}), \end{aligned} \tag{A.4.16}$$

where we used the usual decomposition $p^\mu = mv^\mu + l^\mu$. We see that $iG_0^{\text{heavy}}(p)$ is suppressed by one more power of m and that we get what we expected for $iG_0^{\text{light}}(p)$.

Appendix B

Density loops

Here we analyze how to solve loop integrals with the finite density part of the nucleon propagator. For a fermion tadpole we get

$$(-)\int \frac{d^4k}{(2\pi)^4}(-)2\pi\delta(k^0)\Theta(k_f-|\vec{k}|) = \int_{|\vec{k}|\leq k_f}\frac{d^3\vec{k}}{(2\pi)^3} = \frac{4\pi}{(2\pi)^3}\frac{k_f^3}{3} = \frac{\rho}{4}, \qquad \rho \equiv \frac{2k_f^3}{3\pi^2}. \tag{B.0.1}$$

In the Lagrangian sometimes the terms which can be contracted to a non-vanishing loop at finite density could appear as $\bar{N}N = \bar{N}(\mathbb{I}^{\text{Spin}}_{2\times 2}\otimes\mathbb{I}^{\text{ISO}}_{2\times 2})N = \bar{N}(\mathbb{I}_{4\times 4})N$. This results in an additional factor of 4 from the trace:

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\langle 0|\bar{N}_i(k)\mathbb{I}_{ij}N_j(k)|0\rangle \\
=& -\int \frac{d^4k}{(2\pi)^4}\langle 0|N_i(k)\mathbb{I}_{ij}\bar{N}_i(k)|0\rangle \\
=& -\int \frac{d^4k}{(2\pi)^4}(-)\delta_{ii}2\pi\delta(k^0)\Theta(k_f-|\vec{k}|) \\
=& 4\int \frac{d^4k}{(2\pi)^4}2\pi\delta(k^0)\Theta(k_f-|\vec{k}|) \\
=& 4\frac{4\pi}{(2\pi)^3}\frac{k_f^3}{3} = \rho.
\end{aligned} \tag{B.0.2}$$

Appendix C

Useful integrals

Here we report the explicit form of the integrals that appear in our computations.

C.1 Density integrals

We define

$$\Gamma_0(p) = \int_0^{k_f} dk \int_{-1}^{1} dx \frac{k^2}{m_\pi^2 + p^2 + k^2 + 2pkx},$$
$$\Gamma_1(p) = \int_0^{k_f} dk \int_{-1}^{1} dx \frac{k^3 x/p}{m_\pi^2 + p^2 + k^2 + 2pkx},$$
$$\Gamma_2(p) = \int_0^{k_f} dk \int_{-1}^{1} dx \frac{k^4 \left(1 - x^2\right)/2}{m_\pi^2 + p^2 + k^2 + 2pkx},$$
$$\Gamma_3(p) = \int_0^{k_f} dk \int_{-1}^{1} dx \frac{k^4 \left(3x^2 - 1\right)/\left(2p^2\right)}{m_\pi^2 + p^2 + k^2 + 2pkx},$$

and

$$\tilde{\Gamma}_0(p) = \int_0^{k_f} dk \int_{-1}^{1} dx \frac{k^2}{\tilde{m}_\pi^2 + p^2 + k^2 + 2pkx},$$
$$\tilde{\Gamma}_1(p) = \int_0^{k_f} dk \int_{-1}^{1} dx \frac{k^3 x/p}{\tilde{m}_\pi^2 + p^2 + k^2 + 2pkx},$$
$$\tilde{\Gamma}_2(p) = \int_0^{k_f} dk \int_{-1}^{1} dx \frac{k^4 \left(1 - x^2\right)/2}{\tilde{m}_\pi^2 + p^2 + k^2 + 2pkx},$$
$$\tilde{\Gamma}_3(p) = \int_0^{k_f} dk \int_{-1}^{1} dx \frac{k^4 \left(3x^2 - 1\right)/\left(2p^2\right)}{\tilde{m}_\pi^2 + p^2 + k^2 + 2pkx},$$

