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The surjectivity of the period map

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Abstract

Consider a smooth closed oriented four-manifold, its period map assigns to a Riemannian metric the space of self-dual harmonic 2-forms. This map is from the space of metrics to the Grassmannian of maximal positive subspaces in the second cohomology, where positivity is defined by the cup product.

In this thesis, following "Metric stretching and the period map for smooth 4-manifolds" by Scaduto we show that the period map has a dense image for every four-manifold, and that it is surjective if $b^+ = 1$. We also show surjectivity for manifolds which admit Ricci-flat metrics (four-torus and $K3$ surfaces). The proofs here involve, on one hand, the construction of families of metrics constructed by stretching along various hypersurfaces and, on the other, the properties of a specific subset of metrics on the manifolds considered.

Contents

Introduction

Ever since the publication of Noether's theorem in 1918, a fundamental goal in mathematical physics has been to find symmetries in the action of a physical system. This theorem indeed relates symmetries in a theory to a physical invariant. To generalise this result to a global theory and give it a sound mathematical background, gauge theory was introduced. From this, mathematicians started to study mathematical gauge theory as a branch of differential geometry and topology of its own right.

For physical reasons, gauge theory pays particular attention to four-manifolds and at the same time, gauge theory methods are used to study topological properties of four-manifolds, in particular by defining their invariants. Gauge theory was also used to study the peculiar characteristics of four-manifolds.

In particular, ideas from Yang-Mills theory shaped the direction of the research in geometry at the end of last century. This theory focuses on self-dual and anti-self-dual 2-forms with respect to the Hodge star operator, since the critical points of the Yang-Mills functional are when the corresponding curvature is (anti-)self-dual, also called instantons.

The Hodge star operator and, thus, duality, depend on the choice of metric on the four-manifold. The period map considers smooth, closed, connected, oriented four-manifolds and associates to a Riemannian metric the subspace of self-dual 2-forms in the second cohomology. The period map and the question of its surjectivity arose from the definition of invariants of four-manifolds in the context of gauge theory. In particular several proofs needed to construct elements with respect to their space of (anti-)self-dual forms and to be well-defined this construction needs surjectivity.

This thesis will illustrate and discuss the question of the surjectivity of the period map. The starting point and main focus of this thesis is "Metric stretching and the period map for smooth 4-manifolds" by Christopher Scaduto [Sca23]. Specifically, we will start by exploring the origins of the period map in the literature. Then we will continue by explaining the proof of density of the image in general and of the surjectivity for $b^+ = 1$ as done in Scaduto's paper. We will conclude by expanding the proof of surjectivity to other examples not already covered. This topic was particularly interesting for me, as it provided an opportunity to explore and apply various mathematical concepts and methodologies.

Before Scaduto's paper, there was no general proof of the surjectivity of the period map, even though some proofs of the surjectivity or density of its image under specific circumstances appeared in the literature. The problem of the surjectivity of the period map was introduced as an hypothesis to be used in the proof of some theorems. Indeed, some authors encountered this problem in their proofs, but in those cases, either they went around the problem without relying on the surjectivity or conjured it as a hypothesis and left it as an open question. These approaches will be discussed more in-depth later in this thesis.

In Chapter 1, we introduce the problem of the surjectivity of the period map and provide a comprehensive overview of its state of the art. Here lies the ground for the whole discussion by giving the main definitions used in this thesis. We will conclude by discussing the origins of the problem of surjectivity and how it was treated in previous articles.

In Chapter 2, we will focus to explain and add some context to Scaduto's paper, following its original structure.

In Chapter 4, we expand the proof of surjectivity of the period map focusing on four-manifolds which admit Ricci-flat metrics, which are the four-torus T^4 and $K3$ surfaces. We then use some results from complex geometry to extend the surjectivity of the period map for all minimal compact complex surfaces and we explain why it is not possible to do the same for non-minimal ones.

Keep in mind that the choices made regarding which pieces of background material to include and which to leave out reflect the prior knowledge of the author. Thus, we do not assume much background in topology and Riemannian geometry beyond an introductory course, but nevertheless expect the reader to be familiar with the fundamentals of differential geometry. Prior knowledge of complex geometry is not assumed and will be introduced in Subsection 4.2.1. This work is meant to be readable for geometrically-minded graduate students.

The images in this thesis were either drawn by the author through Tikz or they were taken from Scaduto [Sca23]. The tables in this thesis were taken from Scaduto [Sca23] and Barth, Hulek, Peters, and Van de Ven [BHPV15]. The origins of all images and tables shown in this thesis are enumerated in the List of Figures and List of Tables at the end of the thesis.

Chapter 1

State of the art

This chapter aims to provide a comprehensive overview of the state of the art regarding the *period map*, from its origins and introduction to the most recent results before Scaduto's paper [Sca23], which will be considered in Chapter 2.

Since there are various distinct definitions in the literature referred as the *period map*, we begin by clarifying the specific definition adopted in this thesis. After that we will confront it with the definition of the original term and the connection with each other. Finally, we outline the origins of the problem of the surjectivity of the period map and previous attempts to solve it.

1.1 Definitions and properties of the period map

There are different maps in the literature called *period map*, so here we will clarify the definition we will use in this thesis. Here we will follow closely the presentation given in Scaduto [Sca23], LeBrun [LeB04], and Shevchishin and Smirnov [SS23], where not otherwise stated.

Notation. In this thesis, the bundle of 2-forms over X four-manifold will be denoted by Λ^2, where the manifold considered is clear in the context. Recall that the usual notation is $\Omega^2(X) = \bigwedge^2 \Omega^1(X)$.

Definition 1.1. Let (X, g) be a closed, oriented Riemannian four-manifold. The bundle of 2-forms over X invariantly decomposes as $\Lambda^2 = \Lambda_2^+ \oplus \Lambda_2^-$, where $\Lambda_2^\pm$ are the (± 1)-eigenspaces of the Hodge star operator $\star_g : \Lambda^2 \to \Lambda^2$. Sections φ of Λ^2 are called *self-dual (SD)* if $\star_g \varphi = \varphi$ and *anti-self-dual (ASD)* if $\star_g \varphi = -\varphi$.

Notation. Denote by $d^* : \Omega^k(X) \to \Omega^{k-1}(X)$ the formal adjoint of the exterior differential d with respect to the pairing:

$$(d^*\omega, \nu) = \int_X g(d^*\omega, \nu)\, \mathrm{dvol} = \int_X g(\omega, d\nu)\, \mathrm{dvol} = (\omega, d\nu)$$

On four-manifolds, we have $d^* = - \star_g d \star_g$. [Sco05, Chapt. 9]

Definition 1.2. Let (X, g) be a Riemannian manifold, the *Hodge Laplacian* is defined as

$$\begin{aligned} \triangle_g : \Omega^k(X) &\to \Omega^k(X) \\ \omega &\mapsto (dd^* + d^*d)\omega \end{aligned}$$

[Pet06, Chapt. 9]

Proposition 1.3. $\triangle_g \omega = 0$ *if and only if* $d\omega = 0 = d^*\omega$ *or equivalently* $d\omega = 0 = d \star_g \omega$. *[Pet06, Chapt. 9]*

Proof. From definition of $\triangle_g$, the if direction is immediate. To prove the other direction, we use the definition of adjoints, which yields

$$\begin{aligned}(\triangle_g \omega, \omega) &= (dd^*\omega, \omega) + (d^*d\omega, \omega)\\ &= (d^*\omega, d^*\omega) + (d\omega, d\omega)\end{aligned}$$

Assuming $\triangle_g \omega = 0$, it implies $(d^*\omega, d^*\omega) = (d\omega, d\omega) = 0$, since the Hodge Laplacian on a compact manifold has a non-negative spectrum. Thus, $d\omega = 0$ and $d^*\omega = 0$ □

By the Hodge theorem, each cohomology class in $H^2(X;\mathbb{R})$ has a unique harmonic representative, giving a canonical isomorphism

$$H^2_{dR}(X) \cong \mathscr{H}^2(X) = \{\varphi \in \Gamma(\Lambda^2) \mid \triangle_g \varphi = 0\}$$

The Hodge star operator $\star_g$ commutes with $\triangle_g$ and thus $\mathscr{H}^2(X)$ inherits a direct-sum decomposition from Λ^2

$$H^2(X;\mathbb{R}) = \mathscr{H}^+_g \oplus \mathscr{H}^-_g$$

where $\mathscr{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid \triangle_g \varphi = 0\}$ are the spaces of self-dual and anti-self-dual harmonic forms.

Definition 1.4. The de-Rham cohomology $H^2_{dR}(X)$ has a symmetric bilinear form, called the *intersection form*, defined as

$$\begin{aligned} q : H^2_{dR}(X) \times H^2_{dR}(X) &\longrightarrow H^4_{dR}(X) \cong \mathbb{R}\\ ([\omega_1], [\omega_2]) &\longmapsto \int_X \omega_1 \wedge \omega_2\end{aligned}$$

This form is equivalent to the cup-product on the de Rham cohomology.

Definition 1.5. A subspace $H \subset H^2(X)$ is *positive* (or *negative*) if the bilinear form restricted on H is positive- (or negative-) definite. It is also *maximal* if it is not properly contained inside a positive subspace.

Remark 1.6. The intersection form is positive- (or negative-) definite when restricted to $\mathscr{H}^\pm_g$. Moreover, these two subspaces are mutually orthogonal with respect to q. The subspace $\mathscr{H}^+_g(X)$ is a maximal positive subspace of $H^2(X)$. All maximal positive subspaces have the same dimension, denoted $b^+(X)$, which is an oriented homotopy invariant.

Notation. Denote with $\mathrm{Gr}^+(H^2(X))$ the space of maximal positive subspaces of $H^2(X)$, which is an open subset of the Grassmannian of all $b^+(X)$-dimensional planes in $H^2(X)$.

Denote with $\mathrm{Met}(X)$ the space of all smooth Riemannian metrics on X, equipped with the C^∞-topology.

Definition 1.7. The assignment

$$\begin{aligned}\Pi_X : \mathrm{Met}(X) &\longrightarrow \mathrm{Gr}^+(H^2(X))\\ g &\longmapsto \mathscr{H}^+_g(X)\end{aligned}$$

is called the *period map* of X.

Notation. In some sources, e.g. Shevchishin and Smirnov [SS23], this map is also called *Riemannian period map*. Throughout this thesis, the term will be employed only in contexts where its omission could lead to confusion in the notation.

Remark 1.8.
- There is a one-to-one identification between the spaces $\mathrm{Gr}^+(H^2(X))$ and $\mathrm{Gr}^-(H^2(X))$ through $H \mapsto H^\perp$, where the orthogonality is defined with respect to the intersection form. This also implies a correspondence between the period maps of X and of $\overline{X}$, the orientation-reversal of X.
- Take two conformally equivalent metrics $g, \tilde{g}$, then by definition of the Hodge star operator, for any $\omega, \nu \in H^2(X)$ we get

$$\begin{aligned}\omega \wedge \star_{\tilde{g}} \nu = \tilde{g}(\omega, \nu)\, \mathrm{dvol}_{\tilde{g}} &= \lambda^2 g(\omega, \nu) \frac{1}{\lambda^2}\, \mathrm{dvol}_g \\ &= g(\omega, \nu)\, \mathrm{dvol} = \omega \wedge \star_g \nu\end{aligned}$$

  This implies that $\star_{\tilde{g}} = \star_g$ and thus the subset $\mathscr{H}^+_g(X)$ remains unchanged under any conformally equivalent metric. This implies that Π_X is a well-defined map also from the space of conformal classes of metrics.
- The period map is an open map (in the C^∞-topology) [SS23], smooth and has no critical points [LeB04].

Convention. The decision to focus on maximal positive subsets rather than maximal negative ones is merely a matter of convention, related to its first definitions, as we will see in the next section. Reversing the orientation swaps the positive and negative cones, so throughout the thesis, we will adopt the convention that ensures $b^-(X) \geq b^+(X)$, following the convention usually found in the literature.

1.2 Origins of the term “period map”

In this section, we will first explain how the term “period map” first appeared and explain its correlation and differences with the Riemannian one. In Subsection 1.2.2, we will outline the first definitions of the period map, the origins of the problem of surjectivity and previous attempts to solve it.

1.2.1 Disambiguation of the term

The concept of *periods* of a form on a manifold first appeared with respect to Riemann surfaces and then expanded to consider other complex manifolds, in particular Kähler manifolds. There will be a more in-depth introduction on complex manifolds later in Chapter 4. The main sources of this subsection are Carlson and Griffiths [CG10] and Voisin [Voi02, Chapt. 10], where not otherwise stated.

We first restrict our discussion to Riemann surfaces, and then we will generalize these definitions to higher dimension complex manifolds.

Definition 1.9. Let a_i, b_i , $1 \leq p \leq g$, be a canonical homology basis of a Riemann surface X of genus g. The *A-periods* and *B-periods* of a closed C^∞ 1-form ϕ are respectively

$$A_i(\phi) = \int_{a_i} \phi \qquad B_i(\phi) = \int_{b_i} \phi$$

Let $A(\phi) = (A_1(\phi), \cdots, A_g(\phi))$ and $B(\phi) = (B_1(\phi), \cdots, B_g(\phi))$, then the vector

$$P(\phi) = (A(\phi), B(\phi)) = (A_1(\phi), \cdots, A_g(\phi), B_1(\phi), \cdots, B_g(\phi))$$

is called the *vector period* of ϕ. [Fal24, Chapt. 9]

Definition 1.10. Let α_p, $1 \leq p \leq g$, be a basis of holomorphic 1-forms on a Riemann surface X of genus g, then the *period matrix* P of X is the $g \times 2g$ matrix

$$P = \begin{pmatrix} P(\alpha_1) \\ \vdots \\ P(\alpha_g) \end{pmatrix} = \begin{pmatrix} A(\alpha_1) & B(\alpha_1) \\ \vdots & \vdots \\ A(\alpha_g) & B(\alpha_g) \end{pmatrix}$$

[Fal24, Chapt. 9]

To introduce the concepts of period maps and period domains on Riemann surfaces, we will first start with an introductory example: elliptic curves.

Definition 1.11. An *elliptic curve* is a compact Riemann surface of genus one, defined as the complex solutions of $y^2 = x^3 + ax + b$, plus one point at infinity.

Since an elliptic curve E has genus $g = 1$, it has as homology basis $\{\delta, \gamma\}$, where the intersection number of the two cycles is $\delta \cdot \gamma = 1$. Consider the differential 1-form $\omega = dx/y$, which is holomorphic in local coordinates on E. Following Definition 1.10, the period matrix of E is given by the integrals

$$P = (A, B) = \left(\int_\delta \omega, \int_\gamma \omega \right)$$

By multiplying ω with a suitable constant, A-periods can be normalized to one. In this case, we can show that the *normalized B-periods* have positive imaginary part and thus their space is parametrized by the upper half plane $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$. The upper half plane is the first example of a *period domain.* Both an elliptic curve and a choice of integer homology basis such that $\delta \cdot \gamma = 1$ are necessary to determine a point on this period domain. From these examples, the definition follows.

Definition 1.12. Let X be a Riemann surface of genus g and let α_i, $1 \leq i \leq g$, be a basis of holomorphic 1-forms on X. By multiplying the 1-forms α_i by a suitable constant, the determinant of the A-period matrix $(A(\alpha_1), \cdots, A(\alpha_g))^\top$ can be set to 1. Then the corresponding B-periods are called *normalized B-periods.*

The space of normalised B-periods is called the *period domain.*

In the setting as above, any two normalised homology bases are related by an element of the group Γ of unimodular matrices with integer entries, and thus the two corresponding normalised B periods are linked by the corresponding fractional linear transformation. Given a family of elliptic curves E_t which depends holomorphically on t, its corresponding B-period $B(t)$ is locally defined and varies holomorphically. The map $t \to B(t)$ is called the *local period map.*

Remark 1.13. To generalize to Riemann surfaces of higher genus we set the period domain as the Hermitian symmetric space of $\mathrm{Sp}(2g, \mathbb{R})$: the Siegel upper half space of genus g, given by $g \times g$ complex symmetric matrices with positive definite imaginary part. It will be denoted by $\mathcal{H}_g$. The group acting on it is $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$.

To further generalize to algebraic manifolds of higher dimension, we use the concepts of Hodge structures and Hodge decomposition, see Subsection 4.2.1. For a projective algebraic manifold one has $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$ where $\overline{H^{p,q}} = H^{p,q}$. Such a decomposition, together with the lattice given by the integer cohomology modulo torsion, is a *Hodge structure of weight k*.

General period domains are parameter spaces for polarised Hodge structures of weight k. The model is the subspace of the k-th cohomology of a complex projective algebraic manifold of dimension k which is annihilated by cup-product with the hyperplane class. The resulting parameter space D can be represented as a complex homogeneous space G/V, where G is a Lie group and V is a compact subgroup. Keep in mind that V is rarely a maximal compact subgroup, and so D is rarely hermitian symmetric. Important special cases in which D is of weight $k > 1$ but is nonetheless hermitian symmetric are the period domains of $K3$ surfaces and of the cyclic cubic three-folds associated to cubic surfaces.

In particular, if X is the underlying smooth manifold of a complex $K3$ surface and we restrict to the space of Kähler metrics, then Π_X from Definition 1.7 corresponds to this period ma, see [SS23]. This follows from the strict connection between the Riemannian period map on $K3$ surfaces and its Kähler metrics, as it will be shown in Section 4.2.

To give a more technical definition of the "original" period map, we will focus only on Kähler manifolds. Among other things, this map is used to study the variations of the Hodge structure and to prove through transversality that a set of Hodge structures with a fixed Hodge number is always a complex manifold. Other applications of the "original" period map appear in the study of Riemann surfaces and Calabi-Yau manifolds (see Definition 4.19).

To properly define the period map, we introduce some definition related to the Hodge structure of Kähler manifolds.

Definition 1.14. Let X be Kähler, then the *Hodge filtration* is

$$0 = F^{k+1}H^k(X) \subset \cdots \subset F^pH^k(X) \subset F^{p-1}H^k(X) \subset \cdots \subset F^0H^k(X) = H^k(X,\mathbb{C})$$

by complex subspaces of dimension $b^{p,k}$. Here $b^{p,k} := \dim F^pH^k(X,\mathbb{C})$ and

$$F^pH^k(X,\mathbb{C}) := \bigoplus_{\substack{r+q=k \\ r\geq p}} H^{r,q}(X).$$

Definition 1.15. Consider a family of Kähler manifolds parametrised by a simply connected base B, then the *local period map* $\mathcal{P}^k : B \to \mathcal{D}$ associates to $t \in B$ the Hodge filtration on $H^k(X_t,\mathbb{C})$, viewed as a constant space V.

Let X be a Kähler manifold and $\phi : \mathcal{X} \to B$ a family of deformations of X. Up to restricting B, we may assume that B is contractible, which gives a canonical identification $H^k(X_b,\mathbb{C}) \cong H^k(X_0,\mathbb{C})$, for $b \in B$, coming from the restrictions

$$H^k(\mathcal{X},\mathbb{C}) \cong H^k(X_0,\mathbb{C}) \qquad H^k(\mathcal{X},\mathbb{C}) \cong H^k(X_b,\mathbb{C})$$

These considerations yield the final definition. The theorem following outlines the properties needed for the study of Hodge structures, which is the reason the period map was introduced in the first place.

Definition 1.16. The *period map* is defined such that

$$\begin{aligned}\mathcal{P}^{p,k} : B &\longrightarrow \mathrm{Gr}(b^{p,k}, H^k(X,\mathbb{C})) \\ b &\longmapsto F^pH^k(X_b,\mathbb{C}) \subset H^k(X_b,\mathbb{C}) \cong H^k(X,\mathbb{C}).\end{aligned}$$

Theorem 1.17. *The period map $\mathcal{P}^{p,k}$ is holomorphic for all p, k, such that $p \leq k$, and satisfies the transversality condition.*

1.2.2 Origin of the Riemannian period map

The "original" period map as defined in 1.16 is in general different from the Riemannian one from Definition 1.7. The similar name is justified by the fact that both maps map into a Grassmannian of the cohomology and that they coincide in specific cases, e.g. $K3$ surfaces. The former influenced the name of the latter, following from the similarities.

Notation. To maintain consistency in the discussion, in cases of conflicting notation Scaduto's paper's notation will be preferred over the notation used in the original papers.

First definition of the period map The use of the name *period map* as in Definition 1.7 first appeared in "Connections, cohomology and the intersection forms of 4- manifolds", Donaldson [Don86, Sect. VI]. In this case, the manifolds considered in the article were smooth, compact, simply-connected, and four-dimensional.

The goal of this article was to study some properties of smooth, simply-connected four-manifolds with indefinite intersection form, as a generalisation of the definite case. In particular, the three main theorems give an equivalent shape of the intersection form over the integers for $b^+ = 0, 1, 2$; the structure of the proof is not applicable when $b^+ \geq 3$. The proof is based on the associated moduli spaces of instanton solutions to Yang-Mills differential equations.

Let denote with M_k the moduli spaces of anti-self-dual (ASD) on a bundle with second Chern class $c_2 = k \geq 0$ over a compact Riemannian manifold X. Since the goal is to generalise the proof from the $b^+ = 0$ case, the article uses the fact that on a simply-connected four-manifold X the associated SU(2) moduli spaces have "virtual" dimension given by the Atiyah-Hitchin-Singer formula

$$\dim M_k = 8k - 3(1 + b^+(X)) \qquad k > 0.$$

Using the period map and transversality properties, it can be shown that for manifolds with indefinite intersection form, M_k is a smooth manifold of this dimension, for $k > 0$. Instead, M_0 is a point.

To finish the proof of the two cases when $b^+ = 1, 2$, the article checks some characteristics of the moduli spaces when the intersection form is indefinite. It is needed to check:

1. the presence of quotient singularities associated to reductions of the bundle, which in the case of $b^+ = 0$ are inherited from the ambient space;
2. the behaviour of the ends of the moduli spaces, in particular of the "links" L where a neighbourhood of the ends has a shape like $\mathbb{R}^+ \times L$.

which can be both proven by M_k being a manifold. Indeed, to prove these things, the article introduces the period map and proves that it is transverse to some submanifolds W_e. From transversality properties, the required characteristics follow.

The interest in ASD forms follows from the work in gauge theory. Given L a line bundle over X, the unique Yang-Mills connection on L has curvature the harmonic representative of $c_1(L)$. Any line bundle L admits an ASD connection if and only if $c_1(L)$ lies in the subspace $\mathscr{H}^2_-$, the annihilator of $\mathscr{H}^2_+$ under the fixed cup product form on $H^2(X)$. There are no self-dual reductions of a bundle with $c_2 = h$ if $\mathscr{H}^2_-$ contains no lattice points e with $e^2 = -h$.

Now we need to define these submanifolds W_e. Let the open subset G of the Grassmannian $\mathrm{Gr}(b^+, H^2(X, \mathbb{R}))$ consisting of the b^+-planes on which the cup product form is positive definite. In Scaduto's notation $G = \mathrm{Gr}^+(H^2(X))$. Each lattice point $e \in H^2(X)_{\mathbb{Z}}$ with $e^2 < 0$ defines a codimension b^+ submanifold $W_e \subset G$ made up of the b^+-planes annihilating e.

The goal is to show that the period map is transverse to the submanifolds W_e by computing the derivative of the period map Π_X. It then follows that if the form is indefinite, any metric can be perturbed slightly to avoid singularities from reductions or, if $b^+ = 1$, then any path of metrics can be perturbed so that reductions occur at a discrete set of points. Through transversality properties, we get that all moduli spaces are manifolds and that if there is a path γ between the metrics g_0 and g_1, then there is a cobordism between the two corresponding moduli spaces. A more detailed treatment of this can be found in Donaldson and Kronheimer [DK97, Chapt. 4].

Nonetheless, this article did not pose any question on the surjectivity of the period map, focusing only on the transversality properties of its image.

First appearance of the problem of the surjectivity The problem of the surjectivity of the period map appeared implicitly in Kotschick's articles "SO(3)-Invariants for 4-Manifolds with $b_2^+ = 1$" part I [Kot91] and II [KM94] (this as a joint work with Morgan). These two consecutive articles aimed to introduce and discuss an infinite family of invariants for arbitrary smooth four-manifolds with $b_1 = 0$ and $b^+ = 1$, as an extension of Donaldson polynomial invariants. These invariants give interesting information about the differential topology of specific kinds of four-manifolds.

A previous paper by Kotschick [Kot89] considered the particular case of simply-connected four-manifolds with $b^+ = 1$ and $b^- = 8k$ and $P \to X$ the SO(3)-bundle with $w_2(P) = w_2(TX)$ and $p_1(P) = -3$. In this article, it was crucial to show independence of the metric, and this was a consequence of the fact that P is topologically irreducible, together with the compactness of the moduli space.

The first article [Kot91] defines polynomial invariants for smooth four-manifolds, following Donaldson's construction in [Don90], which defined polynomial invariants using moduli spaces of ASD connections on SO(3)-bundles for $b^+ > 1$ odd. Donaldson's construction considered a generic metric g on X and its corresponding smooth moduli space M of dimension $-2p_1(X) - 3(1 - b_1 + b^+)$ inside $\mathscr{B}^*$. Here $\mathscr{B}^*$ is the space of irreducible connections up to gauge equivalence. Donaldson's construction considers the classes $\Sigma_1, \cdots, \Sigma_d \in H_2(X; \mathbb{Z})$, one can choose representatives V_j for the $\mu(\Sigma_j)$ in general position. Here the map μ is

$$\begin{aligned} \mu : H_2(X; \mathbb{Z}) &\longrightarrow H^2(\mathscr{B}^*; \mathbb{Q}) \\ \Sigma &\longmapsto -\frac{1}{4} p_1(\mathbb{P})/\Sigma \end{aligned}$$

where the bundle $\mathbb{P} \to \mathscr{B}^* \times X$ is the quotient by the free gauge action of the pullback bundle $\pi_2^*(P) \to \mathscr{A}^* \times X$.

