MASTER'S THESIS

Towards a Topological Theory for Domain Walls in (Super-)Yang-Mills Theories

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Abstract

The presence of domain walls in (Super-)Yang-Mills theories was predicted almost 20 years ago. Nevertheless, some of their properties still lack a satisfactory field theoretical interpretation. String theory constructions suggest the appearance of a $U(1)$ Chern-Simons term of level N on the worldvolume of such walls. This has so far not been obtained explicitly in field theory.

Disregarding all dynamical degrees of freedom we are able to derive a topological field theory that consistently incorporates the Aharonov-Bohm phases between non-local operators and the Witten effect. The line and surface operators in our model correspond to electric/magnetic charges and fluxes parametrized by the center $\mathbb{Z}_N$ of the full gauge group $SU(N)$. Introducing domain walls while preserving an additional emergent 1-form gauge symmetry demands precisely the Chern-Simons term seen in string theoretical models.

We are able to generalize the topological field theory to supersymmetric Yang-Mills theories and recover some qualitative statements about the domain wall behavior.

Finally, two possible approaches to extract dynamical information using analogies to other physical systems are briefly outlined.

Contents

Chapter 1

Introduction

Since their discovery in 1954 Yang-Mills theories have been the foundation of many models in physics describing the fundamental interactions of nature, [1]. Their most prominent application is the theories of the weak and the strong interactions, where they have led to new, revolutionary ideas as gauge theories with gauge group $SU(N)$. Their topological properties opened up new possibilities in constructing non-perturbative configurations, such as monopoles, [2] and [3], or instantons, [4]. Instantons are intimately related to the occurrence of a so-called θ-term in the Lagrangian density of Yang-Mills theories, [5], which is of topological nature. Furthermore, the θ-term has implications for the electric charges of monopoles in the model, [6].

Besides all its fascinating new results, some properties of Yang-Mills theories still remain unexplained. The chiral symmetry breaking observed in QCD due to a quark condensate is only one example. The most intriguing feature certainly is confinement, connected to the running coupling discovered in [7]. The coupling constant in Yang-Mills theories grows stronger at lower energies. Thus, the fundamental degrees of freedom, the gauge bosons, are not a good description for the low energy effective theory. Instead, composite objects, such as mesons, baryons, and glueballs are the natural excitations in this limit. All of them are colorless, i.e. do not carry an open index of the gauge group. This is known as confinement. Furthermore, all these objects are massive even though the gauge bosons themselves are massless, leading to a dynamical generation of a mass gap in confining theories.

Many models have been constructed in order to illuminate these properties of non-Abelian gauge theories. One of the most interesting is 't Hooft's large N limit, see section 2.3, and [8]. In this setup radiative corrections are under better control simplifying calculations in the limit $N \to \infty$. Nevertheless, the mechanism causing confinement remained obscure.

Another very powerful tool which sheds some light on the dynamics of Yang-Mills theories is the study of their supersymmetric extensions. The inclusion of this new form of space-time symmetry allows to carry out exact calculations even in the strong coupling regime. In these Super-Yang-Mills theories the β-function was calculated exactly in [9]. Another exact result was the calculation of the gaugino condensate and the corresponding vacuum structure in the $\mathcal{N} = 1$ supersymmetric Yang-Mills models, [10]. This setup is highly related to QCD, observed in nature, and serves as a tool to study the strong interaction, [11].

It needed an extended supersymmetry $\mathcal{N} = 2$, however, to obtain evidence of the confining mechanism. The famous paper by Seiberg and Witten in 1994 [12] presented an explicit construction of confinement via monopole condensation. In the main text we will apply this model of condensation of magnetic charges.

In the 90's these supersymmetric theories experienced another revolution. It was found by Dvali and Shifman, [13], that supersymmetric gauge theories allowed a central extension in the form of domain wall solutions. Under the assumption of a BPS nature of these configurations it is possible to derive their exact tension in the large N limit. This created another puzzle because instead of the typical solitonic behaviour of topological field configurations the energy of the supersymmetric walls rather resembled D-branes in string theory. Further, it was found that chromoelectric flux tubes should be able to end on the walls, just as open strings end on D-branes. This analogy led to another method of investigation and has been an interesting research area for both quantum field theory and string theory, [14], [15], [16].

Soon new phenomena connected to the behavior of this BPS states and even to analogous configurations in non-supersymmetric theories were revealed, [17]. One of the new results from string theory suggested that the worldvolume theory of the domain walls contains a $U(1)$ Chern-Simons theory at level N. This topological field theory, plausible as well in a $\mathcal{N} = 2$ supersymmetry approach [18], has not been seen in field theoretical constructions. In the following we will work out a model exactly uncovering the occurrence of such a term on fundamental domain walls in (Super-)Yang-Mills theories.

As mentioned above, the dynamical behavior of non-Abelian gauge theories is highly complex and one needs to consider simplified models in order to carry out calculations. Because a Chern-Simons theory is by its nature topological, our approach is the derivation of a topological field theory for the entire Yang-Mills setup. The strategy is supported by the classification of confining phases via non-local operators in [19], [20], and [21] as well as their topological description in [22]. The topological field theories reduce correlations of the non-local operators to mere Aharonov-Bohm phases, requiring an additional 1-form gauge symmetry, see chapter 4. All dynamical issues are invisible to the topological constructions, which tremendously simplifies the models. The remaining action encodes the statistics of electric/magnetic flux tubes and charges. If the systems were to keep all the non-Abelian exchange phases it would still be too complicated. Luckily, [19] suggests that it is enough to parametrized the present operators in a discrete charge lattice corresponding to the center of the full gauge group, originating from ideas in [23], [24], and [25]. This discrete gauge group permits a description in terms of the Abelian-Higgs model and finally leads to the topological field theory describing the vacuum structure in (Super-)Yang-Mills theories.

Our model is based on two assumptions. First, confinement is due to the condensation of charge N monopoles. Second, that the Witten effect, [6], which endows monopoles with an electric charge in the presence of a non-vanishing vacuum angle, is present in this approach. These assumptions lead to an intuitive and consistent way of encoding the topological properties of the theories of interest.

Finally, including domain walls in the topological picture while maintaining the extended gauge symmetry demands the appearance of a level N Chern-Simons term on the wall worldvolume. To our knowledge this is the first field theoretical derivation requiring such a term. Parts of the derivation and a similar action are used for a different purpose in [21]. Moreover, axionic considerations illuminate the dynamics of (Super-)Yang-Mills domain walls and recover the correct behavior for non-supersymmetric and supersymmetric models respectively.
The thesis is organized as follows. In chapter 2 we present a brief introduction to non-Abelian gauge theories and their dynamical properties in a non-supersymmetric framework. In chapter 3 all the necessary tools for the later topological construction are worked out, including an introduction to non-local operators and topological field theories. Our explicit model is derived in chapter 4 for Yang-Mills theories and the results are described in full detail. The generalization to Super-Yang-Mills theories is carried out in the following two chapters, offering a brief introduction to the topic and stating exact results relevant for later investigations. The results for the topological approach under inclusion of supersymmetry and related topics connected to the dynamics of the domain walls are summarized in chapter 6. The last chapter 7, hints at how to proceed the investigations and concludes the thesis. In the appendix notational conventions are clarified and an explicit construction of a dyonic condensate is elaborated.

Chapter 2

Yang-Mills Theory

In 1954 Yang and Mills introduced a new form of gauge theory, [1]. Since then these theories played a crucial role for the construction of physical models that successfully describe nature. Nowadays they are the building blocks of the standard model describing the strong and electroweak interactions. The work done in this thesis wants to shed some light on the underlying properties such as their vacuum structure and topological properties. For this reason we briefly introduce the concepts and phenomena of Yang-Mills theories with gauge group $SU(N)$ in this chapter. We concentrate on the relevant properties.

2.1 Basics of Yang-Mills theories

In this section we present a collection of basic facts about Yang-Mills theories with gauge group $SU(N)$. First, we introduce some notational conventions and state the Lagrangian density for pure Yang-Mills theories. Subsequently, we discuss another gauge invariant term that has to be added to the Lagrangian density, the so-called θ-term, and elaborate some of its consequences. Finally, we comment on the introduction of charged particles in the fundamental representation, usually called quarks.
In contrast to later chapters, which deal with topological issues, here we work in the more familiar index notation that is mainly used in the literature, e.g. [5].

2.1.1 The Lagrangian Density of Pure Yang-Mills

To describe a pure gauge theory with gauge group $SU(N)$ we need to introduce a non-Abelian gauge field A. This gauge field takes values in the Lie algebra $su(N)$ and transforms in the adjoint representation. Decomposing the gauge field in component fields with respect to the group structure we can write it as

$$A_\mu = A_\mu^a T^a \quad , \tag{2.1}$$

where T^a denote the generators of the Lie group in the fundamental representation. For gauge group $SU(N)$ the index a runs from 1 to N^2-1. We choose the normalization of the generators to be

$$\mathrm{Tr}\big(T^a T^b\big) = \frac{1}{2}\delta^{ab} \ . \tag{2.2}$$

Moreover, they satisfy the commutation relations

$$\big[T^a, T^b\big] = i f^{abc} T^c \ , \tag{2.3}$$

f^{abc} denoting the structure constants of the Lie algebra.
Under gauge transformations with the group element U the gauge field transforms as

$$A_\mu \to U A_\mu U^{-1} + U \partial_\mu U^{-1} \ . \tag{2.4}$$

The antisymmetric field strength tensor of a non-Abelian gauge field reads

$$F_{\mu\nu} = F^a_{\mu\nu} T^a = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu\right) T^a \ . \tag{2.5}$$

Note that we have rescaled the gauge field incorporating the coupling constant which changes the notation slightly compared to many textbooks, e.g. [26]. On the other hand, it is more convenient in the area of non-perturbative field theory. The dual field strength tensor (interchanging electric and magnetic components) is defined as

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \ . \tag{2.6}$$

The Lagrangian density for pure Yang-Mills theory can be written as

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} \ , \tag{2.7}$$

with the gauge coupling constant g. The equations of motion are derived by the Euler-Lagrange equations

$$D_\mu F^{\mu\nu} = \left(\partial_\mu F^{a\mu\nu} + f^{abc} A^b_\mu F^{c\mu\nu}\right) T^a = 0 \ , \tag{2.8}$$

with covariant derivative

$$D_\mu = \partial_\mu - i A^a_\mu T^a \ . \tag{2.9}$$

Furthermore, the field strength tensor fulfills the Bianchi identity

$$D_\mu \tilde{F}^{\mu\nu} = 0 \ . \tag{2.10}$$

One of the most intriguing properties of Yang-Mills theories is its asymptotic freedom, [7]. For high energies the coupling constant decreases and one encounters a quasi-free theory. This can be parametrized by the running of the coupling, at one loop and UV cutoff scale M this reads

$$\frac{g^2(M)}{8\pi^2} \approx \frac{1}{\beta_0 \log\left(M/\Lambda\right)} \ , \quad \text{with } \beta_0 = \frac{11N}{3} \ . \tag{2.11}$$

The dynamically generated mass scale Λ is another very interesting phenomenon of Yang-Mills theories

$$\Lambda \approx M \exp\left(-\frac{24\pi^2}{11Ng^2}\right) \ . \tag{2.12}$$

Even though no dimensionfull parameter has to be specified for defining the action, the theory creates its own mass scale by the running of the coupling constant. Λ marks the transition to the strong coupling regime in the low energy theory. At this transition point the excitation spectrum of the theory changes dramatically. At high energies the pure gauge theory is well described by weakly interacting gluons (excitations of the field A), at low energies, however, complex bound states of gauge bosons will develop and serve as the fundamental excitations. These are called glueballs. The lightest glueball has an energy $E = \mathcal{O}(\Lambda)$ and the theory develops a mass gap.
The bound states of gluons will always assemble themselves to be colorless (no open gauge group index). This phenomenon is called confinement. Confinement has not yet been solved at a fundamental level, but there are several models, e.g. [27]. One of them is the condensation of monopoles and we will come back to it later.
The action (2.7) does not contain all allowed terms, which is subject of the following section.

2.1.2 The θ-Term

Another gauge and Lorentz invariant term for the Lagrangian is the so-called θ-term

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \ . \tag{2.13}$$

θ is often called the vacuum angle. This term does not change the equations of motion because it can be rewritten as total derivative, e.g. [5]

$$F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = \partial_\mu K^\mu \ , \tag{2.14}$$

with the topological Chern-Simons current

$$K^\mu = 2\epsilon^{\mu\nu\alpha\beta}\left(A^a_\nu \partial_\alpha A^a_\beta + \frac{1}{3} f^{abc} A^a_\nu A^b_\alpha A^c_\beta\right) \ . \tag{2.15}$$

The corresponding topological charge, Q_{top}, measures the winding or instanton number of the field configurations with finite action, see [28]

$$Q_{\text{top}} = \frac{1}{32\pi^2}\int \mathrm{d}^3x K_0 \ . \tag{2.16}$$

For an $SU(N)$ gauge theory this number is an integer and consequently we find a 2π periodicity in the vacuum angle θ. This periodicity will be discussed much deeper in the subsequent chapters 4 and 6.
With the following notation for colorelectric and colormagnetic fields

$$E^a_i = F^a_{0i} \ , \quad B^a_i = \epsilon_{ijk}F^{ajk}, \quad i,j,k \in \{1,2,3\} \ . \tag{2.17}$$

$\mathcal{L}_\theta$ can be rewritten in the form

$$\mathcal{L}_\theta = -\frac{\theta}{8\pi^2}\vec{E}^a \cdot \vec{B}^a \ . \tag{2.18}$$

This suggests that under a CP transformation ($\vec{E} \stackrel{CP}{\to} -\vec{E}$ and $\vec{B} \stackrel{CP}{\to} \vec{B}$) this term transforms under a change of sign and hence violates the symmetry.
The role of this term for supersymmetric gauge theories will be discussed in chapter 5.

2.1.3 Inclusion of Quarks

In order to establish a connection to QCD and to simplify the comparison to supersymmetric gauge theories we include fermionic degrees of freedom. We restrict the discussion to the case of massless fermions, which is of greatest importance for the relation to Super-Yang-Mills (SYM) theories. For convenience we refer to the fermions as quarks.
These quarks, denoted as ψ_f, are Dirac fermions and are labeled by a flavor index f, which indicates their quantity. The $SU(N)$ gauge index is called color index. They transform under the fundamental representation of the gauge group, with group element U

$$\psi_f \to U\psi_f \ . \tag{2.19}$$

The fermionic part of the Lagrangian density (sum over flavor index implied) is

$$\mathcal{L}_{\text{ferm}} = i\bar{\psi}_f \not{D}\psi_f \ , \tag{2.20}$$

with the usual notation $\bar{\psi}_f = \psi_f^\dagger \gamma_0$ and the slash notation for contraction with the Dirac matrices $\not{D} = \gamma_\mu D^\mu$.
There are several changes in the properties of Yang-Mills theories with matter fields. First of all, the β-function for the running coupling changes because the quarks screen the color charges. For n_f flavors we find (see [5])

$$\beta_0 = \frac{11}{3}N - \frac{2}{3}n_f \ . \tag{2.21}$$

Moreover, for massless quarks in the Weyl basis, where γ_5 becomes diagonal, the Lagrangian density splits into a left-handed and right-handed part for the fermions that do not mix. This indicates a further flavor symmetry of the type

$$U(n_f)_L \times U(n_f)_R \ . \tag{2.22}$$

In nature this chiral symmetry is broken down to a diagonal subgroup by the condensation of (anti)quarks. Some of the mesons, e.g. the pions can be understood as Goldstone bosons. The too heavy η' meson already hints at a further breaking of the chiral symmetry through an anomaly, see below. For supersymmetric theories there is an analogous effect, the gaugino condensation, which will be explicitly worked out in chapter 5.
The last consequence due to the inclusion of massless quarks which we want to mention is the chiral anomaly. The splitting of the Lagrangian density in non-mixing chiral terms

suggests that it is invariant under the chiral rotation

$$\psi_f \to e^{i\gamma_5 \alpha}\psi_f \quad . \tag{2.23}$$

Classically this is true, but quantum corrections do not conserve the chiral current

$$j_5^\mu = \bar{\psi}_f \gamma^\mu \gamma_5 \psi_f \quad . \tag{2.24}$$

The effect is due to the triangle diagram depicted in figure 2.1. Its contribution can be

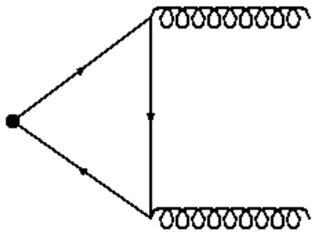


Figure 2.1: Triangle diagram causing the chiral anomaly, the black dot marks a divergence of the chiral current, $\partial_\mu j_5^\mu$

calculated, see e.g. [29], for n_f quarks in the fundamental representation we find

$$\partial_\mu j_5^\mu = \frac{n_f}{16\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \quad . \tag{2.25}$$

The resemblance to the θ-term is direct. This is the reason why for realistic QCD the effective θ-angle is a combination of the θ-term and the imaginary phase of the quark mass matrix, breaking the chiral symmetry explicitly. The chiral anomaly also has far reaching consequences for the supersymmetric case. We refer to chapter 5 for a further discussion.

2.2 Axion Physics

The inclusion of a θ-term in a Yang-Mills theory violates the CP symmetry. Because this CP violation is not seen in experiments, the vacuum angle has to be unnaturally small, $\mathcal{O}(10^{-9})$, see [30]. This apparent fine tuning is known as the strong CP problem.
One solution to this problem is the introduction of a new pseudoscalar field a, the axion. The axion couples to the same combination of gauge fields as the θ-angle and dynamically sets the relevant parameter $\theta + a$ to zero. It was first suggested by Weinberg [31] and Wilczek [32] based on the work of Peccei and Quinn (see [33] and [34]). In this section we explain the introduction of such a field.
Moreover, we encounter another very interesting and important phenomenon for our later conclusions, the so-called Witten effect, [6]. This effect contributes an electric charge to magnetic monopoles for non-vanishing θ-angle (or axion vacuum expectation value), elucidated in the latter for the simplified setup of axionic QED, [35]. This reduction to the Abelian $U(1)$ case proves sufficient for the later construction.

2.2.1 The Axion and its Consequences

As pseudoscalar particle the axion transforms odd under parity transformations and it couples to the same field configuration as the vacuum angle

$$\frac{a}{32\pi^2 f} F^a_{\mu\nu} F^{a\mu\nu} \ , \tag{2.26}$$

where the axion decay constant f, allows for a canonical mass dimension of the axion field.
The standard axionic kinetic term is

$$\mathcal{L}_{\rm ax} \propto \partial_\mu a \partial^\mu a \ , \tag{2.27}$$

and we recognize the shift symmetry

$$a \to a + c \ , \text{with} \ c \in \mathbb{R} \ . \tag{2.28}$$

In the later chapters this shift symmetry is broken down to a discrete subgroup corresponding to the periodicity of θ.
Rendering the axion massive without creating new degrees of freedom or a potential for the field itself is a subtle issue. One way is to use a reverse Higgs mechanism including a 3-form field, see [36]. In $3+1$ dimension a 3-from does not contain any propagating degrees of freedom. Nevertheless, the 3-form can mediate a long-range interaction by a constant field strength acting as a static electric field, similar to the Schwinger model of electrodynamics in $1+1$ dimensions. The axion is eaten up by the 3-form field.
The simplest way to see that form of the Higgs mechanism is by dualizing the axion to a 2-form field B_a. Thereby the relation to the standard Higgsing of a 1-form field by a scalar becomes more apparent.
The free Lagrangian density in analogy to (2.27) is (square brackets encode antisymmetrization)

$$\mathcal{L}_{\rm ax} \propto \partial_{[\mu} B_{a\nu\rho]} \partial^{[\mu} B_a^{\nu\rho]} \equiv G_{\mu\nu\rho} G^{\mu\nu\rho} \ . \tag{2.29}$$

Again, this possesses a shift symmetry

$$B_a \to B_a + \Omega \ , \tag{2.30}$$

with an arbitrary 2-from Ω. The Lagrangian density for a 3-form field C and its field strength $F_4 = \mathrm{d}C$ simply reads

$$\mathcal{L}_C \propto F_{\mu\nu\alpha\beta} F^{\mu\nu\alpha\beta} \ . \tag{2.31}$$

The 3-form field couples electrically to currents with three indices, which describe the worldvolume of domain walls. In the presence of such a source the electric field F_4 jumps in its value and generates a long-range Coulomb-type interaction.
Combining the equations for the fields B_a and C we identify the Higgs-type Lagrangian

density for a massive 3-form

$$\mathcal{L}'_C \propto m^2(G_{\mu\nu\rho} - C_{\mu\nu\rho})(G^{\mu\nu\rho} - C^{\mu\nu\rho}) + \frac{1}{2}F_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta} \ , \tag{2.32}$$

with appropriately normalized mass m. The shift symmetry of B_a is preserved, as are the number of degrees of freedom. The massive 3-form has eaten up the dualized axion and consequently propagates one degree of freedom. The mass leads to a screening of the long-range electric field (exponential decay). By dualizing back one explicitly identifies the axion, see [36]

$$\mathcal{L}_{\text{ax}} \propto g_1\partial_\mu a\partial^\mu a - g_2 a\partial_{[\mu}C_{\nu\alpha\beta]}\epsilon^{\mu\nu\alpha\beta} - \frac{1}{2}F_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta} \ , \tag{2.33}$$

where g_i are appropriately chosen constants that arise in the dualization process, but are of no concern to us in this elementary discussion.
In the background of Yang-Mills theories the 3-form C, can be identified with the Chern-Simons current (2.15)

$$C_{\mu\nu\rho} \propto \epsilon_{\mu\nu\rho\sigma}K^\sigma \ , \tag{2.34}$$

leading to the QCD coupling described in (2.26).