with

$$\tilde{m}_\pi^2 = m_\pi^2 - \left(q^0\right)^2.$$

There are three combinations of those integrals that appear, let's begin with

$$I_1(\sigma,p,q)=\int_{|\vec{k}|<k_f}\frac{d^3k}{(2\pi)^3}\frac{\vec{\sigma}\cdot(\vec{k}+\vec{p})\vec{q}\cdot(\vec{k}+\vec{p})}{m_\pi^2+(\vec{k}+\vec{p})^2}$$
$$=\frac{1}{(2\pi)^3}\int_0^{k_f}\int_{-1}^{1}\int_0^{2\pi}dkdxd\phi k^2\frac{\vec{\sigma}\cdot(\vec{k}+\vec{p})\vec{q}\cdot(\vec{k}+\vec{p})}{m_\pi^2+p^2+k^2+2pkx}$$
$$=\frac{1}{(2\pi)^3}\int_0^{k_f}\int_{-1}^{1}\int_0^{2\pi}dkdxd\phi k^2\frac{(\vec{\sigma}\cdot\vec{k})(\vec{q}\cdot\vec{k})+(\vec{\sigma}\cdot\vec{k})(\vec{q}\cdot\vec{p})+(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{k})+(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})}{m_\pi^2+p^2+k^2+2pkx}.$$

Using that some of the $d\phi$ integrals vanish we can simplify the numerator

$$\int_0^{2\pi}d\phi(\vec{\sigma}\cdot\vec{k})(\vec{q}\cdot\vec{k})=\int_0^{2\pi}d\phi\left[(\sigma_xk_x)(q_xk_x)+(\sigma_yk_y)(q_yk_y)+(\sigma_zk_z)(q_zk_z)\right]$$
$$=\int_0^{2\pi}d\phi\left[\sigma_xq_xk^2\left(1-x^2\right)\cos^2\phi+\sigma_xq_xk^2\left(1-x^2\right)\sin^2\phi+\sigma_zq_zk^2x^2\right]$$
$$=\sigma_xq_xk^2\left(1-x^2\right)\pi+\sigma_yq_yk^2\left(1-x^2\right)\pi+\sigma_zq_zk^2x^22\pi$$
$$=\sigma_xq_xk^2\left(1-x^2\right)\pi+\sigma_yq_yk^2\left(1-x^2\right)\pi+\sigma_zq_zk^2\left(1-x^2\right)\pi+\sigma_zq_zk^2\pi\left(3x^2-1\right)$$
$$=\vec{\sigma}\cdot\vec{q}k^2\left(1-x^2\right)\pi+\vec{\sigma}\cdot\vec{p}\vec{q}\cdot\vec{p}\frac{k^2\pi\left(3x^2-1\right)}{p^2},$$

$$\int_0^{2\pi}d\phi[(\vec{\sigma}\cdot\vec{k})(\vec{q}\cdot\vec{p})+(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{k})]=\int_0^{2\pi}d\phi kx\frac{(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})+(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})}{p}$$
$$=\int_0^{2\pi}d\phi 2kx\frac{(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})}{p}$$
$$=4\pi kx\frac{(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})}{p}.$$

Therefore we are left with

$$I_1(\sigma,p,q)=\frac{1}{4\pi^2}\int_0^{k_f}\int_{-1}^{1}dkdx\frac{(\vec{\sigma}\cdot\vec{q})\frac{k^4\left(1-x^2\right)}{2}+(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})\frac{k^4\left(3x^2-1\right)}{2p^2}+\left(\frac{2k^3x}{p}+k^2\right)(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})}{m_\pi^2+p^2+k^2+2pkx}$$
$$=\frac{1}{4\pi^2}\left[(\vec{\sigma}\cdot\vec{q})\Gamma_2(p)+(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})\left(\Gamma_0(p)+2\Gamma_1(p)+\Gamma_3(p)\right)\right].$$

The same procedure can be done to show explicitly that for $I_1(\sigma,p+q,q)$ one indeed just has to replace $\vec{p}$ with $\vec{p}+\vec{q}$. So to summarize we found

$$\boxed{I_1(\sigma,p,q)=\frac{1}{4\pi^2}\left[(\vec{\sigma}\cdot\vec{q})\Gamma_2(p)+(\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})\left(\Gamma_0(p)+2\Gamma_1(p)+\Gamma_3(p)\right)\right]},\qquad\text{(C.1.1)}$$

$$\boxed{I_1(\sigma,p+q,q)=\frac{1}{4\pi^2}\left[(\vec{\sigma}\cdot\vec{q})\Gamma_2(p+q)+\vec{\sigma}\cdot(\vec{p}+\vec{q})\vec{q}\cdot(\vec{p}+\vec{q})\left(\Gamma_0(p+q)+2\Gamma_1(p+q)+\Gamma_3(p+q)\right)\right].}\qquad\text{(C.1.2)}$$