Counting the intersection points in $V_1 \cap \cdots \cap V_d \cap M$ with signs given by a choice of orientation depending on some value related by the cohomology, denoted by α_X and c, one obtains a number which can be thought of as the value of a polynomial $q^X_{d,c,\alpha_X} \in S^d(H^2(X, \mathbb{Q}))$ on $\Sigma_1, \cdots, \Sigma_d$. There is the need to check that this number is independent of the particular choice of representatives, and independent of the choice of generic metric.

By construction, these polynomials seem to depend on the choice of Riemannian metric. If it is possible to connect two metrics with paths of metrics g_t such that for all t, all the moduli spaces M_{p+4k} for $0 \leq k \leq [-p/4]$ are smooth and cut out transversely by the ASD equation, then it would be possible to construct a cobordism between their two corresponding moduli spaces. Since the moduli spaces for various metrics connected this way are cobordant, these invariants do not depend on the metric chosen, but only on the base-manifold X. Since the codimension of the set of metrics with reducible connections is b^+, such connectivity condition is satisfied when $b^+ > 1$ for dimensionality reasons.

In the case of $b^+ = 1$, this is not necessarily the case since reducible connections may occur in 1-parameter families of metrics. These families might either be on the bundle P itself, or on one of the bundles defining the moduli spaces M_{p+4k}, with $k > 0$, labelled by those $e \in H^2(X;\mathbb{Z})$ such that

$$e = w_2 \bmod 2 \qquad p_1 \leq q(e,e) < 0 \tag{1.1}$$

Definition 1.18. Take a complex-plane bundle $E_k \to X$ with structure group G. A *reducible solution* is an anti-self-dual connection that preserves a splitting of E_k into a sum of two complex-line bundles $E = L \oplus L^{-1}$. [Sco05, Sect. 9.4]

The new invariants introduced by Kotschick depend on the concepts of walls an chambers in the the Grassmannian of the cohomology. These aspects are dependent on the specific characteristics of the case $b^+ = 1$.

Definition 1.19. When $b^+ = 1$, the subset of positive forms in $H^2(X;\mathbb{R})$ is disconnected, and the orientation is just a choice of a connected component, see Figure 1.1. Consider the two copies of the hyperbolic b^--space

$$\mathcal{H} = \{x \in H^2(X;\mathbb{R}) \mid q(x,x) = 1\}$$

and define Δ_d the set of *chambers* on $\mathcal{H}$ partitioned by hyperplanes $W_e = \{x \in H^2(X;\mathbb{R}) \mid q(x,e) = 0\}$, where $e \in H^2(X;\mathbb{Z})$ are all such elements satisfying conditions in Equation 1.1. These hyperplanes W_e are called *walls*. See Figure 1.1 for a graphic representation.

The polynomial invariants we are interested in are defined in [Kot91, Thm. 3.2] as $\Phi^X_{d,c} : \Delta_d \to S^d(H^2(X;\mathbb{Z}))$ following the construction outlined above. The proof of existence and properties of $\Phi^X_{d,c}$ first constructs polynomials dependent on the choice of metric and then uses path-connectedness inside the chamber to have a single value for each element in Δ_d, through the construction of a cobordism as explained above. The problem arises because this approach relies on the fact that the period map Π_X is a fibration, and it is not known whether two metrics whose period points lie in the same chamber can be connected by a path of metrics whose period points lie in that chamber. Thus, in the first article, it is only conjectured, not proven, that the value of an invariant on every chamber is determined by its value on any of the chambers. This would require the surjectivity of the period map, which is not immediately true and was not proven until Scaduto's article [Sca23].

The problem given by the surjectivity of Π_X is circumvented in [KM94] by defining a formula for the polynomial when the walls W_e are crossed. Here, without loss of generality, a wall is defined as the space perpendicular to an integral class of square -1. This article is a continuation of the first article: it proves that the values of these invariants depend only on the chamber containing the self-dual harmonic 2-form for the metric used to define the anti-self-dual (ASD) equation. Actually, it shows some of the general properties of the differences of the values of the invariants as the self-dual 2-form crosses a wall. This

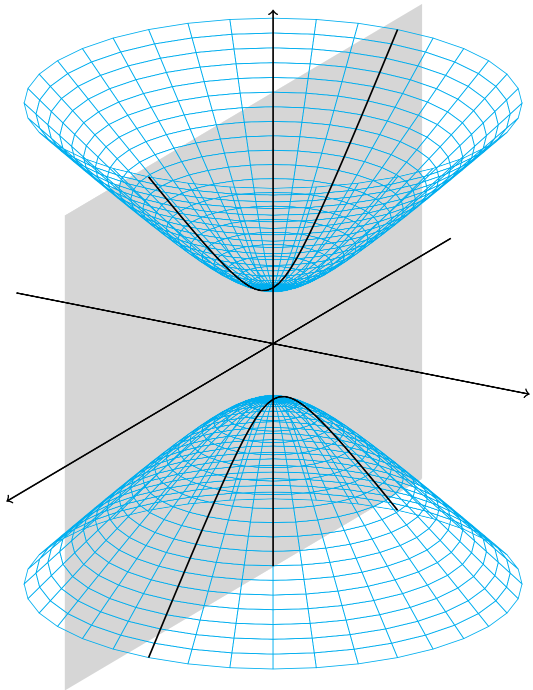

Figure 1.1: For the case with signature $(1, 2)$, we can see that the hyperplane W_e for $e = (0, 1|0)$ partitions into two chambers each of the two connected components of the hyperbolic b^--space $\mathbb{H}^2$ (in blue). The wall between them is drawn in black.

implies that the polynomial invariant is defined on the whole chamber, independently of the path-connectedness of any chamber. In particular, the invariant is defined for all chambers regardless of whether they contain forms which are self-dual harmonic for some metric.

The proof in [KM94] proceeds around the problem of surjectivity, giving an argument which makes no use of detailed properties of the period map. It is based on a glueing construction of concentrated ASD connections over S^4, not resulting necessarily in ASD connections on X. There is no use of the period map later in the article.

Conjectured surjectivity Another article that had to deal with the problem of the surjectivity of the period map is Katz's "Four-manifold systoles and surjectivity of period map" [Kat03]. The goal of this article was to provide a lower bound for the analogous 2-dimensional conformal systolic invariant for a four-manifold X with indefinite intersection form, specifically focusing on the manifold $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$. This lower bound is found to be polynomial in b_2. The proof to be valid requires the surjectivity of the period map, which is only conjured by the article.

Assuming the period map to be surjective, this implies the flexibility in choosing the position of the integer lattice in $H^2_{dR}(X)$ with respect to the L^2-norm. In particular, it is shown in the article that the least norm, λ_1 in $H^p(X, \mathbb{Z})_{\mathbb{R}}$ of a non-zero lattice element, can be made arbitrarily large as the Betti number grows. This relies on the existence of Euclidean unimodular lattices L with arbitrarily high least norm $\lambda_1(L)$, and on the (elementary) classification of indefinite odd unimodular forms.

The surjectivity of the period map is used to prove the main theorem. It is not applied to the line $CT_n^{\perp}$, but rather to the image of $CT_n^{\perp} \subset H^2_{dR}(X)$ under a suitable "Lorentz deformation", and thus it is actually sufficient to have the density of the image of the period map. Such Lorentz transformation $A_s : \pi \to \pi$ is defined by the matrix

$$A_s = \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$$

with respect to the standard basis, where π is the xy-plane.

Let $\mathrm{Met}(X)$ be the space of all Riemannian metrics on X, and let G be the projectivisation of the negative cone of the form ω. In this article, the period map is defined as $\mathcal{P} : M \to G$, the map assigning to each metric its anti-self-dual direction. By assumed surjectivity of $\mathcal{P}$, it is possible to define

$$g_n = \mathcal{P}^{-1} A_s(CT_n^{\perp})$$

which is then used to get the lower bound results.

In the article, Katz questions the possibility of proving the result without relying the hypothesis of surjectivity and he conjectures that it might be possible to do so, at least in the case of the blow-ups of the projective plane. The result is not immediate and it would need some further proving. As shown above, during the proof, the requirement of the surjectivity of the period map is relaxed to assume instead the density of the image.

Previous proofs of surjectivity Before Scaduto's paper [Sca23], there was no proof of the surjectivity of the period map in the general case, according to Shevchishin and Smirnov [SS23], which was posted online a month before Scaduto's. Nonetheless, there ware some proofs of surjectivity or density of the image of the period map under specific cases, specifically for symplectic manifolds.

Li and Liu [LL01] proved the surjectivity of Π_X when $b^+(X) = 1$ and X is a minimal closed oriented smooth 4-manifold admitting symplectic structures. If X is minimal, any $\omega \in H^2(X)$ with $\omega^2 > 0$ is represented by a symplectic form and since symplectic forms are related to metrics on X, the surjectivity of the period map follows from it. This work also implies $\mathrm{Im}(\Pi_X)$ is dense, for non-minimal closed oriented smooth 4-manifolds admitting symplectic structures. These results are consequences of Theorems 1.24 and 1.25, introduced by the following definitions, taken from the article.

Definition 1.20. Let $\mathcal{E}_X$ be the set of cohomology classes whose Poincaré duals are represented by smoothly embedded spheres of square -1, then X is said to be *(smoothly) minimal* if $\mathcal{E}_X$ is the empty set.

Notation. Denote by Ω_X the moduli space of those 2-forms, which are orientation-compatible symplectic structure on X, i.e. any closed 2-form ω such that $\omega \wedge \omega$ is nowhere vanishing and agrees with the orientation.

Definition 1.21. Any symplectic structure determines a homotopy class of compatible almost complex structures on the cotangent bundle, whose first Chern class is called the *symplectic canonical class.*

Definition 1.22. Consider the projection $CC : \Omega_X \to H^2(X; \mathbb{R})$, the image of this projection is called the *symplectic cone of* X, denoted by $\mathcal{C}_X$. For each symplectic canonical class K, there is the K*-symplectic cone*

$$\mathcal{C}_{X,K} = \{e \in \mathcal{C}_X | e = [\omega] \text{ for } \omega \in \Omega_K\}.$$

From this definition, $\mathcal{C}_X$ is the union of the $\mathcal{C}_{X,K}$.

Definition 1.23. Given an orientation-compatible symplectic form ω, the connected component of the positive cone $H^2(X;\mathbb{R})$ containing $[\omega]$ is called the *forward cone* associated to ω, denoted by $\mathcal{FP}$.

Given a symplectic canonical class K, the component containing $\mathcal{C}_K$ is called the *forward cone associated to* K, denoted by $\mathcal{FP}(K)$.

Theorem 1.24. *Let X be a minimal closed, oriented four-manifold with $b^+ = 1$ and K be a symplectic canonical class. Then $\mathcal{C}_{X,K} = \mathcal{FP}(K)$. Thus, any real cohomology class of positive square is represented by an orientation-compatible symplectic form.*

For non-minimal four-manifolds, it is no longer true that any real cohomology class of positive square is represented by an orientation-compatible symplectic form, due to the presence of the set $\mathcal{E}_X$. This is shown is the following theorem, which implies the density of the image of Π_X.

Theorem 1.25. *Let X be a closed, oriented four-manifold with $b^+ = 1$ and $\mathcal{C}_X$ non-empty. Then*

$$\mathcal{C}_X = \{e \in P \mid 0 < |e \cdot E| \ \forall E \in \mathcal{E}_X\}.$$

Another article that also uses the properties of symplectic forms to study the image of the period map is "Period map image density" by Solomon [Sol07], that considered the case of n blow-ups of $\mathbb{CP}^2$ ($X = \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$) and proved the density of the image.

The sketch of the proof proceeds as follows. Consider $\alpha \in \mathcal{P}_g$, where $\mathcal{P}_g$ is the $+1$ eigenspace of the operator $\star_g$. First, it is shown that it suffices to construct a symplectic form

$$\omega \in \Omega^2(X),\ d\omega = 0,\ \omega^2 \neq 0$$

such that $[\omega] = [\alpha]$. Note that the diffeomorphism group $\mathrm{Diff}(X)$ acts by pullback both on the set of symplectic forms on X and on the positive cone $\mathcal{P}$ in $H^2(X;\mathbb{R})$. Thus, it is sufficient to construct ω such that $[\omega]$ is in a dense subset of a fundamental domain for the action of $\mathrm{Diff}(X)$ on $\mathcal{P}$. The proof explicitly constructs a finite set of diffeomorphisms of X which generate a suitable subgroup $G_{(-1,-2)} \subset \mathrm{Diff}(X)$, and then proves that a subset $\mathcal{S} \subset \mathcal{P}$ contains a fundamental domain for the action of $G_{(-1,-2)}$ on $\mathcal{P}$ through techniques from hyperbolic geometry. The subset $\mathcal{S}$ is constructed such that the image of the map Π_X is dense even in $\mathcal{S}$. Throug techniques combining gauge theory and the theory of holomorphic curves on X, the density of the image of Π_X follows.

Remark 1.26. In all the articles cited, the question of the surjectivity of the period map was restricted to the case when $b^+ = 1$. Even though the problem was introduced only for this specific case, but the question gained traction on its own, expanding to consider also cases $b^+ > 1$, without any relation to any of the problems outlined above.

Chapter 2

An Exposition of Scaduto's Article

The scope of this chapter is to explain in depth the results of Scaduto's "Metric stretching and the period map for smooth 4-manifolds" [Sca23] and the reasoning behind the proofs. The article focuses on proving two main theorems:

Theorem 2.1. *For a smooth, closed, connected, oriented four-manifold, the period map* Π_X *has dense image.*

Theorem 2.2. *If, in addition,* $b^+(X) = 1$*, then the period map* Π_X *is surjective.*

Remark 2.3. These two theorems do not require X to be symplectic, and the proofs do not use any gauge theory or pseudo-holomorphic curve theory.

2.1 Foundations and Proof of Theorem 2.1

In this section, we will thoroughly revise the proof of Theorem 2.1. We will define the splitting of a manifold and use it to determine a 1-parameter family of metrics stretching in a cylindrical fashion. These will be the two main ingredients of the proof.

Notation. Let X be a manifold with boundary, then $H^i_c(X) = H^i_c(\text{int}(X))$ is de Rham cohomology with compact supports on the interior of X. We define

$$\begin{aligned}\widehat{H}^i(X) &= \text{Im}\left(H^i(X, \partial X) \to H^i(X)\right)\\ &= \text{Im}\left(H^i_c(X) \to H^i(X)\right)\end{aligned}$$

and $\widehat{b}_i(X) = \dim \widehat{H}^i(X)$.

Recall. If a manifold X^n is orientable, the definition of the pairing is related to the Poincaré duality of forms (see [GH14, Sect. 0.4]) through

$$\begin{aligned}q : H^i(X) \times H^{n-i}(X) &\longrightarrow \mathbb{R}\\ ([\alpha], [\beta]) &\longmapsto \int_X \alpha \wedge \beta = \int_B \alpha = \#(A \cdot B)\end{aligned} \tag{2.1}$$

where $A = \text{PD}([\alpha])$, $B = \text{PD}([\beta])$ and $\#$ is the intersection form in the homology.

2.1.1 Splitting of the manifold

Consider a closed four-manifold X, and $Y \subset X$ a closed three-manifold separating X into two pieces

$$X = X_1 \cup_Y X_2, \tag{2.2}$$

where X_1, X_2 are compact with $\partial X_1 = Y = -\partial X_2$. It is not necessary for any of these manifolds to be connected.

In de Rham cohomology, we have the following description of the map δ in the Mayer-Vietoris sequence

$$\cdots H^1(Y) \xrightarrow{\delta} H^2(X) \xrightarrow{j} H^2(X_1) \oplus H^2(X_2) \xrightarrow{k} H^2(Y) \cdots . \tag{2.3}$$

Let $[\alpha] \in H^1(Y)$, and choose a collar neighbourhood $Y \times (-1,1)$ of Y in X. Let $\rho \in C^\infty(\mathbb{R})$ such that $\int \rho(t)dt = 1$ and $\operatorname{supp}(\rho) \subset (-1,1)$, then we have

$$\delta[\alpha] = [\rho(t)\alpha \wedge dt] \tag{2.4}$$

where via extension by zero $\rho(t)\alpha \wedge dt \in \Omega^2(X)$ is a 2-form on X.

Proposition 2.4. *Let X be a four-manifold separated as in* (2.2) *and $\iota : Y \to X$ the inclusion map. Then we have*

$$\int_X \omega \wedge \delta(\alpha) = \int_Y \iota^*(\omega) \wedge \alpha$$

for all closed forms $\omega \in \Omega^2(X)$.

Proof. Using the definition of $\delta(\alpha)$ from Equation (2.4), we get

$$\begin{aligned}\int_X \omega \wedge \delta(\alpha) &= \int_X \omega \wedge (\rho(t)\alpha \wedge dt) = \int_{Y\times(-1,1)} \rho(t)\iota^*(\omega) \wedge \alpha \wedge dt = \\ &= \int_Y \iota^*(\omega) \wedge \alpha \int_{(-1,1)} \rho(t)dt = \int_Y \iota^*(\omega) \wedge \alpha.\end{aligned}$$

□

Notation. In the calculations during the proofs, we will write $|V| = \dim V$.

Lemma 2.5. *Let X be a four-manifold separated as in* (2.2)*, then*

$$b_2(X) = \widehat{b}_2(X_1) + \widehat{b}_2(X_2) + 2\dim(\operatorname{Im}(\delta)).$$

Proof. By exactness of the Mayer-Vietoris sequence (2.3), we have

$$\begin{aligned} b_2(X) = |\ker(j)| + |\operatorname{Im}(j)| = |\operatorname{Im}(\delta)| + |\ker(k)|, \\ b_2(X_1) + b_2(X_2) = |\operatorname{Im}(k)| + |\ker(k)|, \end{aligned}$$

which yields

$$b_2(X) = b_2(X_1) + b_2(X_2) + |\operatorname{Im}(\delta)| - |\operatorname{Im}(k)|. \tag{2.5}$$

To finish the proof, we first need to prove some claims.

<u>Claim:</u> *Write $\iota_i : Y \to X_i$ for the inclusion maps, then*

$$|\operatorname{Im}(k)| + |\operatorname{Im}(\iota^*)| = |\operatorname{Im}(\iota_1^*)| + |\operatorname{Im}(\iota_2^*)|. \tag{2.6}$$

Proof: By definition $\operatorname{Im}(k) = \operatorname{Im}(\iota_1^*) + \operatorname{Im}(\iota_2^*)$ and $\operatorname{Im}(\iota_1^*) \cap \operatorname{Im}(\iota_2^*) = \operatorname{Im}(\iota^*)$. This implies,

$$\begin{aligned} |\operatorname{Im}(k)| &= |\operatorname{Im}(\iota_1^*) + \operatorname{Im}(\iota_2^*)| = |\operatorname{Im}(\iota_1^*)| + |\operatorname{Im}(\iota_2^*)| - |\operatorname{Im}(\iota_1^*) \cap \operatorname{Im}(\iota_2^*)| \\ &= |\operatorname{Im}(\iota_1^*)| + |\operatorname{Im}(\iota_2^*)| - |\operatorname{Im}(\iota^*)|. \end{aligned}$$

■

Claim:

$$|\operatorname{Im}(\iota_i^*)| = b_2(X_i) - \widehat{b}_2(X_i) \qquad i = 1, 2. \tag{2.7}$$

Proof: Since Y is the boundary of both X_i, there is a long exact sequence for both the pairs (X_i, Y) for $i = 1, 2$:

$$\cdots H^{n+1}(Y) \longrightarrow H^n(X_i, Y) \longrightarrow H^n(X_i) \xrightarrow{\iota_i^*} H^2(Y) \cdots .$$

Thus, exactness yields $\widehat{b}_2(X_i) = |\operatorname{Im}(H^2(X_i, Y) \to H^2(X_i))| = |\ker(\iota_i^*)|$, and it implies $b_2(X_i) = |\ker(\iota_i^*)| + |\operatorname{Im}(\iota_i^*)| = \widehat{b}_2(X_i) + |\operatorname{Im}(\iota_i^*)|$. ■

Claim:

$$|\operatorname{Im}(\iota^*)| = |\operatorname{Im}(\delta)|. \tag{2.8}$$

Proof: Consider the three-dimensional submanifold $Y \subset X$. Since the pairing (2.1) is non-degenerate, we get $(H^2(Y))^* = H^1(Y)$. Let

$$\begin{aligned} l : H^1(Y) &\longrightarrow \operatorname{Im}(\iota^*)^* \\ \alpha &\longmapsto l_\alpha := \left(\beta \mapsto \int_Y \alpha \wedge \beta\right) \end{aligned}$$

The inclusion $I : \operatorname{Im}(\iota^*) \hookrightarrow H^2(Y)$ is injective, making $I^* = l : (H^2(Y))^* = H^1(Y) \to \operatorname{Im}(\iota^*)^*$ surjective. This implies that $|\operatorname{Im}(\iota^*)| = |\operatorname{Im}(\iota^*)^*| = |\operatorname{Im}(l)|$, since these spaces are finite dimensional.

Observe that, by Proposition 2.4, $l_\alpha = 0$ if and only if $\int_X \alpha \wedge \iota^*(\beta) = \int_X \delta(\alpha) \wedge \beta = 0$ for all $[\beta] \in H^2(X)$. This happens if and only if $\delta[\alpha] = 0$, since the pairing above is non-degenerate. This implies $\ker(l) \cong \ker(\delta)$ and thus $\operatorname{Im}(l) \cong H^1(Y)/\ker(l) \cong H^1(Y)/\ker(\delta) \cong \operatorname{Im}(\delta)$. By surjectivity of l, we get the claim. ■

The lemma now follows from Equations (2.5)–(2.8):

$$\begin{aligned} b_2(X) &\overset{(2.5)}{=} b_2(X_1) + b_2(X_2) + |\operatorname{Im}(\delta)| - |\operatorname{Im}(k)| \\ &\overset{(2.6)}{=} b_2(X_1) + b_2(X_2) + |\operatorname{Im}(\delta)| + |\operatorname{Im}(\iota^*)| - |\operatorname{Im}(\iota_1^*)| - |\operatorname{Im}(\iota_2^*)| \\ &\overset{(2.7),\,(2.8)}{=} b_2(X_1) + b_2(X_2) + 2|\operatorname{Im}(\delta)| - b_2(X_1) + \widehat{b}_2(X_1) - b_2(X_2) + \widehat{b}_2(X_2) \\ &= \widehat{b}_2(X_1) + \widehat{b}_2(X_2) + 2|\operatorname{Im}(\delta)|. \end{aligned}$$

□

Remark 2.6. Through integration of the wedge product as in (2.1), we can define a pairing on $H_c^2(X_i)$, which is, in general, degenerate. The degeneracy follows from the fact that the map $H_c^i(X) \to H^i(X)$ is neither surjective nor injective, since exact forms might be mapped to non-exact forms. Thus, the pairing as in (2.1) is degenerate since it factors through this map:

$$\begin{array}{ccc} H_c^2(X) \times H^2(X) & \longrightarrow & \mathbb{R} \\ \uparrow & \nearrow & \\ H_c^2(X) \times H_c^2(X) & & \end{array} \tag{2.9}$$

The induced pairing on $\widehat{H}^2(X_i)$ is instead non-degenerate, since by definition $\widehat{H}^2(X_i) \subset H^2(X_i)$, where it is non-degenerate. Denote by $b^\pm(X_i)$ the dimensions of maximal positive/negative subspaces in $\widehat{H}^2(X_i)$.

Lemma 2.7. *Let X be a four-manifold separated as in* (2.2)*, then*

$$b^\pm(X) = b^\pm(X_1) + b^\pm(X_2) + \dim(\mathrm{Im}(\delta))$$

Proof. Let $H_i \subset H_c^2(X_i)$ be a subspace which maps isomorphically to $\widehat{H}^2(X_i)$; surjectivity is made possible by the definition of $\widehat{H}^2(X_i)$, injectivity by the choice of elements in H_i. In particular, the pairing on H_i is non-degenerate, since the diagram (2.9) restricted on these subsets is non-degenerate.

We have a natural injection $H_1 \oplus H_2 \oplus \mathrm{Im}(\delta) \to H^2(X)$ since these subspaces do not share elements with each other. The decomposition is orthogonal with respect to the pairing: for any $\omega_1 \in H_1$, $\omega_2 \in H_2$ we have $\mathrm{supp}(\omega_1) \cap \mathrm{supp}(\omega_2) = \emptyset$, and $\mathrm{supp}(\omega_i) \cap \mathrm{Im}(\delta) = \emptyset$ for $i = 1, 2$, following from (2.4). This means we can diagonalise the form so that on this subspace it is

$$\underbrace{\langle +1 \rangle^{b^+(X_1)} \oplus \langle -1 \rangle^{b^-(X_1)}}_{\text{on } H_1} \oplus \underbrace{\langle +1 \rangle^{b^+(X_2)} \oplus \langle -1 \rangle^{b^-(X_2)}}_{\text{on } H_2} \oplus \underbrace{\langle 0 \rangle^{|\,\mathrm{Im}(\delta)|}}_{\text{on } \mathrm{Im}(\delta)}$$

By basic algebra of non-degenerate symmetric bilinear forms over $\mathbb{R}$ and Lemma 2.5, there exists $W \subset H^2(X)$ such that

1. $H^2(X) = H_1 \oplus H_2 \oplus \mathrm{Im}(\delta) \oplus W$ with $|W| = |\,\mathrm{Im}(\delta)|$
2. the complement of W with respect to the pairing is $W^\perp = H_1 \oplus H_2 \oplus W$
3. on $\mathrm{Im}(\delta) \oplus W$ the pairing is equivalent to a sum of hyperbolic planes,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Each hyperbolic plane is equivalent over $\mathbb{R}$ to $\langle +1 \rangle \oplus \langle -1 \rangle$. Then

$$b^\pm(X) = b^\pm(X_1) + b^\pm(X_2) + \frac{1}{2}\left(|\,\mathrm{Im}(\delta)| + |W|\right) \qquad \square$$

2.1.2 Metric stretching

Let X_1 and X_2 be compact four-manifolds with boundaries $\partial X_1 = Y = -\partial X_2$, where Y is a three-manifold. For $r \geq 0$, let

$$X(r) = X_1 \cup (Y \times [-r, r]) \cup X_2$$

where $Y \times \{-r\} \cong \partial X_1$ and $Y \times \{r\} \cong -\partial X_2$, as shown in Figure 2.1. We denote $X = X(1)$ to align the notation with the proof of Proposition 2.9, as presented in the appendix of Scaduto's paper.