2.2.2 The Witten Effect

In 1979 Witten showed that a non-zero value of the θ-angle induces an electric charge for magnetic monopoles, turning them into dyons. This mechanism will be relevant in later chapters and we illustrate it in the simple setup of axionic QED discussed by Wilczek (see [35]). The axion plays the same role as the θ-angle.
For that purpose we consider the standard QED Lagrangian density under addition of a topological term coupled to the axion (rescaled to incorporate its decay constant)

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{32\pi^2}a\tilde{F}_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\text{ax}} \ , \tag{2.35}$$

$\mathcal{L}_{\text{ax}}$ encodes the kinetic and potential terms, depending only on a. The usual Maxwell equations get modified under this change. The relevant equation in the absence of a source is

$$-\frac{1}{e^2}\partial_\mu F^{\mu\nu} + \frac{1}{8\pi^2}\partial_\mu\left(a\tilde{F}^{\mu\nu}\right) = 0 \ . \tag{2.36}$$

Using the Bianchi identity, $\partial_\mu\tilde{F}^{\mu\nu} = 0$, and the antisymmetry of the field strength, the $\nu = 0$ equation becomes

$$\vec{\nabla}\cdot\vec{E} = \frac{e^2}{8\pi^2}(\vec{\nabla}a)\cdot\vec{B} \ , \tag{2.37}$$

where $\vec{E}$ and $\vec{B}$ represent the electric and magnetic field respectively.
Let us consider a spherical axionic domain wall with $\langle a\rangle = 0$ inside and $\langle a\rangle = 2\pi$ outside the wall, depicted in figure 2.2. For our consideration we can neglect the thickness of the transition region. This is valid since only the integrated value of $\vec{\nabla}a$ is important. The integration over the full sphere further erases the distance and shape dependence of the wall. We then place a magnetic monopole of magnetic charge m in the center of

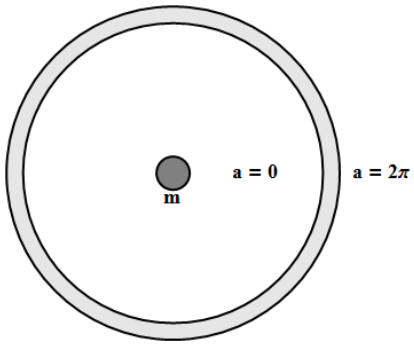


Figure 2.2: Setup for the illustration of the Witten effect

the sphere. In the region $\langle a \rangle = 0$ only a radial magnetic field is present. Outside of the domain wall equation (2.37) suggests that an electric field is generated. In the outer region the monopole acts like a dyon with additional electric charge ($\vec{B}$ parallel to $\vec{\nabla} a$ and V containing the whole sphere with radius R and normal vector $\hat{e}_r$)

$$\begin{aligned} q &= \int_V \mathrm{d}^3x \left[\vec{\nabla} \cdot \vec{E}\right] = \int_V \mathrm{d}^3x \left[\frac{e^2}{8\pi^2} (\vec{\nabla} a) \cdot \vec{B}\right] \\ &= \frac{e^2}{8\pi^2} \int_V \mathrm{d}^3x \left[2\pi\delta(r-R) \frac{m}{4\pi} \frac{\hat{e}_r \cdot \hat{e}_r}{r^2}\right] = \frac{1}{4\pi} m e^2 \ . \end{aligned} \tag{2.38}$$

As in [6] we assume the fundamental magnetic charge to be that of a 't Hooft-Polyakov monopole ([2] and [3]), which occurs in the case of a breakdown of a compact $SU(2)$ gauge group to $U(1)$ and is $m = \frac{4\pi}{e}$. Hence, for $\langle a \rangle \to \langle a \rangle + 2\pi$ it generates a fundamental unit of electric charge

$$q = \frac{1}{4\pi} m e^2 = e \ . \tag{2.39}$$

In general, this considerations can be extended to other gauge groups. We discuss the effect for electric center charges of the gauge group $SU(N)$ in chapter 4.

2.3 Large N Yang-Mills Theory

In this section we outline the large N expansion of Yang-Mills theories (meaning N colors of the gauge group $SU(N)$) and some of its interesting consequences.

The lack of a small expansion parameter in strongly coupled Yang-Mills theories at low energy, such as the coupling constant in QED, is a huge problem for a treatment in perturbation theory. Therefore, we need to consider other parameters for this purpose. The large N limit of these theories introduced by 't Hooft in [8] offers a prudent combination of the gauge coupling g and the number of colors N,

$$\lambda \equiv g^2 N \ , \tag{2.40}$$

called the 't Hooft coupling, which is kept fixed in the limit $N \to \infty$.

The dynamically generated scale (2.12) thus only depends on λ, for pure Yang-Mills theories

$$\Lambda = M \exp\left(-\frac{24\pi^2}{11\lambda}\right) . \qquad (2.41)$$

Therefore, this scale is held fixed in the large N limit. In the following we illustrate the crucial points of this construction.

2.3.1 Planar Theory

The simplifications connected to this limit are due to the fact that the first order diagrams are planar. In order to see that, we introduce the double line notation of this specific form of diagrams.

Because the gauge fields are in the adjoint representation, they carry one fundamental and one anti-fundamental index, $(A_\mu)^i_j$. Therefore, we can think of the gluon as comprised of a quark-antiquark pair (not taking into account the tracelessness of the adjoint representation is a negligible effect).

Let us consider a simple example. The diagram depicted in figure 2.3 has two vertices ($\propto g^2$) and one loop. The ingoing and outgoing gluons must posses the same color de-

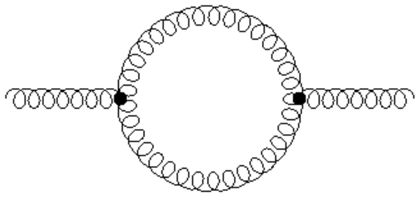


Figure 2.3: Gluon loop diagram

composition. This leaves one free fundamental color index for the loop that has to be summed over. Therefore, the overall contribution is of order $g^2N = \lambda$. In the double line notation the diagram transforms to figure 2.4 and the summation over the single color index in the inner loop becomes apparent. Moreover, the diagram 2.4 can be drawn

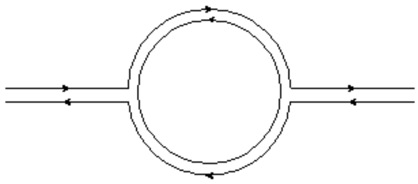


Figure 2.4: Double line representation of the diagram in figure 2.3, the directions of the arrows differ for quark and antiquark

without crossings of the lines in a plane or on a sphere which refers to the name planar. Non-planar diagrams are suppressed in the large N limit (for gluons by $1/N^2$ for quarks by $1/N$, see the next section). These higher order diagrams cannot be drawn without intersections on a plane but on higher genus surfaces. Consequently, the expansion can be characterized by the topology of the various diagrams. The first order are the planar ones, followed by $1/N$ suppressed subsequent contributions. An explicit description of higher order diagrams is given in the next section.

2.3.2 Relation to String Theory

The characterization of interactions by the topology of the diagrams already hints at string theories with their perturbative expansion in Riemann surfaces of the string worldsheet (e.g. [37]). In the following we want to make this analogy even more explicit.
Rewriting the kinetic part of the Yang-Mills Lagrangian density as

$$\mathcal{L} = -\frac{N}{4\lambda} F^a_{\mu\nu} F^{a\mu\nu} \ , \tag{2.42}$$

we can extract the Feynman rules for this model.
A gluon propagator scales as $1/N$ whereas each vertex contributes a factor N. As discussed above every loop demands a summation over color indices and hence also produces a factor of N. Denoting the number of vertices, gluon propagators and loops by v, p, and l respectively, the order in N of vacuum to vacuum diagrams (no external legs) can hence be determined from the diagram by the simple rule

$$N^{v+l-p} \ . \tag{2.43}$$

Reinterpreting the vertices as corners, the propagators as edges, and the loops as faces, the number $v + l - p$ encodes the Euler characteristic χ of the diagram regarded as a polygon. Therefore, we find the order of each diagram to be

$$N^{\chi} \ . \tag{2.44}$$

The Euler characteristic of Riemann surfaces, on the other hand, is fixed by the genus g

$$\chi = 2 - 2g \ . \tag{2.45}$$

This is exactly the counting that would arise in a string theory with closed string coupling $g_s \equiv 1/N$.
To illustrate the planar expansion we would like to consider the standard diagrams for the correction of the gluon propagator and calculate their N and λ dependence, see e.g. [5] or [38]. The 1-loop and 2-loop planar diagrams are depicted in figure 2.5. The left diagram

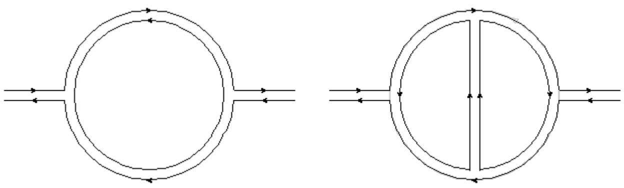


Figure 2.5: 1-loop and 2-loop contributions to gluon propagator in doubleline notation

contains two vertices, two inner propagators, and one loop. Thus, its contribution is $g^2 N = \lambda$. The diagram on the right has four vertices, four inner propagators, and two loops, and is of order $g^4 N^2 = \lambda^2$. Both of the diagrams are not $1/N$ suppressed since they are planar. In figure 2.6 the simplest non-planar diagram is presented. It cannot be drawn in a plane without intersections and there are six vertices, six inner gluon propagators,

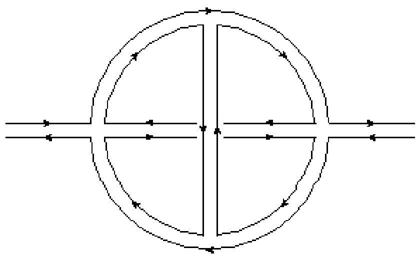


Figure 2.6: Simplest non-planar diagram in doubleline notation

but only one single loop, which determines the contribution $g^6 N = \frac{1}{N^2}\lambda^3$. It is indeed suppressed by a factor $1/N^2$.

The vacuum energy density is calculated to first order by the contribution of vacuum-vacuum amplitudes. The dominant term is due to genus $g = 0$ surfaces and leads to

$$N^{\chi} = N^{2-2g} = N^2 \quad . \tag{2.46}$$

This coincides with our expectation that the vacuum energy density is proportional to the degrees of freedom of the theories (again the difference to $N^2 - 1$ is negligible).

The inclusion of quarks leads to several extensions of the model and we state some of them in the following.

Because quarks carry only one color index, their radiative corrections are of subleading order. In figure 2.7 a correction to the gluon propagator induced by a quark loop is depicted. In double line notation it becomes obvious that the summation over a color

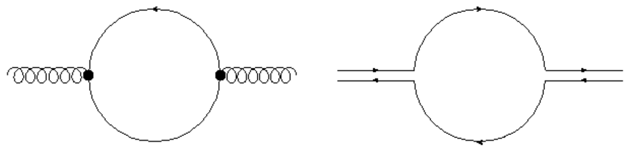


Figure 2.7: Radiative correction to gluon propagator through quarks

index as in 2.4 is missing. This leads to a $1/N$ suppression, which is the case for all diagrams with internal quark loops, see [38].

Under the assumption that confinement is still present in the large N limit the model becomes a theory of free, stable, and non-interacting colorless bound states, see [38], [8], and references therein. These can be purely built up by gluons and are referred to as glueballs, or they might contain quarks. In analogy to QCD a colorless quark-antiquark pair is a meson and a bound state of N quarks a baryon.

We would like to identify the analog of closed strings in the large N theory. For that we study the process of elastic scattering. A 2-2 closed string amplitude is of order g_s^2. The open string coupling is $\sqrt{g_s}$ and consequently the 2-2 amplitude for open strings scales as g_s. With the identification $g_s \equiv 1/N$ we thus search for elastic scattering processes of colorless bound states that scale as $1/N^2$ and $1/N$ respectively.

In [38] (and its references) the elastic scattering amplitudes for colorless bound states have been investigated. It was shown that for glueballs the amplitude scales as $1/N^2$ and for mesons it scales as $1/N$. This leads to the identification of closed strings with glueballs

and open strings with mesons. In pure Yang-Mills theories, however, we do not have quarks and mesons, but, as we will see later, stable electric flux tubes that are not closed and carry a center charge under the gauge group exist. For pure Yang-Mills theories we want to identify them as the fundamental open strings.
A different kind of interesting objects in string theory are D-branes (see [37]) and we examine if the above analogy includes them. D-branes are composite objects of string theory but they behave in a peculiar way under scaling of the string coupling. Usually, the energy of solitonic objects depends on the coupling constant as (see [39])

$$E_{\text{soliton}} \propto 1/g^2 \quad . \tag{2.47}$$

In contrast to solitons D-branes scale as (see [37])

$$E_{\text{D-brane}} \propto 1/g_s \quad . \tag{2.48}$$

In order to find composite objects that behave as D-branes, we have to search for mass $1/g_s = N$ states.
Instantons, for example, have an action of the form ([4])

$$S_{\text{inst}} = \frac{8\pi^2}{g^2} \quad . \tag{2.49}$$

In the large N limit this transforms to

$$S_{\text{inst}} = \frac{8\pi^2 N}{\lambda} \quad , \tag{2.50}$$

which scales as N. Because instantons are points in spacetime, the most natural object for them to be identified with are $D(-1)$-branes. This is supported by many considerations in string theory (e.g. [37]).
In [38] it was shown that the mass of baryons in the large N limit also scales as N. Baryons are points in space and there is the possibility that they might be identified with $D0$-branes, although especially for the gauge group $SU(N)$ this topic is subtle, [40].
For the (Super-)Yang-Mills theory there is an additional object with a characteristic energy scale proportional to N. These are the fundamental domain walls, see chapter 6, and they might be identified with $D2$-branes. Another phenomenon supporting this is that flux tubes can end on these domain walls, just as open strings do end on D-branes. For a deeper discussion we refer to chapter 5 and 7.
This analogy between string theory and large N Yang-Mills theory offers further interesting possibilities and we will encounter it several times in the later discussion.
It is important to mention that we deal with a non-critical sting theory in this context, which means that we are in the wrong number of spacetime dimensions to cancel the conformal anomaly, [41]. Considering that, this approach might first of all yield new insights in strongly coupled theories by ideas developed in string theory and also might lead to new results for non-critical strings from the field theory approach.
In later chapters further parallels to string theory including specific fields living on D-branes are obtained, but first we consider the topological θ-term in the large N expansion.

2.3.3 The Vacuum Angle in Large N

In the previous section we discussed the dependence of the gauge kinetic term in the 't Hooft limit. But there is a second term that appears in the action, the θ-term. The fate of this term is somewhat different to the discussion above and was discussed by Witten in [17] (see also [42] and [43]). Here, we recall the most important implications that also will receive importance in later chapters.

In the large N limit one would naively expect that contributions of instantons in the action diverge and thus they are exponentially suppressed in the path integral,

$$S_{\text{inst}} = \frac{8\pi^2 N}{\lambda} \overset{N\to\infty}{\longrightarrow} \infty \ . \tag{2.51}$$

Nevertheless, the θ-dependence prevails in the large N limit, which was discussed for the non-supersymmetric case in [44]. This might be explained by the divergent contribution of large instantons in the confining theory.

Including the θ-term, the large N Lagrangian density reads

$$-\frac{N}{4\lambda} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \ . \tag{2.52}$$

The usual procedure for the 't Hooft limit states that the parameter to be kept fixed for $N \to \infty$ is θ/N rather than θ itself. Hence the energy density of the groundstate would be a smooth function of θ/N with a general prefactor of N^2 accounting for the order N^2 degrees of freedom in the theory (see equation (2.46)),

$$E(\theta) = N^2 f\left(\frac{\theta}{N}\right) \ . \tag{2.53}$$

If, in addition, we want to maintain the 2π-periodicity of θ we would end up with a constant function

$$E(\theta) = E(\theta + 2\pi) \Rightarrow f\left(\frac{\theta}{N}\right) = f\left(\frac{\theta}{N} + \frac{2\pi}{N}\right) \overset{N\to\infty}{\to} \text{const} \ . \tag{2.54}$$

The solution suggested in [17] is that there are several branches for $E(\theta)$. Each of these

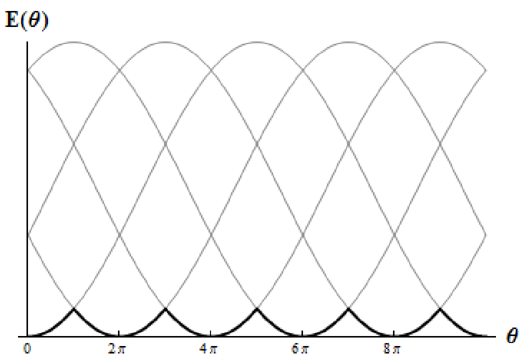


Figure 2.8: Schematical picture of the branches of $E(\theta)$

branches is by itself $2\pi N$ periodic, but their collection restores the 2π symmetry as schematically shown in figure 2.8. The θ-dependence takes the form

$$E(\theta) = N^2 \mathrm{min}_k f\left(\frac{\theta + 2\pi k}{N}\right) \quad . \tag{2.55}$$

The vacua are achieved for multiples of 2π with cusps for $\theta \in \{\pi, 3\pi, 5\pi, \dots\}$. The solution of the $U(1)$ problem connected with the large mass of the η' meson (e.g. [42]) further suggests that

$$\left.\frac{\mathrm{d}^2 E(\theta)}{\mathrm{d}\theta^2}\right|_{\theta=0} \neq 0 \quad , \tag{2.56}$$

which is realized in this picture.

For Yang-Mills theories we accordingly have N branches, labeled by k, which are non-degenerate in energy, but turn out to be quasi-stable in the large N limit.

In the string theory setup of [17] the domain walls interpolating between two adjacent values of k are identified with D-branes in a type IIA theory. This supports the previous claim in section 2.3.2. Furthermore, the colorfluxes can end on these Yang-Mills domain walls and correspond to open strings. The energy density of the brane follows the usual scaling behavior of D-branes and is of order N. Expanding the vacuum energy around a minimum (quadratic term dominant) we find

$$E(\theta) = \mathcal{O}(1)\mathrm{min}_k (\theta + 2\pi k)^2 \quad . \tag{2.57}$$

The energy difference of two neighboring vacua does not depend on N. In the large N limit vacuum transitions accordingly are exponentially suppressed and the vacua, even non-degenerate in energy, are quasi-stable, [45].

For the discussion of the consequences of the θ-term in supersymmetric theories see chapter 5.

Chapter 3

Non-Local Operators and TFT

An important tool to investigate gauge theories are non-local operators. The line and surface operators are of special importance in four dimensions and are briefly reviewed in the following.
Subsequently, we state the relevant facts and mechanisms of topological field theories in various dimensions, starting from $2+1$ dimensional Chern-Simons theories. After some generalizations we describe the $3+1$ dimensional BF theories that will be the foundation of our later construction.
To simplify the notation and avoid confusion caused by the amount of indices we switch to coordinate independent notation using differential forms in this chapter. A dictionary for that is given in appendix B.

3.1 Non-Local Operators

The classification of gauge theories with respect to their different phases is an interesting topic by itself, e.g. [5]. In the present case our theories are believed to be in the confining phase, which dynamically generates a mass gap.
Nevertheless, there are further possibilities to distinguish various realizations of this confining phase. A powerful tool to investigate the fundamental structure and properties of the theory are non-local operators, i.e. line and surface operators ([22] and [19]). We first review the electric line and surface operators connected to Wilson loops, introduced in [46] and their magnetic analogs, called 't Hooft loops. These allow us to distinguish between different global gauge groups that exhibit the same local degrees of freedom.
The differential geometry notation proves to be very useful, since p-forms are the natural objects to be integrated over p-dimensional manifolds. Our notation will follow that in [20] and [22].

3.1.1 Electric Line and Surface Operators

For a general gauge group G the Wilson loop operator, with loop C embedded in the spacetime manifold, reads

$$\mathcal{W}_R(C) = \mathrm{Tr}\left[\mathcal{P}\exp\left(iq\oint_C A_R\right)\right] \ , \tag{3.1}$$

where R labels the representation of the gauge algebra and $\mathcal{P}$ the path ordering. In the following, path ordering is implied and we will omit the explicit notation of $\mathcal{P}$.
The Wilson loop can be pictured by introducing infinitely heavy charges in the representation R with charge q. Then $\mathcal{W}$ is the operator that executes a pair creation of such probe particles, drags one of them around a loop C and annihilates them again, see figure 3.1. For QCD the most interesting case is for probe particles in the fundamental repre-

Figure 3.1: Illustration of a Wilson loop with infinitely heavy probe particles

sentation (quarks). In fact, the expectation value of the Wilson loop is a powerful tool to describe confinement.
Choosing a loop in Euclidean time, where the probe particles are fixed at a distance d for the time T (the transition time from pair creation to the fixed positions can be neglected for large T), the expectation value of the Wilson loop directly relates to the potential energy $V(d)$ between the two charges (see for example [26])

$$\langle \mathcal{W}_R \rangle = \langle \exp\left(-V(d)T\right)\rangle \ . \tag{3.2}$$

There are two possibilities, either the exponent is proportional to the length of the path C, which is roughly $2(T+d)$ (perimeter law), or it is proportional to the area that is enclosed by C, roughly RT (area law). In the second case the potential energy is proportional to the distance d, which can be understood as a flux tube of finite tension connecting the two charges (e.g. [5]), presenting evidence for confinement.
We would like to mention that there is a difficulty related to light fundamental charges in confining theories. Imagine that the energy stored in the tension of the flux tube connecting two heavy probe charges exceeds the energy which is needed to create a light quark-antiquark pair. At this specific distance the flux tube will break and two mesons of the heavy and light matter fields will form. Thus, light matter fields complicate the detection of confinement in this formalism. In our further discussion, however, pure gauge

theories are considered, where no matter fields are present. The probe particles are not physical, but mere auxiliary concepts simplifying the description of line operators.
An Abelian gauge theory will be sufficient for the later construction and we thus constrain the further statements to the case $G = U(1)$. The trace over the gauge group elements and the classification of the representation is not necessary for $G = U(1)$. The Wilson loop is fully described by the loop C and the charge q,

$$\mathcal{W}(C, q) = \exp\left(iq \oint_C A\right) \ . \tag{3.3}$$

We further restrict the discussion to the case of compact $U(1)$, so the admissible charges q are quantized. If C is the boundary of a two dimensional surface Σ, i.e. $C = \partial\Sigma$, which is always the case for base manifolds without holes, it can be recast into the form

$$\mathcal{W}(C, q) = \exp\left(iq \int_\Sigma \mathrm{d}A\right) = \exp\left(iq \int_\Sigma F\right) \ , \tag{3.4}$$

using the generalized Stoke's theorem for p-forms Ω and p+1-dimensional surfaces Σ_{p+1}

$$\int_{\Sigma_{p+1}} \mathrm{d}\Omega = \oint_{\partial\Sigma_{p+1}} \Omega \ . \tag{3.5}$$

This form of the Wilson loop is related to surface operators.
The electric charge in a spacelike region $\mathcal{V}$ bounded by the 2-surface $\partial\mathcal{V}$ can be measured by integrating the dual field strength, $\tilde{F} = *F$, over $\partial\mathcal{V}$, see [47]

$$Q \propto \oint_{\partial\mathcal{V}} *F \ . \tag{3.6}$$

In general, the analog of the electric line operators of a connection 1-form A for 2-forms B is the surface operator over an arbitrary 2-manifold Σ, denoted by

$$\mathcal{W}_S(\Sigma, \eta) = \exp\left(i\eta \int_\Sigma B\right) \ . \tag{3.7}$$

For loop-like constructions we specify Σ to be a closed 2-surface (i.e. $\partial\Sigma = 0$). Instead of a particle this can be pictured by a flux tube sweeping out a two dimensional worldsheet embedded in the spacetime manifold (note the analogy to strings), see figure 3.2. As

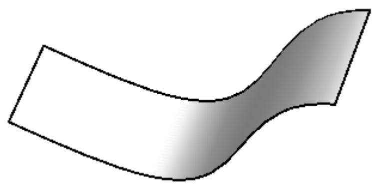


Figure 3.2: 2-dimensional worldsheet of a flux tube embedded in higher dimensional spacetime

discussed in section 3.2.2, there might be a non-trivial phase for linking a surface operator with a line operator in $D = 4$.