For the case where the mass in the integral is replaced by $\tilde{m}_\pi^2 = m_\pi^2 - \left(q^0\right)^2$ we define the integrals

$$\begin{aligned}\tilde{I}_1(\sigma,p,q) &= \int_{|\vec{k}|<k_f} \frac{d^3k}{(2\pi)^3} \frac{\vec{\sigma}\cdot(\vec{k}+\vec{p})\vec{q}\cdot(\vec{k}+\vec{p})}{\tilde{m}_\pi^2 + (\vec{k}+\vec{p})^2}, \\ \tilde{I}_1(\sigma,p+q,q) &= \int_{|\vec{k}|<k_f} \frac{d^3k}{(2\pi)^3} \frac{\vec{\sigma}\cdot(\vec{k}+\vec{p}+\vec{q})\vec{q}\cdot(\vec{k}+\vec{p}+\vec{q})}{\tilde{m}_\pi^2 + (\vec{k}+\vec{p}+\vec{q})^2},\end{aligned}$$

which evaluate to

$$\boxed{\tilde{I}_1(\sigma,p,q) = \frac{1}{4\pi^2}\left[(\vec{\sigma}\cdot\vec{q})\tilde{\Gamma}_2(p) + (\vec{\sigma}\cdot\vec{p})(\vec{q}\cdot\vec{p})\left(\tilde{\Gamma}_0(p) + 2\tilde{\Gamma}_1(p) + \tilde{\Gamma}_3(p)\right)\right]}, \tag{C.1.3}$$

$$\boxed{\tilde{I}_1(\sigma,p+q,q) = \frac{1}{4\pi^2}\left[(\vec{\sigma}\cdot\vec{q})\tilde{\Gamma}_2(p+q) + \vec{\sigma}\cdot(\vec{p}+\vec{q})\vec{q}\cdot(\vec{p}+\vec{q})\left(\tilde{\Gamma}_0(p+q) + 2\tilde{\Gamma}_1(p+q) + \tilde{\Gamma}_3(p+q)\right)\right]}. \tag{C.1.4}$$

Next we analyze

$$\begin{aligned}I_2(p) &= \int_{|\vec{k}|<k_f} \frac{d^3k}{(2\pi)^3} \frac{(\vec{k}+\vec{p})\cdot(\vec{k}+\vec{p})}{m_\pi^2 + (\vec{k}+\vec{p})^2} \\ &= \int_{|\vec{k}|<k_f} \frac{d^3k}{(2\pi)^3} \left[\frac{m_\pi^2 + (\vec{k}+\vec{p})^2}{m_\pi^2 + (\vec{k}+\vec{p})^2} - \frac{m_\pi^2}{m_\pi^2 + (\vec{k}+\vec{p})^2}\right] \\ &= \int_{|\vec{k}|<k_f} \frac{d^3k}{(2\pi)^3} \left[1 - \frac{m_\pi^2}{m_\pi^2 + (\vec{k}+\vec{p})^2}\right] \\ &= \frac{1}{(2\pi)^3}\frac{k_f^3}{3}4\pi - m_\pi^2\Gamma_0(p)\frac{1}{4\pi^2} \\ &= \frac{1}{4\pi^2}\left(\frac{2k_f^3}{3} - m_\pi^2\Gamma_0(p)\right).\end{aligned}$$

Analogous for

$$\tilde{I}_2(p) = \int_{|\vec{k}|<k_f} \frac{d^3k}{(2\pi)^3} \frac{(\vec{k}+\vec{p})\cdot(\vec{k}+\vec{p})}{\tilde{m}_\pi^2 + (\vec{k}+\vec{p})^2}.$$