On $X(r)$ we define a metric $h(r)$ by

$$h(r) = \begin{cases} g_Y + dt^2 & \text{on } Y \times [-r, r] \\ g_i & \text{on } X_i \end{cases} \tag{2.10}$$

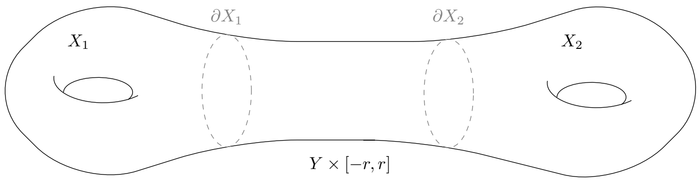

Figure 2.1: Idea of the stretching of a split manifold

for some fixed metric g_Y on Y, and fixed metrics g_i on X_i, so that they both agree with $g_Y + dt^2$ in collar neighbourhoods of the boundaries ∂X_i. There are diffeomorphisms $f_r : X \to X(r)$ natural up to isotopy, which can be considered as a homotopy stretching along r. Thus, we can define a 1-parameter family of metrics $g(r) = f_r^* h(r)$ on X, stretching along Y in a cylindrical manner.

Let $X_1(\infty) = X_1 \cup (Y \times [0, \infty))$ with $Y \times \{0\} \cong \partial X_1$. This manifold comes with a metric $g_1(\infty)$, equal to g_1 on X_1 and $g_Y + dt^2$ on the cylinder, considered as a limit of the metric $h(r)$ defined above. Similar remarks and definitions hold for X_2 and $X_2(\infty)$.

Remark 2.8. We define $\mathscr{H}^2_{X_1}$ as the vector space of the L^2 harmonic 2-forms on $X_1(\infty)$, with the L^2 metric induced by $g_1(\infty)$. There is a natural identification $\mathscr{H}^2_{X_1} = \widehat{H}(X_1)$ following from several properties of L^2 harmonic forms, due to Atiyah, Patodi, and Singer [APS75, Prop. 4.9]. The space of self-dual L^2 harmonic 2-forms $\mathscr{H}^+_{X_1} \subset \mathscr{H}_{X_1}$ gives a maximal positive subspace for the pairing on $\widehat{H}(X_1)$, similarly as in the case of four-manifolds with boundary.

Notation. Denote the space of maximal semi-positive subspaces in $H^2(X)$ as $\overline{\mathrm{Gr}}^+(H^2(X))$, the closure of $\mathrm{Gr}^+(H^2(X))$ in the ambient Grassmannian.

Proposition 2.9. *Let $g(r)$ be a 1-parameter family of metrics which, as $r \to \infty$, stretches along a three-manifold $Y \subset X$ in a cylindrical fashion as described above. Then, as elements of $\overline{\mathrm{Gr}}^+(H^2(X))$, we have*

$$\lim_{r\to\infty} \mathscr{H}^+_{g(r)} = j^{-1}\left(\mathscr{H}^+_{X_1} \oplus \mathscr{H}^+_{X_2}\right) \tag{2.11}$$

where $j : H^2(X) \to H^2(X_1) \oplus H^2(X_2)$ is the map in the Mayer-Vietoris sequence.

Proof. Proof of this proposition will be omitted, but it can be found in the Appendix of Scaduto's paper [Sca23]. □

An equivalent way to state Equation (2.11) is as follows. For $i = 1, 2$, choose subspaces $H_i^+ \subset H_c^2(X_i)$ which map isomorphically to $\mathscr{H}^+_{X_i} \subset \widehat{H}^2(X_i)$, the same notation as in the proof of Proposition 2.7. Then, identifying H_i^+ with its image in $H^2(X)$, we have

$$\lim_{r\to\infty} \mathscr{H}^+_{g(r)} = H_1^+ + \mathrm{Im}(\delta) + H_2^+. \tag{2.12}$$

and this is a direct sum of subspaces. As in the proof of Proposition 2.7, the possible choices of isometries that define $H_i^+ \subset H^2(X)$ differ by the addition of elements in $\mathrm{Im}(\delta)$. This guarantees that (2.12) is independent of the choices of isomorphisms, and thus well-defined.

Remark 2.10. Proposition 2.9 is the main tool used to prove Theorem 2.1. It will not be used in the most general form as written in (2.11) and (2.12), but we will use two simpler cases: when $\mathrm{Im}(\delta) = 0$ and when both X_i are definite.

When $\mathrm{Im}(\delta) = 0$, we have that $j : H^2(X) \to H^2(X_1) \oplus H^2(X_2)$ is an isomorphism, and thus (2.11) may be written as

$$\lim_{r\to\infty} \mathscr{H}^+_{g(r)} = \mathscr{H}^+_{X_1} \oplus \mathscr{H}^+_{X_2}.$$

When X_1 and X_2 are definite, the right side of (2.11) is independent of metrics since we have $\mathscr{H}^+_{X_i} = \mathscr{H}_{X_i}$.

We illustrate the use of Proposition 2.9 with two simple examples.

Example 1. Let $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} = (\mathbb{CP}^2 \setminus D^4) \cup_{S^3} (\overline{\mathbb{CP}^2} \setminus D^4)$, then $H^2(X) = H^2(\mathbb{CP}^2) \oplus H^2(\overline{\mathbb{CP}^2}) \cong \mathbb{R}^2$ with signature $(1, 1)$. The intersection form is the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.13}$$

with respect to the splitting. Let h an hyperplane, i.e. the line $\mathbb{CP}^1 \subset \mathbb{CP}^2 \setminus D^4 \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, and s an exceptional sphere, i.e. the line $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2} \setminus D^4 \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Let $H = \mathrm{PD}^{-1}([h]) \in H^2(X)$, and $E = \mathrm{PD}^{-1}([s]) \in H^2(X)$, they are by construction such that $H^2 = 1$, $E^2 = -1$, $H \cdot E = 0$. For details, see [GH14, Sect. 4.1].

Given the intersection form as in 2.13 and the splitting of $H^2(X)$, we can represent $H^2(X)$ as $\mathbb{R}^2$, with x-axis $\mathbb{R} \cdot E \cong H^2(\overline{\mathbb{CP}^2})$, and y-axis $\mathbb{R} \cdot H \cong H^2(\mathbb{CP}^2)$, see Figure 2.2. The positive subspaces are lines with slope s where $|s| > 1$ (including $|s| = \infty$). Stretching a metric along the connected sum three-sphere S^3, we limit to the y-axis, which corresponds with $H^2(\mathbb{CP}^2) = \mathscr{H}^+_{\mathbb{CP}^2}$.

Example 2. Let $X = S^2 \times S^2$, then $H^2(X) = (H^2(S^2) \otimes H^0(S^2)) \oplus (H^0(S^2) \otimes H^2(S^2)) \cong \mathbb{R}^2$ with again signature $(1, 1)$. The intersection form is the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.14}$$

Thus the duals of $S^2 \times \{*\}$ and $\{*\} \times S^2$ in $H^2(X)$ give a basis for the x- and y-axes, see Figure 2.3. The positive subspaces are lines with positive slope. There are two possible decompositions, which are

$$X \cong (S^2 \times D^2) \cup_Y (S^2 \times D^2) \quad \text{with } Y = S^2 \times S^1, \tag{2.15}$$

$$X \cong (D^2 \times S^2) \cup_Y (D^2 \times S^2) \quad \text{with } Y = S^1 \times S^2; \tag{2.16}$$

they both follow from considering the disk D^2 as a half-sphere. Stretching Y in a cylindrical fashion, we limit to the x-axis and y-axis, respectively for (2.15) and (2.16). In each case, the chosen axis coincides with the line $\mathrm{Im}(\delta) \subset H^2(X)$.

2.1.3 Formal proof of Theorem 2.1

Now we will use these constructions to prove Theorem 2.1.

Proof of Theorem 2.1. Let $n = b_2^+(X) > 0$ and take $H \in \mathrm{Gr}^+(H^2(X))$ such that $H = \langle \sigma_1, \dots, \sigma_n \rangle$ where $\sigma_i \in H^2(X; \mathbb{Q})$ are pairwise orthogonal rational classes. Since rational subspaces are dense in a real space, such H are dense in $\mathrm{Gr}^+(H^2(X))$, so it is sufficient to prove that any such H have a corresponding metric g such that $H = \mathscr{H}^+_g$.

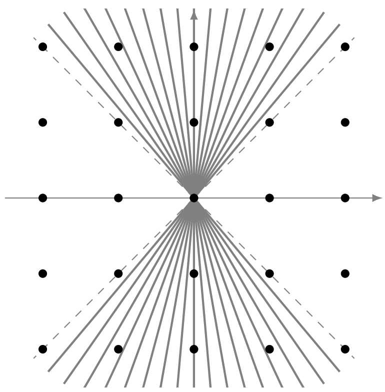

Figure 2.2: $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$

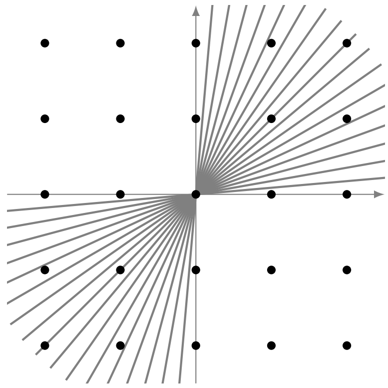

Figure 2.3: $S^2 \times S^2$

Take $N \in \mathbb{Z}_+$ such that $N\sigma_i \in H^2(X)_{\mathbb{Z}} \subset H^2(X;\mathbb{R})$ for all i. Let $\Sigma_i \subset X$ be a closed oriented connected surface such that $[\Sigma_i] \in H_2(X;\mathbb{Z})/\mathrm{tors} \subset H_2(X;\mathbb{R})$ is Poincaré dual to $N\sigma_i$. Here, we use that for a closed four-manifold, every class in $H_2(X;\mathbb{Z})$ is represented by an embedded closed surface (see [GS23, Prop. 1.2.3]). By orthogonality of σ_i, we can arrange $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$.

We also define $W_i \subset X$ a closed disk bundle neighbourhood of Σ_i ($D^2 \hookrightarrow W_i \to \Sigma_i$) such that $W_i \cap W_j = \emptyset$ for all $i \neq j$. Let $Y_i = \partial W_i$ circle bundle over Σ_i ($S^1 \hookrightarrow Y_i \to \Sigma_i$). We construct the bundle Y_i such that it has a non-zero Euler class $e_i \in H^2(\Sigma_i)$. Then we can consider the Gysin exact sequence for the circle bundle $\pi : Y_i \to \Sigma_i$

$$\cdots H^1(Y_i) \xrightarrow{\pi_*} H^0(\Sigma_i) \xrightarrow{e_i \wedge} H^2(\Sigma_i) \xrightarrow{\pi^*} H^2(Y_i) \cdots$$

By dimensionality reasons, we can consider the Euler class as a generator of $H^2(\Sigma_i)$, so the map $e_i\wedge$ is surjective, and we get that $\pi^* : H^2(\Sigma_i) \to H^2(Y_i)$ is zero. Thus, the restriction map $H^2(W_i) \to H^2(Y_i)$ vanishes for any i since it factors as

$$\begin{array}{ccc} H^2(W_i) & \longrightarrow & H^2(Y_i) \\ \big\downarrow & \nearrow_{\pi^*} & \\ H^2(\Sigma_i) & & \end{array}$$

We can split the manifold X as in (2.2) into $X_2 = W_1 \cup \cdots \cup W_n$ and $X_1 = \overline{X \setminus X_2}$ with $Y = \partial X_2 = Y_1 \cup \cdots \cup Y_n$. Observe that with respect to the intersection pairing, X_1 is negative definite, i.e. $\mathscr{H}^+_{X_1} = \emptyset$, and X_2 is positive definite, i.e. $\mathscr{H}^+_{X_2} = \mathscr{H}^2_{X_2}$, as per Remark 2.10. As in Proposition 2.9, we construct a family of metrics $g(r)$ by stretching along Y in a cylindrical fashion.

The vanishing of the map $H^2(W_i) \to H^2(Y_i)$ implies that $\iota_2^* : H^2(X_2) \to H^2(Y)$ is zero. Thus, also $\iota^* = 0$, since $\iota^* = \iota_2^* \circ j_2^*$, where $j_i : X_i \hookrightarrow X$ is the inclusion. As seen when proving Equation (2.8) in the proof of Lemma 2.5, by finite dimensionality we have $\operatorname{Im}\delta \cong (\operatorname{Im}\iota^*)^* \cong \operatorname{Im}\iota^* = 0$. Thus, $j^{-1}(\mathscr{H}^+_{X_1} \oplus \mathscr{H}^+_{X_2}) = \mathscr{H}^+_{X_2} = \widehat{H}^2(X_2)$, following from Remarks 2.8 and 2.10.

Recall the fact that $\partial W_i = \emptyset$, which implies $\widehat{H}^2(W_i) = H^2(W_i)$ and use Equation (2.11) to get

$$\lim_{r\to\infty} \mathscr{H}^+_{g(r)} = j^{-1}(\mathscr{H}^+_{X_2}) = \widehat{H}^2(X_2) = \bigoplus_{i=1}^n \widehat{H}^2(W_i)$$
$$= \bigoplus_{i=1}^n H^2(W_i) = \bigoplus_{i=1}^n H^2(\Sigma_i) = \bigoplus_{i=1}^n \mathbb{R}\cdot\sigma_i = H$$

This completes the proof. □

Remark 2.11.

1. Theorem 2.1 can be extended to closed connected $4k$-manifolds. This can be done since every rational class $s \in H_i(X;\mathbb{Q})$ has a multiple $n \cdot s$ that is represented by an embedded submanifold, see [Tho54, Chapt. II]

2. Another generalization of Theorem 2.1 is done allowing X to have a boundary. Denote by $\mathrm{Met}_{\mathrm{cyl}}(X)$ the space of metrics on $\mathrm{int}(X)$ which are conformally equivalent near the boundary to cylindrical end product metrics, which means that in a neighbourhood of the boundary $U \cong V \times [0,1)$, we have $g \in \mathrm{Met}_{\mathrm{cyl}}(X)$, such that $g|_U \cong g_Y + ds^2$, where g_Y is a smooth metric on the three-manifold $Y = \partial X$. The period map is then $\Pi_X : \mathrm{Met}_{\mathrm{cyl}}(X) \to \mathrm{Gr}^+(\widehat{H}^2(X))$. The proof above adapts to show that if X is connected, then Π_X has a dense image.

3. Given H a rational maximal *semi*-positive definite subspace of $H^2(X)$, which is not positive. In particular, we have

$$H \in \partial\overline{\mathrm{Gr}}^+(H^2(X)).$$

   We can find surfaces Σ_i as in the above proof, but now requiring only that $\Sigma_i \cdot \Sigma_i \geq 0$, with equality holding for certain i. Constructing the family of metrics $g(r)$ as before still gives $\lim_{r\to\infty} \mathcal{H}^+_{g(r)} = H$.

2.2 Foundations and Proof of Theorem 2.2

In this section, we will go into detail about the proof of Theorem 2.2. In Subsection 2.2.1, there is a construction of families of metrics parametrised by polyhedra, specifically the permutahedron. In Subsection 2.2.2, we apply this construction to hypersurfaces related to vectors in the cohomology, and we study the behaviour of the period map on the faces of these polyhedra. In Subsection 2.2.3, we focus on the case $b^+ = 1$ and use these constructions to prove Theorem 2.2.

As motivation for the outline of the proof, we will first consider the two cases with the lowest b_2 possible.

Example 3. Consider the case in which $b_2(X) = 2$ and $b^+(X) = 1$, such as $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $S^2 \times S^2$, as shown in Examples 1 and 2. Proving surjectivity of Π_X is straightforward in this case, as the positive Grassmannian is simply an interval for dimensionality reasons, see Figure 2.4. Choose two non-zero vectors v_1 and v_2 in $H^2(X)_\mathbb{Z}$ which square to zero ($v_i \cup v_i = 0$) and whose spans in $H^2(X)$ give the two endpoints of the compactified Grassmannian. With reference to Figure 2.4, we represent the positive Grassmannian as a line in the positive cone connecting two square-zero elements. This implies that, in the figure, we can take the couple v_1, v_2, but not v_1, v_3 or v_2, v_3, since their connecting line do not cross the positive Grassmannian.

Let Σ_1 and Σ_2 be surfaces in X such that they are Poincaré dual to v_1 and v_2, respectively. Form a family of metrics $g(r)$ parametrised by $\mathbb{R}$ which stretches in a cylindrical fashion along the boundary of a regular neighbourhood of Σ_1 (resp. Σ_2) as $r \to -\infty$ (resp. $+\infty$). An application of Proposition 2.9 shows that the period map on this family induces a map from an interval to an interval which is bijective on endpoints. Since the stretching of metrics is continuous, the period map on this family is surjective.

Example 4. Consider the case $b_2(X) = 3$ and $b^+(X) = 1$. Here, the positive Grassmannian may be identified with the hyperbolic plane $\mathbb{H}^2$, which is, in reference to Figure 2.5, one of the two connected components of the blue hyperboloid. In this case, we will construct 2-dimensional families of metrics, parametrised by polygons, where the parametrisation along the edges corresponds to metrics stretched along some three-manifold, a generalisation of what was done for the proof of Theorem 2.1. This follows what was done in Example 3, where the line connecting v_1 and v_2 can be considered as a one-dimensional polytope. Using Proposition 2.9, we can determine the position of where these edges are mapped into $\mathbb{H}^2 \cong \mathrm{Gr}^+(H^2(X))$. It can be shown that there is a polygonal metric family which maps onto any given (rational) hyperbolic simplex in $\mathbb{H}^2$, implying the surjectivity of Π_X by tassellation.

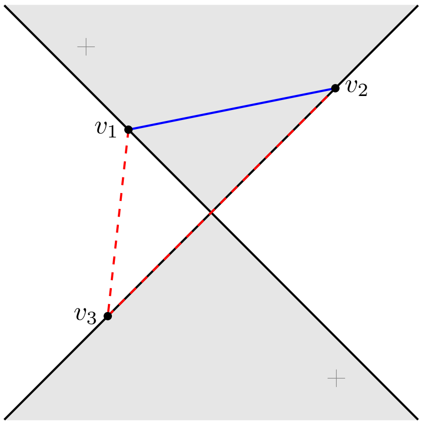

Figure 2.4: $H^2(X)$ with signature $(1, 1)$

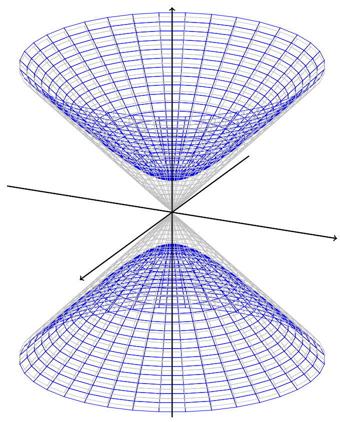

Figure 2.5: $H^2(X)$ with signature $(1, 2)$

These two examples give an idea of the proof in the general case where $b^+(X) = 1$ and $b_2(X) = n + 1$: the goal is to show there is an n-dimensional polytope (specifically the permutahedron P_n) which parametrize a family of metrics which surjects onto any given (rational) hyperbolic simplex in $\mathbb{H}^n$. Then it is sufficient to prove that these simplices tessellate the whole space to prove surjectivity.

2.2.1 Families of metrics parametrised by polyhedra

We will start with constructing a family of metrics parametrised by polyhedra. Then we will focus on a specific polyhedra: the permutahedron.

Construction. Take a neighbourhood of 0 $U \subset (-1, 1) \subset \mathbb{R}$ and consider a family of smooth functions $\psi_r : \mathbb{R} \to \mathbb{R}$ depending smoothly on $r \in [0, \infty]$, such that:

(i) $\psi_0(s) = 1$ for all s,

(ii) $\psi_r(s) = 1$ for $|s| \geq 1$,

(iii) $\psi_r(s) = \frac{1+1/r^2}{s^2+1/r^2}$ for all $s \in U$

(iv) ψ_r is uniformly bounded on $\mathbb{R} \setminus U$.

Definition 2.12. Consider a collection $\mathcal{C} = \{Y_1, Y_2, \dots, Y_m\}$ of closed embedded pairwise disjoint three-submanifolds $Y_i \subset X$, a closed and connected four-manifold. Each Y_i may be disconnected. Choose disjoint collar neighbourhoods $Y_i \times [-1,1] \subset X$, and a metric $g_{\mathbf{0}}$ on X such that $g_{\mathbf{0}} \mid_{Y_i \times [-1,1]} = g_{Y_i} + ds^2$, where all g_{Y_i} are some given metrics on Y_i. Let $Y := Y_1 \cup \cdots \cup Y_m$.

Recall the construction above for the family of smooth functions $\{\psi_r\}_{r \in [0,\infty]}$. For $\mathbf{r} = (r_1, \dots, r_m) \in [0,\infty)^m$, with $g_{\mathbf{0}}$ and g_{Y_i} as given above, define a smooth metric $g_{\mathbf{r}}$ on X by

$$g_{\mathbf{r}}|_{Y_i \times [-1,1]} = g_{Y_i} + \psi_{r_i}(s) ds^2 \qquad \forall i = 1, \dots, m$$
$$g_{\mathbf{r}}|_{X \setminus (Y \times [-1,1])} = g_{\mathbf{0}}$$

Such metrics $g_{\mathbf{r}}$ are smooth on X by definition of $\psi_{\mathbf{r}}$. If we extend the definition in the limit $r_i \to \infty$ for some or all i, we get a family of metrics $g_{\mathbf{r}}$ for $\mathbf{r} \in [0,\infty]^m$, called *broken metrics.*

The broken metric for $\mathbf{r} = (\infty, \dots, \infty)$, restricted to $X \setminus Y$, is conformally equivalent to a metric with cylindrical end metrics on $(-1,0) \times Y$ and $(0,1) \times Y$, and we say that this metric is *broken along* $\mathcal{C}$. More generally, any $g_{\mathbf{r}}$ for $\mathbf{r} \in [0,\infty]^m$ is broken along some subcollection of $\mathcal{C}$, determined by those i for which $r_i = \infty$.

Remark 2.13. This is similar to the construction described in Subsection 2.1.3, except that here we use a different stretching parameter for each Y_i, and also, we do not assume that these Y_i separate X into disconnected manifolds X_i, as in (2.2).

Definition 2.14. Consider a metric $\mathbf{g}$ broken along $\mathcal{C} = \{Y_1, \dots, Y_m\}$ as in Definition 2.12, then its space of self-dual harmonic 2-forms is

$$\mathscr{H}^+_{\mathbf{g}}(X) := j^{-1}\left(\mathscr{H}^+_{X \setminus Y}\right)$$

where $j : H^2(X) \to H^2(X \setminus Y)$ is the map induced by inclusion, and

$$\mathscr{H}^+_{X \setminus Y} \subset \widehat{H}^2(X \setminus Y) = \operatorname{Im}\left(H^2_c(X \setminus Y) \to H^2(X \setminus Y)\right)$$

is the space of self-dual harmonic 2-forms on $X \setminus Y$ under a cylindrical end metric $\mathbf{g}|_{X \setminus Y}$ (a specific kind of metric).

Remark 2.15.

1. Applying Proposition 2.9 to the splitting $X = X \setminus (Y \times (-\varepsilon, \varepsilon)) \cup_{Y \sqcup Y} Y \times (-\varepsilon, \varepsilon)$ guarantees the well-definiteness of Definition 2.14. This is proven since $X \setminus Y \cong X \setminus (Y \times (-\varepsilon, \varepsilon))$ are homotopy equivalent and $H^*(Y \times (-\varepsilon, \varepsilon)) \cong H^*(Y)$ as graded commutative algebras, thus the cup product on $H^2(Y \times (-\varepsilon, \varepsilon))$ is zero by dimensionality reasons. Thus, $\mathscr{H}^+_{X \setminus (Y \times (-\varepsilon,\varepsilon))} = \mathscr{H}^+_{X \setminus Y}$ and $\mathscr{H}^+_{Y \times (-\varepsilon,\varepsilon)} = 0$, which implies that $\mathscr{H}^+_{\mathbf{g}}(X)$ is actually a self-dual subset of a corresponding metric.

2. Through Definition 2.14, we can extend the definition of the period map Π_X to include also broken metrics by setting $\Pi_X(\mathbf{g}) = \mathscr{H}^+_{\mathbf{g}}(X)$.

Proposition 2.16. *Consider a disjoint collection of m hypersurfaces $\mathcal{C}$ in the closed four-manifold X, and construct an associated model family of broken metrics $g_{\mathbf{r}}$ as in Definition*

2.12, where $\mathbf{r} \in [0,\infty]^m$*. Then the period map extends to define a continuous map on this family:*

$$\begin{aligned} \tilde{\Pi} : [0,\infty]^m &\longrightarrow \overline{\mathrm{Gr}}^+(H^2(X)) \\ \mathbf{r} &\longmapsto \Pi_X(g_{\mathbf{r}}) \end{aligned} \tag{2.17}$$

Proof. Observe that the Grassmannian $\overline{\mathrm{Gr}}^+(H^2(X))$ inherits the topology from $H^2(X) \cong \mathbb{R}^{b_2}$ through quotienting.