3.1.2 Magnetic Line and Surface Operators

Instead of creating a pair of electric charges carrying one of them around a loop and finally annihilating both, one can consider the production of a pair of magnetic charges (monopole-antimonopole pair) and perform the same procedure. These operators are called 't Hooft operators and were first considered in [48]. They take a particularly simple form upon using the dual, magnetic gauge field $\tilde{A}$ ($\tilde{F} = \mathrm{d}\tilde{A}$ magnetic and electric fields are interchanged compared to $F = \mathrm{d}A$). The 't Hooft loop, in perfect equivalence to the Wilson loop, reads

$$\mathcal{H}(C,m) = \exp\left(im \oint_C \tilde{A} \right) \ . \tag{3.8}$$

There is an equivalent way to depict a 't Hooft operator. The magnetic charge in a spacelike region $\mathcal{V}$ can be evaluated by an integral similar to equation (3.6) (see [47])

$$Q_m = \frac{1}{2\pi} \oint_{\partial\mathcal{V}} F \ . \tag{3.9}$$

Thus, by tracking the path of a magnetic charge through spacetime we can remove a cylindrical neighborhood, which locally looks like $\mathbb{R}$ along the worldline times S^2 enclosing the monopole, and demand that (see [20])

$$\frac{1}{2\pi} \oint_{S^2} F = m \ . \tag{3.10}$$

Moreover, a Dirac quantization condition for any closed 2-surface $\partial\mathcal{V}$ states, [49]

$$\frac{1}{2\pi} \oint_{\partial\mathcal{V}} F \in \mathbb{Z} \ . \tag{3.11}$$

Equation (3.10) further suggests that the relation $F = \mathrm{d}A$ in the presence of magnetic charges is not true everywhere, but singularities along the worldlines of monopoles have to be included.
These singularities are also present for magnetic surface operators. They represent the worldsheets of magnetic flux tubes and are conveniently parametrized by a singularity of the field strength on a surface Σ, [22]

$$F = 2\pi\alpha\delta_\Sigma \ , \tag{3.12}$$

where possible smooth contributions to F have been omitted. The prefactor α fulfills an equivalent quantization as the magnetic charge m. The δ_Σ should be itself regarded as a 2-form and its action on any other 2-form Ω by integration is

$$\int \delta_\Sigma \wedge \Omega = \int_\Sigma \Omega \ . \tag{3.13}$$

3.1.3 Dyonic Operators

With the notation of 't Hooft and Wilson line operators it is possible to introduce another class of non-local operators along a one dimensional worldline. For these the probe particles carry magnetic as well as electric charges, i.e. they are dyons.
With the established notation for purely electric and magnetic loops the dyonic operators are linear combinations of them, characterized by the electric charge q and the magnetic charge m

$$\mathcal{D}(C, q, m) = \exp\left(iq \oint_C A + im \oint_C \tilde{A}\right) \ . \tag{3.14}$$

Dyonic surface operators that encode the worldsheets of flux tubes carrying both electric and magnetic flux are possible in principle, but are not relevant in the following.
Non-local operators are natural objects to be considered in the framework of topological field theories, because they do not need a specification of the spacetime metric. These theories are subject of the following section.

3.2 Topological Field Theories

Pure topological field theories do not contain any propagating degrees of freedom and consequently are non-dynamical. After performing the Legendre transformation to inspect the Hamiltonian density, one discovers that for topological field theories it vanishes identically. Nevertheless, these theories are not trivial since they encode possible Aharonov-Bohm phases, groundstate degeneracies, and vacuum structures. For example, gauge theories with a discrete gauge group are effectively described by the use of topological theories.
In the following section some properties of topological field theories, that will be relevant later, are worked out. First, we discuss the Chern-Simons theory in $2+1$ dimensions introduced in [50]. A concise review of Chern-Simons theory is presented in [51]. Afterwards, more fields are included allowing for higher spacetime dimensions in the context of BF theories, discussed by Horowitz in [52].

3.2.1 Chern-Simons Theory

The pure Chern-Simons theory is probably the most prominent example of a topological field theory and has a plenty of applications in various areas of physics. It contains a 1-form gauge connection A (of a general gauge group G) which is coupled to itself. This kind of action is only consistent in three spacetime dimensions. Further, it does not depend on the metric of the 3-manifold. This is the reason why these theories are called topological. In fact, they only depend on topological invariants of the base manifold. Nevertheless, they have some intriguing properties that are explained in the following.

Abelian Chern-Simons Theory

If the gauge group is $U(1)$, the pure Chern-Simons action with source term is, see [53]

$$S_{\mathrm{CS}} = \int \left[\frac{k}{4\pi} A \wedge \mathrm{d}A - A \wedge *j \right] \ . \tag{3.15}$$

The equations of motion read

$$*\, j = \frac{k}{2\pi}\, \mathrm{d}A \ . \tag{3.16}$$

If there is no source present, $\mathrm{d}A = F = 0$ and thus the viable states are flat gauge connections. Moreover, there are no propagating degrees of freedom in the pure Chern-Simons theory, because the action is only first order in spacetime derivatives.
Choosing a point charge q as source, $j = q\delta^2(\vec{x})\mathrm{d}t$, we recognize that it induces a magnetic field

$$B = \frac{2\pi q}{k} \delta^2(\vec{x}) \ , \tag{3.17}$$

which is attached to electric charges. Note that in $2+1$ dimensions the magnetic field is a pseudoscalar rather than a vector field. An intuitive way to depict this is in a $3+1$ dimensional space. There, electric charges acquire a magnetic field perpendicular to that plane, see figure 3.3. Due to this phenomenon a phase will be generated if two

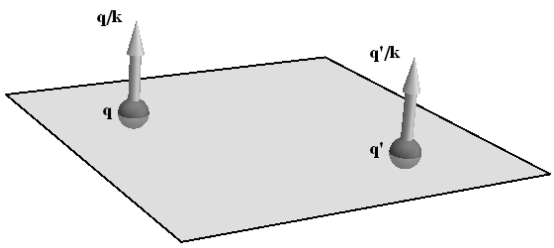


Figure 3.3: Magnetic field attached to electric charges due to Chern-Simons term

electrically charged particles are interchanged, see [54]. This is in perfect equivalence to the Aharonov-Bohm effect in a $2+1$ dimensional setup. This phase shift does not have to add up to an overall factor of ± 1 (bosons and fermions). In $2+1$ dimensions there is an arbitrary exchange phase $e^{i\alpha}$ determined by the prefactor k of the Chern-Simons term (anyons, [55]). The concrete relation for two particles of charge q and q' under their exchange is

$$\alpha = \frac{\pi}{k} q q' \ . \tag{3.18}$$

It is these phases and the correlated quantum statistics that are encoded by topological field theories.
In order to see that, one can calculate the regulated expectation value of two line operators (section 3.1), see [56]

$$\langle \exp\left(iq \oint_{C_1} A \right) \exp\left(iq' \oint_{C_2} A \right) \rangle = \exp\left(\frac{2\pi i}{k} q q' L(C_1, C_2) \right) \ , \tag{3.19}$$

with $L(C_1, C_2)$ the linking number of the two lines, see figure 3.4 for illustration. The

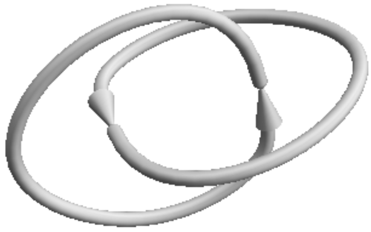


Figure 3.4: Example of two paths with linking number 1

configuration discussed above, of moving one charge around another, is equivalent to a linking number of $L = 1$. The Chern-Simons theory thus automatically calculates the linking number of chosen line operators and multiplies the wavefunction by the appropriate phase. Since only the linking number enters equation (3.19), the line operators can be deformed if $L(C_1, C_2)$ is not altered.

It remains to mention, that after the addition of a kinetic term for the gauge field the presence of the Chern-Simons term leads to a topological mass term. This, however, does not change the number of propagating degrees of freedom as the Higgs mechanism would. The photon mass is

$$m_{\mathrm{CS}} = ke^2 \quad , \tag{3.20}$$

with e^2 the coupling constant appearing in front of the kinetic term, see [51].

Under gauge transformations

$$A \to A + \mathrm{d}f \quad , \tag{3.21}$$

with f a 2π-periodic function (0-form), the Chern-Simons action changes by a total derivative

$$\Delta\mathcal{L}_{\mathrm{CS}} = \frac{ik}{2}\mathrm{d}\left(f \wedge \mathrm{d}A\right) \quad . \tag{3.22}$$

Therefore, the equations of motion do not change. The action itself, however, might change if a boundary of the spacetime manifold is present. In that case one has to include boundary degrees of freedom in order to retrieve the gauge invariance, see for example [51]. This effect that boundaries demand the introduction of further degrees of freedom will also be a crucial observation later.

Non-Abelian Chern-Simons Theory

For Chern-Simons theory with a non-Abelian gauge group there is an additional trilinear coupling of the gauge fields and the trace has to be taken

$$\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right) \quad . \tag{3.23}$$

Under gauge transformations, with U the element of the non-Abelian group, the gauge field A transforms as

$$A = A_\mu \mathrm{d}x^\mu \to \left(UA_\mu U^{-1} + U\partial_\mu U^{-1}\right) \mathrm{d}x^\mu = UAU^{-1} - U^{-1}\mathrm{d}U \ . \tag{3.24}$$

The Chern-Simons Lagrangian density consequently develops an additional contribution which can not be written as a total derivative and reads (see [51])

$$\Delta\mathcal{L}_{\mathrm{CS}} = -\frac{k}{12\pi}\mathrm{Tr}\big(U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U\big) \ . \tag{3.25}$$

For a non-Abelian gauge group the integral of the Pontryagin density is quantized and represents the winding number of the gauge transformation as discussed in [57]

$$\frac{1}{24\pi^2}\int \mathrm{Tr}\big(U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U\big) \in \mathbb{Z} \ . \tag{3.26}$$

Thus, for the theory to be invariant under gauge transformations the coefficient of the Chern-Simons theory has to be quantized

$$k \in \mathbb{N} \ , \tag{3.27}$$

and it is called the level of the Chern-Simons term. In anticipation this notation was already used for the Abelian case, for arguments supporting this notation see e.g. [58].

3.2.2 Two Field Topological Theories (BF Theories)

In Chern-Simons theories the topological action is constructed with one single gauge field A, but this can be generalized to topological actions for more than one field.
In analogy to the Abelian Chern-Simons term we start with the action (the term $B \wedge F$ giving rise to the name BF), see [59]

$$S_{\mathrm{BF}} = \frac{k}{2\pi}\int B \wedge \mathrm{d}A = \frac{k}{2\pi}\int B \wedge F \ . \tag{3.28}$$

The factor of 2 accounts for the doubled contribution of A in the equations of motion for the Chern-Simons theory.
Again, the action does not depend on the spacetime metric and therefore is called topological. Additionally, the Hilbert space is spanned by flat connections and the theory incorporates no propagating degrees of freedom. This can be seen by calculating the equations of motion in the absence of sources

$$B = F = 0 \ . \tag{3.29}$$

These type of theories were first considered in [52].
Since now there are two distinct fields, these models are not restricted to three spacetime dimensions as the Chern-Simons theory. Fixing A to be a 1-form gauge field, in D dimensions B has to be a (D-2)-form for the action to make sense.

Subsequently, we discuss the most interesting cases of $D = 3$ and $D = 4$ applying the results of the previous sections.

BF Theory in D=3

In three spacetime dimensions A and B are 1-form fields and transform under 0-form gauge transformations

$$\begin{aligned} A &\to A + \mathrm{d}f \\ B &\to B + \mathrm{d}g \ , \end{aligned} \tag{3.30}$$

with two 2π-periodic 0-forms f and g. Two currents are introduced that couple to the two fields

$$S_{\mathrm{BF}} = \int \left[\frac{k}{2\pi} B \wedge \mathrm{d}A - A \wedge *j - B \wedge *J \right] \ . \tag{3.31}$$

The equations of motion can be read off

$$\begin{aligned} \mathrm{d}A &= \frac{2\pi}{k} * J \ , \\ \mathrm{d}B &= \frac{2\pi}{k} * j \ . \end{aligned} \tag{3.32}$$

The field configuration is completely determined by the currents.
The magnetic flux of A is attached to the charges of B and vice versa. Thus, a non-trivial exchange phase occurs

$$\alpha = \frac{\pi q m}{k} \ , \tag{3.33}$$

where m and q are the charges of the sources J and j respectively. In general, these phases can be deduced for arbitrary worldlines of the sources, as in the case of pure Chern-Simons theory

$$\langle \exp\left(iq \oint_{C_1} A \right) \exp\left(im \oint_{C_2} B \right) \rangle = \exp\left(\frac{2\pi i}{k} q m L(C_1, C_2) \right) \ . \tag{3.34}$$

The question is, if these theories do arise naturally for specific systems. And indeed, one concrete example is the type II superconductor, i.e. a superconductor with the possibility of magnetic vortices. The usual framework for that is the Abelian-Higgs model, see e.g. [5] and section 3.2.4. A derivation of the emergent BF theory for the dual superconductor can be found in the same section. An approach via the path integral formalism is presented in [59]. In this framework j couples to the electric charges and J to the magnetic vortices, which immediately pinpoints the analogy to the Aharonov-Bohm effect.
For the later work a generalization of the BF theories to four spacetime dimension is necessary.

BF Theory in D=4

For $D = 4$ the field B is a 2-form field and transforms under 1-form gauge transformations λ as follows

$$B \to B + \mathrm{d}\lambda \ . \tag{3.35}$$

Extending our analogy to superconductors, B now couples to the worldsheet of magnetic flux tubes rather than the worldlines of vortices, see [59] and [60]. This extension is natural for topological field theories in $D = 4$, because there cannot be any Aharonov-Bohm phases between two point particles in four spacetime dimensions. The linking number of two line operators in $D \geq 4$ vanishes since two loops always can be unwound. Instead, the phases are present for a point particle moving in the field of a infinitely long solenoid, i.e. a flux tube. In other words, there is a linking of a line and a surface operator, see section 3.1. One can picture this by compactification of one space dimension. By winding the surface operator around the compact dimension one arrives at the same situation as in $D = 3$.

In [52] it was noted that another term of the form

$$\Delta S_{\mathrm{BF}} \propto \int B \wedge B \ , \tag{3.36}$$

is possible in the action for even spacetime dimensions. The topological meaning of this term is to count the number of intersections, [61]. Note that in four spacetime dimensions two dimensional worldsheets intersect in points rather than lines. Again the compactification of one dimension is a nice way to illustrate that.

In the flux tube analogy this leads to a phase generation for perpendicularly crossing flux tubes parametrized by the $B \wedge B$-term. This creates an interaction of the form $\vec{B} \cdot \vec{E}$ resembling the $F \wedge F$-term. Later we will see that the inclusion of the $B \wedge B$-term is necessary in order to encode the properties of the θ-term and its consequences in topological gauge theories, see chapter 4.

Groundstate Degeneracy in Topological Theories

For topological field theories on compact, topologically non-trivial space manifolds one encounters a degeneracy of the groudstates. For simplicity we consider the $2+1$ dimensional Chern-Simons theory at level k in this section, where space is compactified to the torus T^2.

On the torus there are two fundamental non-contractible loops γ_1 and γ_2. Line operators of the field A along these directions are denoted by

$$\mathcal{W}_i = \exp\left(i \oint_{\gamma_i} A \right) \ . \tag{3.37}$$

$\mathcal{W}_i^{-1}$ describes the line operator with reversed path $-\gamma_i$. Adiabatically switching on a unit of flux, $\frac{2\pi}{k}$, through the hole of the torus, see figure 3.5 leads to an additional phase of $\mathcal{W}_1$, [62]. The Hamiltonian of the theory does not change by adding an exterior flux adiabatically. This induces a transition of the groundstate to a state with equal energy, i.e. another groundstate. These two states are only distinguished by the expectation value of the various Wilson loops, which helps to classify gauge theories and their phases in general, see chapter 4. After inserting k elementary fluxes the phase is a multiple of 2π and hence unobservable. Consequently, there are k different groundstates on the torus.

In general, one could argue that by adding a flux through the tube of the torus one creates

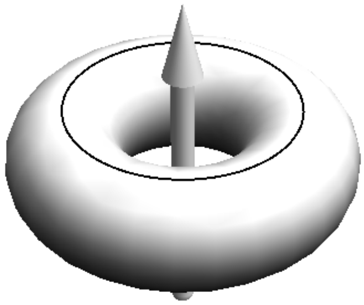


Figure 3.5: Torus with additional flux through the hole and fundamental line operator around it

even more degenerate groundstates, altering the expectation values of $\mathcal{W}_2$. But this is not the case because we find

$$\langle \mathcal{W}_2^{-1} \mathcal{W}_1^{-1} \mathcal{W}_2 \mathcal{W}_1 \rangle = \exp\left(\frac{2\pi i}{k}\right) , \tag{3.38}$$

since it can be deformed to two loops of linking number $L = 1$. This immediately implies

$$\langle \mathcal{W}_2 \mathcal{W}_1 \rangle = \langle \mathcal{W}_1 \mathcal{W}_2 \rangle \exp\left(\frac{2\pi i}{k}\right) . \tag{3.39}$$

Thus, inserting fluxes in the tube of the torus and probing with $\mathcal{W}_2$ is the same as inserting flux trough the hole of the torus and probing with the operator $\mathcal{W}_1$. All groundstates are hence sufficiently characterized by the amount of flux through one of the holes.
For an Abelian Chern-Simons theory at level k the groundstate degeneracy is k, which can be pictured by adding $q \in \{0, \ldots, k-1\}$ flux quanta through to hole of the torus. This points towards a $\mathbb{Z}_k$ symmetry of the theory discussed in the following section.
In general, if we compactify the space manifold on a genus g surface the groundstate degeneracy is k^g for the level k Chern-Simons theory.
For the BF theory in the previous section the situation changes slightly. Instead of two different line operators we have four

$$\mathcal{W}_i(\phi) = \exp\left(i \oint_{\gamma_i} \phi\right), \quad \text{with } \phi \in \{A, B\} . \tag{3.40}$$

There are two different Heisenberg algebras, similar to (3.39), and we find 2^g as many groundstates $(2k)^g$. These correspond to adding $q \in \{0, \ldots, k-1\}$ fluxes of A and B through the holes of the genus g surface, see [59].

3.2.3 Gauge Theories with Discrete Gauge Group

Another perspective on topological field theories is from the point of view of gauge theories with discrete gauge group, for a review on that topic see [63].
We utilize the above mentioned model of charges and vortices in $2+1$ dimensions. Far in the Higgs regime the vortices become pointlike particles at positions $\{\vec{x}_i\}$. Apart from

the vortex positions the gauge fields are in a pure gauge configuration

$$A = \mathrm{d}\alpha \ . \tag{3.41}$$

With the constraint (using the gauge $A_0 \equiv 0$)

$$\oint_C A = 2\pi i n \ , \tag{3.42}$$

where n is the number of vortices minus the number of antivortices in the spacelike region enclosed by C. All field configurations that fulfill this property are gauge equivalent (only considering small gauge transformation). The equivalence classes of the configurations are the states of the theory. They are completely described by the winding number of the gauge fields along closed loops, corresponding to the enclosed number of (anti)vortices. Therefore, a characterization by a discrete $\mathbb{Z}$-symmetry is possible.
If the number of vortices is only conserved modulo the integer number k, this theory descends to a $\mathbb{Z}_k$ theory.
This can be constructed by breaking a continuous $U(1)$ symmetry down to a discrete subgroup, for example by the condensation of charged particles.
A construction of this type in form of the aforementioned Abelian-Higgs model is carried out in the next section.

3.2.4 Continuum Description of a TFT

We explicitly derive a continuum description of a topological field theory (gauge group $\mathbb{Z}_k$) starting from the Abelian-Higgs model, compare [59] for a derivation with index notation and [20] for the differential notation. We will work in Euclidean four dimensional spacetime, see A. This model is used for our later construction of a TFT for Yang-Mills theories in chapter 4.
The Abelian-Higgs model is a theory of one complex scalar field ϕ and a $U(1)$ gauge field A. In order to implement the $\mathbb{Z}_k$-symmetry, the charge of the scalar field is set to k. The gauge transformations read

$$\begin{aligned} \phi &\to e^{ikf}\phi \\ A &\to A + \mathrm{d}f \ . \end{aligned} \tag{3.43}$$

After splitting ϕ into modulus part and phase

$$\phi = \rho e^{i\varphi} \ , \tag{3.44}$$

the gauge transformation acts as shift of the phase φ

$$\varphi \to \varphi + kf \ . \tag{3.45}$$

The covariant derivative for ϕ is

$$D\phi = (\mathrm{d} - ikA)\phi \ \to \ e^{ikf} D\phi \ . \tag{3.46}$$

Hence, the kinetic part of the action for the field ϕ and the gauge field A can be written as

$$S_{\text{kin}} = -\frac{1}{2g^2} F \wedge *F + D\phi \wedge (*D\phi)^* \equiv -\frac{1}{2g^2} F \wedge *F + |D\phi|^2 \ , \tag{3.47}$$

where * represents complex conjugation. We choose the usual Higgs potential for the scalar field

$$V(\phi) = \frac{\lambda}{4}\left(|\phi|^2 - v^2\right)^2 \ , \tag{3.48}$$

with coupling constant λ and a real, positive v. The potential energy vanishes for $\rho = v$. This vacuum expectation value marks the condensation of charge k particles. The phase φ is eaten up by the gauge field, which acquires a mass

$$m_A^2 = 2v^2k^2 \ . \tag{3.49}$$

The excitations of the real Higgs field $H = \rho - v$ are massive as well

$$m_H^2 = 2\lambda v^2 \ . \tag{3.50}$$

Far in the Higgs regime, i.e. for $v \gg 1$, one can set $\rho = v$ everywhere except at the locations of possible flux tubes that are characterized by

$$\oint_C A \neq 0 \ , \tag{3.51}$$

if the path C encloses the worldsheet of the flux. They are correlated to singularities in the phase φ. In this limit these flux tubes are infinitely thin, because their radius scales as the Compton wavelength, $\propto \frac{1}{m}$, of the fundamental particles, [57]

$$m_H, m_A \overset{v\to\infty}{\to} \infty \ . \tag{3.52}$$

Since the charge k condensate should not develop a non-trivial phase in the presence of flux tubes, the flux Φ, has to be quantized similar to the Dirac quantization

$$k \oint_C A \in 2\pi\mathbb{Z} \ \Rightarrow \ \Phi \in \frac{2\pi}{k}\mathbb{Z} \ . \tag{3.53}$$

k flux tubes produce a trivial phase even for fundamental charges (charge 1) and cannot be detected. In other words the fluxes take values in $\mathbb{Z}_k$ as desired.