So we summarize

$$\boxed{I_2(p) = \frac{1}{4\pi^2}\left(\frac{2k_f^3}{3} - m_\pi^2\Gamma_0(p)\right)}, \tag{C.1.5}$$

$$\boxed{\tilde{I}_2(p) = \frac{1}{4\pi^2}\left(\frac{2k_f^3}{3} - \tilde{m}_\pi^2\tilde{\Gamma}_0(p)\right)}. \tag{C.1.6}$$

Another important integral is

$$I_3(\sigma,p) = \int_{|\vec{k}|<k_f} \frac{d^3k}{(2\pi)^3} \frac{\vec{\sigma}\cdot(\vec{k}+\vec{p})}{m_\pi^2+(\vec{k}+\vec{p})^2},$$
$$\tilde{I}_3(\sigma,p) = \int_{|\vec{k}|<k_f} \frac{d^3k}{(2\pi)^3} \frac{\vec{\sigma}\cdot(\vec{k}+\vec{p})}{\tilde{m}_\pi^2+(\vec{k}+\vec{p})^2},$$

which evaluates to

$$\boxed{I_3(\sigma,p) = \vec{\sigma}\cdot\vec{p}\frac{1}{4\pi^2}\left(\Gamma_1(p)+\Gamma_0(p)\right)}, \tag{C.1.7}$$

$$\boxed{\tilde{I}_3(\sigma,p) = \vec{\sigma}\cdot\vec{p}\frac{1}{4\pi^2}\left(\tilde{\Gamma}_1(p)+\tilde{\Gamma}_0(p)\right)}. \tag{C.1.8}$$

C.2 Vacuum integrals

As in [43] we define

$$\Delta_\pi = -\frac{1}{i}\int \frac{d^dk}{(2\pi)^2}\frac{1}{k^2-m_\pi^2+i\epsilon} = 2m_\pi^2\left(L(\lambda)+\frac{1}{(4\pi)^2}\ln\frac{m_\pi}{\lambda}\right)+\mathcal{O}(d-4), \tag{C.2.1}$$

$$\frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\frac{\{1,k_\mu,k_\mu k_\nu\}}{(k^2-m_\pi^2+i\epsilon)(\omega-v\cdot k+i\epsilon)} = \{J_0(\omega), v_\mu J_1(\omega), g_{\mu\nu}J_2(\omega)+v_\mu v_\nu J_3(\omega)\}, \tag{C.2.2}$$

where

$$L(\lambda) = \frac{\lambda^{d-4}}{(4\pi)^2}\left(\frac{1}{d-4}-\frac{1}{2}[\ln 4\pi+\Gamma'(1)+1]\right) \tag{C.2.3}$$

and

$$\begin{aligned}
J_0(\omega) &= -4\omega L + \frac{2\omega}{(4\pi)^2}\left(1-2\ln\frac{m_\pi}{\lambda}\right) - \frac{1}{4\pi^2}\sqrt{m_\pi^2-\omega^2}\arccos\frac{-\omega}{m_\pi}+\mathcal{O}(d-4),\\
J_1(\omega) &= \omega J_0(\omega)+\Delta_\pi,\\
J_2(\omega) &= \frac{1}{d-1}[(m_\pi^2-\omega^2)J_0(\omega)-\omega\Delta_\pi],\\
J_3(\omega) &= \omega J_1(\omega)-J_2(\omega).
\end{aligned} \tag{C.2.4}$$

We can notice that the expression for $J_0(\omega)$ is valid for $\omega < m_\pi$ that is the region we are interested in since we can stay in the reference frame of the outgoing nucleon. Even if we are not in the nucleon reference frame and we encounter $v\cdot(p+q)$, this would still be $\ll m_\pi$ since we have a small momentum propagating in the scattering.
Another integral often appearing is

$$\frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu}{(k^2-m_\pi^2+i\epsilon)}\frac{1}{(\omega'-v\cdot k+i\epsilon)(\omega-v\cdot k+i\epsilon)}. \tag{C.2.5}$$