Claim: *The set* $\mathcal{B} = \{\{\Pi_X(g_{\mathbf{r}}) \mid \mathbf{r} \in \mathrm{B}_\varepsilon(\mathbf{r}_0) \cap [0,\infty]^m\} \mid \mathbf{r}_0 \in [0,\infty]^m, \varepsilon > 0\}$ *forms a basis of the topology on* $\overline{\mathrm{Gr}}^+(H^2(X))$*.*

Proof: We will use the definition of the Hodge star-operator $*_g : \Omega^p(X) \to \Omega^{4-p}(X)$, where $*_g\beta$ is such that $\alpha \wedge *_g\beta = g(\alpha,\beta)\,\mathrm{dvol}_g$ for any $\alpha \in \Omega^p(X)$ where dvol_g is the Riemannian volume element, see [DK97, Sect. 1.1].

Consider any $\mathbf{r}_0 \in [0,\infty]^m$, $\mathbf{n} \in S^{b_2-1}$ and $\varepsilon > 0$ small enough and such that $\mathbf{r}_0 + \varepsilon\mathbf{n} \in [0,\infty]^m$. From Definition 2.12 and smoothness of functions ψ_i, we get $g_{\mathbf{r}_0+\varepsilon\mathbf{n}} \simeq g_{\mathbf{r}_0} + \varepsilon g(\mathbf{n})$, where $g(\mathbf{n}) : \Omega^p(X) \times \Omega^p(X) \to \mathbb{R}$ is such that $g(\mathbf{n})|_{Y_i \times [-1,1]} = \psi'_{r_i}(s)ds^2$ and $g(\mathbf{n})|_{X \setminus Y \times [-1,1]} = 0$. Then, given $\{dy^1, \cdots, dy^4\}$ oriented local coordinates, the volume form of $g_{\mathbf{r}_0+\varepsilon\mathbf{n}}$ is such that

$$\begin{aligned} \mathrm{dvol}_{g_{\mathbf{r}_0+\varepsilon\mathbf{n}}} &= \sqrt{\det g_{\mathbf{r}_0+\varepsilon\mathbf{n}}}\,dy^1 \wedge \cdots \wedge dy^4 \\ &\simeq \sqrt{\det g_{\mathbf{r}_0} + \varepsilon \det g(\mathbf{n})}\,dy^1 \wedge \cdots \wedge dy^4 \\ &\simeq \left(\sqrt{\det g_{\mathbf{r}_0}} + \frac{\varepsilon}{2}\sqrt{\det g(\mathbf{n})}\right) dy^1 \wedge \cdots \wedge dy^4 \\ &\simeq \mathrm{dvol}_{g_{\mathbf{r}_0}} + \frac{\varepsilon}{2}\sqrt{\det g(\mathbf{n})}\,dy^1 \wedge \cdots \wedge dy^4 =: \mathrm{dvol}_{g_{\mathbf{r}_0}} + \varepsilon\,\mathrm{dvol}(\mathbf{n}) \end{aligned}$$

Observe that despite the notion $g(\mathbf{n})$ is not a metric on X and $\mathrm{dvol}(\mathbf{n}) \in \Omega^4(X)$ is not a volume form of X. Thus, applying these approximations to the Hodge star definition, we get

$$\begin{aligned} \alpha \wedge *_{g_{\mathbf{r}_0+\varepsilon\mathbf{n}}}\beta &= g_{\mathbf{r}_0+\varepsilon\mathbf{n}}(\alpha,\beta)\,\mathrm{dvol}_{g_{\mathbf{r}_0+\varepsilon\mathbf{n}}} \\ &\simeq \left(g_{\mathbf{r}_0}(\alpha,\beta) + \varepsilon g(\mathbf{n})(\alpha,\beta)\right)\left(\mathrm{dvol}_{g_{\mathbf{r}_0}} + \varepsilon\,\mathrm{dvol}(\mathbf{n})\right) \\ &\simeq g_{\mathbf{r}_0}(\alpha,\beta)\,\mathrm{dvol}_{g_{\mathbf{r}_0}} + \varepsilon\left(g(\mathbf{n})(\alpha,\beta)\,\mathrm{dvol}_{g_{\mathbf{r}_0}} + g_{\mathbf{r}_0}(\alpha,\beta)\,\mathrm{dvol}(\mathbf{n})\right) \\ &\simeq \alpha \wedge *_{g_{\mathbf{r}_0}}\beta + \varepsilon\left(g(\mathbf{n})(\alpha,\beta)\,\mathrm{dvol}_{g_{\mathbf{r}_0}} + g_{\mathbf{r}_0}(\alpha,\beta)\,\mathrm{dvol}(\mathbf{n})\right) \end{aligned}$$

This implies that for any $\beta \in \Omega^p(X)$ we have $*_{g_{\mathbf{r}_0+\varepsilon\mathbf{n}}}\beta \simeq *_{g_{\mathbf{r}_0}}\beta + \varepsilon f(\beta)$, for some smooth function $f : \Omega^p(X) \to \Omega^{4-p}(X)$. Let ω_ε be one of the self-dual 2-forms with respect to the metric $g_{\mathbf{r}_0+\varepsilon\mathbf{n}}$, we get

$$\omega_\varepsilon = *_{g_{\mathbf{r}_0+\varepsilon\mathbf{n}}}\omega_\varepsilon \simeq *_{g_{\mathbf{r}_0}}\omega_\varepsilon + \varepsilon f(\omega_\varepsilon)$$

which implies that ω_ε is a self-dual 2-form with respect to $g_{\mathbf{r}_0}$ up to an ε-term. Thus, there exists a 2-form ω_0 self-dual with respect to the metric $g_{\mathbf{r}_0}$, so that we can write $\omega_\varepsilon \simeq \omega_0 + \varepsilon\tilde{f}(\omega_0)$ with sum following from $H^2(X) \cong \mathbb{R}^{b_2}$ where $\tilde{f} : H^2(X) \to H^2(X)$ is such that

$$\begin{aligned} \omega_\varepsilon &\simeq *_{g_{\mathbf{r}_0+\varepsilon\mathbf{n}}}\left(\omega_0 + \varepsilon\tilde{f}(\omega_0)\right) \simeq *_{g_{\mathbf{r}_0}}\left(\omega_0 + \varepsilon\tilde{f}(\omega_0)\right) + \varepsilon f(\omega_0 + \varepsilon\tilde{f}(\omega_0)) \\ &\simeq *_{g_{\mathbf{r}_0}}\omega_0 + \varepsilon *_{g_{\mathbf{r}_0}}\tilde{f}(\omega_0) + \varepsilon f(\omega_0 + \varepsilon\tilde{f}(\omega_0)) \simeq \omega_0 + \varepsilon\left(*_{g_{\mathbf{r}_0}}\tilde{f}(\omega_0) + f(\omega_0)\right) \end{aligned}$$

Consider $\Gamma_\varepsilon \in \overline{\mathrm{Gr}}^+(H^2(X))$ generated by $\Gamma_\varepsilon = \langle \omega_\varepsilon^1, \cdots, \omega_\varepsilon^{b^+} \rangle$ self-dual b^+-plane with respect to the metric $g_{\mathbf{r}_0+\varepsilon\mathbf{n}}$, as above we have a corresponding 2-form ω_0^i such that $\omega_\varepsilon^i \simeq \omega_0^i + \varepsilon\tilde{f}(\omega_0^i)$ for all $i = 1, \cdots, b^+$. Then we can define $\Gamma_0 = \langle \omega_0^1, \cdots, \omega_0^{b^+} \rangle$ as the self-dual b^+-plane with respect to the metric $g_{\mathbf{r}_0}$, and by construction it is such that $\Gamma_\varepsilon \simeq \Gamma_0 + \varepsilon h(\Gamma_0)$ where $h : \overline{\mathrm{Gr}}^+(H^2(X)) \to \overline{\mathrm{Gr}}^+(H^2(X))$.

Thus, any $U \in \mathcal{B}$ is an open subset in $\overline{\mathrm{Gr}}^+(H^2(X))$, by construction. Since ε can be shown as small as possible, we can make any subset as a union of these elements. Thus, the claim follows. ■

Given the claim, the map $\tilde{\Pi}$ in (2.17) is such that

$$\tilde{\Pi}^{-1}\left(\{\Pi_X(g_{\mathbf{r}}) \mid \mathbf{r} \in \mathrm{B}_\varepsilon(\mathbf{r}_0) \cap [0,\infty]^m\}\right) = \mathrm{B}_\varepsilon(\mathbf{r}_0) \cap [0,\infty]^m$$

which is open in $[0,\infty]^m$, which guarantees continuity. □

Definition 2.17. Consider an n-dimensional convex polytope P. We say that P is *labelled by hypersurfaces in X* if to each face $F \subset \partial P$ on the boundary of P with codimension m is associated $\mathcal{C}_F$, a collection of m pairwise disjoint hypersurfaces, such that to the face intersection of F and F' is associated $\mathcal{C}_{F\cap F'} \subset \mathcal{C}_F \cup \mathcal{C}_{F'}$.

Construction. The goal is to construct $G_P = \{g_p\}_{p\in P}$, a family of broken metrics parametrised by the polytope P, by induction on $\dim P$. If $\dim P = 0$, then $G_P = \{g\}$ with g any smooth metric.

For the inductive step, let $\dim P = n$ and assume that all $\dim < n$ families of metrics are constructed. Consider a point p on the interior of a proper face F of P of codimension m. Since P is labelled by hypersurfaces, the proper face F has associated the collection $\mathcal{C}_F = \{Y_1, \cdots, Y_m\}$. By definition, the point p has a neighbourhood of the form $(0,\infty]^m \times P'$, where $P' \subset F$ is a convex polytope. To describe a neighbourhood of g_p in G_P, let $p = (\mathbf{r}, p') \in (0,\infty]^m \times P' \subset P$. From the inductive hypothesis we get $g_{p'}$ the associated metric to p', which implies the definition of g_p:

$$g_p|_{Y_i\times[-1,1]} = g_{Y_i,p'} + \psi_{r_i}(\mathbf{s})d\mathbf{s}^2 \qquad \forall i = 1,\dots,m$$
$$g_p|_{X\setminus(Y\times[-1,1])} = g_{\mathbf{0},p'}$$

where $g_{Y_i,p'}$ and $g_{\mathbf{0},p'}$ follow from the definition $g_{p'}$.

This gives a family of broken metrics parametrised by $[0,\infty]^m \times P'$, which is our model for a neighbourhood of $g_p \in G_P$. Using convexity properties of metrics, we can construct a family parametrised by a collar neighbourhood of the whole boundary $\partial P \subset P$ satisfying these model conditions. Extending this to all of P through again convexity properties defines G_P.

Remark 2.18. The family of metrics G_P parametrised by the polytope P gives a map

$$\Pi_{X,P} : P \longrightarrow \overline{\mathrm{Gr}}^+(H^2(X))$$
$$p \longmapsto \Pi_X(g_p)$$

This map is continuous by Proposition 2.16 and the construction above. Furthermore, since g_p for $p \in \mathrm{int}(P)$ is a smooth (unbroken) metric by definition, the image of $\mathrm{int}(P)$ under the *extended period map* $\Pi_{X,P}$ is contained in the image of the non-extended period map Π_X from Definition 1.7.

Definition 2.19. The *n-dimensional permutahedron* P_n is a polytope, whose proper faces are labelled by nested sequences $\mathbf{I} = \{I_i\}_{i=1}^{l}$ $(1 \leq l \leq n)$ of the form

$$\emptyset \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_l \subsetneq \{1, \ldots, n+1\}. \tag{2.18}$$

Such nested sequences form a poset by declaring $\mathbf{I} \leq \mathbf{I}'$ if and only if $\mathbf{I}' \subset \mathbf{I}$. Then the poset of proper non-empty faces of P_n is isomorphic to the poset of such nested sequences.

Notation. Geometrically, we can consider $P_n \subset \mathbb{R}^{n+1}$ as a polytope in the hyperplane $\{x_1 + \cdots x_{n+1} = 1 + 2 + \cdots + n + (n+1)\}$ with vertex coordinates as permutations of the first $(n+1)$ natural numbers. The notation P_n is non-standard here, as n refers to the dimension of the permutahedron as a polytope, while we are working on the set $\{1, \ldots, n+1\}$. For reference, see Figure 2.6, where P_2 is shown.

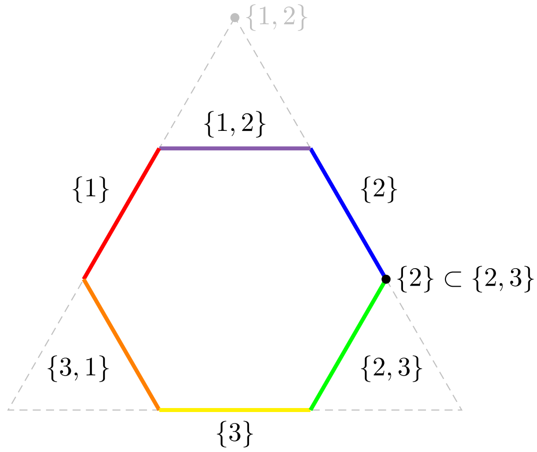

Figure 2.6: Here we consider the case of $n = 2$: to the collection of three surfaces $\Sigma_1, \Sigma_2, \Sigma_3 \subset X$ is associated a metric family parametrized by the permutahedron P_2, which is a hexagon. Each face is labelled by a non-empty proper subset of $\{1, 2, 3\}$ (a nested sequence with one subset). Faces $\{i\}$ and $\{i, j\}$ correspond to metrics broken along boundaries of regular neighborhoods of $\bigcup_{k \in \{i\}} \Sigma_k = \Sigma_i$ and $\bigcup_{k \in \{i,j\}} \Sigma_k = \Sigma_i \cup \Sigma_j$, respectively, as defined in the construction. The vertex joining $\{i\}$ and $\{i, j\}$ is the nested sequence $\{i\} \subset \{i, j\}$.
The triangle in grey is a 2-simplex Δ_2 is here to show how the forgetful map $\mathfrak{F} : P_n \to \Delta_n$ acts for $n = 2$: the purple face $\{1, 2\}$ is sent to the vertex $\{1, 2\}$.

Remark 2.20. Consider a collection of closed, embedded surfaces $\mathcal{S} = \{\Sigma_1, \ldots, \Sigma_{n+1}\}$ in X, not necessarily pairwise disjoint. For each $I \subset \{1, \ldots, n+1\}$, let X_I is a regular neighbourhood of $\cup_{i \in I} \Sigma_i$, and let $Y_I = \partial X_I \subset X$ be the three-manifold. Without loss of generality, these Y_I can be chosen so that $I \subset J$ implies $Y_I \cap Y_J = \emptyset$.

Following the same notation as in Definition 2.19, the face $F_{\mathbf{I}} \subset P_n$ corresponding to $\mathbf{I} = \{I_i\}_{i=1}^{l}$ has codimension l in P_n. Following the construction above, since there is a one-to-one correspondence between a poset of nested sequences and the set of three-manifolds Y_I, given by

$$\mathbf{I} \longleftrightarrow \{Y_I \mid I \in \mathbf{I}\}$$

Then we associate to $F_{\mathbf{I}}$ the collection $\mathcal{C}_{\mathbf{I}} = \{Y_I \mid I \in \mathbf{I}\}$ which is a disjoint set of closed embedded three-manifolds in X, making P_n a polytope with faces labelled by hypersurfaces in X.

Thus, applying the metric family construction to any collection of closed, embedded surfaces $\mathcal{S} = \{\Sigma_1, \dots, \Sigma_{n+1}\}$ in X, we obtain a family of metrics on X parametrised by the n-dimensional permutahedron P_n. We remark that for a nested sequence as in (2.18), by convention we define $I_{l+1} = \{1, \dots, n+1\}$.

Construction. The general construction of this kind of model family of metrics can be summarised as follows. To any collection of closed, embedded surfaces $\mathcal{S} = \{\Sigma_1, \dots, \Sigma_{n+1}\}$ in X, we can define a corresponding set of three-manifolds Y_I defined above. To these posets of manifolds, we associate posets of nested sequences, which are associated to posets of proper non-empty faces of P_n. These posts of faces yield a model family of metrics.

2.2.2 Application to cohomology

We apply the construction above to connect permutahedra with bases in the second cohomology space, and we use these to define our collection of embedded surfaces.

Given any subset $V = \{v_1, \dots, v_{n+1}\} \subset H^2(X)_{\mathbb{Z}}$, we can choose closed, connected embedded surfaces $\Sigma_i \subset X$ such that $[\Sigma_i]$ is Poincaré dual to v_i, as explained in Remark 2.11 point 1. We assume that the surfaces are pairwise transverse, and that $\Sigma_i \cap \Sigma_j \cap \Sigma_k = \emptyset$ for distinct i, j, k. Since we can apply the construction of the previous paragraph to the collection $\mathcal{S} = \{\Sigma_1, \dots, \Sigma_{n+1}\}$, associated to any subset $V \subset H^2(X)_{\mathbb{Z}}$ of size $n+1$ we construct a family of metrics parametrized by a permutahedron P_n, and obtain an associated period map

$$\Pi_{X,V} : P_n \longrightarrow \overline{\mathrm{Gr}}^+(H^2(X)). \tag{2.19}$$

Notation. For a subspace $W \subset H^2(X)$, we write $W^\perp$ for its complement with respect to the intersection pairing, and $\mathrm{null}(W) = W \cap W^\perp$. Write $V_I = \mathrm{span}\{v_i \mid i \in I\}$ for the subspace of $H^2(X)$ spanned by $\{v_i\}_{i \in I}$.

Proposition 2.21. *Let $\Pi_{X,V} : P_n \to \overline{\mathrm{Gr}}^+(H^2(X))$ be the period map associated to a family of metrics parametrized by the permutathedron P_n, constructed from $V = \{v_1, \dots, v_{n+1}\} \subset H^2(X)_{\mathbb{Z}}$ as above. For a face $F_{\mathbf{I}} \subset \partial P_n$ corresponding to $\mathbf{I} = \{I_i\}_{i=1}^l$ as in (2.18), and $g \in \mathrm{int}(F_{\mathbf{I}}) \subset \partial P_n$, we have*

$$\Pi_{X,V}(g) = H_{l+1}^+ + \sum_{i=1}^{l} H_i^+ + \mathrm{null}(V_{I_i}) \tag{2.20}$$

for some collection of $H_i^+ \in \mathrm{Gr}^+(V_{I_i} \cap V_{I_{i-1}}^\perp)$, for $i = 1, \dots, l$, and $H_{l+1}^+ \in \mathrm{Gr}^+(V_{I_l}^\perp)$.

Remark 2.22. The requirement for $g \in \mathrm{int}(F_{\mathbf{I}})$ (parametrized by an element in $\mathrm{int}(F_{\mathbf{I}})$) is here to guarantee that a neighbourhood of g is of shape $(0, \infty]^l \times F'$, where F' is a face of a given codimension l.

Proof. Recall that we defined X_I as a regular neighbourhood of $\cup_{i \in I} \Sigma_i$, where $[\Sigma_i] = \mathrm{PD}(v_i)$. We also set $Y_I = \partial X_I \subset X$.

Observe that the hypersurfaces $Y_i := Y_{I_i}$ decompose X into the union of the $l+1$ submanifolds $X_i := X_{I_i} \setminus \mathrm{int}(X_{I_{i-1}})$ and thus that $\partial X_i = Y_i \cup (-Y_{i-1})$. By convention we define $Y_0 = Y_{l+1} = \emptyset$ and $X_{I_{l+1}} = X$. To conclude let $\delta_i : H^1(Y_i) \to H^2(X)$ the map from the Mayer-Vietoris sequence for X decomposed along Y_i.

Let $g \in \mathrm{int}(F_{\mathbf{I}})$, since it is broken, it is the limit of a family of smooth metrics $g_{\mathbf{r}}$ for some r_i. Thus, applying Proposition 2.9 in the form of Equation (2.12) iteratively on these

r_i, we get

$$\Pi_{X,V}(g) = \sum_{i=1}^{l+1} H_i^+ + \operatorname{Im}(\delta) \tag{2.21}$$

where $H_i^+ \subset H_c^2(X_i)$ maps isomorphically to $\widehat{H}^2(X_i)$ as the space of L^2 self-dual harmonic 2-forms on X_i, and $\delta : H^1(Y) \to H^2(X)$, where $Y = \cup_{i=1}^l Y_i$ and $\delta = \sum \delta_i$. So now, we have to prove that $\operatorname{Im}(\delta) = \sum_{i=1}^l \operatorname{null}(V_{I_i})$, starting from connecting V_i to $H_c^2(X_{I_i})$.

<u>Claim:</u> *The image of*

$$H_c^2(X_{I_i}) \to H^2(X) \tag{2.22}$$

is equal to V_i, where $V_i := V_{I_i}$ for $i = 1, \dots, l$ and $V_{l+1} = H^2(X)$.

<u>Proof:</u> If $i = l+1$, the claim follows directly from the definition: we defined X to be closed and connected, so $H_c^2(X_{I_{l+1}}) = H_c^2(X) \cong H^2(X)$. Thus, $\operatorname{Im}(H_c^2(X_{I_{l+1}}) \to H^2(X)) = H^2(X) = V_{l+1}$.

Otherwise, X_{I_i} is defined as the regular neighbourhood of $\cup_{j \in I_i} \Sigma_j$, so locally we have $X_{I_i} \cong D^2 \times \cup_{j\in I_i} \Sigma_j$. This implies $H_c^2(X_{I_i}) \cong H^2(D^2) \otimes H_c^2(\cup_{j \in I_i} \Sigma_j) \cong H_c^2(\cup_{j\in I_i}\Sigma_j)$ and so $H_c^2(X_{I_i})$ is generated by the Poincaré duals of the surfaces Σ_j. By definition of Poincaré duality, the classes $[\Sigma_j]$ are sent under (2.22) to the v_j, which proves the claim. We remark here that to be able to define $[\Sigma_j]$ as homology classes, the surfaces Σ_i must be connected. To have the corresponding v_j orthogonal to each other, we also require that the Σ_i are pairwise transverse and $\Sigma_i \cap \Sigma_j \cap \Sigma_k = \emptyset$ for distinct i, j, k. ■

Now we want to connect V_i and H_i^+. Observe that $H_c^2(X_i) \to H^2(X)$ factors through (2.22): considering the inclusions $X_i \overset{i_i}{\hookrightarrow} X_{I_i} \overset{j_i}{\hookrightarrow} X$ we get

$$\begin{array}{ccccc} H_c^2(X_i) & \longrightarrow & H_c^2(X) & \hookrightarrow & H^2(X) \\ \downarrow {\scriptstyle i_{i\#}} & \nearrow {\scriptstyle j_{i\#}} & & & \\ H_c^2(X_{I_i}) & & & & \end{array}$$

which implies that $\operatorname{Im}(incl \circ j_{i\#} \circ i_{i\#}) \subset \operatorname{Im}(incl \circ j_{i\#})$. Furthermore, this map is injective when restricted to H_i^+, since H_i^+ is chosen to map isomorphically to $\mathscr{H}_{X_i}^+$. Thus, $H_i^+ \subset V_i$, following from claim above.

By construction, forms in H_i^+ are compactly supported outside of $X_{I_{i-1}}$, thus given $\iota_i : X_i \hookrightarrow X$, for any $\alpha \in H_i^+$ and $\beta \in V_{i-1}$ we have by definition $\iota_i^*(\alpha) = \alpha_i$ and $\iota_i^*(\beta) = 0$. Thus, their intersection form is

$$\begin{aligned} q(\alpha,\beta) &= \alpha \cap \operatorname{PD}(\beta) = \alpha \cap \iota_{i*}(\operatorname{PD}({\iota_i}^*\beta)) = \iota_{i*}({\iota_i}^*\alpha \cap (\operatorname{PD}({\iota_i}^*\beta))) \\ &= \iota_{i*}({\iota_i}^*\alpha \cap (\operatorname{PD}(0))) = 0 \end{aligned}$$

and so we also have $H_i^+ \subset V_{i-1}^\perp$. Considering these results, we get $H_i^+ \subset V_i \cap V_{i-1}^\perp$. Since $H_c^2(X_i) \to H^2(X)$ does not change the positivity of the forms with respect to the self-intersection, we also get $|H_i^+| = b^+(V_{i-1}^\perp \cap V_i)$.

Further, $\operatorname{Im}(\delta_i)$ is in the nullspace of the pairing on $H_c^2(X_{I_i})$, and so it gets mapped into the nullspace of V_i, $\operatorname{null}(V_i) \subset H^2(X)$. We obtain $\operatorname{Im}(\delta) = \operatorname{Im}(\sum \delta_i) = \sum_{i=1}^l \operatorname{Im}(\delta_i) \subset \sum_{i=1}^l \operatorname{null}(V_i)$. This, together with (2.21), implies that the left side of (2.20) is contained in the right side. To complete the proof, it suffices to show that the dimension of the right-hand side is equal to $b^+(X)$. To do so, we first need $b^+(V_i)$.