Now that there are only massive degrees of freedom the electric field is screened and we can consider the Lagrangian density that survives for large distances. This only encodes the Aharonov-Bohm phases and hence is a topological field theory.

As mentioned in [59] this integrating out degrees of freedom is fundamentally different from the Wilsonian approach. In the Wilsonian effective action the coupling constants are varied in order to keep the correlation length constant. In the topological limit the couplings are held fixed and the correlations due to dynamical interactions vanish.

The only term surviving this procedure is

$$S_{\text{top}} = v^2 \int (\mathrm{d}\varphi - kA) \wedge *(\mathrm{d}\varphi - kA) \ . \tag{3.54}$$

For large v the Euclidean action is dominated by the classical contributions and in fact equation (3.54) can be rewritten as a constraint with Lagrange multiplier 3-form h, see [22]

$$S_{\text{top}} = \int h \wedge (\mathrm{d}\varphi - kA) \ . \tag{3.55}$$

This constraint demands that apart from the positions of the vortex lines the gauge field is pure gauge. At the flux tubes the phase φ, as well as the gauge field A, are singular. All dynamical interactions are integrated out and only the topological long-distance physics connected to exchange phases of line operators of charge in $\mathbb{Z}_k$ and fluxes in $\frac{1}{k}\mathbb{Z}_k$ are conserved.

This is the same structure as for Yang-Mills gauge theories discussed in the next chapter.

Chapter 4

TFT for Yang-Mills Theories

Now the necessary tools for the construction of a TFT for Yang-Mills theories with gauge group $SU(N)$ are established and it is derived step by step in this chapter. The only assumptions needed are that the mechanism for confinement is the condensation of charge N monopoles and that the Witten effect takes place. Utilizing the Abelian-Higgs model as continuum description of a $\mathbb{Z}_k$ gauge theory we succeed in finding a topological model of the Yang-Mills theory encoding all relevant Aharonov-Bohm phases in the presence of a θ-term. Further, gauge invariance of the action demands the inclusion of a Chern-Simons term on the Yang-Mills domain walls discussed in this chapter. This confirms some conclusions that are suggested by string theory.
Before the explicit derivation we point out why it is possible to work with a discrete gauge group $\mathbb{Z}_N$ rather than the full $SU(N)$. One of the consequences is the classification of the confining phase in gauge theories by the spectrum of line operators, analogous to that in [19].

4.1 Electric and Magnetic Charges in Yang-Mills

The local degrees of freedom of pure Yang-Mills theories, i.e. the gauge fields A, take values in the Lie algebra g and hence are insensitive to the global structure of the gauge group. If the universal cover of the Lie group with algebra g is denoted by $\tilde{G}$, the theories giving rise to the same local degrees of freedom can be written as quotient

$$G = \tilde{G}/H \ . \tag{4.1}$$

H denotes a subgroup of the center $\mathcal{Z}$, of the universal cover. The center is the subgroup of the elements in G that commute with all group elements.
In order not to confuse the important implications by keeping the abstract notations we will restrict this general formulation to the Lie algebra $su(N)$. The universal cover with this algebra is the Lie group $SU(N)$, its center is

$$\mathcal{Z}_{SU(N)} = \mathbb{Z}_N \ . \tag{4.2}$$

Subsequently, we are only interested in the two cases

$$G = SU(N), \quad \text{and} \quad G = SU(N)/\mathbb{Z}_N \ . \tag{4.3}$$

The center of $SU(N)$ consists of the elements

$$\mathcal{Z}_{SU(N)} = \left\{ \mathbb{1}_N \exp\left(2\pi i \frac{p}{N}\right), \ p = 0, \ldots, N-1 \right\} \ , \tag{4.4}$$

where $\mathbb{1}_N$ is the $N \times N$ identity matrix.
As mentioned before, the local degrees of freedom of both theories are the same but we will be able to distinguish the theories by non-local operators discussed in the previous chapter.
The physical and gauge invariant electric charges and fluxes of a group are labeled by elements of the center $\mathcal{Z}_G$, which can be seen by a construction via the weight lattice of the Lie group modulo the root lattice [19]. In fact, even in dynamical confining models the string tension of the electric flux tube exclusively depends on the center charge of the probe particles.
To illustrate this circumstance, we recall [64]. Imagine a flux tube stretched between two infinitely heavy probe charges in an arbitrary representation R of the gauge group G, see figure 4.1. Naively, the string has a tension T_R that depends on the representation

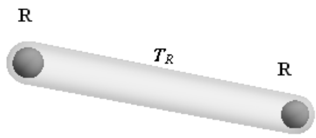


Figure 4.1: Flux tube spanned between heavy probe particles in representation R

R. But upon binding light gluons in the adjoint representation to the probe particles we can scan the whole spectrum of representations of the gauge group. If the flux tube is sufficiently long these gluons can be neglected for the calculation the energy of the whole configuration which is dominated by the string tension T_R times the string length. Consequently, the string tension does not depend on the representation of the probe particles. Adding adjoint fields does not change the charge under the center of the gauge group (adjoints are neutral under the center), hence the only value characterizing the flux tube is its center charge.
Another argument is that physical flux tubes (e.g. chromoelectric fluxes in QCD) must not carry colorflux, because it is a gauge dependent quantity. The center charges and fluxes on the other hand are gauge invariant because of their commutation properties, and parametrize the physical objects.
This implies that the Wilson loops are sufficiently characterized by the center charge of the probe particles and a representation does not have to be specified.
The magnetic charges and fluxes can be constructed similarly from the weights of the Langland-dual Lie algebra modulo their root lattice, see [65]. This is once more isomorphic to the center $\mathcal{Z}_G$. It also can be pictured as the first homotopy class of the gauge group $\pi_1(G)$.

For the two interesting cases we find

$$\mathcal{Z}_{SU(N)} \cong \mathbb{Z}_N, \qquad \pi_1(SU(N)) \cong 0 \ , \tag{4.5}$$

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_N} \cong 0, \qquad \pi_1(SU(N)/\mathbb{Z}_N) \cong \mathbb{Z}_N \ . \tag{4.6}$$

Therefore, non-local operators are specified by their electric/magnetic charges and fluxes in a $\mathbb{Z}_N \times \mathbb{Z}_N$ lattice. That is characteristic for $\mathbb{Z}_N$ theories and leads to a classification of gauge theories by the spectrum of line operators.

4.2 Quasi-Vacua in Yang-Mills

In the following the quasi-stable vacua of Yang-Mills models and their relations are discussed, which leads to some consequences for the dynamics of possible domain wall configurations.
Pure Yang-Mills theories develop N quasi-stable vacua that are labeled by an integer number k and exchange their roles for different vacuum angle θ

$$\theta = 2\pi k, \quad k \in \{0, \ldots, N-1\} \ . \tag{4.7}$$

The true vacuum is determined by minimizing the term $\theta - 2\pi k$, see chapter 2. We recalled that the energy difference between adjacent vacua scales as $\mathcal{O}(N^0)$ and the tension of interpolating domain walls as $\mathcal{O}(N)$.
In order to trace what differs in these vacua we have to discuss the mechanism of confinement. In our model we apply the picture of a dual superconductor. This means that in the vacuum there is a condensate of magnetic charges confining the electric flux (as in the normal superconductor magnetic charges are confined). For a review of that mechanism of creating confinement see for example [66]. In the following sections we will explicitly carry out the derivation of our topological action, starting from the assumption of a monopole condensate. This approach is further supported by supersymmetric models such as the Seiberg-Witten model in $\mathcal{N} = 2$ supersymmetry, see [12].
The true vacuum in a Yang-Mills theory exhibits a condensate of monopoles. This should be the case for each minimal energy solution depending on θ. Consequently, at $\theta = 2\pi$, where the true vacuum is parametrized by $k = 1$, the condensate consists of purely magnetically charged configurations. The Witten effect, discussed in section 2.2.2, dictates the situation of the $k = 1$ vacuum at $\theta = 0$. The condensed monopoles acquire an additional electric charge. For the present shift $\theta \to \theta - 2\pi$ the electric charge is minus the magnetic charge (in fundamental units). The different vacua represent differently charged condensates. For a fundamental wall that interpolates between the $k = 0$ and $k = 1$ vacuum for $\theta = 0$ the condensate is made of monopoles with charge $(Q_e, Q_m) = (0, m)$ on one side, and dyons with charge $(-m/N, m)$ on the other. This configuration is by no means static, the difference in vacuum energy extends the true vacuum and accelerates the wall. This does not influence our discussion, however.
To understand the difference of the energy in both theories we come back to the 3-form dynamics considered in 2.2.1. In pure Yang-Mills theories, there is no dynamic axion.

Instead, we are only left with the massless 3-from C, coupling to the domain walls and one discrete θ-like parameter k. The jump in k, indicating the domain wall, see 4.4.3, sources the 3-form via the term

$$\mathrm{d}k \wedge C \tag{4.8}$$

On the domain wall worldvolume k jumps by one and creates an electric field via the 3-form field strength $*F_4 \propto F_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\alpha\beta}$. The equations of motion read

$$\mathrm{d}(*F_4) - \kappa \mathrm{d}k = 0 \ , \tag{4.9}$$

where κ is a non-zero constant. In the true vacuum $k = \theta = 0$ there is no 4-form electric field F_4. But the wall produces F_4 in the quasi-stable state of the dyonic condensate. This makes a contribution to the energy proportional to F_4^2 and lifts the energy degeneracy of the quasi-stable configurations.
The finite 4-form field strength can be made plausible by inspecting its functional from. In section 2.1.2 we saw that $\mathrm{d}C$ is proportional to $\vec{E} \cdot \vec{B}$. While this quantity vanishes for purely electrically or magnetically charged condensate, it does not vanish in the presence of dyonic charges. Thus, if the condensate consists of dyons, we expect a non-vanishing vacuum expectation value for the 4-form field strength. The connection of a dyonic condensate and the θ-term is further confirmed in the explicit construction in appendix C.
For a wall-antiwall configuration, where k first jumps by one and then back to the original value the long-range interaction via the non-screened electric field causes a strong Coulomb attraction between the walls. Eventually they annihilate. For supersymmetric domain walls of this type the picture changes, see chapter 6.
The dynamics, however, do not influence a topological construction and the classification of gauge theories.

4.3 Classification by the Choice of Line Operators

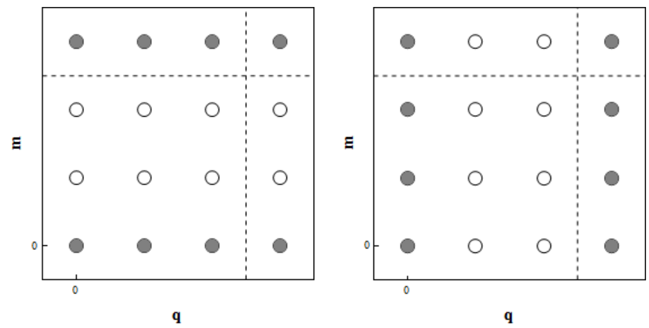
The line operators are sufficiently described by their electric and magnetic charges (q, m) taking values in $\mathbb{Z}_N \times \mathbb{Z}_N$. Seiberg et al. use the notion of genuine line operators to identify the allowed spectrum of probe particles and classify the gauge theory, [21]. These genuine line operators do not need a surface attached to them.
This means that using the generalization of the Dirac quantization for non-local operators, [67]

$$qm' - mq' \in N\mathbb{Z} \ , \tag{4.10}$$

only operators that generate a trivial phase are allowed, i.e. a multiple of 2π. In other words, one admits lines in the spectrum which are not able to detect the flux produced by other allowed probe particles.
For later convenience we rescale our electric charges by a factor of $1/N$ resembling the center charges of fundamental quarks. The new Dirac quantization reads

$$qm' - mq' \in \mathbb{Z} \ . \tag{4.11}$$

Let us consider two examples. If the fundamental Wilson line $\mathcal{W}$ of the $SU(N)$ theory with a center charge $(1/N, 0)$ is chosen to be in the spectrum of genuine line operators, the Dirac quantization (4.10) directly tells us that only magnetic charges are allowed that are a multiple of N. For $SU(3)$ this is pictured in the left panel of figure 4.2, equivalent to [19]. If instead we choose the 't Hooft lines $\mathcal{H}$ to be genuine, only integer electric charges,



(a) $SU(3)$, genuine $\mathcal{W}$ (b) $SU(3)$, genuine $\mathcal{H}$

Figure 4.2: Spectrum of non-local operators in $SU(3)$ depending on the choice of genuine line operators, denoted by solid points in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ lattice, the dashed lines mark the periodicity of $\mathbb{Z}_3$

i.e. N multiples of the fundamental charge, are allowed (right panel of figure 4.2).
All other operators have to be supplemented by surface operators to account for the non-trivial phases and hence physical interaction between the corresponding non-local operators.
The set of allowed operators classifies the theory beyond the mere notion of confinement. Further implications are induced by the Witten effect, discussed in section 2.2.2. The Witten effect implies that the condensates in the quasi-stable vacua of the Yang-Mills theory differ in their electric charge. This influences the notion of genuine line operators. The same holds for other non-local operators in the theory. A line operator that is characterized by the charges $(p/N, m)$ transforms under the change $\theta \to \theta + 2\pi$ as follows

$$\left(\frac{p}{N}, m\right) \overset{\theta\to\theta+2\pi}{\longrightarrow} \left(\frac{1}{N}\left[(p+m) \bmod N\right], m\right) \quad , \tag{4.12}$$

Equivalently, the change of the label of the vacua changes the identification of genuine line operators.
Let us consider this effect for the spectra of figure 4.2 for $\theta = 0$. In section 4.2 we found that a charge $(\frac{p}{N}, m)$ in fundamental units on the $k = 0$ branch corresponds to a charge

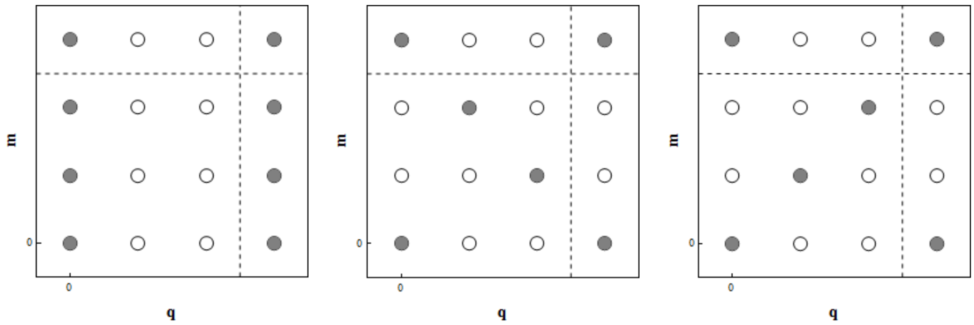
$$\left(\frac{1}{N}\left[(p-m) \bmod N\right], m\right) \quad , \tag{4.13}$$

in the quasi-stable state at $k = 1$.
For genuine line operators $\mathcal{W}$ at $k = 0$ the genuine line operators at $k = 1$ are unchanged, because only monopoles with magnetic charge in $N\mathbb{Z}$ are allowed, but the contribution is canceled by mod N.

For genuine line operators $\mathcal{H}$ at $k=0$ their analogs on the branch $k=1$ are

$$(0,m) \;\text{ at }\; k=0 \leftrightarrow \left(\frac{1}{N}(-m \bmod N), 0\right) \;\text{ at }\; k=1 \;\; . \tag{4.14}$$

In figure 4.3 the situation is depicted for $N=3$. The spectrum does change indeed. Its



(a) $k=0$ (b) $k=1$ (c) $k=2$

Figure 4.3: Change of the spectrum of genuine line operators for $SU(3)$ in different quasi-stable vacua labeled by k

periodicity in k is N as suggested by [17]. The allowed line operators become dyonic under the change of k, as was suspected by the Witten effect.

The picture of flux tubes connecting charges as a consequence of confinement suggests the following choice of the spectrum. In the true vacuum at $\theta = 0$ (monopole condensate and $k=0$) all electrically charged probe particles are confined. On the other hand all purely magnetically charged operators are screened by the condensate. These flux tubes carrying center fluxes of the $SU(N)$ gauge theory naturally serve as surface operators, see section 3.1. Thus, we choose the 't Hooft lines $\mathcal{H}$ to be genuine line operators and consequently the Wilson $\mathcal{W}$ and dyonic lines $\mathcal{D}$ have to be supplemented by surface operators carrying the confined flux. In a dynamical picture these surface operators may literally become the confining strings of finite tension. In the topological framework, however, where the Hamiltonian vanishes identically, they do not contribute to the energy, but encode the Aharonov-Bohm phases between the non-local operators. This also explains why there are no constraints such as minimization of the area in the TFT model (enlarging the area does not cost energy). Only the topological properties, i.e. intersections or linking numbers, are of importance and deformations changing them are not allowed.

4.4 Construction of the TFT

With the choice made above and all tools at hand we now construct a topological field theory of $SU(N)$ Yang-Mills theories. These models are governed by a $\mathbb{Z}_N \times \mathbb{Z}_N$ charge lattice for the line and surface operators.

We further consistently implement the Witten effect and the different condensate charges in the quasi-stable vacua. Introducing Yang-Mills domain walls, interpolating between the branches for fixed θ, a level N Chern-Simons action on the wall has to be included in order to preserve gauge invariance. This mechanism supports the string theory consideration

of [68] and yields a new field theoretical interpretation.
In the following we work in four dimensional Euclidean spacetime. Since our theory only contains the phases, all terms are purely imaginary reflecting the phase $-i|S_E|$ in the path integral for specific configurations.

4.4.1 The Topological Action for the Dual Superconductor

The fact that all our charges and fluxes are described by $\mathbb{Z}_N$ allows us to use an Abelian construction of the theory which tremendously simplifies the task, see [22]. In section 3.2.4 the topological action for a $\mathbb{Z}_k$ theory was derived by condensing a charge k field and starting from the Abelian-Higgs model. We use the same action in the following but with one crucial difference. In order to create confinement of electric charges, not electrically but magnetically charged fields are condensed. To preserve the discrete symmetry $\mathbb{Z}_N$, charge N monopoles are the constituents of the condensate.
The topological action to start with is (compare equation (3.55) and [22])

$$S = \int h \wedge (\mathrm{d}\varphi - N\tilde{A}) \ , \tag{4.15}$$

with Lagrange multiplier 3-form h. The gauge transformations read (see equation (3.43) and (3.45))

$$\begin{aligned} \varphi &\to \varphi + Nf \ , \\ A &\to A + \mathrm{d}f \ , \end{aligned} \tag{4.16}$$

with a 0-form gauge function f which is 2π periodic, $f \sim f + 2\pi$. The magnetic scalar field φ is dualized with a 2-form field B and the Lagrange multiplier is integrated out

$$S \to \int \left[h \wedge (\mathrm{d}\varphi - N\tilde{A}) + \frac{i}{2\pi}\mathrm{d}\varphi \wedge \mathrm{d}B \right] \to \frac{iN}{2\pi}\int \tilde{A} \wedge \mathrm{d}B \ . \tag{4.17}$$

B couples to the vortices of φ, i.e. exactly to the electric flux tubes of the confining theory. We assume that the spacetime manifold does not have a boundary and integrate the above action by parts

$$S = -\frac{iN}{2\pi}\int \mathrm{d}\tilde{A} \wedge B = -\frac{iN}{2\pi}\int \tilde{F} \wedge B \ . \tag{4.18}$$

This is the BF action discussed in section 3.2.2 describing the topological properties of charges and fluxes in four dimensional spacetime.
In order to recover the more familiar gauge field A, the field $\tilde{A}$ is dualized as well

$$S \to \int \left[-\frac{iN}{2\pi}\tilde{F} \wedge B + \frac{i}{2\pi}\mathrm{d}\tilde{A} \wedge \mathrm{d}A \right] \to \frac{i}{2\pi}\int \tilde{F} \wedge (F - NB) \ . \tag{4.19}$$

F and B are proportional to each other on the equations of motion, considering $\tilde{F}$ as Lagrange multiplier, just as one would expect for a field B that couples to electric flux tubes. The gauge field A has its usual 0-form gauge transformation properties, since it only appears as $\mathrm{d}A$. But there is an additional 1-form gauge transformation λ

$$\begin{aligned} B &\to B + \mathrm{d}\lambda \ , \\ A &\to A + N\lambda \ , \\ \tilde{A} &\to \tilde{A} \ . \end{aligned} \tag{4.20}$$

The field strength F transforms to $F + N\mathrm{d}\lambda$ thus it might produce a phase for other operators in the spectrum similar to the effect of an insertion of 't Hooft lines, see equation (3.10) (in a region with closed boundary Σ, i.e. $\partial\Sigma = 0$)

$$Q_m = \frac{N}{2\pi} \oint_\Sigma \mathrm{d}\lambda \tag{4.21}$$

Phases induced by a gauge transformation certainly should not be visible in the topological field theory. Thus, it has to be a trivial phase, that is a multiple of 2π. Equivalently, the corresponding magnetic charge of the 't Hooft loop, creating the same effect, has to be a multiple of N. This leads to a quantization condition for the 1-form transformations λ

$$\frac{1}{2\pi} \oint \mathrm{d}\lambda \in \mathbb{Z} \ . \tag{4.22}$$

This action, together with the gauge transformations, describes the dual superconductor and its spectrum of non-local operators. We check that in the following section.