This can be rewritten using the following identity

$$\frac{1}{(\omega' - v\cdot k + i\epsilon)(\omega - v\cdot k + i\epsilon)} = \frac{1}{\omega' - \omega}\left[\frac{1}{\omega - v\cdot k + i\epsilon} - \frac{1}{\omega' - v\cdot k + i\epsilon}\right], \tag{C.2.6}$$

as

$$\begin{aligned}&\frac{1}{\omega' - \omega}\left(\frac{1}{i}\int \frac{d^d k}{(2\pi)^d}\frac{k_\mu k_\nu}{(k^2 - m_\pi^2 + i\epsilon)(\omega - v\cdot k + i\epsilon)} - \omega \to \omega'\right)\\ &\frac{1}{\omega' - \omega}(g_{\mu\nu}[J_2(\omega) - J_2(\omega')] + v_\mu v_\nu[J_3(\omega) - J_3(\omega')]).\end{aligned} \tag{C.2.7}$$

We can define a scale dependent and divergent quantity as

$$\Lambda(\lambda) = 2m_\pi^2\left(L(\lambda) + \frac{1}{(4\pi)^2}\ln\frac{m_\pi}{\lambda}\right), \tag{C.2.8}$$

and report a combination of functions that will often appear in the following

$$J_2(\omega) \pm J_2(0) = \frac{1}{d-1}\left[(m_\pi^2 - \omega^2)\left(\frac{\omega}{8\pi^2} - \frac{1}{4\pi^2}\sqrt{m_\pi^2 - \omega^2}\arccos\left(-\frac{\omega}{m_\pi}\right)\right) \pm \frac{m_\pi^3}{8\pi} - (6m_\pi^2\omega - 4\omega^3)\Lambda(\lambda)\right], \tag{C.2.9}$$

that in $d = 4$ reads

$$J_2(\omega) \pm J_2(0) = \frac{1}{3}\left[(m_\pi^2 - \omega^2)\left(\frac{\omega}{8\pi^2} - \frac{1}{4\pi^2}\sqrt{m_\pi^2 - \omega^2}\arccos\left(-\frac{\omega}{m_\pi}\right)\right) \pm \frac{m_\pi^3}{8\pi} - (6m_\pi^2\omega - 4\omega^3)\Lambda(\lambda)\right]. \tag{C.2.10}$$

Bibliography

[1] Enrico Cappellaro. Supernovae and their cosmological implications. *Nuovo Cimento Rivista Serie*, 45(8):549–586, August 2022. doi: 10.1007/s40766-022-00034-1.

[2] Guido D'Amico, Alessandro Strumia, Alfredo Urbano, and Wei Xue. Direct anthropic bound on the weak scale from supernovæ explosions. *Phys. Rev. D*, 100(8): 083013, 2019. doi: 10.1103/PhysRevD.100.083013.

[3] Steen Hannestad and Georg Raffelt. Supernova Neutrino Opacity from Nucleon-Nucleon Bremsstrahlung and Related Processes. , 507(1):339–352, November 1998. doi: 10.1086/306303.

[4] Hannah Yasin. *Core-collapse supernovae and the equation of state.* PhD thesis, Technische Universität, Darmstadt, 2019. URL `http://tuprints.ulb.tu-darmstadt.de/8840/`.

[5] Tobias Melson, Hans-Thomas Janka, Robert Bollig, Florian Hanke, Andreas Marek, and Bernhard Mueller. Neutrino-driven explosion of a 20 solar-mass star in three dimensions enabled by strange-quark contributions to neutrino-nucleon scattering. *The Astrophysical Journal*, 808, 04 2015. doi: 10.1088/2041-8205/808/2/L42.

[6] Sanjay Reddy, Madappa Prakash, James M. Lattimer, and Jose A. Pons. Effects of strong and electromagnetic correlations on neutrino interactions in dense matter. *Phys. Rev. C*, 59:2888–2918, 1999. doi: 10.1103/PhysRevC.59.2888.

[7] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model.* Cambridge University Press, 3 2014. ISBN 978-1-107-03473-0, 978-1-107-03473-0.

[8] Giuliano Panico and Andrea Wulzer. *The Composite Nambu-Goldstone Higgs.* Springer International Publishing, 2016. doi: 10.1007/978-3-319-22617-0. URL `https://doi.org/10.1007%2F978-3-319-22617-0`.

[9] Mark Srednicki. *Quantum Field Theory.* Cambridge University Press, 2007. doi: 10.1017/CBO9780511813917.

[10] H. Georgi. *Weak Interactions and Modern Particle Theory.* 1984. ISBN 978-0-8053-3163-9.