Claim: *$b^+(V_i) = b^+(V_{i-1}) + b^+(V_{i-1}^\perp \cap V_i) + |N_{i-1}| - |N_{i-1} \cap N_i|$ for $1 \leq i \leq l+1$ where we denoted $N_i = \mathrm{null}(V_i)$, and $b^+(W)$ is the dimension of a maximal positive subspace of W.*

Proof: Observe that we have a decomposition with respect to the orthogonal pairing

$$V_i \setminus N_i = (V_{i-1} \setminus N_{i-1}) \oplus \left(V_{i-1}^\perp \setminus N_{i-1} \cap V_i \setminus N_i\right) \oplus (N_{i-1} \setminus N_i)$$

Similarly as in the proof of Lemma 2.7, we get

$$\begin{aligned} b^+(V_i) &= b^+(V_i \setminus N_i) \\ &= b^+(V_{i-1} \setminus N_{i-1}) + b^+(V_{i-1}^\perp \setminus N_{i-1} \cap V_i \setminus N_i) + \mid N_{i-1} \setminus N_i \mid \\ &= b^+(V_{i-1}) + b^+(V_{i-1}^\perp \cap V_i) + \mid N_{i-1} \mid - \mid N_{i-1} \cap N_i \mid \end{aligned}$$

■

Now we need to connect these $b^+(V_i)$ to $b^+(X)$. Since we have defined $X = X_{I_{l+1}}$ and $V_0 = N_0 = \emptyset = N_{l+1}$, iterating the identity from the claim yields

$$\begin{aligned} b^+(X) = b^+(V_{l+1}) &= b^+(V_l) + b^+(V_l^\perp \cap V_{l+1}) + \mid N_l \mid - \mid N_l \cap N_{l+1} \mid \\ &= b^+(V_{l-1}) + b^+(V_{l-1}^\perp \cap V_l) + \mid N_{l-1} \mid - \mid N_{l-1} \cap N_l \mid + \\ &\qquad + b^+(V_l^\perp \cap V_{l+1}) + \mid N_l \mid - \mid N_l \cap N_{l+1} \mid \\ &= \cdots = b^+(V_0) + \sum_{i=1}^{l+1} b^+(V_{i-1}^\perp \cap V_i) + \sum_{i=1}^{l+1} (|N_{i-1}| - |N_{i-1} \cap N_i|) \\ &= \sum_{i=1}^{l+1} b^+(V_{i-1}^\perp \cap V_i) + \sum_{i=1}^{l} (|N_i| - |N_i \cap N_{i+1}|) \,. \end{aligned} \tag{2.23}$$

Thus, we need to write the second part in (2.23) in a different shape.

Claim: *$N_{i-1} \cap N_i = N_{i-1} \cap \sum_{j=i}^{l} N_j$ for any $i = 2, \cdots, l$.*

Proof: Use the fact that $N_i = V_i \cap V_i^\perp$, $V_{i-1} \subset V_i$ and $V_{i+1}^\perp \subset V_i^\perp$. Then

$$\begin{aligned} N_{i-1} \cap N_i &= V_{i-1} \cap V_{i-1}^\perp \cap V_i \cap V_i^\perp = V_{i-1} \cap V_i^\perp \\ N_{i-1} \cap \sum_{j=i}^{l} N_j &= V_{i-1} \cap V_{i-1}^\perp \cap \left(V_i \cap V_i^\perp + \cdots + V_l \cap V_l^\perp\right) \\ &= V_{i-1} \cap \left(V_i^\perp + \cdots + V_l^\perp\right) = V_{i-1} \cap V_i^\perp \end{aligned}$$

■

Then we finish the proof by applying the claims:

$$\begin{aligned} \left|\sum_{i=1}^{l} N_i\right| &= |N_1| + \left|\sum_{i=2}^{l} N_i\right| - \left|N_1 \cap \sum_{i=2}^{l} N_i\right| = |N_1| - |N_1 \cap N_2| + \left|\sum_{i=2}^{l} N_i\right| \\ &= \cdots = \sum_{i=1}^{l-1} (|N_i| - |N_i \cap N_{i+1}|) + |N_l| = \sum_{i=1}^{l} (|N_i| - |N_i \cap N_{i+1}|) \end{aligned}$$

which is equal to last part of (2.23). The H_i^+ are pairwise orthogonal, since they all have pairwise disjoint support, and similarly, each H_i^+ is orthogonal to all N_i. This implies

$$\begin{aligned} b^+(X) &= \sum_{i=1}^{l+1} b^+(V_{i-1}^\perp \cap V_i) + \sum_{i=1}^{l} (|N_i| - |N_i \cap N_{i+1}|) \\ &= \sum_{i=1}^{l+1} \left|H_i^+\right| + \left|\sum_{i=1}^{l} N_i\right| = \left|\sum_{i=1}^{l+1} H_i^+\right| + \left|\sum_{i=1}^{l+1} N_i\right| = \left|\sum_{i=1}^{l+1} (H_i^+ + N_i)\right| \end{aligned}$$ □

Recall. In general, if W is a vector space with a negative definite inner product, then $\mathrm{Gr}^+(W)$ is a point, corresponding to the zero subspace of W.

Remark 2.23. The statement is more general than what is needed for Theorem 2.2. Consider the case when the intersection pairing on V_i is non-degenerate for each i, i.e. $N_i = \emptyset$, with notation as above. Then we have a direct sum decomposition

$$H^2(X) = \bigoplus_{i=1}^{l+1} V_i \cap V_{i-1}^\perp,$$

and (2.20) implies that an element in the face $F_\mathbf{I}$ associated to the nested sequence $\mathbf{I} = \{I_i\}_{i=1}^l$ is

$$\Pi_{X,V}(g) = H_{l+1}^+ + \sum_{i=1}^{l} H_i^+$$

with respect to this decomposition. Thus we have

$$\begin{aligned} \Pi_{X,V}(\operatorname{int} F_\mathbf{I}) &= \bigcup_{g \in \operatorname{int} F_\mathbf{I}} \left(H_{l+1}^+ + \sum_{i=1}^{l} H_i^+ \right) \\ &\subset \prod_{i=1}^{l+1} \mathrm{Gr}^+(V_i \cap V_{i-1}^\perp) \subset \mathrm{Gr}^+(H^2(X)). \end{aligned} \tag{2.24}$$

If both $b^+(X)$ and $b^-(X)$ are greater than 1, then the first manifold in (2.24) is of codimension at least 2 in $\mathrm{Gr}^+(H^2(X))$, whenever it is proper. A similar remark holds in the case in which one or more of the V_i are degenerate. This is one reason that the proof of Theorem 2.2 given below does not directly generalise to the case when both $b^+(X)$ and $b^-(X)$ are greater than 1. This is why the codimension of this submanifold is the minimum codimension of the face of the simplex as built in the proof, and codimension 2 and bigger impedes tesselation of the Grassmannian.

2.2.3 The case $b^+ = 1$ and the formal proof of Theorem 2.2

We restrict to the case in which $b^+(X) = 1$ and let $n = b_2(X) - 1 \geq 2$. The requirement for $n \geq 2$ is justified by the fact that for $n = 1$ we have already considered the two cases in Example 3. We will follow the structure of proof outlined in Example 4.

Recall. By $\mathbb{R}^{1,n}$ we mean the vector space $\mathbb{R}^{n+1}$ equipped with the bilinear form

$$x \cdot y = x_0 y_0 - x_1 y_1 - \cdots - x_n y_n. \tag{2.25}$$

On $\mathbb{R}^{1,n}$ any n-dimensional hyperbolic space may be defined as the hyperboloid

$$\mathbb{H}^n = \{x \in \mathbb{R}^{1,n} \mid x \cdot x = 1,\ x_0 > 0\}$$

with metric induced by the negative of (2.25). Using this correspondence, the space of positive lines $\ell \subset \mathbb{R}^{1,n}$ is identified with $\mathbb{H}^n$ by sending ℓ to $\ell \cap \mathbb{H}^n$, see the case $n = 2$ shown in Figure 2.5. Through this, we identify $\mathrm{Gr}^+(H^2(X))$ with $\mathbb{H}^n$, where rational points (points corresponding to lines spanned by rational vectors) are preserved.

Choose an isometry between $H^2(X)$, equipped with its intersection pairing, and $\mathbb{R}^{1,n}$ If possible, we also require that the integral lattice $H^2(X)_\mathbb{Z}$ is sent to $\mathbb{Z}^{1,n} \subset \mathbb{R}^{1,n}$ under this isomorphism. Such requirement is not always possible, as a counterexample of this, consider the case of $H \oplus E_8$ where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad E_8 = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} \in \mathbb{R}^{8\times 8}$$

with $\mathrm{sgn}(H) = (1, 1)$, $\mathrm{sgn}(E_8) = (0, 8)$. The matrix $H \oplus E_8$ is equivalent to

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{pmatrix}$$

on $\mathbb{R}$ but not on $\mathbb{Z}$.

In the isomorphism $H^2(X) \to \mathbb{R}^{1,n}$ it may not be possible to arrange that $H^2(X)_\mathbb{Z}$ maps onto $\mathbb{Z}^{1,n}$. For the proof of Theorem 2.2, we need enough simplices in $\mathbb{H}^n$ defined over these points. So, it is sufficient to prove that $\mathbb{R}\cdot H^2(X)_\mathbb{Z}$ is dense in the positive Grassmannian. To check this, take the inclusion $i : H^k(X;\mathbb{Z}) \hookrightarrow H^k(X;\mathbb{R})$. Using the fact that the homology groups of closed manifolds are finitely generated, by the universal coefficient theorem

$$H^k(X;\mathbb{Z}) \otimes \mathbb{R} \cong H^k(X;\mathbb{R})$$
$$[f]_\mathbb{Z} \otimes c \mapsto c \cdot [f]_\mathbb{Z}$$

and with this identification we have $i([f]_\mathbb{Z}) = [f]_\mathbb{Z} \otimes 1$. Take norm 1 representative of a class in $\mathbb{R} \cdot H^2(X;\mathbb{Z}) \subset \mathrm{Gr}^+(H^2(X))$. Then we get $[\mathbb{R} \cdot H^2(X;\mathbb{Z})] \cong \mathbb{Q}^{1,n} \cap S^n \subset \mathbb{R}^{1,n} \cap S^n \cong \mathrm{Gr}^+(H^2(X))$ which is dense, as required.

Let $V = \{v_1, \dots, v_{n+1}\} \subset H^2(X)_\mathbb{Z}$ be a linearly independent set, and construct an associated family of metrics parametrised by P_n. As defined in (2.19), we then have the period map

$$\Pi_{X,V} : P_n \longrightarrow \mathbb{H}^n$$

and our goal is to show that this is surjective. To achieve this, we will choose V such that it determines a simplex $\Delta_V \subset \mathbb{H}^n$, and show that $\Pi_{X,V}$ maps P_n onto Δ_V.

To do so, first we have to connect Δ_n to P_n. The n-simplex Δ_n may be described combinatorially as having its proper faces labelled by nonempty proper subsets $I \subset \{1, \dots, n+1\}$. The correspondence is such that the face F_I is of codimension $|I|$, and $F_I \subset F_J$ if and only if $J \subset I$. This can be connected to the description of Δ_n that arises from realizing it as the region in n-dimensional space bounded by a collection of hyperplanes labelled by $1, \dots, n+1$, where each proper face is labelled by the set of the numbers of the hypersurfaces they are the intersection of, for reference see Figure 2.6.

There is a "forgetful" map

$$\mathfrak{F} : P_n \longrightarrow \Delta_n$$

which on the boundary maps the face $F_\mathbf{I}$ to F_{I_l} where I_l is the maximal proper subset appearing in the nested sequence $\mathbf{I}$. In fact, geometrically the permutahedron P_n can be viewed as a truncation of the simplex Δ_n, see Figures 2.6 and 2.7. From this viewpoint, the map $\mathfrak{F}$ is the result of collapsing faces on P_n introduced by truncation onto corresponding faces in Δ_n.

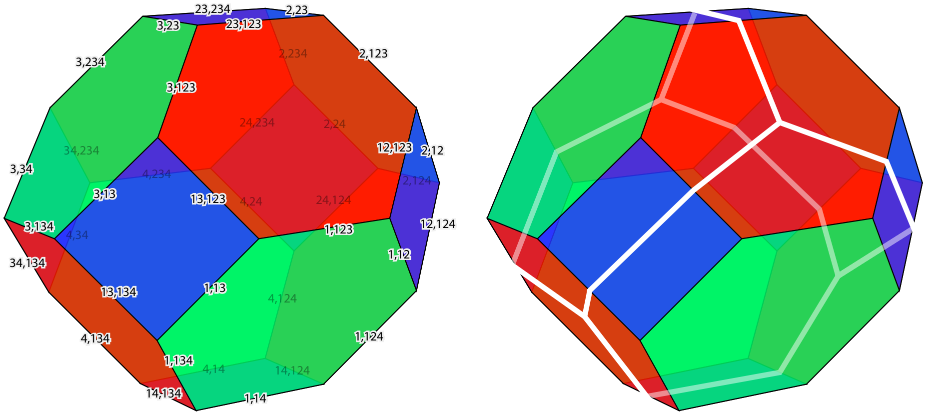

Figure 2.7: Description of the permutahedron P_3. **Left:** Here, each edge is labelled, where "34, 134" is shorthand for the nested sequence $\{3, 4\} \subset \{1, 3, 4\}$. The green (resp. red) hexagonal faces correspond to nested sequences with one subset of size 1 (resp. 3), such as $\{1\}$ (resp. $\{1, 2, 3\}$). Square blue faces correspond to nested sequences with one subset of size 2. Paint the faces of the 3-simplex Δ_3 green. Then P_3 is obtained from Δ_3 by cutting off corners (revealing red), and further truncating (revealing blue). **Right:** The map $\mathbf{i} : \Delta_3 \to P_3$ sends the 1-skeleton of Δ_3 into ∂P_3 as shown. The map $\mathfrak{F} : P_3 \to \Delta_3$ is the result of collapsing the truncated faces (red collapses to vertices, and blue collapses to edges).

We can also define another map

$$\mathbf{i} : \Delta_n \longrightarrow P_n$$

which is described as follows: viewing P_n inside of Δ_n as the result of truncation, $\mathbf{i}$ sends a given point in Δ_n to its closest point in P_n, see the explanation in the caption of Figure 2.7. For any face $F_I \subset \partial\Delta_n$, we have

$$\mathbf{i}(F_I) \subset \bigcup F_{\mathbf{I}} \subset \partial P_n \tag{2.26}$$

where the union is over all $\mathbf{I} = \{I_i\}_{i=1}^{l}$ such that $I \subset I_l$.

The last ingredient for the actual proof of Theorem 2.2 is Lemma 2.25, which implies the surjectivity of maps behaving like $\mathfrak{F}$. We will just recall the following Theorem, since it will be used in the proof of the Lemma.

Theorem 2.24 (Surjectivity theorem)**.** *Let $g : D^n \to D^n$ be a continuous map such that $g(S^{n-1}) \subseteq S^{n-1}$ and the resulting map $f = g|_{S^{n-1}}$ has non-trivial degree. Then g is surjective.*

Lemma 2.25. *Let $f : P_n \to \Delta_n$ be a continuous map such that for each face $F_{\mathbf{I}} \subset \partial P_n$ corresponding to a nested sequence $\mathbf{I} = \{I_i\}_{i=1}^{l}$ we have $f(F_{\mathbf{I}}) \subset F_{I_l} \subset \partial\Delta_n$. Then f is surjective.*

Proof. Let F_I be a face of Δ_n corresponding to $I \subset \{1, \ldots, n+1\}$. Then we get

$$f(\mathbf{i}(F_I)) \subset f\left(\bigcup F_{\mathbf{I}}\right) \subset \bigcup f\left(F_{\mathbf{I}}\right)$$

where the union is over all nested sequences $\mathbf{I}$ with $I \subset I_l$, as in (2.26). By assumption, each $f(F_{\mathbf{I}}) \subset F_{I_l}$. On the other hand, $F_{I_l} \subset F_I$ since $I \subset I_l$. Thus putting all together $f(\mathbf{i}(F_I)) \subset F_I$.

Claim: *The map* $f \circ \mathbf{i}$ *has degree* 1 *as a map on* $(\Delta_n, \partial\Delta_n)$.

Proof: Since $f(\mathbf{i}(F_I)) \subset F_I$, $f \circ \mathbf{i}$ is a self-map on the simplex Δ_n which maps each face into itself. By induction on n and continuity, we get $f \circ \mathbf{i}|_{\partial\Delta_n} = \mathrm{id}_{\partial\Delta_n}$. Thus, it is homotopic rel $\partial\Delta_n$ to the identity map. Since the degree map is homotopy invariant we get $\deg(f \circ \mathbf{i}) = \deg(\mathrm{id}_{\Delta_n}) = 1$. ■

Using the claim and the fact that $f(\mathbf{i}(F_I)) \subset F_I$, by Theorem 2.24 the map $f \circ \mathbf{i}$ is surjective. The surjectivity of f follows immediately, since $\mathbf{i}$ is surjective by construction. □

Proof of Theorem 2.2. Let $V = \{v_1, \ldots, v_{n+1}\} \subset H^2(X)_{\mathbb{Z}}$ be a linearly independent set such that V_I is negative definite for all subsets $I \subset \{1, \ldots, n+1\}$ of cardinality n. We will use the identification $\mathrm{Gr}^+(H^2(X)) \cong \mathbb{H}^n$ as explained above. The hyperplanes

$$H_i := v_i^{\perp} \cap \mathbb{H}^n$$

are not-empty by the requirement that the V_I are negative definite and bound an n-simplex $\Delta_V \subset \mathbb{H}^n$. Observe that every compact hyperbolic n-simplex in $\mathbb{H}^n$ with rationally defined vertices arises in this fashion, since given the corresponding rational hypersurfaces that bound them, it is always possible to find their orthogonal vector $v_i \in H^2(X)_{\mathbb{Z}}$. We remark that every point in $\mathbb{H}^n$ is in the interior of such a simplex, by density of rational points. Thus, to show that Π_X is onto, it suffices to show that

$$\mathrm{int}(\Delta_V) \subset \Pi_{X,V}(\mathrm{int}\,(P_n)) \tag{2.27}$$

where $\Pi_{X,V}$ is the period map restricted to the n-dimensional permutahedron family of metrics associated to V. Since Theorem 2.2 focuses on smooth metrics, here it is important that $\Pi_{X,V}(\mathrm{int}\,(P_n))$ is in the image of the period map, as the interior of P_n parametrizes smooth (unbroken) metrics.

We first need to determine the behaviour of $\Pi_{X,V}$ on the boundary of P_n. Let $\mathbf{I} = \{I_i\}_{i=1}^{l}$ be a nested sequence of proper non-empty subsets of $\{1, \ldots, n+1\}$, and $F_{\mathbf{I}}$ the corresponding codimension l face of P_n. Since V_{I_i} is negative definite by construction, we have $\mathrm{Gr}^+(V_{I_i}) = \{0\}$ and $\mathrm{null}(V_{I_i}) = \{0\}$ and since $V_{I_i} \cap V_{I_{i-1}}^{\perp} \in V_{I_i}$, also $\mathrm{Gr}^+(V_{I_i} \cap V_{I_{i-1}}^{\perp}) = \{0\}$ for all $i = 1, \ldots, l$. Then an application of Proposition 2.21 yields

$$\begin{aligned}
\Pi_{X,V}(F_{\mathbf{I}}) \subset \mathrm{cl}\,[\Pi_{X,V}(\mathrm{int}\, F_I)] &\subset \mathrm{cl}\left[\bigcup_{g \in \mathrm{int}\, F_I}\left(H_{l+1}^{+} + \sum_{i=1}^{l}(H_i^{+} + \mathrm{null}\, V_{I_i})\right)\right] \\
&\subset \mathrm{cl}\left[\mathrm{Gr}^+(V_{I_l}^{\perp}) \times \prod_{i=1}^{l}(\mathrm{Gr}^+(V_{I_i} \cap V_{I_{i-1}}^{\perp}) \times \mathrm{null}\ V_{I_i})\right] \\
&\subset \mathrm{cl}\left[\mathrm{Gr}^+\left(\langle v_i \mid i \in I_l\rangle^{\perp}\right)\right]
\end{aligned}$$

Using the identification stated above, we describe a point in the Grassmannian as the intersection with $\mathbb{H}^n$, and so we get

$$\begin{aligned}\Pi_{X,V}(F_{\mathbf{I}}) \subset \mathrm{cl}\left[\langle v_i \mid i \in I_l\rangle^{\perp} \cap \mathbb{H}^n\right] &= \mathrm{cl}\left(\bigcap_{i\in I_l} v_i^{\perp} \cap \mathbb{H}^n\right) \\ \subseteq \bigcap_{i\in I_l} \mathrm{cl}\left(v_i^{\perp} \cap \mathbb{H}^n\right) \subseteq \bigcap_{i\in I_l} \mathrm{cl}(v_i^{\perp}) \cap \mathrm{cl}(\mathbb{H}^n) &= \bigcap_{i\in I_l} H_i \subset \partial\Delta_n\end{aligned}$$

Since it is true for all faces $F_{\mathbf{I}}$ of the permutahedron P_n, this implies that $\Pi_{X,V}(\partial P_n) \subseteq \partial\Delta_V$.

Next, choose a retraction $r : \mathbb{H}^n \to \Delta_V$ with the property that $r(H_i) \subset \Delta_V \cap H_i$. For example, r can be the map which sends a point in $\mathbb{H}^n$ to its closest point in Δ_V. Then, by construction, the map

$$\begin{aligned} r \circ \Pi_{X,V} : P_n &\longrightarrow \Delta_V \\ F_{\mathbf{I}} &\longmapsto \bigcap_{i\in I_l} H_i \cap \Delta_V\end{aligned}$$

Thus, by Lemma 2.25 $r \circ \Pi_{X,V}$ surjects onto Δ_V and since $r|_{\Delta_V} = \mathrm{id}$ by construction, we get $\Delta_V \subset \Pi_{X,V}(P_n)$. This leads to

$$\begin{aligned}\mathrm{int}\,\Delta_n = \Delta_n \setminus \partial\Delta_n \subseteq \Pi_{X,V}(P_n) \setminus \partial\Delta_n &\subset \Pi_{X,V}(P_n) \setminus \Pi_{X,V}(\partial P_n) \\ &\subset \Pi_{X,V}(P_n \setminus \partial P_n) = \Pi_{X,V}(\mathrm{int}\, P_n)\end{aligned}$$

which gives (2.27), as desired. □

Remark 2.26. Following Remark 2.11 point 1, the proof of Theorem 2.2 given above adapts to the case in which $\dim X = 4k$ for any positive integer k, with no essential changes.

Following Remark 2.11 point 2, the above proof also shows that for connected X with boundary, the period map $\mathrm{Met}_{\mathrm{cyl}}(X) \to \mathrm{Gr}^+(\widehat{H}^2(X))$ is surjective when $b^+(X) = 1$. As explained at the beginning of this subsection, it is again necessary to check that $\mathbb{R} \cdot H^2(X)_{\mathbb{Z}}$ is dense in the Grassmannian.

Remark 2.27. Let $V = \{v_1, \ldots, v_{k+1}\} \subset H^2(X)_{\mathbb{Z}}$ be a linearly independent set, and $\Pi_{X,V}$ the associated period map. In the above proof, Proposition 2.21 was used in the simplified case in which V determines a bounded simplex in $\mathbb{H}^n$. More generally, for a nested sequence $\mathbf{I} = \{I_i\}_{i=1}^{l}$ set

$$i^+ := \min\{j \mid V_{I_j} \text{ not negative definite }\}.$$

Let $g \in F_{\mathbf{I}}$. If V_{I^+} is degenerate, it has a 1-d null space, and $\Pi_{X,V}(g) = \mathrm{null}(V_{I^+})$. Otherwise,

$$\Pi_{X,V}(g) \subset V_{I_{i^+}} \cap V_{I_{i^+-1}}^{\perp}.$$

In many cases, these conditions cut out a region in $\mathbb{H}^n$ which contains an n-dimensional arbitrary polytope, possibly with ideal points on the sphere at infinity, and the above proof carries over to show that $\Pi_{X,V}(P_n)$ contains this polyhedron. In the next subsection, we will see this in some examples.

Remark 2.28. The choice of permutahedra to parametrise the metrics is fundamental for keeping the proof as general as possible. These polytopes are universal from our viewpoint, in that they parametrise a family of metrics for any collection of surfaces in X, regardless of the intersection pairings of the surfaces. On the other hand, if certain subcollections of surfaces do not intersect, one can often work with simpler polytopes. For example, if $k+1$ surfaces are pairwise disjoint, one can construct an associated metric family which is a k-simplex.

Along the same lines, one might try to prove Theorem 2.2 using pairwise disjoint surfaces with negative self-intersection, with polytopes whose codimension 1 faces are labelled by these surfaces. One is led to search for what are called right-angled (finite volume) hyperbolic polyhedra. However, such polyhedra do not exist in $\mathbb{H}^n$ for $n > 12$ [Duf10]. This illustrates the utility of working with the general construction, avoiding conditions on how the surfaces intersect.

2.2.4 Examples in the hyperbolic plane

To illustrate the construction used in the proof of Theorem 2.2, we consider examples in the case that $b^+(X) = 1$ and $b_2(X) = 3$. Upon choosing an isometry of $H^2(X)$ with $\mathbb{R}^{1,2}$, the positive Grassmannian is identified with the hyperbolic plane $\mathbb{H}^2$, as shown in Figure 2.5. As before, we try to make these choices so that the integral lattice $H^2(X)_{\mathbb{Z}}$ is sent to $\mathbb{Z}^{1,2} \subset \mathbb{R}^{1,2}$.

Let $V = \{v_1, v_2, v_3\} \subset H^2(X)_{\mathbb{Z}}$, which we also view as a subset of $\mathbb{Z}^{1,2} \subset \mathbb{R}^{1,2}$. For simplicity, we assume that V is a linearly independent set. By choosing connected embedded surfaces $\Sigma_1, \Sigma_2, \Sigma_3 \subset X$ which are Poincaré dual to v_1, v_2, v_3, we construct a family of (broken) metrics on X parametrized by the permutahedron P_2, see Figure 2.6. We have an associated period map $\Pi_{X,V} : P_2 \to \mathbb{H}^2$ for this family of metrics.

The codimension 1 faces of P_2 are represented by the sides of the hexagon and are $F_{\{i\}}$ and $F_{\{i,j\}}$ where $i, j \in \{1, 2, 3\}$ and $i \neq j$. Proposition 2.21 determines the behaviour of the period map on these faces. The application of Proposition 2.21 is given by the following calculations, and the results are summarised in Table 2.1. Briefly, the image of face $F_{\{i\}}$ (resp. $F_{\{i,j\}}$) under $\Pi_{X,V}$ is constrained in three possible ways, depending on the isomorphism type of the bilinear form restricted to the subspace $\langle v_i \rangle$ (resp. $\langle v_i, v_j \rangle$).