4.4.2 Non-Local Operators for the Dual Superconductor

In order to investigate the allowed electric charges for the Wilson line operators we examine the electric surface operators integrated over a closed 2-surface Σ (glueballs). For all allowed fluxes these should be invariant under 1-form gauge transformations

$$\begin{aligned} \exp\left(i\eta \oint_\Sigma F\right) \to& \exp\left(i\eta \oint_\Sigma [F + N\mathrm{d}\lambda]\right) \\ =& \exp\left(i\eta \oint_\Sigma F + 2\pi i\eta N k\right) \stackrel{!}{=} \exp\left(i\eta \oint_\Sigma F\right), \quad \text{for } k \in \mathbb{Z} \ . \end{aligned} \tag{4.23}$$

Thus, the electric fluxes, and consequently the charges producing them are quantized

$$\eta = \frac{k}{N}, \ \ q = \frac{k'}{N}, \ \text{ for } k, k' \in \mathbb{Z}_N \ . \tag{4.24}$$

This is the same situation as for the Abelian-Higgs model in section 3.2.4. The valid Wilson loops and their gauge transformed versions can be parametrized as

$$\exp\left(i\frac{p}{N} \oint_C A\right) \to \exp\left(i\frac{p}{N} \oint_C [A + N\lambda]\right), \ \ p \in \{0, \ldots, N-1\} \ . \tag{4.25}$$

As expected, in general they are not gauge invariant. Under the addition of a surface operator over an open 2-surface Σ, with $\partial\Sigma = C$, they become invariant

$$\Delta\left(i\frac{p}{N}\oint_C A - ip\int_\Sigma B\right) = i\frac{p}{N}\oint_C N\lambda - ip\int_\Sigma \mathrm{d}\lambda = 0 \ , \tag{4.26}$$

the last equality follows from Stokes' theorem, equation (3.5). This surface marks the worldsheet of the electric flux tube stretched between two charges.
The 't Hooft loops, on the other hand, are gauge invariant on their own, exactly reproducing our choice of genuine line operators.

4.4.3 Inclusion of the Witten Effect

In the action in equation (4.19) a dual superconductor is described but there is no possibility to see the Witten effect by changing some parameter θ in the theory or to see the branches with different condensates. In order to include the Witten effect and the occurrence of different quasi-stable vacua in our action, we make use of the change of the electric charges of the non-local operators under a shift of θ. See also the explicit construction of a dyonic condensate in appendix C.
Starting with a condensate of charge $(0, N)$, the Witten effect suggests that under the shift $\theta \to \theta + 2\pi$ it develops N times the fundamental electric charge and now acts as $(1, N)$. Pure 't Hooft loops should not be gauge invariant anymore but should also require the attachment of a surface operator. This can be achieved by modifying the 1-form gauge transformations of the dual gauge field $\tilde{A}$, see as well [21]

$$\tilde{A} \to \tilde{A} - \frac{\theta}{2\pi}\lambda \ . \tag{4.27}$$

For $\theta = 0$, 't Hooft loops are invariant. For $\theta = 2\pi$ the following combination is invariant $(C = \partial\Sigma)$

$$\exp\left(im\oint_C \tilde{A} + im\int_\Sigma B\right) \ . \tag{4.28}$$

As desired, an electric surface operator has to be included. Moreover, loop operators in line with the charge of the condensate should be gauge invariant on their own and unconfined. For $\theta = 2\pi$ this has to be the case for the charge combination $(\frac{m}{N}, m)$ and sure enough the transformations cancel.
But now the action (4.19) does not stay invariant after the inclusion of the modified transformations for $\tilde{A}$

$$\Delta S = -\frac{i\theta}{4\pi^2}\int \mathrm{d}\lambda \wedge (F - NB) \ . \tag{4.29}$$

The term $\propto \mathrm{d}\lambda \wedge F$ is of no concern, because for $\theta \in 2\pi\mathbb{Z}$ its integrated value adds up to a multiple phase of 2π and leaves the action unaltered. The term $\propto \mathrm{d}\lambda \wedge B$ has to be canceled by the inclusion of additional terms. In section 3.2.2 we stated that for even spacetime dimensions a term of the form $B \wedge B$ is allowed, see [52]. Indeed, this has the right transformation properties. The combination

$$\frac{i\theta N}{8\pi^2}B \wedge B \to \frac{i\theta N}{8\pi^2}B \wedge B + \frac{i\theta N}{4\pi^2}\mathrm{d}\lambda \wedge B + \frac{i\theta N}{8\pi^2}\mathrm{d}(\lambda \wedge \mathrm{d}\lambda) \ , \tag{4.30}$$

yields exactly the right contribution to cancel the additional term. The total derivative does not alter this result for spacetime manifolds without a boundary.
For a gauge invariant action that incorporates the Witten effect we arrive at

$$S = \frac{i}{2\pi} \int \left[\tilde{F} \wedge (F - NB) - \frac{N\theta}{4\pi} B \wedge B \right] \ , \tag{4.31}$$

with 1-form gauge transformations

$$\begin{aligned} B &\to B + \mathrm{d}\lambda \ , \\ A &\to A + N\lambda \ , \\ \tilde{A} &\to \tilde{A} - \frac{\theta}{2\pi}\lambda \ . \end{aligned} \tag{4.32}$$

To inspect the changes induced by the $B \wedge B$-term we exploit the other possibility of describing 't Hooft loops discussed in chapter 3 and check if it leads to the same result. Removing a cylindrical neighborhood around the worldline of the monopole (locally $\mathbb{R} \times S^2$), we fix the magnetic flux through the sphere and by the equations of motion find a correlated constraint for B

$$\frac{1}{2\pi} \oint_{S^2} F = m \overset{\text{eom}}{\Rightarrow} \frac{1}{2\pi} \oint_{S^2} B = \frac{m}{N} \ . \tag{4.33}$$

Splitting the 2-form B into a singular part, which corresponds to magnetic charges, and a smooth part for electric configurations

$$B = B_{\text{sing}} + B_{\text{sm}} \ , \tag{4.34}$$

the part of the $B \wedge B$-term containing the singular contribution reads (mind that $B_{\text{sing}} \wedge B_{\text{sing}}$ vanishes due to the properties of the wedge product)

$$\frac{iN\theta}{8\pi^2} \int 2B_{\text{sing}} \wedge B_{\text{sm}} = \frac{im\theta}{2\pi} \int_{S^2_\perp} B_{\text{sm}} \ . \tag{4.35}$$

It transforms as

$$\frac{im\theta}{2\pi} \int_{S^2_\perp} B_{\text{sm}} \to \frac{im\theta}{2\pi} \int_{S^2_\perp} (B_{\text{sm}} + \mathrm{d}\lambda) \ . \tag{4.36}$$

For $\theta \neq 0$ it needs to be supplemented by a surface operator which depends on the worldline of the monopole (locally but not globally $\mathbb{R}$). This surface carries the appropriate electric flux $-m/N$, as discovered before. This compliance shows the correct behavior of the $B \wedge B$ term.
Moreover, to incorporate the presence of N different quasi-stable vacua labeled by k, one can use the correspondence

$$k \to k + 1 \ \leftrightarrow \ \theta \to \theta - 2\pi \ . \tag{4.37}$$

The change of the vacuum energy cannot be quantified in the topological framework. The spectrum of line operators, however, is well described by this relation. The action describing a dual superconductor with θ term and vacua labeled by k is

$$S = \frac{i}{2\pi}\int \left[\tilde{F}\wedge(F-NB) - \frac{N\theta}{4\pi}B\wedge B + \frac{Nk}{2}B\wedge B\right] \ . \tag{4.38}$$

The effect of the change in the condensate charge for different k alters the gauge transformations for $\tilde{A}$ as well

$$\tilde{A} \to \tilde{A} - \frac{\theta}{2\pi}\lambda + k\lambda \ . \tag{4.39}$$

Almost the same action and transformations were presented without derivation in [21], but with a different application. There the authors coupled this type of TFT to a dynamical $SU(N)$ theory in order to obtain a $SU(N)/\mathbb{Z}_N$ theory. Our interpretation is different, we regard the action as a topological field theory for the $SU(N)$ Yang-Mills theory itself. Other similar actions are found for the theories of oblique confinement in [22] and for lattice models of Yang-Mills theories in [69].
The effect of the $B\wedge B$-term in the gauge theory can be seen by integrating out $\tilde{F}$. Recall that by dualizing $\tilde{A}$, $\tilde{F}$ is regarded as an independent field fulfilling the Bianchi identity induced by the dualizing term, see [21]. Therefore, we are allowed to use the equations of motion of $\tilde{F}$ rather than $\tilde{A}$ and obtain the constraint equation

$$B = \frac{1}{N}F \ . \tag{4.40}$$

Plugging this back into the action (4.31) the result is

$$S = -\frac{i\theta}{8\pi^2 N}\int F\wedge F + \frac{ik}{4\pi N}\int F\wedge F \ . \tag{4.41}$$

This exactly reproduced the θ-term in Yang-Mills gauge theory up to a factor of $1/N$. The supplemental inclusion of the N different vacua restores the 2π symmetry which is needed in the $SU(N)$ theory. In the following discussion we keep the vacuum angle θ fixed at zero. In pure Yang-Mills theory without axions, θ is a free parameter and this approach is valid. The action therefore is

$$S = \frac{i}{2\pi}\int \left[\tilde{F}\wedge(F-NB) + \frac{Nk}{2}B\wedge B\right] \ , \tag{4.42}$$

with gauge transformations

$$\begin{aligned} B \to & B + \mathrm{d}\lambda \ , \\ A \to & A + N\lambda \ , \\ \tilde{A} \to & \tilde{A} + k\lambda \ . \end{aligned} \tag{4.43}$$

Compare this to [21].

Yang-Mills Domain Walls

With the action (4.42) and the corresponding gauge transformations we are ready to investigate properties of Yang-Mills domain walls in this topological setting. To this end

one adds a jump in k on a codimension one surface, i.e. a domain wall. Only fundamental walls, for which $\Delta k = 1$, are considered. The three dimensional worldvolume of the wall is denoted by $\mathcal{V}$.
This jump in k has consequences for the properties under the gauge transformations of the action. The first term in (4.42) changes under 1-form transformations (4.43) to

$$\frac{i}{2\pi}\tilde{F}\wedge(F-NB)\to\frac{i}{2\pi}\tilde{F}\wedge(F-NB)+\frac{i}{2\pi}(\mathrm{d}k\wedge\lambda+k\mathrm{d}\lambda)\wedge(F-NB)\ . \tag{4.44}$$

The term $\mathrm{d}k$ can be regarded as δ function on the worldvolume $\mathcal{V}$ in the following sense

$$\int \mathrm{d}k\wedge\Omega=\int_{\mathcal{V}}\Omega\ , \tag{4.45}$$

for arbitrary 3-from Ω. The second term in (4.42) changes as well

$$\frac{iNk}{4\pi}B\wedge B\to\frac{iNk}{4\pi}B\wedge B+\frac{iNk}{2\pi}\mathrm{d}\lambda\wedge B+\frac{iNk}{4\pi}\mathrm{d}(\lambda\wedge\mathrm{d}\lambda)\ . \tag{4.46}$$

The total change of the action reads

$$\Delta S=\int\left[\frac{i}{2\pi}\mathrm{d}k\wedge\lambda\wedge(F-NB)+\frac{ik}{2\pi}\mathrm{d}\lambda\wedge F+\frac{iNk}{4\pi}\mathrm{d}(\lambda\wedge\mathrm{d}\lambda)\right] \tag{4.47}$$

Splitting the single terms in a total derivative and a $\mathrm{d}k$ contribution

$$\begin{aligned}\frac{iNk}{4\pi}\mathrm{d}(\lambda\wedge\mathrm{d}\lambda)\ &=\ \mathrm{d}\left(\frac{iNk}{4\pi}\lambda\wedge\mathrm{d}\lambda\right)-\frac{iN}{4\pi}\mathrm{d}k\wedge\lambda\wedge\mathrm{d}\lambda\ ,\\ \frac{ik}{2\pi}\mathrm{d}\lambda\wedge F\ &\overset{\mathrm{d}F=0}{=}\ \mathrm{d}\left(\frac{ik}{2\pi}\lambda\wedge F\right)-\frac{i}{2\pi}\mathrm{d}k\wedge\lambda\wedge F\ ,\end{aligned} \tag{4.48}$$

the total change in the action becomes

$$\Delta S=\int\left[\frac{i}{2\pi}\mathrm{d}\left(k\lambda\wedge F+\frac{Nk}{2}\lambda\wedge\mathrm{d}\lambda\right)-\frac{iN}{4\pi}\mathrm{d}k\wedge(2\lambda\wedge B+\lambda\wedge\mathrm{d}\lambda)\right]\ . \tag{4.49}$$

The first term is a total derivative and hence only relevant if the spacetime manifold has a boundary. More interesting is the second term, it implies that Yang-Mills domain walls generate a contribution to the action on their worldvolume ($\Delta k = 1$)

$$\Delta S_{\mathrm{wall}}=-\frac{iN}{4\pi}\int_{\mathcal{V}}[2\lambda\wedge B+\lambda\wedge\mathrm{d}\lambda]\ . \tag{4.50}$$

This situation is similar to the case of a boundary of the pure Chern-Simons theory in chapter 3. To allow for domain walls in the topological theory and simultaneously retain the gauge symmetry of the action we consequently have to introduce degrees of freedom on the worldvolume of the domain wall. These new degrees of freedom have to transform under the 1-form gauge transformation. The natural choice is a 1-form field $\mathcal{A}$ transforming under shift symmetry

$$\mathcal{A}\to\mathcal{A}-\lambda\ . \tag{4.51}$$

This is very similar to statistical gauge fields often used in condensed matter physics, e.g. for the fractional quantum Hall effect, see [70].
In order to cancel the contribution of the fundamental domain wall, the appropriate worldvolume action for $\mathcal{A}$ is

$$S_{\mathcal{V}} = -\frac{iN}{4\pi} \int_{\mathcal{V}} [2\mathcal{A} \wedge B + \mathcal{A} \wedge \mathrm{d}\mathcal{A}] \ . \tag{4.52}$$

This action contains a coupling of the 2-form field B to $\mathcal{A}$ and a $U(1)$ Chern-Simons term at level N. An equivalent action arises for boundaries of the spacetime manifold, see [21]. The full action in the presence of domain walls is the sum of both contributions, $S + S_{\mathcal{V}}$. Gauge invariance is thus only possible if one includes a level N Chern-Simons term on the domain wall. Precisely this term was predicted by string theory investigations for Super-Yang-Mills theories in [68] and made plausible by breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry in [18]. Our construction now shows that such a term should as well be present on large N Yang-Mills domain walls, even without supersymmetry. An equivalent construction with some modifications also works for Super-Yang-Mills domain walls and we will elaborate that in chapter 6. This is to our knowledge the first direct construction of the level N Chern-Simons theory for Yang-Mills domain walls and because it only uses topological characteristics of the theory it should be unaltered under various deformations of the dynamical theory.
The walls source a non-vanishing 4-from field strength as described in 4.2, leading to their strong long-range interaction.

Flux Tubes Ending on Domain Walls

The string theory constructions of [14] for Super-Yang-Mills and [17] for Yang-Mills theories suggest that electric flux tubes can end on these domain walls, in the same way fundamental strings end on D-branes. In this section we investigate if this is also true in our formalism.
The criterion for the existence of certain operators in the topological theory is the gauge invariance under 1-form gauge transformations. The 2-form field B couples to the electric fluxes, but the pure surface operator of B is not gauge invariant on its own and has to be extended by a Wilson loop, $\mathcal{W}$, of the gauge field A. With the domain wall on the other hand there is yet another field transforming under the 1-form transformations, i.e. $\mathcal{A}$.
This opens the following possibility. Consider an open electric surface operator of B over the 2-surface Σ, where the boundary $\partial\Sigma$ is located in the worldvolume of the domain wall $\mathcal{V}$, $(\partial\Sigma \subset \mathcal{V})$. Then the operator

$$\exp\left(iN\eta \int_{\Sigma} B + iN\eta \oint_{\partial\Sigma} \mathcal{A} \right) \ , \tag{4.53}$$

is gauge invariant. This mechanism allows the electric flux tubes, coupling to B, to end on a domain wall, see figure 4.4. The topological theory successfully comprises the occurrence of the level N Chern-Simons term as well as the possibility for electric flux tubes to end on the domain wall. These properties are very difficult to reconstruct in dynamical models

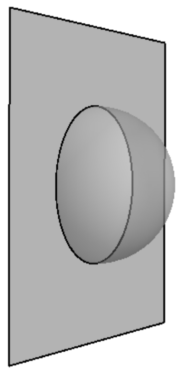


Figure 4.4: An electric surface operator ending on a Yang-Mills domain wall

but become rather simple in this topological framework, both describing the properties of a statistical gauge field $\mathcal{A}$.
The downside of the topological theory, however, is that all dynamical and energetical phenomena can not be investigated. Such notions like the string tension, the domain wall tension, or dynamical interactions are out of reach in the TFT approach.
Nevertheless, in chapter 7 we will present an outlook how one might be able to obtain information about these properties by using an analogy to string theory or the fractional quantum Hall effect.

Chapter 5

Super-Yang-Mills Theories

Now we turn to the investigation of $\mathcal{N} = 1$ Super-Yang-Mills theories. The use of supersymmetric properties allows to obtain some insights and exact results for the strong coupling regime by expanding from weak coupling results. Therefore, the understanding of these supersymmetric theories is highly important also for the non-supersymmetric models we considered so far.

There is strong evidence that Super-Yang-Mills theories share some of the most important properties of their non-supersymmetric relatives. First, there should be only colorless asymptotic states in the spectrum at low energies. Second, fundamental electric charges should be confined, resulting in the area law of Wilson loops in the fundamental representation. And finally, the theory should dynamically generate a mass gap, so there are no massless degrees of freedom in the spectrum, [71].

Again we restrict our discussion to the gauge group $SU(N)$. The notational conventions concerning supersymmetry are adopted from [5] and are summarized in appendix D. To simplify a comparison with the literature we switch back to the widely used index notation. The notations connected to gauge theory are the same as in chapter 2.

5.1 Lagrangian density of SYM

For the supersymmetric extension of pure gauge theories there is a gauge field in the adjoint representation A^a_μ, and its fermionic superpartners λ^a, called gauginos, also transforming in the adjoint representation.

After constructing a vector superfield with these component fields and using the Wess-Zumino gauge (see e.g. [72]) to get rid of the additional components we find

$$\begin{aligned} V &= V^a T^a \quad , \quad \text{with} \\ V^a &= -2\theta^\alpha \bar{\theta}^{\dot{\alpha}} A^a_{\alpha\dot{\alpha}} - 2i\bar{\theta}^2(\theta\lambda^a) + 2i\theta^2(\bar{\theta}\bar{\lambda}^a) + \theta^2\bar{\theta}^2 D^a \quad . \end{aligned} \tag{5.1}$$

The scalar field in the adjoint representation D^a is auxiliary and non-dynamical but will be relevant for the determination of the vacua. It should not be confused with D_μ or $D_{\alpha\dot{\alpha}}$ which denote the covariant derivative in Lorentz-vector or spinorial notation respectively

and do not carry color indices.
The non-Abelian field strength tensor superfield reads

$$W^a_\alpha = i\left(\lambda^a_\alpha + i\theta_\alpha D^a - \theta^\beta F^a_{\alpha\beta} - i\theta^2 D_{\alpha\dot{\alpha}}\bar{\lambda}^{a\dot{\alpha}}\right) \quad , \tag{5.2}$$

where

$$\begin{aligned} F^a_{\alpha\beta} &= -\frac{1}{2}F^a_{\mu\nu}\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}\left(\sigma^\nu\right)_\beta{}^{\dot{\alpha}} \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu \\ D_\mu X^a &= \partial_\mu X^a + f^{abc}A^b_\mu X^c, \quad X \in \{\lambda, A\} \ . \end{aligned} \tag{5.3}$$

The gauge invariant term appearing in the Lagrangian density which creates the kinetic terms for the gauge fields and gauginos is

$$\begin{aligned} W^{a\alpha}W^a_\alpha =& -\lambda^a\lambda^a - 2i(\lambda^a\theta)D^a + 2\lambda^{a\alpha}F^a_{\alpha\beta}\theta^\beta + \\ &+ \theta^2\left(D^aD^a - \frac{1}{2}F^{a\alpha\beta}F^a_{\alpha\beta}\right) + 2i\theta^2\bar{\lambda}^a_{\dot{\alpha}}D^{\dot{\alpha}\alpha}\lambda^a_\alpha \ . \end{aligned} \tag{5.4}$$

For a pure Super-Yang-Mills theory (no additional matter fields) this is enough to describe the action of the theory.
The θ-term of the gauge theory occurs naturally in supersymmetric theories by the complexification of the coupling constant

$$\frac{1}{g^2} \to \frac{1}{g^2} - i\frac{\theta}{8\pi^2} \ . \tag{5.5}$$

The full Lagrangian density of the pure gluodynamics reads (h.c. stands for hermitian conjugate)

$$\begin{aligned} \mathcal{L} &= \frac{1}{4g^2}\int \mathrm{d}^2\theta W^{a\alpha}W^a_\alpha + h.c. \\ &= -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{i}{g^2}\lambda^{a\alpha}D_{\alpha\dot{\beta}}\bar{\lambda}^{a\dot{\beta}} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \quad , \end{aligned} \tag{5.6}$$

with dual field strength tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The only difference to the non-supersymmetric term is the gauge invariant kinetic terms for the gauginos.
The supersymmetric action is closely related to the usual QCD action with one flavor. The main difference is that quarks transform under the fundamental representation of the gauge group, whereas the gauginos transform under the adjoint representation.
Nevertheless, the analogy promises generally valid results for both theories. One interesting outcome is the occurrence of the gluino condensate (in QCD this corresponds to the observed quark condensate).

5.2 Gluino Condensation

After we provided the setup in the last section, we want to state some interesting results that can be derived in the context of Super-Yang-Mills theories.