[11] Stefan Scherer. Introduction to chiral perturbation theory. *Adv. Nucl. Phys.*, 27: 277, 2003.

[12] Nadia Fettes, Ulf-G. Meißner, Martin Mojžiš, and Sven Steininger. The chiral effective pion-nucleon lagrangian of order p4. *Annals of Physics*, 283(2):273–307, aug 2000. doi: 10.1006/aphy.2000.6059. URL `https://doi.org/10.1006%2Faphy.2000.6059`.

[13] Nadia Fettes, Ulf-G. Meissner, and Sven Steininger. Pion - nucleon scattering in chiral perturbation theory. 1. Isospin symmetric case. *Nucl. Phys. A*, 640:199–234, 1998. doi: 10.1016/S0375-9474(98)00452-7.

[14] E. Epelbaum. The nucleon nucleon interaction in a chiral effective field theory. Other thesis, 7 2000.

[15] E. Epelbaum, A. Nogga, Walter Gloeckle, H. Kamada, Ulf G. Meissner, and H. Witala. Three nucleon forces from chiral effective field theory. *Phys. Rev. C*, 66: 064001, 2002. doi: 10.1103/PhysRevC.66.064001.

[16] E. Epelbaum, H.-W. Hammer, and Ulf-G. Meißner. Modern theory of nuclear forces. *Reviews of Modern Physics*, 81(4):1773–1825, dec 2009. doi: 10.1103/revmodphys.81.1773. URL `https://doi.org/10.1103%2Frevmodphys.81.1773`.

[17] Steven Weinberg. Phenomenological Lagrangians. *Physica A*, 96(1-2):327–340, 1979. doi: 10.1016/0378-4371(79)90223-1.

[18] Steven Weinberg. Nuclear forces from chiral Lagrangians. *Phys. Lett. B*, 251:288–292, 1990. doi: 10.1016/0370-2693(90)90938-3.

[19] Steven Weinberg. Effective chiral Lagrangians for nucleon - pion interactions and nuclear forces. *Nucl. Phys. B*, 363:3–18, 1991. doi: 10.1016/0550-3213(91)90231-L.

[20] Steven Weinberg. Three body interactions among nucleons and pions. *Phys. Lett. B*, 295:114–121, 1992. doi: 10.1016/0370-2693(92)90099-P.

[21] Hermann Krebs. Nuclear Currents in Chiral Effective Field Theory. *Eur. Phys. J. A*, 56(9):234, 2020. doi: 10.1140/epja/s10050-020-00230-9.

[22] E. Epelbaum. Four-nucleon force using the method of unitary transformation. *Eur. Phys. J. A*, 34:197–214, 2007. doi: 10.1140/epja/i2007-10496-0.

[23] U. van Kolck. Few nucleon forces from chiral Lagrangians. *Phys. Rev. C*, 49:2932–2941, 1994. doi: 10.1103/PhysRevC.49.2932.

[24] N. Kaiser and V. Niessner. Density-dependent NN-interaction from subleading chiral 3N-forces: short-range terms and relativistic corrections. *Phys. Rev. C*, 98 (5):054002, 2018. doi: 10.1103/PhysRevC.98.054002.

[25] V. Bernard, E. Epelbaum, H. Krebs, and Ulf-G. Meissner. Subleading contributions to the chiral three-nucleon force. I. Long-range terms. *Phys. Rev. C*, 77:064004, 2008. doi: 10.1103/PhysRevC.77.064004.

[26] V. Bernard, E. Epelbaum, H. Krebs, and U. G. Meissner. Subleading contributions to the chiral three-nucleon force II: Short-range terms and relativistic corrections. *Phys. Rev. C*, 84:054001, 2011. doi: 10.1103/PhysRevC.84.054001.

[27] V. Bernard, Norbert Kaiser, and Ulf-G. Meissner. Chiral dynamics in nucleons and nuclei. *Int. J. Mod. Phys. E*, 4:193–346, 1995. doi: 10.1142/S0218301395000092.

[28] U G Meissner. Chiral perturbation theory with nucleons. 9 1991. doi: 10.2172/10107296. URL `https://www.osti.gov/biblio/10107296`.