Construction. By Proposition 2.21, given $\mathbf{I} = \{I_1\}$, for any $g \in \operatorname{int} F_{\mathbf{I}}$ we get $\Pi_{X,V}(g) = H_2^+ + H_1^+ + \operatorname{null}(V_{I_1})$ where $H_1^+ \in \operatorname{Gr}^+(V_{I_1})$ and $H_2^+ \in \operatorname{Gr}^+(V_{I_1}^\perp)$. This implies that $\Pi_{X,V}(g) \subset \operatorname{Gr}^+(V_{I_1}) + \operatorname{Gr}^+(V_{I_1}^\perp) + \operatorname{null}(V_{I_1})$.

Consider the case $I_1 = \{i\}$, so $V_{I_1} = \langle v_i \rangle$ and $\dim V_{I_1}^\perp = 2$, then

- if $v_i \cdot v_i < 0$, we have $\operatorname{Gr}^+(V_{I_1}) = \{0\}$, $\operatorname{null}(V_{I_1}) = \{0\}$ and $\operatorname{Gr}^+(V_{I_1}^\perp) = \langle v_i \rangle^\perp \cap \mathbb{H}^2$. Here $\langle v_i \rangle^\perp \cap \mathbb{H}^2$ is a geodesic for dimensionality reasons.
- if $v_i \cdot v_i = 0$, we have $\operatorname{Gr}^+(V_{I_1}) = \{0\}$, $\operatorname{null}(V_{I_1}) = \langle v_i \rangle \cap \partial\mathbb{H}^2$ and $\operatorname{Gr}^+(V_{I_1}^\perp) = \{0\}$.
- if $v_i \cdot v_i > 0$, we have $\operatorname{Gr}^+(V_{I_1}) = \langle v_i \rangle \cap \mathbb{H}^2$, $\operatorname{null}(V_{I_1}) = \{0\}$ and $\operatorname{Gr}^+(V_{I_1}^\perp) = \{0\}$. Here $\langle v_i \rangle \cap \mathbb{H}^2$ is a point for dimensionality reasons.

Consider the case $I_1 = \{i, j\}$, so $V_{I_1} = \langle v_i, v_j \rangle$ and $\dim V_{I_1}^\perp = 1$, then

- if all $v \in \langle v_i, v_j \rangle \setminus \{0\}$ are such that $v \cdot v < 0$, we have $\operatorname{Gr}^+(V_{I_1}) = \{0\}$, $\operatorname{null}(V_{I_1}) = \{0\}$ and $\operatorname{Gr}^+(V_{I_1}^\perp) = \langle v_i, v_j \rangle^\perp \cap \mathbb{H}^2$. Here $\langle v_i, v_j \rangle^\perp \cap \mathbb{H}^2$ is a point for dimensionality reasons.
- if there is a $v \in \langle v_i, v_j \rangle \setminus \{0\}$ such that $v \cdot v = 0$ and none of them is such that $v \cdot v > 0$, we have $\operatorname{Gr}^+(V_{I_1}) = \{0\}$, $\operatorname{null}(V_{I_1}) = \langle v \rangle \cap \partial\mathbb{H}^2$ and $\operatorname{Gr}^+(V_{I_1}^\perp) = \{0\}$.
- if there is a $v \in \langle v_i, v_j \rangle \setminus \{0\}$ such that $v \cdot v > 0$, we have $\operatorname{Gr}^+(V_{I_1}) = \langle v_i, v_j \rangle \cap \mathbb{H}^2$, $\operatorname{null}(V_{I_1}) = \{0\}$ and $\operatorname{Gr}^+(V_{I_1}^\perp) = \{0\}$. Here $\langle v_i, v_j \rangle \cap \mathbb{H}^2$ is a geodesic for dimensionality reasons.

Now, we will check these results on some example families. Here we will write vectors in $\mathbb{R}^{1,2}$ as $x = (x_0 \mid x_1, x_2)$, to emphasize the signature of the bilinear form.

Condition	Constraint on period map	Type
$v_i \cdot v_i < 0$	$\Pi_{X,V}(F_{\{i\}}) \subset \langle v_i \rangle^\perp \cap \mathbb{H}^2$	geodesic
$v_i \cdot v_i = 0$	$\Pi_{X,V}(F_{\{i\}}) = \langle v_i \rangle \cap \partial\mathbb{H}^2$	ideal point
$v_i \cdot v_i > 0$	$\Pi_{X,V}(F_{\{i\}}) = \langle v_i \rangle \cap \mathbb{H}^2$	point
$\forall v \in \langle v_i, v_j \rangle \setminus \{0\} \ : \ v \cdot v < 0$	$\Pi_{X,V}(F_{\{i,j\}}) = \langle v_i, v_j \rangle^\perp \cap \mathbb{H}^2$	point
$\exists v \in \langle v_i, v_j \rangle \setminus \{0\} : \ v \cdot v = 0$	$\Pi_{X,V}(F_{\{i,j\}}) = \langle v \rangle \cap \partial\mathbb{H}^2$	ideal point
$\exists v \in \langle v_i, v_j \rangle \ : \ v \cdot v > 0$	$\Pi_{X,V}(F_{\{i,j\}}) \subset \langle v \rangle \cap \mathbb{H}^2$	geodesic

Table 2.1: Each condition in the left column gives a constraint on where the period map sends a face of the hexagon P_2. The types of subsets of $\mathbb{H}^2$ appearing are described in the right column.

Example 5. A particularly symmetric family of examples $V = \{v_1, v_2, v_3\}$ is given by

$$v_{i+1} = (1 \mid a\cos(2i\pi/3),\ a\sin(2i\pi/3))\,, \qquad i \in \{0,1,2\}. \tag{2.28}$$

where $a \in \mathbb{R} \setminus \{0\}$. Each V in this family is not integral (nor rational), but nonetheless illustrates the behaviour of the face constraints of the period map. When $a > 2$, the $v_i^\perp$ give geodesics forming a hyperbolic triangle. However, for all $a \neq 0$, the constraints on the period map give some finite area polygon in $\mathbb{H}^2$. See Figure 2.8, where illustations are given in the Poincaré disk for $a > 0$. When $a < 0$, there are similar pictures, with some colors interchanged.

Example 6. There are many other types of configurations of geodesics and points that arise from the face constraints given in Table 2.1. For example, here are some sets $V = \{v_1, v_2, v_3\} \subset H^2(X)_{\mathbb{Z}} = \mathbb{Z}^{1,2}$, which give rise to the configurations depicted in Figure 2.9:

(i): $v_1 = (0 \,|\, 1, 0)$ $\quad v_2 = (0 \,|\, 0, 1)$ $\quad v_3 = (2 \,|\, 1, 3)$

(ii): $v_1 = (0 \,|\, 1, 1)$ $\quad v_2 = (1 \,|\, 1, -1)$ $\quad v_3 = (2 \,|\, 1, 2)$

(iii): $v_1 = (0 \,|\, 1, -1)$ $\quad v_2 = (0 \,|\, 0, 1)$ $\quad v_3 = (2 \,|\, 1, 2)$

(iv): $v_1 = (0 \,|\, 1, 1)$ $\quad v_2 = (3 \,|\, 1, 3)$ $\quad v_3 = (3 \,|\, -3, 1)$

Remark 2.29. Not every linearly independent V gives a configuration which encloses a polygon with non-empty interior (and thus the image of the corresponding period map on P_2 is not guaranteed to have non-empty interior). For a simple example, take $v_1 = (1 \,|\, 1, 1)$, $v_2 = (0 \,|\, 1, 1)$, $v_3 = (0 \,|\, 1, -1)$.

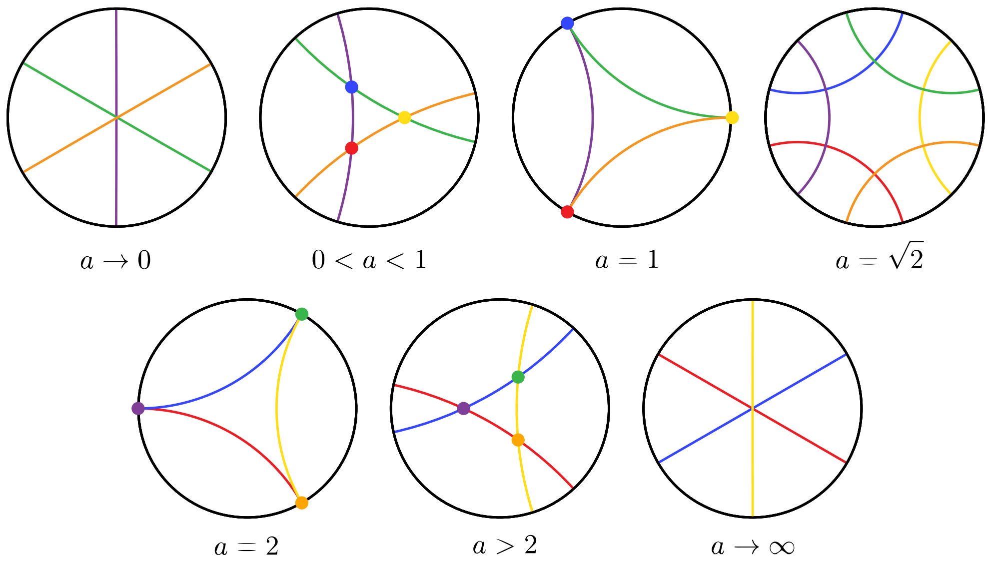

Figure 2.8: The subsets defining the face constraints from Table 2.1, given for the family of $V = \{v_1, v_2, v_3\}$ defined in (2.28). The colors correspond to the face colors of the hexagon in Figure 2.6.

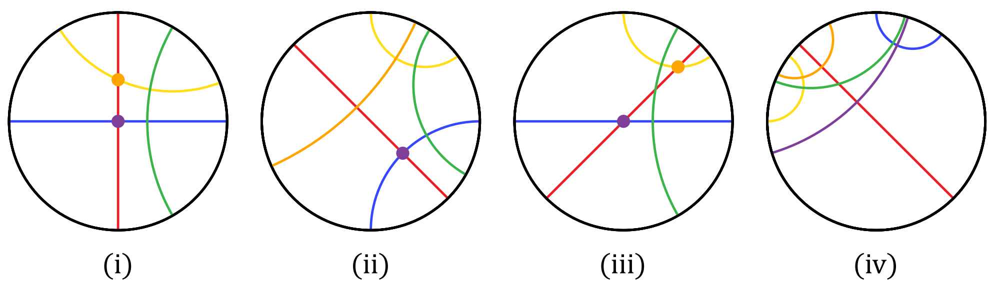

Figure 2.9: Examples of face constraint configurations for some other choices of V.

Chapter 3

Gluing harmonic forms

Here we give a proof of Proposition 2.9, by showing how it follows from well-known gluing results for harmonic forms. In this part, we will follow [CLM96]. For similar gluing constructions, see also [DFK02].

Notation. We will use the same notation introduced in Subsection 2.1.2.

Write $\|\cdot\|_Y$ for the L^2 metric on $\Omega^*(Y)$ induced by g_Y. In this notation, we assume the metric on any cylinder is the product metric: $\|\cdot\|_X$ is defined using the metric $h(1)$ on $X = X(1)$, and $\|\cdot\|_{X(r)}$ using $h(r)$ on $X(r)$.

3.1 The gluing map

We start by recalling some results gluing results from [CLM96]. In this article, all the construction was done for a generic self-adjoint elliptic operator D on $\Gamma(E)$ of a real vector bundle $E \to X$, with the splitting $X = X_1 \cup_Y X_2$. These will lead to the construction of a splicing map:

$$\Phi_r : \mathscr{H}^+_{X_1} \oplus \mathcal{V}_Y \oplus \mathscr{H}^+_{X_2} \longrightarrow \Omega^+_{h(r)}(X(r)) \tag{3.1}$$

Recall $\mathscr{H}^+_{X_i}$ is the space of L^2 harmonic self-dual 2-forms on $X_i(\infty)$ with its cylindrical end metric. To construct this map we first need some definitions.

Notation. Write $\pi : Y \times (0, \infty) \to Y$ for projection, the map $\pi^* : \Omega^*(Y) \to \Omega^*(Y \times (0, \infty))$ can be extended to $\Omega^*(Y) \wedge dt$ through $\pi^*(\alpha \wedge dt) = \pi^*(\alpha) \wedge dt$.

Definition 3.1. A 2-form ω_1 on $X_1(\infty)$ is an *extended* L^2 harmonic form if:

- it is harmonic
- $\|\omega_1\|_{X_1} < \infty$
- $\|\omega_1 - \pi^*(\alpha \wedge dt + \beta)\|_{Y \times (0,\infty)} < \infty$ for some harmonic forms $\alpha \in \mathscr{H}^1_Y$, $\beta \in \mathscr{H}^2_Y$

In this case, we say that ω_1 *extends* $\alpha \wedge dt + \beta$.

A similar notion of extended L^2 harmonic forms is defined for $X_2(\infty)$.

Notation. The space $\mathcal{V}_Y$ is the vector space of harmonic 1-forms α on Y such that there exist extended L^2 harmonic self-dual 2-forms ω_i on $X_i(\infty)$ extending $\alpha \wedge dt + \star_Y \alpha$.

Proposition 3.2. *The map* $\delta : H^1(Y) \to H^2(X)$ *induces*

$$\mathcal{V}_Y \cong \mathrm{Im}(\delta),$$

The map $\iota : Y \hookrightarrow X$ *induces*

$$\begin{aligned} \mathcal{V}_Y &\xrightarrow{\cong} \mathrm{Im}[\iota^* : H^2(X) \to H^2(Y)] \\ \alpha &\longmapsto [\star_Y \alpha] \end{aligned}$$

Proof. These last claims follow from Lemma B.1 of [CLM96], restricted to the case of self-dual harmonic 2-forms on a 4-manifold. □

Construction. Let $\rho : \mathbb{R} \to [0,1]$ be a smooth non-decreasing function satisfying $|\rho'| \leq 4$ and

$$\rho(t) = \begin{cases} 0, & t \leq 1/4 \\ 1, & t \geq 3/4 \end{cases}$$

We use this to describe (3.1). For any element ω in its domain $\mathscr{H}^+_{X_1} \oplus \mathcal{V}_Y \oplus \mathscr{H}^+_{X_2}$, set $\Phi_r(\omega)|_{X_i} = \omega$, for $i = 1, 2$. Now consider $\omega_1 \in \mathscr{H}^+_{X_1}$; similar results happen for $\omega_2 \in \mathscr{H}^+_{X_2}$. We define $\Phi_r(\omega_1)$ elsewhere by:

$$\begin{aligned} \Phi_r(\omega_1)|_{Y\times[-r,0]}(y,t) &= \rho(-t)\omega_1(y, t + r), \\ \Phi_r(\omega_1)|_{X_2(r)} &= 0 \end{aligned}$$

The translation appears by the way in which we identify $X_1(r) = X_1 \cup Y \times [0, r]$ as a subset of $X(r)$. The description of Φ_r for elements of $\mathscr{H}^+_{X_2}$ is similar.

We are left with defining the behaviour for $\alpha \in \mathcal{V}_Y$. By definition of $\mathcal{V}_Y$, there exist unique extended L^2 self-dual harmonic 2-forms ω_i on $X_i(\infty)$, extending $\omega_\alpha := \alpha \wedge dt + \star_Y \alpha$, such that ω_i is L^2 orthogonal to $\mathscr{H}^+_{X_i}$; this gives uniqueness, by the splitting given. Then

$$\Phi_r(\beta)|_{Y\times[-r,0]}(y,t) = \rho(-t)(\omega_1(y, t + r) - \pi^*\omega_\alpha(y)) + \pi^*\omega_\alpha(y)$$

$$\Phi_r(\beta)|_{Y\times[0,+r]}(y,t) = \rho(+t)(\omega_2(y, t - r) - \pi^*\omega_\alpha(y)) + \pi^*\omega_\alpha(y)$$

Let $\mathbf{P}_r : \Omega^2_{h(r)}(X(r)) \to \mathscr{H}^2_{X(r)}$ denote L^2 projection of 2-forms on $X(r)$ to the space of $h(r)$-harmonic 2-forms. Note that by construction the image of $\mathbf{P}_r\Phi_r$ lies in the $h(r)$-self-dual harmonic 2-forms $\mathscr{H}^+_{X(r)}$. It follows from the results in [CLM96] (see Lemma 4.1 and Appendix B) that there exists an R such that for $r \geq R$, we have

$$\|\mathbf{P}_r\Phi_r(\omega) - \Phi_r(\omega)\|_{X(r)} \leq e^{-cr}\|\Phi_r(\omega)\|_{X(r)} \tag{3.2}$$

[add something from [clm]] where $c > 0$ is a constant, independent of r, determined by the spectrum of the g_Y-Laplacian on forms of Y, as shown in [CLM96]. Furthermore, for large r

$$\mathbf{P}_r \circ \Phi_r : \mathscr{H}^+_{X_1} \oplus \mathcal{V}_Y \oplus \mathscr{H}^+_{X_2} \to \mathscr{H}^+_{X(r)}$$

is an isomorphism. This is why in the limit $\mathbf{P}_r$ is the identity and Φ_r has an inverse mapping.

3.2 Comparison of metrics

Let us be more explicit about the diffeomorphisms $f_r : X \to X(r)$, introduced in Subsection 2.1.2. To avoid problems in the following construction, we will assume $r \geq 1$, so that $X = X(1)$.

Construction. Write s for the coordinate in $[-1, 1]$, and t in $[-r, r]$. Choose diffeomorphisms $\chi_r : [-1, 1] \to [-r, r]$ such that

- $\chi_r'(s) = 1$ for $1 - |s| < \varepsilon$,
- $\chi_r'(s) \geq 1$ for all $s \in [-1, 1]$,
- $\chi_r'(s) \leq Cr$ for some fixed $C > 1$, independent of r.

Then we can define the diffeomorphisms

$$f_r : X \to X(r),$$
$$f_r|_{X_i} = \mathrm{id}_{X_i}, \qquad f_r|_{Y\times[-1,1]} = \mathrm{id}_Y \times \chi_r.$$

Let $\omega \in \Omega^2(X(r))$. Write $\omega_i = \omega|_{X_i}$ and $\omega|_{Y\times[-r,r]} = \alpha_t \wedge dt + \beta_t$ where $\alpha_t \in \Omega^1(Y)$ and $\beta_t \in \Omega^2(Y)$ depend smoothly on t. So we can write $\omega = \omega_1 + \omega_2 + (\alpha_t \wedge dt + \beta_t)$.

Proposition 3.3. *Following the construction and notation above, we have*

$$\|\omega\|^2_{X(r)} = \sum_{i=1}^{2} \|\omega_i\|^2_{X_i} + \int_{-1}^{1} \chi_r'(s)(\|\alpha_s\|^2_Y + \|\beta_s\|^2_Y)ds$$

where $s = \chi_r^{-1}(t)$.

Proof. Following notation above

$$\begin{aligned}\|\omega\|^2_{X(r)} &= \|\omega_1 + \omega_2 + (\alpha_t \wedge dt + \beta_t)\|^2_{X(r)} \\ &= \|\omega_1\|^2_{X_1} + \|\omega_2\|^2_{X_2} + \|\alpha_t \wedge dt + \beta_t\|^2_{Y\times[-r,r]}\end{aligned}$$

where the last part becomes

$$\begin{aligned}\|\alpha_t \wedge dt + \beta_t\|^2_{Y\times[-r,r]} &= \int_{Y\times[-r,r]} \langle \alpha_t \wedge dt + \beta_t, \alpha_t \wedge dt + \beta_t\rangle_{g_Y+dt^2}\, \mathrm{dvol}_{g_Y+dt^2} \\ &= \int_{Y\times[-r,r]} \big(\langle \alpha_t \wedge dt, \alpha_t \wedge dt\rangle_{g_Y+dt^2} + \langle \beta_t, \beta_t\rangle_{g_Y+dt^2} + \\ &\qquad + 2\langle \alpha_t \wedge dt, \beta_t\rangle_{g_Y+dt^2}\big)\, \mathrm{dvol}_{g_Y+dt^2} \\ &= \int_{-r}^{r} dt \int_Y (\langle \alpha_t, \alpha_t\rangle_{g_Y} \langle dt, dt\rangle_{dt^2} + \langle \beta_t, \beta_t\rangle_{g_Y})\, \mathrm{dvol}_{g_Y} \\ &= \int_{-r}^{r} dt \left(\|\alpha_t\|^2_Y + \|\beta_t\|^2_Y\right) = \int_{-1}^{1} ds\chi_r'(s) \left(\|\alpha_s\|^2_Y + \|\beta_s\|^2_Y\right)\end{aligned}$$

□

Proposition 3.4. *Following the construction and notation above, we have*

$$\|f_r^*\omega\|^2_X = \sum_{i=1}^{2} \|\omega_i\|^2_{X_i} + \int_{-1}^{1} (\chi_r'(s)^2\|\alpha_s\|^2_Y + \|\beta_s\|^2_Y)ds,$$

which is the squared L^2-norm of ω defined using the metric $g(r)$.

Proof. Following notation above

$$\begin{aligned}\|f_r^*\omega\|^2_{X(r)} &= \|\omega_1 + \omega_2 + f_r^*(\alpha_t \wedge dt + \beta_t)\|^2_{X(r)} \\ &= \|\omega_1\|^2_{X_1} + \|\omega_2\|^2_{X_2} + \|(\mathrm{id}_Y + \chi_r)^*(\alpha_t \wedge dt + \beta_t)\|^2_{Y\times[-r,r]}\end{aligned}$$

where, following a similar calculation as above, the last part becomes

$$\begin{aligned}&\|(\mathrm{id}_Y + \chi_r)^*(\alpha_t \wedge dt + \beta_t)\|^2_{Y\times[-r,r]} = \|\alpha_{\chi_r(s)} \wedge d(\chi_r(s)) + \beta_{\chi_r(s)}\|^2_{Y\times[-r,r]} \\ &= \|\alpha_s \wedge ds\chi_r'(s) + \beta_s\|^2_{Y\times[-1,1]} = \|\alpha_s \wedge ds\chi_r'(s)\|^2_{Y\times[-1,1]} + \|\beta_s\|^2_{Y\times[-1,1]} \\ &= \int_{-1}^1 ds \int_Y \left(\chi_r'(s)^2\langle\alpha_s,\alpha_s\rangle_{g_Y} + \langle\beta_s,\beta_s\rangle_{g_Y}\right) \mathrm{dvol}_{g_Y} \\ &= \int_{-1}^1 ds \left(\chi_r'(s)^2\|\alpha_s\|^2_Y + \|\beta_s\|^2_Y\right)\end{aligned}$$

□

Corollary 3.5. Following the construction and notation above, we have

$$\frac{1}{Cr}\|\omega\|^2_{X(r)} \le \|f_r^*\omega\|^2_X \le Cr\|\omega\|^2_{X(r)}. \tag{3.3}$$

Proof. Recall that for $s \in [-1,1]$ we have $1 \le \chi_r'(s) \le Cr$ and that $Cr > 1$, then applying Propositions 3.3 and 3.4 yields

$$\begin{aligned}\|f_r^*\omega\|^2_X &= \sum_{i=1}^2 \|\omega_i\|^2_{X_i} + \int_{-1}^1 (\chi_r'(s)^2\|\alpha_s\|^2_Y + \|\beta_s\|^2_Y)ds \\ &\le\end{aligned}$$

□

3.3 Proof of Proposition 2.9

With this background established, in this section we prove Proposition 2.9. Note that the composition $f_r^*\mathbf{P}_r : \Omega^+_{h(r)}(X(r)) \to \mathscr{H}^+_{X(r)} \to \mathscr{H}^+_X$ is equal to the projection onto the $g(r)$-self-dual 2-forms of X. Thus our goal is to show that the image of $f_r^*\mathbf{P}_r\Phi_r$ as $r \to \infty$ converges to

$$\lim_{r\to\infty} \mathscr{H}^+_{g(r)} = \lim_{r\to\infty} f_r^*\mathbf{P}_r\Phi_r\left(\mathscr{H}^+_{X_1} \oplus \mathcal{V}_Y \oplus \mathscr{H}^+_{X_2}\right) = j^{-1}\left(\mathscr{H}^+_{X_1} \oplus \mathscr{H}^+_{X_2}\right)$$

We begin with some elementary observations, on which we will base our proof.

Remark 3.6. If $\{\omega_i\}_{i=1}^\infty$ and ω are closed forms in $\Omega^k(X)$, then to show $[\omega_i] \to [\omega]$ in $H^k(X)$, it suffices to show that

$$\lim_{i\to\infty} \int_X \omega_i \wedge \eta = \int_X \omega \wedge \eta \tag{3.4}$$

for every closed form η of complementary degree. Further, suppose X has a metric inducing an L^2 norm on forms, and suppose $\|\omega_i - \omega_i'\|_{L^2} \to 0$ for some other sequence of (not necessarily closed) forms $\{\omega_i'\}_{i=1}^\infty$. Then to show $[\omega_i] \to [\omega]$, it suffices to show (3.4) with ω_i' replacing ω_i.

The proof can be split into 2 steps.