5.2.1 Chiral Anomaly

One of the most important properties is the breaking of the chiral symmetry (also observed for QCD with massless quarks, see chapter 2)

$$\lambda^a \to e^{i\alpha}\lambda^a, \quad \bar{\lambda}^a \to e^{-i\alpha}\bar{\lambda}^a \ , \tag{5.7}$$

via the chiral anomaly. In the pure Super-Yang-Mills theory the chiral current coincides with the R-symmetry current and is written (see [5])

$$R^\mu = \frac{1}{g^2}\bar{\lambda}^a\bar{\sigma}^\mu\lambda^a \ . \tag{5.8}$$

The same triangle diagram as in massless QCD, figure 2.1, leads to the anomaly of the chiral current. Here the gauginos run in the loop. Their different transformation properties (adjoint) cause an additional factor of N for the divergence of the R-current compared to the non-supersymmetric case with massless quarks

$$\partial_\mu R^\mu = \frac{N}{16\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \ . \tag{5.9}$$

The factor N in the above formula shows that there is a remnant unbroken subgroup $\mathbb{Z}_{2N}$ of the chiral $U(1)$ under which the gauginos transform as

$$\lambda^a \to \exp\left(i\pi\frac{j}{N}\right)\lambda^a, \ \ j \in \{0, \dots N-1\} \ . \tag{5.10}$$

This discrete symmetry demonstrates a real difference to the non-supersymmetric case. The breaking of the chiral symmetry can be understood in terms of gaugino condensation. The vacuum expectation value of the gaugino bilinear acquires a non-vanishing value. The gaugino condensate further breaks the $\mathbb{Z}_{2N}$ down to $\mathbb{Z}_2$. Thus, there are N equivalent vacua for $\mathcal{N}=1$ Super-Yang-Mills theories, which are parametrized by the phase of the gaugino condensate

$$\langle\lambda^a\lambda^a\rangle \propto \exp\left(2\pi i\frac{k}{N}\right), \quad k \in \{0, ..., N-1\} \ . \tag{5.11}$$

Moreover, these N vacua are predicted by the Witten index (number of bosonic minus fermionic vacua) of the theory, [73]. In contrast to the pure Yang-Mills theory these are real vacua all with vanishing energy density.
There is a defined correlation between the θ-angle and the phase of the gaugino condensate which was described in [10]. As can be seen in (5.6) the θ-term reads

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \ . \tag{5.12}$$

From this follows that by chiral transformation of the gaugino fields with parameter α the divergence of the R-current is shifted by

$$\partial_\mu R^\mu = \frac{N}{16\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \to (1+\alpha)\frac{N}{16\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \ . \tag{5.13}$$

Hence, $\mathcal{L}_\theta$ vanishes for

$$\alpha = -\frac{\theta}{2N} \quad . \tag{5.14}$$

Comparing this to the transformation properties of the gaugino condensate, we find

$$\langle\lambda^a\lambda^a\rangle_\theta = \langle\lambda^a\lambda^a\rangle_0 \exp\left(i\frac{\theta}{N}\right) \quad . \tag{5.15}$$

For $\theta \to \theta + 2\pi$ the vacua are shifted by one as depicted in figure 5.1. In a sense the vacuum

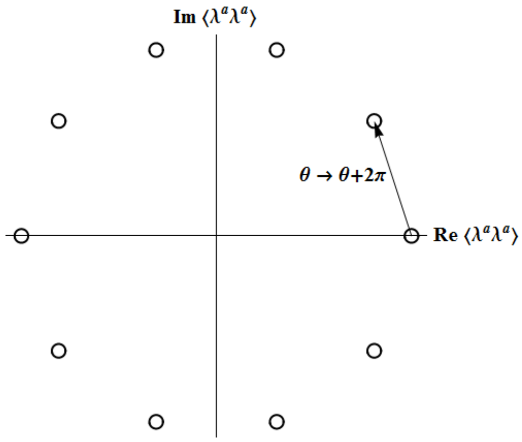


Figure 5.1: Vacua in dependence of gaugino phases for $N = 10$ and their θ-dependence

angle θ is a dynamical variable now influenced by the phase of the gaugino condensate. The phase of the gaugino condensate acts as an axion field a, discussed in section 2.2.

5.2.2 Derivation of the Gaugino Condensate

In 1987 Shifman and Vainshtein provided a direct calculation of the gaugino condensate that we want to briefly recapitulate here, see [10].
We will consider the easiest model with gauge group $SU(2)$, nevertheless the given arguments can be straightforwardly generalized.
First, we introduce two chiral matter fields in the (anti-)fundamental representation Φ_i^c (for $SU(2)$ these two representations are equivalent)

$$\Phi_i^c = \phi_i^c + \sqrt{2}\theta\psi_i^c + \theta^2 F_i^c \quad . \tag{5.16}$$

$i \in \{1, 2\}$ represents the flavor index and $a \in \{1, 2\}$ the color index. Both are $SU(2)$ indices. These two fields are endowed with a mass by the addition of a superpotential

$$\mathcal{W} = \frac{m}{2}\Phi_i^c\Phi_c^i \quad . \tag{5.17}$$

The modified Lagrangian density is (V denotes the vector superfield of the gauge sector)

$$\mathcal{L} = \frac{1}{2g^2}\int \mathrm{d}\theta^2 W^{a\alpha}W_\alpha^a + \frac{1}{4}\int \mathrm{d}\theta^2\mathrm{d}\bar{\theta}^2\,\bar{\Phi}^i e^V \Phi^i + \frac{1}{4}m\left[\int \mathrm{d}^2\theta\,\Phi_i^c\Phi_c^i + h.c.\right] \tag{5.18}$$

For vanishing mass m the scalar potential is due to the D-terms in the kinetic part of the action (ϕ_i denotes the scalar squark fields, the contraction in color indices is understood)

$$V(\phi_i) = \frac{g^2}{8}\left(\sum_i \bar{\phi}_i T^a \phi_i\right)^2 , \quad \text{with} \quad T^a = \frac{1}{2}\sigma^a \ . \tag{5.19}$$

Hence, there is a flat direction for the scalar components of the matter fields which can be parametrized by one arbitrary complex variable v

$$\phi_1 = \begin{pmatrix} v \\ 0 \end{pmatrix} , \quad \phi_2 = \begin{pmatrix} 0 \\ v \end{pmatrix} \ . \tag{5.20}$$

This degeneracy is protected by non-renormalization theorems to all orders in perturbation theory, but might be lifted by non-perturbative effects.

For $vg \gg \Lambda$ (Λ being the dynamically generated mass scale) the gauge group is broken completely. Consequently, all gauge bosons become massive with mass $m_v \propto gv$. The field content in this super-Higgs sector can be reassembled into three massive vector superfields and one light chiral superfield along the flat direction $\Phi^2 = \Phi_i^c \Phi_c^i$. In the low energy limit we can integrate out the heavy vector fields and consider an effective theory of the light superfield Φ^2. Since the vacuum structure and the corresponding condensates are also present in the low energy limit, their physics must be encoded in the action of Φ^2. In the regime $vg \gg \Lambda$ it is perfectly justified to integrate out the heavy vector fields.

If we include the mass term, the scalar potential will change and the flat direction is lifted

$$V_m \propto |mv|^2 \ . \tag{5.21}$$

This would push v to 0, where the full gauge theory is restored. Nevertheless, there is a further non-perturbative contribution by instantons. Its contribution to the superpotential is constraint by an anomaly free R-symmetry as discussed in [74] and [11] and has the functional form

$$\mathcal{W}_{inst} = C\frac{\Lambda^5}{\Phi^2} \ , \tag{5.22}$$

where C is a finite constant. For this calculation the regime $gv \gg \Lambda$ is very important since otherwise one encounters problems due to the contribution of large instantons. Moreover, this non-perturbative effect tends to increase the vacuum expectation value v.

In total the F-term reads

$$F^* = -\frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} = -\frac{m}{2}\Phi + 2C\frac{\Lambda^5}{(\Phi^2)^2}\Phi \ . \tag{5.23}$$

For a vacuum state this term has to vanish and therefore

$$v^2 = \pm 2\left(\frac{C\Lambda^5}{m}\right)^{\frac{1}{2}} \ . \tag{5.24}$$

For small mass parameter the constraint, $gv \gg \Lambda$ is fulfilled.

In order to relate this quantity to the vacuum expectation value of the gaugino condensate we use the so-called Konishi anomaly (see [75])

$$\frac{1}{8}\bar{D}^2\left(\bar{\Phi}^i e^V \Phi^i\right) = \frac{1}{2}m\Phi^2 + \frac{1}{16\pi^2}W^{a\alpha}W^a_\alpha \ , \tag{5.25}$$

with superderivatives $\bar{D}$. Calculating the lowest component in θ (fermionic coordinate in superspace) in a vacuum state the left hand side vanishes and we are left with

$$\frac{1}{16\pi^2}\langle\lambda^a\lambda^a\rangle = \frac{1}{2}m\langle\phi^2\rangle = \pm\left(C\Lambda^5 m\right)^{\frac{1}{2}} \ . \tag{5.26}$$

The gluino condensate in the case of one light flavor is exactly calculable.
To elaborate the limit $m \to \infty$ in which the above theory flows to pure Super-Yang-Mills we use some holomorphicity properties of supersymmetric theories.
This can be done by analyzing the concrete m-dependence of the gaugino condensate, see [76]. For that purpose we extend the mass parameter m to a chiral spurion superfield M, which is not charged under the gauge group. Its lowest component develops a vacuum expectation value serving as usual mass. With this new superfield there is an extended R-symmetry which is not anomalous and survives also in the strong coupling regime

$$W_\alpha \to e^{i\gamma}W_\alpha \ , \ \ \Phi_i \to e^{-i\gamma}\Phi_i \ , \ \ M \to e^{4i\gamma}M \ , \ \ \theta_\alpha \to e^{i\gamma}\theta_\alpha \ . \tag{5.27}$$

Incorporating the chirality dependence and the extended R-symmetry we deduce

$$\langle W^{a\alpha}W^a_\alpha\rangle \propto M^{\frac{1}{2}} \ \ \Rightarrow \ \ \langle\lambda^a\lambda^a\rangle \propto m^{\frac{1}{2}} \ . \tag{5.28}$$

This mass dependence of the gaugino condensate is exact and holds at weak as well as at strong coupling. A similar reasoning leads to

$$\langle\Phi^2\rangle \propto M^{-\frac{1}{2}} \ \ \Rightarrow \ \ \langle\phi^2\rangle \propto m^{-\frac{1}{2}} \ . \tag{5.29}$$

This is consistent with the Konishi anomaly (5.25). Thus, the functional shape of the gaugino condensate is valid also at strong coupling and we can finally consider the limit $m \to \infty$. First we have to compare the dynamically generated scales in theories with and without matter. The first coefficient in the β-function with n_f flavors and N colors for supersymmetric theories (see [5], [77]) is

$$\beta_0 = 3N - \frac{1}{2}n_f \ . \tag{5.30}$$

Note that this differs form the value for non-supersymmetric theories discussed in chapter 2.
The mass scale Λ of the theory is calculated by, see equation (2.12)

$$\frac{\alpha(M)}{2\pi} = \frac{g^2(M)}{8\pi^2} \approx \frac{1}{\beta_0\ln\left(\frac{M}{\Lambda}\right)} \Rightarrow \Lambda = M\exp\left(-\frac{2\pi}{\beta_0\alpha}\right) \ . \tag{5.31}$$

For energies $M > m$ the first coefficient is $\beta_0 = 5$, whereas for $M < m$ it is $\beta_0 = 6$ in our $SU(2)$ example. This means we have two different mass scales, Λ' for the theory with matter and Λ for the theory without matter, which for $M = m$ are

$$\Lambda' = m \exp\left(-\frac{2\pi}{5\alpha}\right), \quad \Lambda = m \exp\left(-\frac{2\pi}{6\alpha}\right) \ . \tag{5.32}$$

Consequently, we find

$$\Lambda'^5 m = \Lambda^6 \ . \tag{5.33}$$

The gaugino condensate in the pure $SU(2)$ gauge theory of Super-Yang-Mills hence is

$$\langle \lambda^a \lambda^a \rangle = \pm \tilde{C} \Lambda^3 \ , \tag{5.34}$$

with non-vanishing constant $\tilde{C}$.
The generalized version in terms of the properly normalized dynamically generated mass scale Λ_G of a pure gauge theory with gauge group $G = SU(N)$ is (see [10])

$$\langle \lambda^a \lambda^a \rangle = C(N) \exp\left(2\pi i \frac{j}{N}\right) \Lambda_G^3 \ , \tag{5.35}$$

leading to N distinct vacua with $j = 0, \ldots, N-1$. It is very likely that the constant C depends on N since it is defined as the trace of a elementary fields and in fact in [14] and [78] it is stated that the value of the gaugino condensate is

$$\langle \lambda^a \lambda^a \rangle = N \Lambda^3 \exp\left(2\pi i \frac{j}{N}\right) \ . \tag{5.36}$$

5.2.3 Vacuum Structure of SYM

The vacuum structure of Super-Yang-Mills theories differs from that in models without supersymmetry.
First of all, there are the N distinct vacua labeled by an integer variable k that describes the phase of the gaugino condensate. In contrast to non-supersymmetric Yang-Mills theories these vacua are truly degenerate in energy. A shift of the vacuum angle θ is equivalent to a change in the gaugino condensate, see equation (5.15). This means that in Super-Yang-Mills theories θ and k are not independent parameters but intimately correlated.
As we have seen for the non-supersymmetric case, the change of vacuum induces a constant electric field due to the 3-form field strength. In pure Yang-Mills theories the 3-form is massless and causes a long-range interaction between a wall and an antiwall. Further, the wall in Yang-Mills theories is no static object, because there is a difference in the vacuum energy on both sides.
In the supersymmetric case the phase of the gluino condensate acts as an axionic field. By eating up the axion, the 3-from field coupling to domain walls becomes massive. Hence, the electric field gets screened and does not create a long-range interaction between walls. Furthermore, we do expect the domain walls in Super-Yang-Mills theories to be perfectly stable and static configurations. This is only possible if the 3-form field is screened in the presence of a domain wall that interpolates between the vacua. Only by this mechanism

the BPS property and connected stability can be developed and we will work out the respective mechanism in the following chapter.

Chapter 6

Super-Yang-Mills Domain Walls

In this chapter we describe some exact results for Super-Yang-Mills domain walls that cannot be obtained in a non-supersymmetric framework. Especially the wall tension can be calculated exactly for the assumption that the walls are BPS saturated states.
Afterwards, we generalize the topological theory to the bosonic sector of the Super-Yang-Mills domain walls and point out the differences.
In the last chapter we have seen that $\mathcal{N} = 1$ Super-Yang-Mills theory with gauge group $SU(N)$ in four dimensional spacetime develops N discrete vacua. These can be characterized by the phase of the gaugino condensate and shifted by changing the θ-angle. Due to the spontaneously broken discrete $\mathbb{Z}_{2N}$ symmetry down to $\mathbb{Z}_2$ it is natural to include the possibility of domain walls.
These domain walls were first discussed by Dvali and Shifman in [79] and [13] and later became subject of intensive investigation. These Super-Yang-Mills domain walls can also be constructed in an M-theory background which was first done by Witten in [14]. One of the most interesting features of the encountered configurations is that they behave very similar to D-branes in string theory. As in pure Yang-Mills theories the chromoelectric flux tubes of the Super-Yang-Mills theories can end on the walls. Furthermore, in 't Hooft's large N limit the tension of these domain walls is proportional to N rather to N^2 which also resembles string theory with the identification of the string coupling $g_s = N^{-1}$. The same was shown in non-supersymmetric theories by a string theory construction. In Super-Yang-Mills it can be directly calculated from the quantum field theoretical point of view.
In order to calculate the tension of the domain walls we proceed by investigating the central extension of the superalgebra. The central charges are thereby related to the topological quantum numbers and these are identified with the tensions for BPS states (see [80]). BPS states are states which preserve part of the supersymmetry, e.g. [39]. For domain walls in four dimensions they preserve two of the four real supercharges. We have to mention that so far there is no direct construction of the BPS walls in Super-Yang-Mills theories. Nevertheless, there is strong evidence that the domain walls in Super-Yang-Mills

in fact are BPS saturated, for reference see for example [81], [15], [82] and [16].

6.1 Exact Domain Wall Tension

In the following we will assume that the Super-Yang-Mills domain walls indeed are BPS saturated. Even if this might not be the case, the presented calculations lead to a perfectly valid lower bound of the energy. The energy of the physical configuration is likely to be close to this bound (see [83]). The centrally extended superalgebra reads, see [5] (with supercharges Q_α)

$$\{Q_\alpha, Q_\beta\} = -4\Sigma_{\alpha\beta}\bar{Z} \ , \tag{6.1}$$

where the wall area tensor is defined by

$$\Sigma_{\alpha\beta} = -\frac{1}{2}\int \mathrm{d}x_{[\mu}\mathrm{d}x_{\nu]}\,(\bar{\sigma}^\mu)_{\alpha\dot{\alpha}}\,(\sigma^\nu)^{\dot{\alpha}}_{\beta} \ , \tag{6.2}$$

with matrices σ^μ, $\bar{\sigma}^\nu$ as defined in appendix D. There are two supercharges in the domain wall background $Q^{(w)}_\alpha$ that fulfill the anticommutation relation

$$\left\{Q^{(w)}_\alpha, Q^{(w)}_\beta\right\} = 8\Sigma_{\alpha\beta}(T - |Z|) \ , \tag{6.3}$$

T denotes the domain wall tension. Thus, if we want to preserve these supercharges, that is we want the anticommutator to vanish, which is necessary for a BPS state, the wall tension is equal to the absolute value of the central charge $|Z|$.
For Super-Yang-Mills theories with matter, this central charge has been calculated in [84] and [85]. Up to total superderivatives (vanishing in supersymmetric vacuum) it reads

$$Z = \frac{2}{3}\Delta\left\{3\mathcal{W} - \frac{3N - \frac{1}{2}n_f}{16\pi^2}W^{a\alpha}W^a_\alpha\right\}_{\theta=0} \ . \tag{6.4}$$

The Δ means the difference of the expression in brackets at spatial infinities perpendicular to the domain wall, the $\theta = 0$ refers to the lowest component in the fermionic coordinates. For pure gluodynamics without superpotential and $n_f = 0$ the central charge is an effect that arises due to the anomaly and cannot be seen at a classical level.
Finally, the wall tension is

$$T = |Z| = \frac{N}{8\pi^2}\,|\langle\lambda^a\lambda^a\rangle_{k_1} - \langle\lambda^a\lambda^a\rangle_{k_2}| \ , \tag{6.5}$$

for a domain wall interpolating between vacua where the gaugino phase is labeled by k_j.
Now we consider the Super-Yang-Mills theory in 't Hooft's large N limit, with the 't Hooft coupling, see chapter 2

$$\lambda \equiv g^2 N \ . \tag{6.6}$$

Thus, the Lagrangian density expressed in terms of λ is

$$\mathcal{L} = \frac{N}{4\lambda}\int \mathrm{d}\theta^2 W^{a\alpha}W^a_\alpha + h.c. \ . \tag{6.7}$$

Plugging in the value of the gaugino condensate (5.36), we derive the exact tension of a domain wall interpolating between two supersymmetric vacua labeled by k_1 and k_2

$$T = \frac{N^2}{8\pi^2}\Lambda^3 \left| \exp\left(2\pi i \frac{k_1}{N}\right) - \exp\left(2\pi i \frac{k_2}{N}\right)\right| \quad . \tag{6.8}$$

We can rewrite this to obtain the result given in [81]

$$\begin{aligned} T =& \frac{N^2}{8\pi^2}\Lambda^3 \left| \exp\left(\pi i \frac{k_1+k_2}{N}\right)\left[\exp\left(\pi i \frac{k_1-k_2}{N}\right) - \exp\left(-\pi i \frac{k_1-k_2}{N}\right)\right]\right| = \\ =& \frac{N^2}{4\pi^2}\Lambda^3 \left|\sin\left(\pi \frac{k_1-k_2}{N}\right)\right| \quad . \end{aligned} \tag{6.9}$$

Thus, considering elementary domain walls with $k_1 - k_2 = 1$

$$T = \frac{N}{4\pi}\Lambda^3 + \mathcal{O}\left(\frac{1}{N}\right) \tag{6.10}$$

With 't Hooft's identification in the large N limit in Yang-Mills theories with a non-critical string theory with string coupling $g_s = \frac{1}{N}$ the tension of the elementary domain walls scales as

$$T \propto \frac{1}{g_s} \quad . \tag{6.11}$$

This is the same scaling behaviour as for D-branes [86].