[29] G. Ecker. Chiral invariant renormalization of the pion-nucleon interaction. *Physics Letters B*, 336(3-4):508–517, sep 1994. doi: 10.1016/0370-2693(94)90565-7. URL `https://doi.org/10.1016%2F0370-2693%2894%2990565-7`.

[30] Ulf-G. Meißner, Guido Müller, and Sven Steininger. Renormalization of the chiral pion–nucleon lagrangian beyond next-to-leading order. *Annals of Physics*, 279(1): 1–64, jan 2000. doi: 10.1006/aphy.1999.5919. URL `https://doi.org/10.1006%2Faphy.1999.5919`.

[31] Jeremy W. Holt, Mannque Rho, and Wolfram Weise. Chiral symmetry and effective field theories for hadronic, nuclear and stellar matter. *Phys. Rept.*, 621:2–75, 2016. doi: 10.1016/j.physrep.2015.10.011.

[32] A. Dedes, W. Materkowska, M. Paraskevas, J. Rosiek, and K. Suxho. Feynman rules for the standard model effective field theory in r -gauges. *Journal of High Energy Physics*, 2017(6), jun 2017. doi: 10.1007/jhep06(2017)143. URL `https://doi.org/10.1007%2Fjhep06%282017%29143`.

[33] Alexander Bartl. *Neutrino Interactions with Supernova Matter*. PhD thesis, Darmstadt, Tech. U., 7 2016.

[34] J. Mené ndez, D. Gazit, and A. Schwenk. Chiral two-body currents in nuclei: Gamow-teller transitions and neutrinoless double-beta decay. *Physical Review Letters*, 107(6), aug 2011. doi: 10.1103/physrevlett.107.062501. URL `https://doi.org/10.1103%2Fphysrevlett.107.062501`.

[35] Nadia Fettes and Ulf-G. Meißner. Complete analysis of pion–nucleon scattering in chiral perturbation theory to third order. *Nuclear Physics A*, 693(3-4):693–709, oct 2001. doi: 10.1016/s0375-9474(01)00977-0. URL `https://doi.org/10.1016%2Fs0375-9474%2801%2900977-0`.

[36] Reuven Balkin, Javi Serra, Konstantin Springmann, and Andreas Weiler. The QCD axion at finite density. *JHEP*, 07:221, 2020. doi: 10.1007/JHEP07(2020)221.

[37] Paulo Bedaque, Sanjay Reddy, Srimoyee Sen, and Neill Warrington. Neutrino-nucleon scattering in the neutrino-sphere. *Physical Review C*, 98, 01 2018. doi: 10.1103/PhysRevC.98.015802.

[38] R. J. Furnstahl and Brian D. Serot. Covariant feynman rules at finite temperature: Time-path formulation. *Phys. Rev. C*, 44:2141–2174, Nov 1991. doi: 10.1103/PhysRevC.44.2141. URL `https://link.aps.org/doi/10.1103/PhysRevC.44.2141`.

[39] Mariano Quiros. Finite temperature field theory and phase transitions. In *ICTP Summer School in High-Energy Physics and Cosmology*, pages 187–259, 1 1999.

[40] S. Stelzl K. Springmann, M. Stadlbauer and A. Weiler Meißner. The qcd-axion–nucleon coupling to n2lo at finite density. To appear.

[41] Pierluca Carenza, Tobias Fischer, Maurizio Giannotti, Gang Guo, Gabriel Martínez-Pinedo, and Alessandro Mirizzi. Improved axion emissivity from a supernova via nucleon-nucleon bremsstrahlung. *JCAP*, 10(10):016, 2019. doi: 10.1088/1475-7516/2019/10/016. [Erratum: JCAP 05, E01 (2020)].

[42] Thomas Vonk, Feng-Kun Guo, and Ulf-G. Meißner. The axion-baryon coupling in SU(3) heavy baryon chiral perturbation theory. *JHEP*, 08:024, 2021. doi: 10.1007/JHEP08(2021)024.

[43] Thomas Vonk, Feng-Kun Guo, and Ulf-G. Meißner. Precision calculation of the axion-nucleon coupling in chiral perturbation theory. *Journal of High Energy Physics*, 2020(3), mar 2020. doi: 10.1007/jhep03(2020)138. URL `https://doi.org/10.1007%2Fjhep03%282020%29138`.