Step 1 Let $\alpha \in \mathcal{V}_Y$. In this part, we will prove that the following holds in $H^2(X)$:

$$\lim_{r\to\infty}\left[\frac{1}{2r}f_r^*\mathbf{P}_r\Phi_r(\alpha)\right] = \delta[\alpha]. \tag{3.5}$$

To do so, we first consider the forms $\frac{1}{2r}f_r^*\Phi_r(\alpha)$ without the projection $\mathbf{P}_r$ and then we will prove that the norm of the difference converges to zero. We have

$$\frac{1}{2r}f_r^*\Phi_r(\alpha) = \frac{1}{2r}\chi_{Y\times(-1,1)}f_r^*\omega' + \frac{1}{2r}\chi_{Y\times(-1,1)}f_r^*\omega_\alpha + \frac{1}{2r}\sum_{i=1,2}\chi_{X_i}\omega_i \tag{3.6}$$

where $\omega' = \Phi_r(\beta) - \pi^*(\omega_\alpha)$, recalling $\omega_\alpha = \alpha\wedge dt + \star_Y\alpha$. The χ_S are characteristic functions. The terms $\chi_{X_i}\omega_i$ have L^2 norm on X independent of r, and so the last term in (3.6) converges in L^2 to zero. Utilizing (3.3), the L^2 norm of the first term in (3.6) satisfies

$$\|\frac{1}{2r}f_r^*\omega'\|_{L^2(Y\times(-1,1))} \leq \left(\frac{C}{r}\right)^{1/2}\left(\|\omega_1 - \pi^*\omega_\alpha\|_{L^2(Y\times(0,\infty))} + \|\omega_2 - \pi^*\omega_\alpha\|_{L^2(Y\times(-\infty,0))}\right)$$

Here ω_i are extended L^2 self-dual harmonic 2-forms on $X_i(\infty)$ extending ω_α. Thus the first term in (3.6) converges in L^2 to zero as $r\to\infty$. The middle term in (3.6) is

$$\omega_r := \frac{1}{2r}\chi_{Y\times(-1,1)}f_r^*\omega_\alpha = \frac{1}{2r}\chi_{Y\times(-1,1)}\left(\star_Y\alpha + \chi_r'(s)\alpha\wedge ds\right).$$

At this point we may ask that the functions χ_r are chosen such that $\chi_r'/2r$ converge in L^2 to a smooth bump function of integral 1, matching the description (2.4). On the other hand, all we need is the weaker convergence (3.4). To this end, let $\eta\in\Omega^2(X)$ be closed. On $Y\times(-1,1)$, we may write $\eta = \alpha_1\wedge ds + \alpha_2$, where $\alpha_i = \alpha_i(s)$ are i-forms on Y depending smoothly on $s\in(-1,1)$. Then

$$\int_X \omega_r\wedge\eta = \frac{1}{2r}\int_{Y\times(-1,1)}\star_Y\alpha\wedge\alpha_1\wedge ds + \frac{1}{2r}\int_{Y\times(-1,1)}\chi_r'(s)\alpha_2\wedge\alpha\wedge ds$$

The integral appearing in the first term on the right side is independent of r, and thus the first term converges to zero as $r\to\infty$. The second term is

$$\frac{1}{2r}\int_{-1}^{1}\chi_r'(s)\left(\int_Y\alpha_2\wedge\alpha\right)ds = \int_Y\alpha_2\wedge\alpha = \int_Y\iota^*(\eta)\wedge\alpha,$$

where we have used that the integral over $Y = Y\times\{s\}$ appearing does not depend on the parameter s. This follows from η being closed, which implies $[\alpha_2(s)]\in H^2(Y)$ is independent of s. We obtain

$$\lim_{r\to\infty}\int_X\omega_r\wedge\eta = \int_X\eta\wedge\delta(\alpha),$$

as desired. Thus far, our argument implies that for all closed $\eta\in\Omega^2(X)$, we have

$$\lim_{r\to\infty}\frac{1}{2r}f_r^*\Phi_r(\alpha)\wedge\eta = \int_X\delta(\alpha)\wedge\eta. \tag{3.7}$$

We next observe the following, using inequalities (3.2) and (3.3):

$$\|\frac{1}{2r}f_r^*\mathbf{P}_r\Phi_r(\alpha) - \frac{1}{2r}f_r^*\Phi_r(\alpha)\|_X \leq Cre^{-cr}\|\frac{1}{2r}f_r^*\Phi_r(\alpha)\|_X. \tag{3.8}$$

From the previous paragraph, we have that $\|\frac{1}{2r}f_r^*\Phi_r(\alpha)-\omega_r\|_X$ converges to zero as $r\to\infty$. Furthermore, by direct calculation, we have

$$\|\omega_r\|_X^2 = \frac{1}{2r^2}\int_{-1}^{1}\chi_r'(s)\|\alpha\|_Y^2 \leq \frac{C}{r}\|\alpha\|_Y^2.$$

It follows from this and (3.8) that $\frac{1}{2r}f_r^*\mathbf{P}_r\Phi_r(\alpha)-\frac{1}{2r}f_r^*\Phi_r(\alpha)$ converges to zero in L^2. With (3.7), this completes the proof of claim (3.5).

Step 2 Let $\omega\in\mathscr{H}_{X_1}^+$. Then $[f_r^*\mathbf{P}_r\Phi_r(\omega)]\in H^2(X)$. To make sense of the de Rham class of ω on X, we recall from [APS75] that on $X_1(\infty)$, ω is cohomologous (in the sense of currents) to a smooth form ω_c with support on $X_1\subset X=X(1)$, say. Then $[\omega_c]\in H^2(X)$ makes sense. A different choice of ω_c will give the same cohomology class in $H^2(X)$ up to $\mathrm{Im}(\delta)$. We claim that

$$\lim_{r\to\infty}[f_r^*\mathbf{P}_r\Phi_r(\omega)] = [\omega_c] \mod \mathrm{Im}(\delta) \tag{3.9}$$

in $H^2(X)/\mathrm{Im}(\delta)$. A similar claim holds for X_2 replacing X_1.

As in Step 1, we use inequalities (3.2) and (3.3) to observe

$$\|f_r^*\mathbf{P}_r\Phi_r(\omega)-f_r^*\Phi_r(\omega)\|_X \leq Cre^{-cr}\|\Phi_r(\omega)\|_X. \tag{3.10}$$

Furthermore, $\|\Phi_r(\omega)\|_X\leq\|\omega\|_{X_1(\infty)}$ by the construction of $\Phi_r(\omega)$, which applies a cutoff function to ω. Thus $\|f_r^*\mathbf{P}_r\Phi_r(\omega)-f_r^*\Phi_r(\omega)\|_X\to 0$ as $r\to\infty$.

Next, suppose we have closed 2-forms η_j such that $[\eta_j]$ gives a basis for $H^2(X)$. Then, to show $\lim_{r\to\infty}[f_r^*\mathbf{P}_r\Phi_r(\omega)]=[\omega_c]$ in $H^2(X)$ from here, it would suffice to show for all j:

$$\lim_{r\to\infty}\int_X f_r^*\Phi_r(\omega)\wedge\eta_j = \int_X \omega_c\wedge\eta_j. \tag{3.11}$$

Choose compactly supported closed 2-forms $\eta_{i,j}$ on X_i such that $[\eta_{i,j}]$ give a basis of $H_i\subset H_c^2(X_i)$, a subspace which maps isomorphically to $\widehat{H}(X_i)$. Note each $\eta_{i,j}$ may be viewed as a form on X. We may further choose closed forms η_k' with support in $Y\times(0,1)\subset X$ that induce a basis of $\mathrm{Im}(\delta)\subset X$. As in the proof of Lemma 2.7, we may then choose W so that

$$H^2(X) = H_1\oplus H_2\oplus\mathrm{Im}(\delta)\oplus W,$$

where the pairing restricted to $H_1\oplus H_2$ is non-degenerate, $H_1\oplus H_2$ is orthogonal to $\mathrm{Im}(\delta)\oplus W$, and there are closed forms η_k'' inducing a basis of W such that

$$\int_X\eta_k'\wedge\eta_l' = \int_X\eta_k''\wedge\eta_l'' = 0, \qquad \int_X\eta_k'\wedge\eta_l'' = \delta_{kl}.$$

We observe that if $[\omega'],[\omega'']\in H^2(X)$ induce the same linear forms in $(H_1\oplus H_2\oplus\mathrm{Im}(\delta))^*$ via the pairing, then $[\omega']\equiv[\omega'']\pmod{\mathrm{Im}(\delta)}$. Thus to establish (3.9) it suffices to check (3.11) for η_j among the forms $\eta_{1,j}$, $\eta_{2,j}$, η_k'. As $\Phi_r(\omega)=\omega$ on X_1, where the support of $\eta_{1,j}$ lies, we have

$$\int_X f_r^*\Phi_r(\omega)\wedge\eta_{1,j} = \int_X\omega_c\wedge\eta_{1,j}.$$

The support of $f_r^*\Phi_r(\omega)$ is disjoint from the supports of $\eta_{2,j}$ and η_k', and so pairs to give zero with these forms, just as is the case for ω_c. Combined with $f_r^*\mathbf{P}_r\Phi_r(\omega)-f_r^*\Phi_r(\omega)\to 0$ in L^2, this proves (3.9). Finally, Steps 1 and 2 combine to prove the proposition in the form (2.12).

Chapter 4

Four manifolds with Ricci-flat metrics

In this chapter, we will consider the behaviour of the period map for the two specific cases of smooth, closed, connected, oriented four-dimensional manifolds that admit Ricci-flat metrics: the four-torus T^4 and $K3$ surfaces.

These examples emphasise particular geometric features of the manifolds: on the four-torus, we focus on the flat metrics inherited from the quotient structure, while on the $K3$ surfaces, we concentrate on their Kähler metrics.

4.1 Four-torus T^4

The four-torus is defined as $T^4 \cong \mathbb{R}^4/\mathbb{Z}^4 \cong S^1 \times S^1 \times S^1 \times S^1$. Observe that T^4 inherits a subset of metrics from $\mathbb{R}^4$ through the quotienting $\mathbb{R}^4/\mathbb{Z}^4$. These metrics have zero Riemann curvature tensor $R_{\mathbb{R}^4} \equiv 0$, which gets passed down to $R_{T^4} \equiv 0$. Thus, these kinds of metrics are Ricci-flat.

Following from these observations, in the following proposition we describe the second cohomology of this manifold.

Proposition 4.1. *Consider the four-torus T^4, then $b_2(T^4) = 6$ and $b^+(T^4) = b^-(T^4) = 3$.*

Proof. Using the isomorphism $T^4 \cong S^1 \times S^1 \times S^1 \times S^1$ and Künneth theorem, we get

$$H^*(T^4;\mathbb{R}) = H^*(S^1;\mathbb{R}) \otimes H^*(S^1;\mathbb{R}) \otimes H^*(S^1;\mathbb{R}) \otimes H^*(S^1;\mathbb{R}) = \bigotimes_{i=1}^{4} \Lambda_{\mathbb{R}}(x_i)$$

as a isomorphism of graded-commutative $\mathbb{R}$-algebras. This implies that the first and second cohomology are $H^1(T^4;\mathbb{R}) \cong \mathbb{R}^4$ and $H^2(T^4;\mathbb{R}) \cong \Lambda^2 H^1(T^4;\mathbb{R}) \cong \mathbb{R}^6$. Thus, given $\omega_1, \cdots, \omega_4$ a basis of $H^1(T^4;\mathbb{R})$, we have $\{\omega_i \wedge \omega_j \pm \epsilon_{ijkl}\omega_k \wedge \omega_l\}_{i,j,k,l}$ as respectively the self-dual and anti-self-dual basis of $H^2(T^4;\mathbb{R})$, proving the claim. □

To prove the surjectivity of the period map for T^4, we first focus on a generic four-vector space, following the outline from Donaldson and Kronheimer [DK97, Chapt. 1]. Consider an oriented four-dimensional vector space U and its associated six-dimensional space $\Lambda^2 = \Lambda^2(U)$ with intrinsic structure inherited from the invariance under SO(4) of U. Recall that the group SO(4) is made of transition matrices between two orthonormal bases in four-dimensional vector spaces. The construction of this wedge product yields a natural indefinite quadratic form q on U, with values in the line Λ^4. Given a choice of volume element on Λ^4, q becomes a real-valued form with signature 0, i.e. with $b^+ = b^-$.

A choice of conformal structure on U singles out maximal positive and negative subspaces Λ^+, Λ^- with respect to q. Then given one of these two subspaces, say Λ^+, it is possible to determine the other through q:

$$\Lambda^- = \{\omega \in \Lambda^2 \mid q(\omega, \eta) = 0 \ \forall \eta \in \Lambda^+\} \tag{4.1}$$

Now, we want to focus on the connection between these conformal structures and the subspaces Λ^+.

Proposition 4.2. *For any three-dimensional positive subspace $\Lambda^+ \subset \Lambda^2$, there is a unique conformal structure on U for which this is the self-dual subspace.*

Proof. The proof follows a sketch from both Donaldson and Kronheimer [DK97, Chapt. 1] and Salamon [Sal89, Chapt. 7]. We identify any conformal structure on $U \cong \mathbb{R}^4$ with a choice of orthonormal basis up to fixed rotations. The set of these bases is made of quadruples of independent vectors and has a one-to-one correspondence with the Lie group $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$.

On U it is possible to choose a reference conformal metric and thus a reference subspace $\Lambda_0^+ \subset \Lambda^2$ as its corresponding maximal positive subspace. Then any other three-dimensional negative subspace Λ^+ can be represented as the rotation from Λ_0^+ to Λ^+ in the six-dimensional vector space Λ^2 with signature (3,3). These rotations are elements of $\mathrm{SO}(3,3)$ that do not flip the time orientation, i.e. elements of its identity connected component, denoted by $\mathrm{SO}_0(3,3)$. This is not a one-to-one correspondence, since Λ^+ and Λ^- are connected through orthogonality from (4.1), and so there is a need to quotient out independent rotations of the basis inside $\Lambda^\pm$. Thus, we identify the space of three-dimensional positive subspaces with the Lie group $\mathrm{SO}_0(3,3)/\mathrm{SO}(3)\times\mathrm{SO}(3)$. Now it is sufficient to check the existence of a bijection between these two Lie groups.

The map $\mathrm{SO}(4) \to \mathrm{SO}(3)\times\mathrm{SO}(3)$ is a double cover since the diagram

$$\begin{array}{ccc} \mathrm{SO}(4) & \longrightarrow & \mathrm{SO}(3)\times\mathrm{SO}(3) \\ {\scriptstyle 2:1}\uparrow & & {\scriptstyle 4:1}\uparrow \\ \mathrm{Spin}(4) & = & \mathrm{Spin}(3)\times\mathrm{Spin}(3) \end{array}$$

commutes. Here, the vertical maps are universal cover maps by definition of the $\mathrm{Spin}(n)$ group. The number of sheets is given by the order of the fundamental groups, which are $\pi_1(\mathrm{SO}(4), e) = \mathbb{Z}_2$ and $\pi_1(\mathrm{SO}(3)\times\mathrm{SO}(3), e) = \mathbb{Z}_2\times\mathbb{Z}_2$.

Topologically, any Lie group G can be written as $G = K\times\mathbb{R}^n$ where $K \subset G$ is its maximal compact subgroup. From this equality G and K share the same topology and specifically the same universal group. Recall that $\mathrm{SO}(3)\times\mathrm{SO}(3) \subset \mathrm{SO}_0(3,3)$ and $\mathrm{SO}(4) \subset \mathrm{SL}(4,\mathbb{R})$ are both maximal compact subgroups. This implies that the map $\mathrm{SL}(4,\mathbb{R}) \xrightarrow{2:1} \mathrm{SO}_0(3,3)$ is also a double cover. Then by quotienting, we get that the map

$$\mathrm{SL}(4,\mathbb{R}) \big/ \mathrm{SO}(4) \to \mathrm{SO}_0(3,3) \big/ \mathrm{SO}(3)\times\mathrm{SO}(3)$$

is well-defined and a bijection, which proves the claim. □

Set $U \cong T_p^*X$ for a generic $p \in X$, any fibre of the vector bundle $T^*X = \Omega^1(X)$ over X, an oriented four-dimensional manifold. Given a Riemannian metric g on X, it induces a conformal structure on any T_p^*X, which has an associated $\Lambda_p^+ \subset \Lambda^2(T_p^*X)$. Observe that the reasoning above is invariant with respect to the choice of frame. Thus, it is possible to define a subspace $\Lambda^+ \subset \Lambda^2 T^*X = \Lambda^2\Omega^1(X) = \Omega^2(X)$ associated to the metric g. This implies that the map sending metrics to maximal positive subspaces of the space of 2-*forms* is surjective. This is true for any oriented four-manifold, not only for the four-torus.

As shown in Proposition 4.1, the specific cohomology of T^4 yields that $\Omega^1(M) \cong H^1_{dR}(T^4)$ and $\Omega^2(M) \cong \Lambda^2 H^1_{dR}(T^4) = H^2_{dR}(T^4)$, and thus we prove the surjectivity of the period map as defined in Definition 1.7.

4.2 $K3$ surfaces

In this section, we will consider the second example of four-manifolds with Ricci-flat Einstein-Kähler metrics: $K3$ surfaces. The $K3$ surfaces admitting Kähler metrics are a fundamental aspect of the proof of the surjectivity of the period map. Thus, the first subsection will give some background facts on complex geometry to make the actual proof more understandable. The second subsection will focus on $K3$ surfaces and prove the surjectivity of the period map. The last subsection extends these results to other minimal complex surfaces, following the Enriques-Kodaira classification.

4.2.1 Background in complex geometry

In this subsection, we first introduce complex manifolds with their additional complex structure, and on them we define the Dolbeault cohomology and harmonic forms. Then we consider specific kinds of complex manifolds: Kähler, Calabi-Yau and hyperkähler. We conclude by enumerating some of their properties.

Main sources of this subsection are Huybrechts [Huy05, Chapt. 2-3], Gross, Huybrechts, and Joyce [GHJ12, Chapt. 4-5-14] and Barth, Hulek, Peters, and Van de Ven [BHPV15, Sect. I.15-IV.2].

Definition 4.3. A *holomorphic atlas* on a differentiable manifold is an atlas of the form $\{(U_i, \varphi_i)\}$ of the form $\varphi_i : U_i \overset{\cong}{\to} \varphi_i(U_i) \subset \mathbb{C}^n$, such that the transition functions φ_{ij} are holomorphic.

Definition 4.4. An *almost complex manifold* is a differentiable manifold X together with a vector bundle endomorphism $J : TX \to TX$ such that $J^2 = -\operatorname{Id}$ where TX is the real tangent bundle of the underlying real manifold.

The endomorphism J is called the *almost complex structure* on the underlying differentiable manifold.

Notation. An (almost) complex manifold X is denoted by $X = (M, J)$, where M is the underlying real manifold and J is the additional structure.

Fact. Any complex manifold X admits a natural almost complex structure.

Definition 4.5. A *complex manifold* X of complex dimension n is a (real) differentiable manifold of real dimension $2n$ endowed with an equivalence class of holomorphic atlases.

The additional structure on complex manifolds induces further structure on the tangent space and, consequently, on the cohomology.

Fact. Let X be an almost complex manifold. Then there exists a direct sum decomposition

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$$

of complex vector bundles on X, such that the $\mathbb{C}$-linear extension of J acts as multiplication by i on $T^{1,0}X$ respectively by $-i$ on $T^{0,1}X$.

Fact. For an almost complex manifold X one defines the complex vector bundles

$$\bigwedge\nolimits_{\mathbb{C}}^{k} X := \bigwedge\nolimits^{k}(T_{\mathbb{C}}X)^*$$
$$\bigwedge\nolimits^{p,q} X := \bigwedge\nolimits^{p}(T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge\nolimits^{q}(T^{0,1}X)^*$$

Their sheaves of sections are denoted by $\mathcal{A}^k_{X,\mathbb{C}}$ and $\mathcal{A}^{p,q}_X$, respectively.

There exist natural direct sum decompositions

$$\bigwedge\nolimits_{\mathbb{C}}^{k} X = \bigoplus_{p+q=k} \bigwedge\nolimits^{p,q} X \qquad \mathcal{A}^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_X$$

and moreover $\overline{\bigwedge^{q,p} X} = \bigwedge^{p,q} X$ and $\overline{\mathcal{A}^{q,p}_X} = \mathcal{A}^{p,q}_X$. Global sections of $\mathcal{A}^{p,q}_X$ are called forms *type* (or *bidegree*) (p,q) and are elements in $\mathcal{A}^{p,q}(X)$.

Definition 4.6. Let X be endowed with an integrable almost complex structure J. Then the (p,q)-*Dolbeault cohomology* is the vector space

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,*}(X), \overline{\partial}) = \frac{\ker(\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\operatorname{Im}(\overline{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))}$$

Fact. The Dolbeault cohomology of X computes the cohomology of the sheaf Ω^p_X, i.e. $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$.

Definition 4.7. The *Hodge numbers* of a compact complex manifold X are the numbers $h^{p,q}(X) := \dim H^{p,q}(X)$.

Complex manifolds, like any manifold, can be endowed with a Riemannian metric. We focus on those which are in agreement with the additional structure of the complex manifolds. Using this kind of metric, we can define formal adjoints and a Hermitian product on global forms.

Definition 4.8. A Riemannian metric g on X is a *Hermitian metric* on X if for any point $x \in X$ the scalar product g_x on T_xX is compatible with the almost complex structure J_x. The induced real $(1,1)$-form $\omega(\cdot,\cdot) := g(J(\cdot),\cdot)$ is called the *Hermitian form.*

The complex manifold X endowed with a Hermitian structure g is called a *Hermitian manifold.* Note that the Hermitian structure g is uniquely determined by the almost complex structure J and the fundamental form ω, since $g(\cdot,\cdot) = \omega(\cdot, J(\cdot))$.

Lemma 4.9. *If (X,g) is a Hermitian manifold then the formal adjoint of the exterior differential d is $d^* = \partial^* + \overline{\partial}^*$ and ${\partial^*}^2 = {\overline{\partial}^*}^2 = 0$.*

Proof. See [Huy05, Lem. 3.1.4]. □

Definition 4.10. If (X,g) is a Hermitian manifold, then the *Laplacians associated to ∂^* and $\overline{\partial}^*$*, respectively, are defined as

$$\triangle_{\partial} := \partial^*\partial + \partial\partial^* \qquad \triangle_{\overline{\partial}} := \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$$

and are such that $\triangle_{\partial}, \triangle_{\overline{\partial}} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X)$.

Definition 4.11. Let (X, g) be a compact Hermitian manifold, then on $\mathcal{A}^*_{\mathbb{C}}(X)$ there is an *Hermitian product* defined as

$$(\alpha, \beta) := \int_X g_{\mathbb{C}}(\alpha, \beta)\, \mathrm{dvol}_g$$

where $g_{\mathbb{C}}$ is the Hermitian extension of the Riemannian metric g and dvol_g is the volume form with respect to g.

Proposition 4.12. *Let (X, g) be a compact Hermitian manifold. Then the degree decomposition $\mathcal{A}^*_{\mathbb{C}}(X) = \bigoplus_k \mathcal{A}^k_{\mathbb{C}}(X)$ and the bidegree decomposition $\mathcal{A}^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$ are orthogonal with respect to $(\cdot,\cdot)$ from Definition 4.11.*

Proof. See [Huy05, Prop. 3.2.2]. □

Lemma 4.13. *Let X be a compact Hermitian manifold, then with respect to the Hermitian product $(\cdot,\cdot)$ the operators ∂^* and $\overline{\partial}^*$ are the formal adjoints of ∂ and $\overline{\partial}$, respectively.*

Proof. See [Huy05, Lem. 3.2.3]. □

Definition 4.14. Let (X, g) be a Hermitian complex manifold. A form $\alpha \in \mathcal{A}^k(X)$ is called *$\overline{\partial}$-harmonic* if $\Delta_{\overline{\partial}}(\alpha) = 0$. Moreover,

$$\mathscr{H}^k_{\overline{\partial}}(X, g) := \{\alpha \in \mathcal{A}^k_{\mathbb{C}}(X) \mid \Delta_{\overline{\partial}}(\alpha) = 0\}$$
$$\mathscr{H}^{p,q}_{\overline{\partial}}(X, g) := \{\alpha \in \mathcal{A}^{p,q}_{\mathbb{C}}(X) \mid \Delta_{\overline{\partial}}(\alpha) = 0\}$$

Analogously, one defines *∂-harmonic* forms and the spaces $\mathscr{H}^k_{\partial}(X, g)$ and $\mathscr{H}^{p,q}_{\partial}(X, g)$.

Lemma 4.15. *Let (X, g) be a compact Hermitian manifold. A form $\alpha \in \mathcal{A}^k(X)$ is $\overline{\partial}$-harmonic (resp. ∂-harmonic) if and only if $\overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$ (resp. $\partial\alpha = \partial^*\alpha = 0$).*

Proof. See [Huy05, Lem. 3.2.5]. □

Theorem 4.16 (Hodge decomposition)**.** *Let (X, g) be a compact Hermitian manifold. Then there exist two natural orthogonal decompositions*

$$\mathcal{A}^{p,q}(X) = \partial\mathcal{A}^{p-1,q}(X) \oplus \mathscr{H}^{p,q}_{\partial}(X) \oplus \partial^*\mathcal{A}^{p+1,q}(X)$$
$$\mathcal{A}^{p,q}(X) = \overline{\partial}\mathcal{A}^{p,q-1}(X) \oplus \mathscr{H}^{p,q}_{\overline{\partial}}(X) \oplus \overline{\partial}^*\mathcal{A}^{p,q+1}(X)$$

The spaces $\mathscr{H}^{p,q}(X)$ are finite-dimensional. If additionally (X, g) is assumed to be Kähler then $\mathscr{H}^{p,q}_{\partial}(X) = \mathscr{H}^{p,q}_{\overline{\partial}}(X)$.

Proof. See [Huy05, Thm. 3.2.8]. □

The Hodge decomposition takes a particular form on Kähler manifolds. Other specific complex manifolds are here defined: Calabi-Yau and hyperkähler. $K3$ surfaces are the only complex manifolds that are both Calabi-Yau and hyperkähler.

Definition 4.17. A Hermitian metric g with ω its Hermitian form on a complex manifold (X, J) is called *Kähler* if one of the following three equivalent conditions holds:

(i) $d\omega = 0$

(ii) $\nabla J = 0$

(iii) $\nabla\omega = 0$

where ∇ is the Levi-Civita connection of g. We then call (X, J, g) a *Kähler manifold*, and ω the *Kähler form*.