6.2 Strings Ending on Domain Walls

The formalism of supersymmetry equips us with the possibility to study string configurations that end on domain walls as explicit solutions. Since these configurations are of great interest, we recall the simplest model to illustrate their properties.
To this end we consider $\mathcal{N} = 2$ models of Super-Yang-Mills with matter. In this theories there are explicit solutions of flux tubes (in this case these are magnetic) that do end on corresponding domain walls, [87]. We briefly outline the mechanism of these junctions following [88].
In order to work in the weak coupling limit, where all effects are under good theoretical control, we demand extended supersymmetry. In the simplest theory of $\mathcal{N} = 2$ Super-QED with two flavors the field content is one $U(1)$ vector superfield (the neutral scalar component will be denoted by a) and two charged hypermultiplets (scalar components ϕ_i). The scalar potential is fixed by the supersymmetry to be

$$V(a, \phi_i) = \sum_{i=1}^{2} (a - m_i)^2 |\phi_i|^2 + \frac{g^2}{2}\left(\sum_{i=1}^{2} |\phi_i|^2 - v^2\right)^2 \quad . \tag{6.12}$$

The m_i are the masses of the charged flavor fields and the Fayet-Iliopoulos term v^2 induces a vacuum expectation value for the matter components. The different masses do explicitly break the flavor symmetry $SU(2)_f \to U(1)_f$. There are two vacua of the theory where the scalar potential vanishes

$$a = m_i, \quad |\phi_j|^2 = \delta^i_j v^2, \quad i \in \{1,2\} \ . \tag{6.13}$$

The gauge symmetry is spontaneously broken by the vacuum expectation value of one of the fields ϕ_i. Thus, the photon becomes massive by the Higgs effect eating up the phase of ϕ_i. The other charged scalar acquires a mass $(m_i - m_j)$.
The existence of isolated vacua permits domain walls for which $a \to m_1$ for example for $z \to -\infty$ and $a \to m_2$ for $z \to \infty$. Furthermore, in one of the vacua the phase of the matter that develops an expectation value can wind and hence gives rise to strings. In this setup there exist $\frac{1}{2}$-BPS saturated domain walls and strings with tensions (see [88])

$$T_{\text{wall}} = v^2(m_2 - m_1) \equiv v^2 \Delta m, \quad T_{\text{string}} = 2\pi v^2 \ . \tag{6.14}$$

The wall exhibits two collective coordinates. One is due to the breaking of translational symmetry and describes the embedding of the domain wall by ζ. The second arises because of the remaining flavor symmetry $U(1)_f$

$$\phi_1 \to e^{i\alpha}\phi_1, \quad \phi_2 \to e^{-i\alpha}\phi_2 \ , \tag{6.15}$$

acting non-trivially in the center of the domain wall where $\phi_i \neq 0$ for $i = 1, 2$. The configuration space of zero modes of the domain walls also called moduli space is

$$\mathcal{M} = \mathbb{R} \times S^1 \ . \tag{6.16}$$

The effective world volume action on the domain walls for the bosonic zero modes to lowest order in derivatives just describes two free scalar fields

$$\mathcal{S}_{\text{eff}} = \frac{1}{2} T_{\text{wall}} \int \mathrm{d}x^3 \left(\partial_a \zeta \partial^a \zeta + \frac{1}{\Delta m^2} \partial_a \alpha \partial^a \alpha \right), \quad a \in \{0,1,2\} \ . \tag{6.17}$$

This can be cast into a more convenient form by dualizing the periodic scalar field α. The result is a $U(1)$ gauge field in $2+1$ dimensions (one propagating degree of freedom) whose field strength is subject to the equation

$$\partial_a \alpha = \frac{T_{\text{wall}}}{4\pi v^4} \epsilon_{abc} F^{bc} \ . \tag{6.18}$$

Moreover, we rescale the field $\zeta = (1/2\pi v^2)\xi$ and are left with the Lagrangian density with the coupling constant $e^2 = (4/\pi^2 v^4) T_{\text{wall}}$

$$\mathcal{L}'_{eff} = \frac{1}{4e^2} F^{ab} F_{ab} + \frac{1}{2e^2} \partial_a \xi \partial^a \xi \ . \tag{6.19}$$

The effective worldvolume theory is dual to a gauge theory. Moreover, in [89] and [90] explicit solutions for strings or flux tubes ending on these walls were found. They obey the equation

$$\Delta m \zeta + i\alpha = \log\left(\Delta m (X^1 + iX^2)\right) \ , \tag{6.20}$$

where X is the complexified position of the endpoint of the string on the wall. The winding of α leads to a radial electric field in the dual picture of the worldsheet gauge theory and hence the end of the flux tubes on the domain walls act like charges.
This can be also described in the bulk picture and the solution preserves $\frac{1}{4}$ of the supercharges, meaning that they are $\frac{1}{4}$-BPS solutions, [87].
There are explicit field theoretical constructions of flux tubes ending on domain walls that serve as prototypes of open strings ending on D-branes.
In order to get more intuition for what happens if a flux tube ends on a domain wall we want to work out one further consideration, done in [13]. It was shown that this phenomenon is strongly related to the localization of gauge fields on lower dimensional surfaces.
The easiest mechanism for such a localization is to have a confining gauge theory in the bulk that is broken down to an Abelian and hence non-confining subgroup on the domain wall. The domain wall action consequently contains a massless gauge field whose lowest excitations are localized on the wall since there is a dynamically generated mass gap in the bulk. Considering heavy test charges of this unbroken Abelian gauge field on the domain wall and adiabatically pulling them into the bulk, there will be a flux tube stretched from the test charge to the domain wall which represents the string discussed above, see figure 6.1. This connecting flux tube is due to the charge conservation of the test charge in combination with the confinement of electric flux in the bulk theory. Thus, assuming a localized gauge field on the domain walls one immediately encounters flux tubes that may end on the wall. One problem of this mechanism is that one needs Higgs fields to

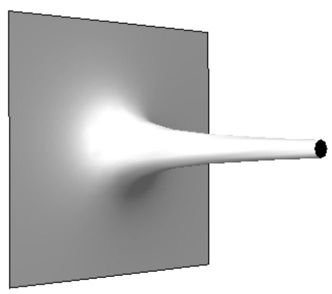


Figure 6.1: Domain wall with test charge that is pulled out of the wall but is still connected to it via a flux tube

break the non-Abelian gauge group down to an Abelian subgroup in order to create the confining phase in the bulk and the Coulomb phase on the wall. In the theory which we are interested in, namely $\mathcal{N} = 1$ Super-Yang-Mills with gauge group $SU(N)$, we would need several real scalar fields to break the gauge group to the maximal torus (the maximal Abelian subgroup in this case $U(1)^N$). In a pure Super-Yang-Mills theory, however, there are no scalar fields at all, therefore the localization mechanism briefly discussed above is not applicable in this context.
Nevertheless, the string theory picture and the thought experiment with heavy test charges tells us that a localized gauge field and flux tubes ending on domain walls are intimately

related. In the following we elaborate a topological model of domain walls and strings in Super-Yang-Mills theories. We apply the same formalism as in chapter 4 with some generalizations.

6.3 TFT for Super-Yang-Mills Domain Walls

There are supersymmetric extensions available for all the topological field theories in chapter 3, see e.g. [91] and [92]. We are, however, interested in the bosonic part of the theory and the topological models worked out are sufficient.

6.3.1 TFT for Super-Yang-Mills

The whole mechanism of categorizing confining theories by the non-local operators that carry center charges remain valid in a supersymmetric setting.
Confinement is once more assumed to be an effect of the condensation of charge N monopoles. In Super-Yang-Mills theories, however, the parameters θ and k are not independent anymore and the action should only include one of the contributions. We choose to describe the domain walls by a change of the vacuum angle θ and therefore the action is (see equation (4.38))

$$S = \frac{i}{2\pi} \int \left[\tilde{F} \wedge (F - NB) - \frac{N\theta}{4\pi} B \wedge B \right] \ , \tag{6.21}$$

with 1-form transformations

$$\begin{aligned} B &\to B + \mathrm{d}\lambda \ , \\ A &\to A + N\lambda \ , \\ \tilde{A} &\to \tilde{A} - \frac{\theta}{2\pi}\lambda \ . \end{aligned} \tag{6.22}$$

By relation (5.15) the fundamental domain walls interpolating between vacua with $\Delta k = 1$ are equally well described by a jump of θ by 2π. With this analogy on both sides of the domain wall monopoles are condensed in contrast to Yang-Mills theories. One can see this using the construction in the Yang-Mills theories. There the two vacua are at the same value of θ and really contain condensates of different charges. Extending this to Super-Yang-Mills the charge of the condensate for fixed θ is a dyonic one, but the phase of the gaugino condensate has the same effect as an effective θ-angle. As a consequence the condensates for fixed effective θ and different charges are equivalent to condensates of the same charges but different effective vacuum angle. The modified gauge transformations of $\tilde{A}$ for Super-Yang-Mills are not an effect of a dyonic condensate but rather the effect of describing the configurations at $\theta \neq 0$ with the same fields as for $\theta = 0$.
Assuming θ to be a variable encoding domain walls, the action changes under 1-form gauge transformations in analogy to (4.49)

$$\Delta S = \int \left[-\frac{i}{8\pi^2} \mathrm{d}(2\theta\lambda \wedge F + N\theta\lambda \wedge \mathrm{d}\lambda) + \frac{iN}{8\pi^2} \mathrm{d}\theta \wedge (2\lambda \wedge B + \lambda \wedge \mathrm{d}\lambda) \right] \ . \tag{6.23}$$

Assuming a sharp jump of θ by 2π on a codimension one surface $\mathcal{V}$

$$\mathrm{d}\theta = 2\pi\delta_{\mathcal{V}} \ , \tag{6.24}$$

and a spacetime manifold without boundaries we find a contribution of the domain wall

$$\Delta S_{\text{wall}} = \frac{iN}{4\pi}\int_{\mathcal{V}} [2\lambda \wedge B + \lambda \wedge \mathrm{d}\lambda] \ . \tag{6.25}$$

Once more wall degrees of freedom $\mathcal{A}$, have to be included with the action

$$S_{\mathcal{V}} = \frac{iN}{4\pi}\int_{\mathcal{V}} [2\mathcal{A} \wedge B + \mathcal{A} \wedge \mathrm{d}\mathcal{A}] \ , \tag{6.26}$$

and the 1-form transformation property

$$\mathcal{A} \to \mathcal{A} - \lambda \ . \tag{6.27}$$

Besides the coupling of the wall degrees of freedom $\mathcal{A}$ to the 2-form field B, the correct $U(1)$ Chern-Simons term of level N is present, as predicted in [68] and [18].
The mechanism for flux tubes ending on the domain wall in a gauge invariant fashion is exactly the same as for pure Yang-Mills theories and we refer to chapter 4. The real difference is the occurrence of a gaugino condensate in Super-Yang-Mills and its consequences.

6.3.2 Consequences of the Gaugino Condensate

The same mechanism as in non-supersymmetric theories localizes a Chern-Simons term on the domain walls in $\mathcal{N} = 1$ Super-Yang-Mills theories. The phase of the condensate describes an axionic field a. This field was not present in the pure Yang-Mills case and has some important consequences for the dynamics of the domain walls.
In chapter 4 it was stated that for a dyonic condensate the vacuum expectation value of the 4-form field strength $F_4 = \mathrm{d}C$ connected to the Chern-Simons 3-form of Yang-Mills theories generates a non-zero expectation value. Thus, this massless 3-form supports a long-range interaction via an electric field. This mechanism leads to a confinement of a wall-antiwall pair by a constant electric field in the intermediate region.
For Super-Yang-Mills domain walls, on the other hand, the condensate on both sides of the walls is comprised of monopoles. Consequently, we expect the vacuum expectation value of F_4 to vanish. This can be understood as a Higgsing of the 3-form field by the axionic field, as described in section 2.2. The massive 3-form does not support long-range interactions and the domain walls are not confined.
The presence of a domain wall creates a jump of the 4-form field strength nonetheless. The electric field, however, is screened and the amplitude vanishes close to the wall, see figure 6.2 and [36].
The topological formalism presents a way for understanding the occurrence of a level N Chern-Simons term on the domain wall worldvolume. Moreover, it suggests a way to understand the properties of interaction between two domain walls by the properties and

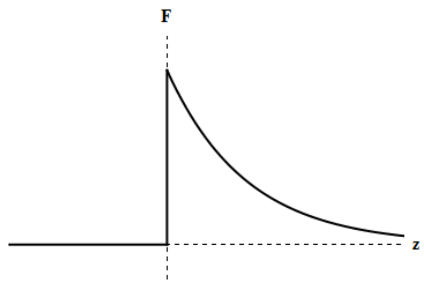


Figure 6.2: Profile of the electric field in the presence of a domain wall

relations of the θ-angle.
The weak point of a topological field theory remains that it cannot describe the dynamical and energetical issues. In the following outlook we describe how analogies to string theory and fractional quantum Hall systems might help us to overcome that problem.

Chapter 7

Outlook and Conclusion

In this chapter possibilities of getting insights into the dynamics of (Super-)Yang-Mills theories by analogies to string theory and fractional quantum Hall systems in solid state physics are demonstrated. Unfortunately, these ideas have not been worked out properly so far and therefore they only give a hint at how one might proceed to obtain dynamical properties.

7.1 Analogy to String Theory

Already the large N construction presents a way to interpret the results of Yang-Mills theories in a string theoretical framework. Thus, it is natural to try to relate some of the conclusions obtained in our topological construction to phenomena in string theory.

First, let us specify which objects are present in a string theory setup. There are Dp-branes which have a (p+1)-dimensional worldvolume and couple to a massless (p+1)-form, the so-called Ramond-Ramond (RR) field denoted by $C_{(p+1)}$. The choice of the type of string theory determines the dimension of allowed D-branes. For example, in type IIB string theory p has to be odd, see [37].

Furthermore, dualizations of the form, $\mathrm{d}C_p \wedge \mathrm{d}C_q$ relate objects with (p+1)-dimensional worldvolume to objects with (D-3-p)-dimensional worldvolume, i.e. for $D = 10$ in superstring theory

$$p + q = 6 \ . \tag{7.1}$$

The fundamental strings are dual to $NS5$-branes, the $D1$-branes are dual to $D5$-branes for ten dimensional superstring theory.

Another object, present in string theory, is the Kalb-Ramond field, a 2-form field coupling to the fundamental strings.

Relations between objects in string theory can be further obtained via S-duality, in a special case the complexified coupling τ (equivalent to the complex coupling in Super-Yang-Mills theories) is transformed to

$$\tau \to -\frac{1}{\tau} \ . \tag{7.2}$$

This S-duality generalizes an electric-magnetic duality for string theories.
The real part of the coupling constant τ is identified with the dilaton field Φ via

$$\mathrm{Re}(\tau) = g_s = e^{\Phi} \ . \tag{7.3}$$

With the field content specified, we try to identify some of the objects in the gauge theory by embedding it in a string theoretical model.
The fundamental strings ending on D-branes should be identified with the electric flux tubes of our confined theory, as discussed in chapters 2 and 5. Their dual objects are $NS5$-branes. The Kalb-Ramond field couples to them and corresponds to the field B in the topological construction.
Under the S-duality described above the Kalb-Ramond field B and the RR-field C_2 get interchanged, [93]. The RR-field C_2 couples to $D1$-branes. On the other hand, under electric-magnetic duality on the field theory side, electric flux tubes should be turned into their magnetic analogs. This suggests to identify magnetic flux tubes with $D1$-branes of string theory.
Further, in string theory descriptions for gauge theories the following term arises, [78]

$$e^{-\Phi}\mathrm{Tr}(F \wedge *F) + C_0 \mathrm{Tr}(F \wedge F) \ , \tag{7.4}$$

naturally leading to a relation between the RR-field C_0 and the axion. Furthermore, this axionic field can be combined with the dilaton to a complexified coupling as in Super-Yang-Mills theories. They relate to the pseudoscalar and scalar glueballs of the field theory respectively, [94]. The RR-form C_0 couples to $D(-1)$-branes that by the above equation correspond to the instantons of the field theory.
However, in our confining field theory there are no massless fields. Moreover, the flux tubes and domain walls carry discrete charges under the center of the gauge group. The analogy of the domain walls in Super-Yang-Mills and axionic domain walls constructed in chapter 6 additionally suggests a duality between instantons and domain walls which does not satisfy equation (7.1) for $D = 4$, namely

$$p + q = 0 \ . \tag{7.5}$$

Consequently, our attempt to relate the domain wall properties to stringy objects seem to fail. But there is a solution discussed in [95]. For string theories in which the fundamental string carries a discrete charge, as in our case, the duality relation (7.1) changes to

$$p + q = 7 \ . \tag{7.6}$$

For $D(-1)$-branes the dual objects in this case are $D8$-branes describing domain walls in string theory. This suggests that we should use an equivalent constraint in $D = 4$

$$p + q = 1 \ . \tag{7.7}$$

String-like objects with $p = 1$ are now dual to non-local operators with $q = 0$, the string theory construction of [94] relates these operators to worldlines of electric and magnetic

charges. This is further supported by [96]. There the $NS5$-branes are the monopole condensate. A dyonic condensate is created by a $NS5$-$D5$ bound state. So, in a sense, we can understand the $NS5$ as comprising the monopoles and the $D5$ as creating the electric charges of the condensate which can be interpreted as baryons.
The instantons, on the other hand, are dual to objects with worldvolumes of dimension three, exactly fitting the domain walls.
Let us try to understand this relation using the arguments of section 2.2. Usually we would expect that the dual object of the field theoretical axion, i.e. the axionic string, couples to two dimensional worldsheets. But in section 2.2 it was shown that the axion is eaten up by the Chern-Simons 3-form, rendering it massive. The axionic string, now part of a massive 3-form, naturally couples to the domain walls of codimension one.
This effect refers to two phenomena in string theory with discrete charges. First of all, the massive 3-form corresponds RR-fields in this special type of string theory. And the Higgsing procedure allows the axion to couple to objects of one dimension higher, which precisely is reflected in (7.6).
These identifications might allow to obtain further dynamical information about the domain wall systems that were constructed in the previous chapter and extend our ability for the investigation of the connected mechanisms.

7.2 Analogy to Fractional Quantum Hall Systems

Another promising analogy is the relation to fractional quantum Hall systems. Since we cannot describe the whole theory of the fractional quantum Hall effect, we restrict the discussion to an intuitive picture pointing out some fascinating relations.
The fractional quantum Hall effect describes a two dimensional electron gas in the presence of an exterior magnetic field perpendicular to the plane, similar to figure 3.3. A fractional quantum Hall system with filling factor ν is described by a Chern-Simons term at level $1/\nu$ (see [97]), such as the worldvolume action of our domain walls

$$S_{\mathrm{FQH}} = \frac{1}{4\pi\nu}\int \mathcal{A}\wedge \mathrm{d}\mathcal{A} \ . \tag{7.8}$$

The missing factor i is due to the fact that we are working in Minkowski spacetime.
The filling factor counts the number of electrons per unit of fundamental magnetic flux $\Phi_0 = \frac{2\pi}{e}$. In this system one electron binds $1/\nu$ fluxes of the magnetic field corresponding to $\mathcal{A}$. The groundstate of the system is well described by a composite of fractionalized electrons. These quasi-particles are comprised of one fundamental flux bound to ν electrons. For the filling factor of $\nu = 1/N$ we recover the level N Chern-Simons term of the (Super-)Yang-Mills domain wall action. In this case the electrons fractionalize to parts of charge $1/N$ each connected with a fundamental flux, [98].
The splitting of charges into fragments has an analog in the topological model of Yang-Mills theories. Consider a baryonic object, i.e. a junction of N fundamental flux tubes. In the bulk the flux tubes have to preserve the junction, otherwise the configuration would not be gauge invariant. On the wall, however, the gauge field $\mathcal{A}$ allows them to split up in single flux tubes ending on the wall, see figure 7.1. The end points of the flux tubes on

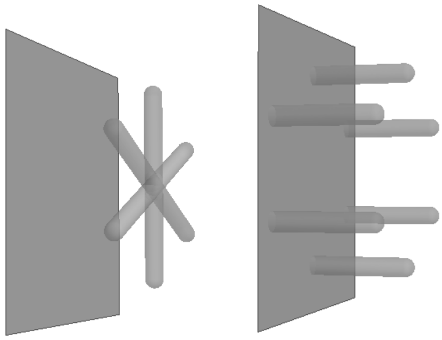


Figure 7.1: Baryonic vertex in the bulk and on the wall respectively

the wall act like electric charges of $1/N$ much like the fractionalized electrons and due to the coinciding Chern-Simons term show the same statistical properties under exchange. The exterior magnetic field in the fractional quantum Hall setup is equivalent to the jump in the θ-angle for the domain walls since this also generates a magnetic flux for electric charges located on the wall, see [53].
The fractional quantum Hall groundstates can be as well described by so-called Laughlin-states, introduced in [99]. The question is if this formulation might be as well applicable for the theory on the domain wall. The fundamental excitations are the fractionalized electrons and might correlate to flux tubes in the (Super-)Yang-Mills picture. This would be definitely an interesting subject of investigation and might yield insights in the dynamics of the domain walls.
The most interesting property is that string theory constructions suggest that on walls interpolating between vacua with $\Delta k = 2$ instead of $\Delta k = 1$, the typical symmetry enhancement of coinciding D-branes should occur. This would transform the $U(1)$ into an $U(2)$ Chern-Simons term at level N. From the topological field theory point of view this cannot be understood in our model, because a factor of two in front of the domain wall action would be sufficient to render the action gauge invariant. This phenomenon on the other hand is observed for fractional quantum Hall systems. In fact, the so-called bilayer configurations with two fractional quantum Hall states develop non-Abelian statistics and in turn are described by a non-Abelian Chern-Simons theory, [100], [101] and [102]. This is due to another index which describes the label of the layer. There are two different kinds of excitations in the bilayer systems. One transforms under a $U(1)$ symmetry by simultaneous phase rotations on both layers. The other has a non-trivial structure in the labels of the layer and transforms under an additional $SU(2)$ symmetry, see [103]. Our hope is to relate these two different excitations to flux tubes that are stretched between the walls.
Although the correspondence has not been elucidated so far, this is a promising candidate for understanding the symmetry enhancement in string theory from a quantum field theoretical standpoint.

7.3 Conclusion

After giving a short introduction to non-supersymmetric Yang-Mills theories, we explained the useful large N limit and the connected dynamics. The change of the vacuum structure in the large N limit was pointed out and led to the notion of Yang-Mills domain walls as mentioned in [17].
In the following chapter we developed the necessary tools for a topological construction of the theory. Especially the properties of electric/magnetic line and surface operators were pointed out including their relation to the generation of Aharonov-Bohm phases. With the classification of confining gauge theories by these non-local operators we succeeded in developing a consistent topological field theory for the vacuum structure of Yang-Mills theories. This was mainly possible, because the important gauge group in the topological setup is $\mathbb{Z}_N$ rather than the full $SU(N)$. This allowed a continuum description of the gauge theory using a broken $U(1)$ symmetry, in analogy to the Abelian-Higgs model.
The action was derived by making two simple and plausible assumptions. First, that confinement is generated by the condensation of charge N monopoles. Second, that the Witten effect modifies the genuine line operators and the transformation properties of the gauge fields in a natural way. The action obtained is

$$S = \frac{i}{2\pi} \int \left[\tilde{F} \wedge (F - NB) - \frac{N\theta}{4\pi} B \wedge B \right] \ . \tag{7.9}$$

Taking into account the N quasi-stable vacua of Yang-Mills theories in the 't Hooft limit there is a further contribution labeling these states

$$S_{\text{YM}} = S + \frac{iNk}{4\pi} \int B \wedge B \ . \tag{7.10}$$

The fields contained in the action are the gauge field A with $\mathrm{d}A = F$, the dual field strength $\tilde{F}$ and a 2-form field B coupling to flux tubes carrying electric flux under the center of the $SU(N)$. This action possesses an additional 1-form gauge freedom

$$\begin{aligned} B &\to B + \mathrm{d}\lambda \ , \\ A &\to A + N\lambda \ , \\ \tilde{A} &\to \tilde{A} - \frac{\theta}{2\pi}\lambda + k\lambda \ . \end{aligned} \tag{7.11}$$

We proceeded by incorporating Yang-Mills domain walls, these are domain walls interpolating between different quasi-vacua for fixed θ (conveniently $\theta = 0$). This means that in one vacuum monopoles of charge $(0, N)$ condense whereas in the other the condensed particles are dyons of charge $(-1, N)$. This is directly related to the change of the genuine line operators in the spectrum, that is in the dyonic condensate monopoles have to be supplemented by a surface operator. For our specific choice the genuine line operators (in contrast to [21]) correspond to the unconfined charges in the theory and hence give an intuitive interpretation.
It is peculiar that in the presence of domain walls we have to include a boundary term in the action on the worldvolume of the domain wall. This contribution introduces a new

gauge field $\mathcal{A}$ living on the domain wall and coupling to electric flux tubes. The action for $\mathcal{A}$ exhibits the level N Chern-Simons term predicted by other constructions in [68] and [18]

$$S_{\mathcal{V}} = -\frac{iN}{4\pi}\int_{\mathcal{V}} [2\mathcal{A}\wedge B + \mathcal{A}\wedge \mathrm{d}\mathcal{A}] \ . \tag{7.12}$$

With the 1-form gauge transformation

$$\mathcal{A} \to \mathcal{A} - \lambda \ , \tag{7.13}$$

this enables the electric flux tubes to end on the wall. A property as well predicted by string theoretical and supersymmetric constructions, [14] and [17].

Furthermore, the dyonic condensate on one side of the wall leads to a non-vanishing 4-form electric field F_4 of the Chern-Simons current. This contributes a finite amount of energy to the quasi-stable configuration and leads to a confinement of a wall-antiwall pair as expected in the non-supersymmetric setup.

We generalized the construction for Yang-Mills theories to Super-Yang-Mills models. In this framework exact results can be obtained and the mechanism of strings ending on domain walls becomes more natural. We reviewed the derivation of the most relevant configurations.

In Super-Yang-Mills theories the vacuum structure changes dramatically, all the quasi-stable vacua of Yang-Mills theories are degenerate and differ in the phase of the occurrent gaugino condensate. This yields a close relation between the phase and the θ-angle by the chiral anomaly, rendering them non-independent values. In this theory domain walls can either be described by a change of the gluino condensate or a change in the θ-angle by 2π. We chose the latter option. Thus, in Super-Yang-Mills the condensates on both sides of the domain walls are purely magnetic charges. The notion of genuine line operators thus has to be taken with care, because we use the same fields in regimes of different θ (the description of monopoles changes). Nevertheless, bearing this in mind the interpretation of non-local operators as in pure Yang-Mills works out. Moreover, since now only monopoles condense, the 4-form electric field is screened and the domain walls are not confined. This is necessary for their interpretation as BPS states in a supersymmetric setup and can be understood as the reverse Higgsing of the axionic gaugino condensate phase by the Chern-Simons 3-form endowing both of them with a mass.

The worldvolume action is present nonetheless, generating the level N Chern-Simons term for boundary degrees of freedom encoded by $\mathcal{A}$. The flux tubes retain their ability to end on the walls as desired, [13].

The topological theories derived in the thesis hence present an intuitive way of understanding the occurrence of a Chern-Simons term and the mechanism of strings ending on (Super-)Yang-Mills domain walls. Unfortunately, besides the reasoning of an energy connected to a 4-form field strength they do not allow to gain dynamical information about the theories. Only the phases for various processes regarding non-local operators are encoded.

Finally, we outlined two possibilities to investigate the elaborated theory in analogy to other physical models which might enable us to gather dynamical information about the configurations and definitely are a promising way to continue the research on this topic.

Appendix A

Euclidean Action for Yang-Mills Theories

In the main part (chapter 4 and 6) we work in Euclidean spacetime. For this reason in this appendix we want to specify the related notations and the connections to the Minkowskian case, used in other chapters.
In minimal supersymmetric models with Weyl or Majorana fermions it is impossible to find a Euclidean description of the theory. This is due to the fact that in four dimensions one cannot find four purely imaginary γ-matrices satisfying the Euclidean Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \quad . \tag{A.1}$$

Theories that exhibit an extended supersymmetry and only contain Dirac fermions allow for such a Euclidean description.
In the case of minimal supersymmetric models we might nevertheless perform an analytic continuation of the time variable (see [5])

$$t = x_0 \equiv -ix_4 \quad . \tag{A.2}$$

After passing to Euclidean coordinates there is no difference between upper and lowercase indices because they are transformed among each other by δ^μ_ν. In Euclidean spacetime Greek indices take the values $1, 2, 3, 4$ whereas in Minkowskian spacetime they take the values $0, 1, 2, 3$ (Latin indices encode the spatial components $1, 2, 3$ in both cases).
The integral measure of the action transforms accordingly

$$-i\int \mathrm{d}^3x\mathrm{d}t \to -\int \mathrm{d}^3x\mathrm{d}x_4 \tag{A.3}$$

Thus, we identify the Euclidean Lagrangian density and action

$$\begin{aligned} \mathcal{L}_E &= \mathcal{L}|_{t=-ix_4} \\ S_E &= -iS \end{aligned} \tag{A.4}$$

Due to the change of the time coordinate we introduce a new notation for the zeroth component of the gauge field

$$A_0^a = iA_4^a \ . \tag{A.5}$$

Consequently, the field strength tensor changes in the timelike components

$$F_{0j}^a = iF_{4j}^a \ . \tag{A.6}$$

The gauge part of the Euclidean action S_E can be rewritten

$$-S_E = iS = -\int \mathrm{d}^4x \left[\frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{i\theta}{32\pi^2} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a\right], \quad \text{with } \epsilon_{1234} = 1 \ . \tag{A.7}$$

Some of the field components might become purely imaginary in this formulation, but the physical relevant quantities such as the action remain real as necessary.
Immediately evident is that the topological part of the action, namely the θ-term, is purely imaginary and accordingly only contributes a phase in the path integral, as expected.

Appendix B

Dictionary for Differential Forms

In the following chapter we want to summarize the notational conventions for the use of differential forms and the corresponding operators. We mostly follow the conventions of [104] and [37]. It is very convenient to rewrite gauge theories in terms of differential forms in order to avoid an extensive use of indices. We present the notation for a non-Abelian gauge theory with gauge field in the adjoint representation of $SU(N)$ in Euclidean spacetime $\mathbb{R}^4$ (in the sense of appendix A).
For the Euclidean spacetime metric and the flat base manifold $\mathbb{R}^4$ differential forms are written in coefficient representation using the canonical basis of the cotangent bundle $\mathrm{d}x^\mu$ dual to $\frac{\partial}{\partial x_\mu}$ of the tangent bundle. For a general p-form Ω our convention is

$$\Omega = \frac{1}{p!}\Omega_{\mu_1 \dots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \quad , \tag{B.1}$$

with antisymmetric coefficients $\Omega_{\mu_1 \dots \mu_p}$.
The gauge field is regarded as a connection 1-form on a $SU(N)$-principle bundle over $\mathbb{R}^4$

$$A = A^a_\mu T^a \mathrm{d}x^\mu \quad . \tag{B.2}$$

The field strength is rewritten as

$$\begin{aligned} F = \mathrm{d}A + A \wedge A &= \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ \text{with} \quad F_{\mu\nu} &= \left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu\right] T^a \quad , \end{aligned} \tag{B.3}$$

f^{abc} denoting the structure constants of the gauge group and T^a the generators of the algebra $su(N)$ in the fundamental representation.
The Hodge star operator dualizes the field strength

$$* F^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{a\alpha\beta} = \tilde{F}^a_{\mu\nu} \Rightarrow *F = \frac{1}{2}\tilde{F}^a_{\mu\nu} T^a \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \quad . \tag{B.4}$$

In general the Hodge star operator acts on the coefficients of p-forms as

$$(*\Omega)_{\mu_1\dots\mu_{4-p}} = \frac{1}{p!}\epsilon_{\mu_1\dots\mu_4}\Omega^{\mu_{4-p+1}\dots\mu_4} \quad . \tag{B.5}$$

The exterior derivative transforms a p-form into a (p+1)-form and is defined as

$$\mathrm{d}\Omega = \frac{1}{p!}\left(\frac{\partial}{\partial x^\nu}\Omega_{\mu_1\dots\mu_p}\right)\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} \quad . \tag{B.6}$$

Consequently $\mathrm{d}^2 = 0$ due to the symmetry of the partial derivatives.
p-forms are the natural objects to be integrated over a p-dimensional manifold, in particular the canonical volume form is

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \quad , \tag{B.7}$$

and we find the identity

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \epsilon^{\mu\nu\rho\sigma}\omega \quad . \tag{B.8}$$

The integral of a p-form over a p-manifold is

$$\int \Omega = \int \left(\frac{1}{p!}\Omega_{\mu_1\dots\mu_p}\right)\mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} = \int (\Omega_{1\dots p})\,\omega \tag{B.9}$$

With the wedge product of a p-form α and a q-form β, defined as

$$(\alpha \wedge \beta)_{\mu_1\dots\mu_{p+q}} = \frac{(p+q)!}{p!q!}\alpha_{[\mu_1\dots\mu_p}\beta_{\mu_{p+1}\dots\mu_{p+q}]} \quad , \tag{B.10}$$

we have ($\mathrm{Tr}\big(T^aT^b\big) = \frac{1}{2}\delta^{ab}$)

$$\begin{aligned}\mathrm{Tr}(F \wedge *F) &= \left[\frac{1}{2}F^a_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right] \wedge \left[\frac{1}{2} * F^b_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\right]\mathrm{Tr}\big(T^aT^b\big) \\ &= \frac{1}{16}F^a_{\mu\nu}F^{a\alpha\beta}\epsilon_{\rho\sigma\alpha\beta}\epsilon^{\mu\nu\rho\sigma}\omega = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\omega \quad .\end{aligned} \tag{B.11}$$

Therefore, the standard action of Yang-Mills theory can be rewritten in the form

$$-\frac{1}{4g^2}\int \mathrm{d}^4x F^a_{\mu\nu}F^a_{\mu\nu} \to -\frac{1}{g^2}\int \mathrm{Tr}(F \wedge *F) \quad . \tag{B.12}$$

For an Abelian $U(1)$ theory the trace is not necessary and thus

$$-\frac{1}{4g^2}\int \mathrm{d}^4x F_{\mu\nu}F_{\mu\nu} \to -\frac{1}{2g^2}\int \mathrm{Tr}(F \wedge *F) \quad . \tag{B.13}$$

The θ-term as well has a formulation in terms of differential forms

$$\begin{aligned}\mathrm{Tr}(F \wedge F) &= \left[\frac{1}{2}F^a_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right] \wedge \left[\frac{1}{2}F^b_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\right]\mathrm{Tr}\big(T^aT^b\big) \\ &= \frac{1}{8}F^a_{\mu\nu}F^a_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\omega = \frac{1}{4}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}\omega \quad ,\end{aligned} \tag{B.14}$$

which means that

$$\mathcal{L}_\theta = \frac{i\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \to \frac{i\theta}{8\pi^2} \mathrm{Tr}(F \wedge F) \ . \tag{B.15}$$

Which can be written as total derivative

$$F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} = \partial_\mu K_\mu \ . \tag{B.16}$$

In differential forms we have, see [53]

$$\mathrm{Tr}(F \wedge F) = \mathrm{d}\,\mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) \ , \tag{B.17}$$

hence the θ term can be recast into the form

$$\frac{i\theta}{8\pi^2} \mathrm{Tr}(F \wedge F) = \frac{i\theta}{8\pi^2} \mathrm{d}\,\mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) \ . \tag{B.18}$$

Appendix C

Direct Construction of a Dyonic Condensate

In chapter 2 we discussed, why the quasi-stable vacua of Yang-Mills theories, labeled by k, differ in the charge of the condensate. In this appendix we want to carry out a direct construction of the condensation of dyons in the framework of the Abelian-Higgs model following the construction of a monopole condensate in chapter 4.
For a dyonic condensate the phase of the scalar field ϕ in the Abelian-Higgs model is eaten up by a combination of the gauge field A and its dual $\tilde{A}$. With the Lagrange multiplier 3-form h the action is in analogy to equation (3.55) where we directly point out the dual relation between A and $\tilde{A}$

$$S_d = \int \left[h \wedge \left(\mathrm{d}\varphi - m\tilde{A} - \frac{q}{N}A \right) + \frac{i}{2\pi} \mathrm{d}A \wedge \mathrm{d}\tilde{A} \right] . \tag{C.1}$$

Consequently, the dyons in the condensate have a charge $(q/N, m)$. Again we dualize the phase φ to a 2-form field B and integrate out the Lagrange multiplier h

$$\begin{aligned} S_d &= \int \left[h \wedge \left(\mathrm{d}\varphi - m\tilde{A} - \frac{q}{N}A \right) + \frac{i}{2\pi} \mathrm{d}\varphi \wedge \mathrm{d}B + \frac{i}{2\pi} \mathrm{d}A \wedge \mathrm{d}\tilde{A} \right] \\ &\to \int \left[\frac{i}{2\pi} \left(m\tilde{A} + \frac{q}{N}A \right) \wedge \mathrm{d}B + \frac{i}{2\pi} \mathrm{d}A \wedge \mathrm{d}\tilde{A} \right] . \end{aligned} \tag{C.2}$$

Since the spacetime manifold has no boundary, we can integrate the first term by parts and find

$$\begin{aligned} S_d &= \int \left[-\frac{im}{2\pi} \tilde{F} \wedge B - \frac{iq}{2\pi N} F \wedge B + \frac{i}{2\pi} F \wedge \tilde{F} \right] \\ &= \int \left[\frac{i}{2\pi} \tilde{F} \wedge (F - mB) - \frac{iq}{2\pi N} F \wedge B \right] . \end{aligned} \tag{C.3}$$

With B coupling to A and $\tilde{A}$ simultaneously it is not allow to regard $\tilde{F}$ as an independent variable as in chapter 4. Thus, we cannot simply use the equations of motion of $\tilde{F}$, i.e.

$F = mB$. If we forget about this for a second and use the equation of motion we obtain

$$S_d = \int \left[\frac{i}{2\pi} \tilde{F} \wedge (F - mB) - \frac{iqm}{2\pi N} B \wedge B \right] \tag{C.4}$$

For charge $(0, N)$ this exactly reproduces the result in chapter 4. If we switch to a dyon condensate of charge $(1, N)$, the action becomes as well similar to the one we would expect from the construction in chapter 4. The factor of two is probably a consequence of double counting using the equations of motion. Nevertheless, the similarity of the two actions confirms our assumption of differently charged condensates in the quasi-stable vacua of Yang-Mills theories.

Appendix D

SUSY Conventions

In chapter 5 we discuss $\mathcal{N} = 1$ Super-Yang-Mills theories. This appendix clarifies our notational conventions and some mechanisms special for supersymmetric theories, and follows [5].

D.1 Spinorial Notation

In 3+1 dimensions one can choose to work in Majorana or Weyl basis. We choose the latter option and differ between left- and right-handed fermions. In analogy to the literature we adopt the dotted indices notation, e.g. [72]. For left-handed spinors with two complex components we write ψ_α, for their right-handed analogs $\bar{\psi}^{\dot{\alpha}}$. Dotted and undotted indices are raised or lowered with the Levi-Civita symbol $\epsilon_{\dot{\alpha}\dot{\beta}}$ and $\epsilon^{\alpha\beta}$ respectively, with

$$\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = 1 \ . \tag{D.1}$$

The following relation between left- and right handed spinors holds

$$(\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}, \quad (\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}} \ , \tag{D.2}$$

where * denotes complex conjugation. For the Lorentz invariant contraction built from a single chirality we use a shorthand notation

$$\psi^\alpha \chi_\alpha = \psi\chi, \quad \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi} \ . \tag{D.3}$$

In order to contract different chiralities and construct a Lorentz vector we use the Pauli matrices

$$(\sigma^\mu)_{\alpha\dot{\beta}} = (\mathbb{1}, \vec{\sigma})_{\alpha\dot{\beta}}, \quad (\bar{\sigma}^\mu)_{\dot{\beta}\alpha} = (\mathbb{1}, -\vec{\sigma})_{\dot{\beta}\alpha} \ , \tag{D.4}$$

with contractions carrying one Lorentz index

$$\bar{\psi}\bar{\sigma}^\mu\chi, \quad \psi\sigma^\mu\bar{\chi} \ . \tag{D.5}$$

Lorentz vectors as the gauge field A_μ can also be decomposed in spinorial $(\frac{1}{2}, \frac{1}{2})$ notation

$$A_{\alpha\dot{\beta}} = A_\mu(\sigma^\mu)_{\alpha\dot{\beta}}, \quad A^\mu = \frac{1}{2}A_{\alpha\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \ . \tag{D.6}$$

D.2 Superspace and Superfields

A very convenient way to implement the extended Poincaré symmetry in supersymmetric theories is the introduction of superspace. For that we include further fermionic coordinates in form of the Grassmann variables θ_α and $\bar{\theta}^{\dot{\alpha}}$. Superspace is hence parametrized by the collection of the fermionic and the usual spacetime coordinates

$$\{x^\mu, \theta, \bar{\theta}\} \ . \tag{D.7}$$

Elements of the extended Poincaré group in superspace can be written as exponentials of the supersymmetry generators Q, $\bar{Q}$, and the momentum operator P_μ

$$G(x^\mu, \theta, \bar{\theta}) = \exp\left(i\theta Q + i\bar{\theta}\bar{Q} - ix^\mu P_\mu\right) \ . \tag{D.8}$$

The spinorial operators fulfill the anticommutation relation

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2P_{\alpha\dot{\beta}} \ , \tag{D.9}$$

with all other (anti)commutators vanishing. In superspace coordinates one can define the supercharges and superderivatives as, see [5]

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} + \bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \qquad \bar{Q}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - \theta^\alpha\partial_{\alpha\dot{\alpha}}, \tag{D.10}$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\partial_{\alpha\dot{\alpha}} \tag{D.11}$$

The element $G(c^\mu, \epsilon, \bar{\epsilon})$ acts on superspace itself by

$$\begin{aligned} x^\mu &\to x^\mu - i\left(\theta_\alpha\bar{\epsilon}_{\dot{\beta}} + \bar{\theta}_{\dot{\beta}}\epsilon_\alpha\right)(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \ , \\ \theta_\alpha &\to \theta_\alpha + \epsilon_\alpha \ , \\ \bar{\theta}_{\dot{\alpha}} &\to \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}} \ . \end{aligned} \tag{D.12}$$

We can distinguish two invariant subspaces (x_L, θ) and $(x_R, \bar{\theta})$, with

$$\begin{aligned} (x_L)_{\alpha\dot{\beta}} &= x_{\alpha\dot{\beta}} - 2i\theta_\alpha\bar{\theta}_{\dot{\beta}} \ , \\ (x_R)_{\alpha\dot{\beta}} &= x_{\alpha\dot{\beta}} + 2i\theta_\alpha\bar{\theta}_{\dot{\beta}} \ . \end{aligned} \tag{D.13}$$

With this notation one can define superfields, i.e. fields on superspace that summarize the usual fields and their superpartners. The most important superfields for our discussion are the chiral superfield and the vector superfield, see [5]. The chiral superfield reads

$$\Phi(x_L) = \phi(x_L) + \sqrt{2}\theta\psi(x_L) + \theta^2 F(x_L) \ . \tag{D.14}$$

It only depends on the left-chiral subspace and comprises a complex scalar field ϕ, a Weyl spinor ψ, and a non-dynamical auxiliary field F. The antichiral superfield which only depends on the right-chiral subspace is denoted by $\bar{\Phi}$.
The vector superfield V, depends on all superspace coordinates and is real. The gauge transformations are parametrized by the chiral superfield Λ

$$V \to V - i(\Lambda - \bar{\Lambda}) \ . \tag{D.15}$$

This allows us to gauge away some of the components and in the Wess-Zumino gauge the vector superfield reduces to

$$V = -2\theta^{\alpha}\bar{\theta}^{\dot{\beta}}A_{\alpha\dot{\beta}} - 2i\bar{\theta}^{2}\theta\lambda + 2i\theta^{2}\bar{\theta}\bar{\lambda} + \theta^{2}\bar{\theta}^{2}D \ . \tag{D.16}$$

With the real gauge field A, gaugino field λ, and real auxiliary field D.
The kinetic terms of the gauge degrees of freedom are summarized in the supersymmetric field strength chiral superfield

$$W_{\alpha} = i(\lambda_{\alpha} + i\theta_{\alpha}D - \theta^{\beta}F_{\alpha\beta} - i\theta^{2}\partial_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}}) \ . \tag{D.17}$$

D.3 Superinvariant Actions

The superfield formalism simplifies the construction of superinvariant actions considerably. The supertransformation suggests that for an arbitrary product of chiral superfields the θ^2-component transforms as a total derivative, [72]. Therefore, it does not change the action on a spacetime manifold without boundaries. These terms are extracted by integration over the Grassmann variables (equivalent to a differentiation).
Terms of this type are the gauge kinetic term, see chapter 5

$$\frac{1}{4g^{2}}\int \mathrm{d}\theta^{2}W^{\alpha}W_{\alpha} + h.c. \ , \tag{D.18}$$

and the superpotential term

$$\int \mathrm{d}\theta^{2}\mathcal{W} + h.c. \ . \tag{D.19}$$

$\mathcal{W}$ is a holomorphic function of the chiral superfields Φ in the theory. For the theory to be renormalizable in $3+1$ dimensions it should be of power smaller than four in the superfields, [5].
Only the $\theta^2\bar{\theta}^2$-component of a real superfield transforms under a total derivative as well and is another possibility to create superinvariant terms, e.g. the gauge invariant couplings of chiral superfields to the gauge vector multiplet

$$\int \mathrm{d}\theta^{2}\mathrm{d}\bar{\theta}^{2}\bar{\Phi}e^{V}\Phi \ . \tag{D.20}$$

The generalization to non-Abelian gauge fields is straightforward and the relevant terms are used in the main section, see chapter 5.
Of great importance for the investigation the vacua of the supersymmetric theory is

the scalar potential, i.e. the potential for the scalar components. For supersymmetric gauge theories with matter (chiral superfields) it consists of two contributions, the D-term contribution of the vector superfield and the F-terms of the matter sectors Φ_i

$$V = \frac{1}{2g^2} D^2 + F_i^* F_i \ , \tag{D.21}$$

summation over the matter index i is implicit.
For non-Abelian gauge theories this is only modified by the inclusion of a group index for the D-terms

$$V = \frac{1}{2g^2} D^a D^a + F_i^* F_i \ . \tag{D.22}$$

The equation of motion for the D-term yields their typical form for this type of theories

$$D^a = -g^2 \phi_i^* T^a \phi_i \ , \tag{D.23}$$

with ϕ_i the scalar field component of Φ_i. The F-terms are connected to the superpotential by their equation of motion

$$F_i^* = -\left(\frac{\partial \mathcal{W}}{\partial \Phi_i}\right)_{\theta=0} \ . \tag{D.24}$$

A typical non-Abelian gauge theory with matter transforming in the fundamental representation has the gauge and super-invariant action

$$\begin{aligned} S_{\text{SUSY}} = &\int \mathrm{d}^4x \left[\left(\frac{1}{4g^2}\int \mathrm{d}^2\theta\, W^{a\alpha} W^a_\alpha + h.c.\right)\right] \\ + &\int \mathrm{d}^4x \left[\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \bar{\Phi}_i e^V \Phi_i + \left(\int \mathrm{d}^2\theta\, \mathcal{W}(\Phi_i) + h.c.\right)\right] \ . \end{aligned} \tag{D.25}$$

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Declaration of authorship

I hereby declare that the submitted thesis is my own original work. All sources used are acknowledged as references.

Hiermit erkläre ich, die vorliegende Arbeit selbständig verfasst zu haben und keine anderen als die in der Arbeit angegebenen Quellen und Hilfsmittel benutzt zu haben.

Munich, March 27, 2014

Markus Dierigl