Corollary 4.18. Let (X, g) be a compact Kähler manifold, then there exists a decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

This decomposition does not depend on the chosen Kähler structure.

Moreover, with respect to complex conjugation on $H^*(X, \mathbb{C}) = H^*(X, \mathbb{R}) \otimes \mathbb{C}$ one has $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Proof. See [Huy05, Cor. 3.2.12]. □

Fact. The set of closed positive real $(1, 1)$-forms $\omega \in \mathcal{A}^{1,1}(X)$ is the set of all Kähler forms, and on a compact complex manifold X is an open convex cone in the linear space $\{\omega \in \mathcal{A}^{1,1}(X) \cap \mathcal{A}^2(X) \mid d\omega = 0\}$.

Definition 4.19. A *Calabi-Yau m-fold* for $m \geq 2$ is a quadruple (X, J, g, Ω) such that (X, J) is a compact m-dimensional complex manifold, g a Kähler metric on (X, J) with holonomy group $\mathrm{Hol}(g) = \mathrm{SU}(m)$, and $\Omega \in \Omega^{(m,0)}(X)$ non-zero constant form called the *holomorphic volume form*, which satisfies

$$\frac{\omega^m}{m!} = (-1)^{m(m-1)/2} \left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega}$$

Definition 4.20. A Riemannian $4m$-manifold (X, g) is called *hyperkähler* if $\mathrm{Hol}(g) = \mathrm{Sp}(m)$. A hyperkähler manifold comes naturally equipped with complex structures J_1, J_2, J_3 with Kähler forms $\omega_1, \omega_2, \omega_3$, such that $\nabla J_j = \nabla\omega_j = 0$ for $j = 1, 2, 3$, where ∇ is the Levi-Civita connection of g.

Fact. If $a_1, a_2, a_3 \in \mathbb{R}$ with $a_1^2 + a_2^2 + a_3^2 = 1$ then $J = a_1J_1 + a_2J_2 + a_3J_3$ is a complex structure on M, and g is Kähler with respect to it, with Kähler form $\omega_g = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$.

Notation. Let X be a compact Kähler manifold, set $H^{1,1}(X; \mathbb{R}) := H^2(X; \mathbb{R}) \cap H^{1,1}(X; \mathbb{C})$.

These particular manifolds carry some interesting properties.

Definition 4.21. Let X be a compact Kähler manifold. The *Kähler class* associated to a Kähler structure on X is the cohomology class of its Kähler form ω.

The *Kähler cone* of X is the cone in $H^{1,1}_{\mathbb{R}}(X)$ given by

$$\mathcal{K}_X = \{c \in H^{1,1}_{\mathbb{R}}(X) \mid c \text{ represented by a Kähler form}\}$$

i.e. the set of all Kähler classes of X.

Proposition 4.22. *Let (X, J) be a compact Kähler manifold with $c_1(X) = 0 \in H^2(X; \mathbb{R})$. Then there is a unique Ricci-flat Kähler metric in each Kähler class on X.*

The Ricci-flat Kähler metrics on X form a smooth family of dimension $h^{1,1}(X)$, isomorphic to the Kähler cone $\mathcal{K}$ of X.

Proof. See [GHJ12, Thm. 5.2] and [BHPV15, Thm. I.15.1]. □

Proposition 4.23. *Let X be a compact surface. Then, for $p + q = 2$ the Dolbeault group $H^{p,q}(X)$ is naturally isomorphic to the subspace of $H^{p+q}(X;\mathbb{C})$, whose elements can be represented by a closed form of type (p, q). In this way, one obtains natural decompositions $H^k(X;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$.*

The same happens for $p + q = 1$, if $b_1(X)$ is even.

Proof. See [BHPV15, Thm. IV.2.9]. □

Proposition 4.24 (Signature theorem)**.** *Let X be a compact complex surface, then the cup-product form on $H^2(X;\mathbb{R})$, restricted to $H^{1,1}_{\mathbb{R}}(X)$, is non-degenerate of type $(1, h^{1,1} - 1)$ if $b^+(X)$ is even and of type $(0, h^{1,1})$ if $b^+(X)$ is odd.*

Proof. See [BHPV15, Thm. IV.2.13]. □

Proposition 4.25. *Let X be a compact complex surface, then the cup-product form on $H^2(X;\mathbb{R})$, restricted to $H^{2,0}_{\mathbb{R}}(X) \oplus H^{0,2}_{\mathbb{R}}(X)$, is positive definite.*

Proof. Without loss of generality, we restrict the proof to elements in $H^{2,0}_{\mathbb{R}}(X)$ or $H^{0,2}_{\mathbb{R}}(X)$ separately. Consider $\omega \in H^{2,0}_{\mathbb{R}}(X)$ and the corresponding complex conjugate $\overline{\omega} \in H^{0,2}_{\mathbb{R}}(X)$, then

$$(\omega, \overline{\omega}) = \int_X \omega \wedge \overline{\omega} = \int_X \|\omega\|^2 \,\mathrm{dvol} \geq 0$$

which concludes the proof. □

4.2.2 $K3$ surfaces and proof of surjectivity of the period map

In this subsection, we will first define $K3$ surfaces and then enumerate some of their properties, focusing on them being both Calabi-Yau and hyperkähler. These properties will then be used to prove the surjectivity of the period map, restricted to the subset of Kähler metrics.

Sources of this subsection are Barth, Hulek, Peters, and Van de Ven [BHPV15] and Gross, Huybrechts, and Joyce [GHJ12].

Definition 4.26. A $K3$ *surface* is a complex surface X with the canonical line bundle $\mathscr{K}_X$ trivial and $b_1(X) = 0$.

Proposition 4.27. *For any $K3$ surface X, we have $c_1(X) = 0$, $c_2(X) = 24$, $\tau(X) = -16$.*

Proof. See [BHPV15, Prop. I.3.1]. □

Notation. Denote

$$L = -E_8 \oplus -E_8 \oplus H \oplus H \oplus H,$$

$$L_{\mathbb{R}} = L \otimes \mathbb{R} \cong H^2(X;\mathbb{R}) \quad \text{with } (\cdot,\cdot) \text{ extended } \mathbb{R}\text{-bilinearly},$$

$$L_{\mathbb{C}} = L \otimes \mathbb{C} \cong H^2(X;\mathbb{C}) \quad \text{with } (\cdot,\cdot) \text{ extended } \mathbb{C}\text{-bilinearly},$$

$$\Omega = \{[\omega] \in \mathbb{P}(L_{\mathbb{C}}) \mid (\omega,\omega) = 0, (\omega,\overline{\omega}) > 0\}.$$

Proposition 4.28. *Let X be a $K3$ surface, then $H^2(X;\mathbb{Z})$ is torsion-free of rank 22 and it is isometric to L, when equipped with the cup-product pairing.*

Proof. See [BHPV15, Prop. I.3.2]. □

Corollary 4.29. Let X be a $K3$ surface, then the second cohomology has signature $(3, 19)$.

Proposition 4.30. *Let X be a $K3$ surface, then its Hodge diamond is*

$$\begin{matrix} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Proof. See [BHPV15, Prop. I.3.3]. □

Definition 4.31. A $K3$ surface X with an isomorphism $\phi : H^2(X;\mathbb{Z}) \to L$ is called a *marked $K3$ surface*. It is denoted by (X, ϕ).

Definition 4.32. Let (X, ϕ) be a marked $K3$ surface and fix a complex structure J on X. Consider $\omega_X \in H^{2,0}(X)$ nowhere vanishing holomorphic 2-form, then the line $\phi_{\mathbb{C}}(\langle\omega_X\rangle) \in \Omega \subset \mathbb{P}(L_{\mathbb{C}})$ is called the *period point* of (X, ϕ).

Remark 4.33. Changes in the complex structure J yield changes in the period point in Ω.

Theorem 4.34. *Every point of Ω occurs as the period point of a marked $K3$ surface.*

Proof. See [BHPV15, Thm. VIII.14.2] □

Remark 4.35. It follows from Theorem 4.34 that, given any 2-form $\omega \in \Omega$, there is a complex structure J_ω so that ω is holomorphic.

Proposition 4.36. *Any two $K3$ surfaces are diffeomorphic. In particular, any $K3$ surface is simply-connected.*

Proof. See [BHPV15, Cor. VIII.8.6]. □

Remark 4.37. From proposition above, all $K3$ surfaces share the same underlying smooth four-manifold, so in this section, we are considering one manifold, not a class of manifolds. Additionally, any of the following considerations will be independent on the choice of marked $K3$ surface and isomorphism ϕ, since diffeomorphisms do not change the cohomology.

Remark 4.38. In complex dimension 2, we have $\mathrm{Sp}(1) = \mathrm{SU}(2)$, so (compact) hyperkähler 4-manifolds and Calabi-Yau 2-folds coincide, as said above. All Calabi-Yau 2-folds are $K3$ surfaces, and conversely, every $K3$ surface (X, J) admits a family of Kähler metrics g making it into a Calabi-Yau 2-fold.

Using Propositions 4.22 and 4.27 together, we get that any $K3$ surface X admits Ricci-flat metrics and, actually, Ricci-flat Kähler metrics.

Proposition 4.39. *Let X be a $K3$ surface. Then, given any $U \in \mathrm{Gr}^+(H^2(X))$ three-dimensional (maximal) positive subspace in $H^2(X)$, there is a complex structure J on X such that U can be split as:*

$$U = H^{2,0}_{\mathbb{R}}(X) \oplus \mathbb{R}\omega \oplus H^{0,2}_{\mathbb{R}}(X) \tag{4.2}$$

where $\omega \in H^{1,1}_{\mathbb{R}}(X)$.

Proof. Consider a marked $K3$ surface (X, ϕ), and recall from Remark 4.37 that the choice of the isomorphism ϕ can be done without loss of generality. Then we have $\phi_{\mathbb{C}}(U) \cap \{\omega \mid (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\} \neq \emptyset$ and this follows from how we defined the isomorphism ϕ. Choose $\omega \in \phi_{\mathbb{C}}(U) \cap \{\omega \mid (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\}$, by Theorem 4.34 ω is a period point with respect to a complex structure J. Fix the complex surface (X, J) for the rest of the proof.

Since U is independent on the complex structure, we must have $\langle \omega, \overline{\omega} \rangle = H^{2,0}_{\mathbb{R}}(X) \oplus H^{0,2}_{\mathbb{R}}(X) \subset U$, which is in agreement with Proposition 4.25. Since $\dim(H^{2,0}_{\mathbb{R}}(X) \oplus H^{0,2}_{\mathbb{R}}(X)) = 2$, see Proposition 4.30, by dimensionality reason $U \cap H^{1,1}_{\mathbb{R}}(X) \neq \emptyset$. Thus, there is $\omega \in H^{1,1}_{\mathbb{R}}(X)$ such that

$$U = H^{2,0}_{\mathbb{R}}(X) \oplus \mathbb{R}\omega \oplus H^{0,2}_{\mathbb{R}}(X). \qquad \square$$

Proposition 4.40. *Let (X, J) be a $K3$ surface with g Kähler metric. Denote with ω_g the corresponding Kähler form. Then the intersection form is positive definite when restricted to the line $\mathbb{R}\omega_g$.*

Proof. By Remark 4.38, $K3$ surfaces are Calabi-Yau 2-folds. Following Definition 4.19 we get that for any choice of complex structure J and for any Kähler metric g, the corresponding Kähler form ω_g is such that

$$\frac{\omega_g^2}{2!} = (-1)\left(\frac{i}{2}\right)^2 \Omega \wedge \overline{\Omega} = \frac{1}{4} \Omega \wedge \overline{\Omega}$$

Thus, applying the definition of holomorphic volume form, we get

$$\int_X \omega_g \wedge \omega_g = \frac{1}{2} \int_X \Omega \wedge \overline{\Omega} > 0. \qquad \square$$

Remark 4.41. It follows from Proposition 4.40 that to prove the surjectivity of the period map, it is sufficient to prove that any $\omega \in H^{1,1}_{\mathbb{R}}(X)$ has a Kähler representative with a corresponding Kähler metric.

Notation. Let X be a Kähler manifold. We denote by $\mathrm{Met}^{\mathrm{K}}(X)$ the subset of Kähler metrics on X. The subset of Ricci-flat Kähler metrics is denoted by ${\mathrm{Met}^{\mathrm{K}}}_0(X)$.

Proposition 4.42. *Let X be a $K3$ surface, then the map*

$$\mathrm{P}^K_X : (\mathrm{Met}^{\mathrm{K}}(X)/\sim) \longrightarrow \mathcal{K}_X$$

is injective, where $g_1 \sim g_2$ if they are in the same Kähler class.

Proof. This follow directly from definition of Kähler class and Kähler cone. $\square$

Corollary 4.43. The period map Π_{K3} of a $K3$ surface is surjective.

Proof. To prove surjectivity, we restrict to the subset of Ricci-flat Kähler metrics ${\mathrm{Met}^{\mathrm{K}}}_0(X)$, as explained in Remark 4.41. From Proposition 4.22, we get that ${\mathrm{Met}^{\mathrm{K}}}_0(X)$ is isomorphic to $(\mathrm{Met}^{\mathrm{K}}(X)/\sim)$ and that $\dim {\mathrm{Met}^{\mathrm{K}}}_0(X) = h^{1,1}(X)$. Then by Proposition 4.42, we have that $\mathrm{P}^K_X : {\mathrm{Met}^{\mathrm{K}}}_0(X) \to \mathcal{K}_X$ is injective, and thus bijective by dimensionality reasons. So we have that $\dim \mathcal{K}_X = \dim {\mathrm{Met}^{\mathrm{K}}}_0(X) = h^{1,1}(X)$.

Since by definition $\mathcal{K}_X \subseteq H^{1,1}_{\mathbb{R}}(X)$, it implies equality. This proves surjectivity as observed in Remark 4.41. $\square$

4.2.3 Generalization to other compact complex surfaces

In this subsection, we will use the elements introduced above to consider other compact connected complex surfaces, following the Enriques-Kodaira classification. We first start with the definition of the *Kodaira dimension*, an invariant of complex manifolds. The main sources of this subsection are Milne [Mil03, Chapt. 9] and Barth, Hulek, Peters, and Van de Ven [BHPV15, Sect. I.7-VI.1], where not otherwise stated.

We will start by giving some algebraic background used in this classification. In the following definitions, we will consider the fields $\Omega \supset F$ with Ω much bigger than F (e.g. $\mathbb{C} \supset \mathbb{Q}$).

Definition 4.44. Let F be a field and E an integral domain containing F as a subring. An element $\alpha \in E$ is *algebraic over* F if there is a non-zero $g \in F[X]$ so that $g(\alpha) = 0$.

Definition 4.45. Let $\gamma \in \Omega$ and let $A \subset \Omega$. When γ is algebraic over $F(A)$, it is said to be *algebraically dependent* on A (over F). A set B is *algebraically dependent* on A if every element of B is algebraically dependent on A.

Definition 4.46. A *transcendence basis* for Ω over F is an algebraically independent set A such that Ω is algebraic over $F(A)$.

Definition 4.47. The cardinality of a transcendence basis for Ω over F is called the *transcendence degree* of Ω over F. It will be denoted by $\text{trdeg}(\Omega : F)$. It is well-defined since it is possible to show that any transcendence bases for Ω over F have the same cardinality.

Notation. Let X be a complex n-manifold. Write $\mathscr{K}_X = \bigwedge^2(T^*X)$ for the canonical line bundle of X.

Construction. Let X be any compact complex manifold, then there is a pairing

$$\Gamma\left(X, \mathscr{K}_X^{\otimes m_1}\right) \otimes \Gamma\left(X, \mathscr{K}_X^{\otimes m_2}\right) \to \Gamma\left(X, \mathscr{K}_X^{\otimes (m_1+m_2)}\right).$$

Thus, we can construct a commutative ring $R(X)$ with unit element from the direct sum $\mathbb{C} \oplus \sum_{m\geq 1} \Gamma(X, \mathscr{K}_X^{\otimes m})$. This ring is called the *canonical ring* of X.

Notation. Denote with $\text{tr}(R(X)) := \text{trdeg}(R(X) : \mathbb{C})$ the degree of transcendency of the ring $R(X)$ over $\mathbb{C}$.

Definition 4.48. Let X be a connected complex manifold of dimension n. The *Kodaira dimension* $\text{kod}(X)$ of X is defined as follows:

$$\text{kod}(X) = \begin{cases} -\infty & \text{if } R(X) \cong \mathbb{C}, \\ \text{tr}(R(X)) - 1 & \text{otherwise.} \end{cases}$$

Remark 4.49. It can be proven that $\text{tr}(R(X))$ is finite and thus $\text{kod}(X)$ to be well-defined. Actually we have that $\text{kod}(X) = -\infty, 0, 1, \cdots, n$, where n is the complex dimension of X.

This invariant is used to classify minimal complex surfaces.

Recall. Recall from Definition 1.20, a smooth surface is called *minimal*, if it does not contain any (-1)-curve.

Definition 4.50. A non-singular surface X_{min} is called a *minimal model* of the non-singular surface X, if X_{min} is minimal itself, and if there is a bimeromorphic map from X onto X_{min}.

Class of surfaces	kod(X)	$b_1(X)$	c_1^2	c_2	Signature
Minimal rational surfaces					
Projective plane	$-\infty$	0	9	3	(1,0)
Hirzebruch surfaces	$-\infty$	0	8	4	(1,1)
Minimal surfaces of class VII	$-\infty$	1	≤ 0	≥ 0	(0,b_2)
Ruled surfaces of genus $g \geq 1$	$-\infty$	$2g$	$8(1-g)$	$4(1-g)$	(1,1)
Enriques surfaces	0	0	0	12	(1,9)
Hyperelliptic surfaces	0	2	0	0	(1,1)
Kodaira surfaces					
Primary	0	3	0	0	(2,2)
Secondary	0	1	0	0	(0,0)
Tori	0	4	0	0	(3,3)
$K3$ surfaces	0	0	0	24	(3,19)
Minimal proper elliptic surfaces	1		0	≥ 0	
Minimal surfaces of general type	2	even	> 0	> 0	

Table 4.1: Minimal complex surfaces according to Enriques-Kodaira classification

Theorem 4.51 (Enriques–Kodaira classification)**.** *Every surface has a minimal model in exactly one of the classes enumerated in Table 4.1. This model is unique (up to isomorphisms) except for the surfaces whose minimal models are either rational surfaces or ruled surfaces of genus $g \geq 1$.*

We will first give a brief definitions of these models:

rational surface: surface that is birationally equivalent to $\mathbb{CP}^2$: these are only $\mathbb{CP}^2$ and an infinite sequence of surfaces, the Hirzebruch surfaces Σ_n for $n = 0, 2, 3, ...$;

surface of class VII: surface X with $\mathrm{kod}(X) = -\infty$ and $b_1(X) = 1$: examples of these can be Hopf surfaces and Inoue surfaces, but a complete classification is still lacking;

ruled surface of genus $g \geq 1$: surface X admitting a ruling, i.e. an analytically locally trivial fibration with fibre $\mathbb{CP}^1$ and a structure group $\mathrm{PGL}(2, \mathbb{C})$ over a smooth curve of genus $g \geq 1$;

Enriques surface: surface X with $b_1(X) = 0$, for which $\mathscr{K}_X^{\otimes 2} \cong \mathcal{O}_X$, but $\mathscr{K}_X \neq \mathcal{O}_X$;

Hyperelliptic surfaces: surface X with $b_1(X) = 2$, admitting a holomorphic, locally trivial fibration over an elliptic curve with an elliptic curve as a typical fibre;

primary Kodaira surface: surface with $b_1(X) = 3$, admitting a holomorphic locally trivial fibration over an elliptic curve with an elliptic curve as typical fibre;

secondary Kodaira surface: surface X, which admits a primary Kodaira surface as unramified covering, but which itself is not a primary Kodaira surface; these are elliptic fibre spaces over rational curves, with $b_1(X) = 1$;

$K3$ **surface:** see Definition 4.26;

torus: see Section 4.1;

properly elliptic surface: elliptic surface X with $\mathrm{kod}(X) = 1$;

surface of general type: surface X with $\mathrm{kod}(X) = 2$.

Remark 4.52. Due to Theorem 4.51, in the following reasoning we will only consider the minimal models. For a discussion about non-minimal surfaces see Remark 4.53.

The surfaces in most of the classes in Table 4.1 are minimal by definition. The minimality of ruled surfaces of genus $g \geq 1$ is a consequence of Lüroth's theorem for curves (the image of a rational curve is again rational). Minimality in classes with $\mathrm{kod}(X) = 1$ is due to the fact that $(\mathscr{K}, E) = -1$ where $\mathscr{K}$ is the canonical divisor of X and E a (-1)-curve.

Table 4.1 was compiled following closely [BHPV15, Tbl. 10] and applying Propositions 4.24 and 4.25 to get the signature, when c_1^2 and c_2 are known. When $\mathrm{kod}(X) = 1, 2$ there are not enough information to be able to calculate the signature.

Looking at values of signature in Table 4.1, we can observe that the surjectivity of the period map is trivial for the projective plane, minimal surfaces of class VII, and secondary Kodaira surfaces, since they all have definite cup-product. Additionally, the surjectivity of the period map for surfaces with $b^+ = 1$ follows from Theorem 2.2.

Since tori and $K3$ surfaces were treated extensively in Sections 4.1 and 4.2, the only surfaces with a known signature that we still need to consider are the primary Kodaira surfaces. Given a complex structure J on a primary Kodaira surface, by Proposition 4.25, we have that the cup product form is positive definite when restricted to $H^{2,0}_{\mathbb{R}}(X) \oplus H^{0,2}_{\mathbb{R}}(X)$. Since $b(X) = 3$ odd, by Proposition 4.24, the cup product is negative definite restricted to $H^{1,1}_{\mathbb{R}}(X)$. These considerations are not enough to prove surjectivity, since we would still need to prove that for every positive definite subspace U there is a complex structure J so that $U = H^{2,0}_{\mathbb{R}}(X) \oplus H^{0,2}_{\mathbb{R}}(X)$.

Remark 4.53 (Non-minimal surfaces). Non-minimal surfaces are obtained from minimal surfaces by repeatedly blowing up points [Pet21]. Thus, they require a separate discussion since the blowing up of a point changes the signature. Without loss of generality, let consider the blow up of a single point.

Surfaces of the shape $X = X_m \# \overline{\mathbb{CP}^2}$ do not directly inherit the surjectivity of the period map. In this case, we get $H^2(X) = H^2(X_m) \oplus H^2(\overline{\mathbb{CP}^2})$ and thus, $\mathrm{sgn}(X) = (b^+(X_m), b^-(X_m)+1)$. By Proposition 2.9, we get that the subspace $H^2(X_m) \oplus \{0\} \subset H^2(X)$ is in the image of the period map Π_X. It is not possible to apply Proposition 2.9 to elements in the positive cone of $H^2(X)$ with non-zero projection into $H^2(\overline{\mathbb{CP}^2})$. Thus, this kind of elements would require an ad-hoc construction to prove that they are in $\mathrm{Im}(\Pi_X)$.

Conclusions

The main question considered in this thesis was: *is the period map surjective?* The answer is found to be positive when $b^+ = 1$, as shown in Scaduto's paper, and when the manifold admits Ricci-flat metrics. Scaduto also proved that, in general, the image of the period map is dense. Regarding other cases not considered here, it is not known whether or not it is surjective, and there are no counterexamples, either.

The results presented in this thesis confirm the conjectured surjectivity of the period map at least in some cases. These things were yet to be proven in the general cases before Scaduto's paper. This thesis gave a deeper understanding of Scaduto's article and provided direct proof for other special manifolds.

We remark here that in all proofs shown in this thesis, we used a specific subset of metrics on the manifold considered in order to exploit their particular properties, e.g. being Kähler, broken... This is why it is not so straightforward to extend the proof of surjectivity to other cases, even though several paths forward remain open. However, each of these potential paths encounters significant obstacles:

- expanding the cases covered by Theorems 2.1 and 2.2: use specific properties related to the characteristic of the cases considered; these characteristics are not expandible further since 2.1 relies on properties of $H^2(X;\mathbb{Z})$, which are not extendable to the real case, and 2.2 needs to tessellate $\mathrm{Gr}^+(H^2(X))$ and to do so we need it to be codimension 1;

- building new manifolds from old ones: consider manifolds which admit the surjectivity of the period map and from these construct new manifolds (e.g $X\#\overline{\mathbb{CP}^2}$, $X_1\#X_2$, $X_1 \cup_Y X_2$) and prove that surjectivity gets directly inherited to these new manifolds; this can not be done, since it is not possible to use only Proposition 2.9: it only covers a subset of the positive second cohomology as already explained in the text;

- from the density of the image: using Theorem 2.2, it is sufficient to prove that $\mathrm{Im}(\Pi_X)$ is closed in $\mathrm{Gr}^+(H^2(X))$; the problem here is that to study the image we need to consider *all* metrics on a manifold and not a subset of them, so it is not possible to use any property of these metrics;

- other special manifolds: focusing on specific manifolds requires the use of specific properties that give us the possibility to consider a specific kind of metrics

Each of these proof strategies requires more complex constructions than the direct approach based on known results, as used in this thesis.

Keep in mind that from previous articles, the proof of surjectivity had only applications for $b^+ = 1$ since all theorems needed only this hypothesis, which was already proven by Scaduto's paper. So the research going forward is not strictly required to prove for the other four-manifolds, since there are no direct applications for this surjectivity. Nonetheless, some of the methods developed in this thesis, or arising from the above proof strategies, may apply to the study of other properties of four-manifolds.

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Selbstständigkeitserklärung

Hiermit versichere ich, die vorliegende Arbeit selbstständig angefertigt und alle Quellen und Hilfsmittel angegeben zu haben.

Giulia Morelli

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