Higgs mechanism as an alternative to confinement in Yang-Mills theories

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Statement of authorship

I hereby declare that the thesis I am submitting is entirely my own original work except where otherwise indicated. Any use of the works of any other author is properly acknowledged in the references. I further declare that I have not submitted this thesis at any other institution in order to obtain a degree.

Contents

Abstract

In this thesis, the Higgs mechanism is explored as an alternative to confinement in $SU(N)$ gauge theories. In [1–3], the saturation of the entropy bound by various systems is discovered. Furthermore, this violation of the entropy bound is accompanied by the violation of unitarity of certain scattering processes. Confinement then can be seen as a built-in mechanism of the theory that avoids both violations. However, the same effect could be achieved by other means, for example by generating a mass gap in the theory through the introduction of Higgs fields in different representations. When the fields are introduced in the fundamental representation, we find that no gapless degrees of freedom remain in the theory, providing a consistent alternative to confinement from this point of view. In contrast, with one field in the adjoint representation no such possibility appears to be feasible.

Introduction

The problem of confinement is a long-standing issue in quantum chromodynamics (QCD). This phenomenon, by which at low energies all the observable degrees of freedom are color singlets, has no fully analytic solution and therefore has been an open question since the discovery of assymptotic freedom by Politzer [4] and Gross and Wilczek [5].

In a series of papers by Dvali [1–3], some previously known to be black hole properties are shown to be shared by non-perturbative solutions in known Yang-Mills theories. In particular, some entropy bounds saturated by black holes are also reached by these objects near a certain critical point. Under this approach, confinement is then proposed to serve as a mechanism to avoid the violation of the aforementioned bounds, as well as consequently respecting unitarity in certain scattering processes. This is necessary in order to preserve the consistency of the theory.

Confinement works because it generates a mass gap in the theory, which lifts the degeneracy of microstates associated to a certain configuration and thereby reduces its entropy. However, other mechanisms can be thought of which can serve the same purpose. In particular, introducing Higgs fields which spontaneously break the symmetry can also generate such a mass gap. Therefore, the objective of this work is, after assimilating the concepts of this theoretical framework, investigate whether different Higgs fields in different representations of the symmetry can consistently replace confinement under this criterion.

In Chapter 1, the fundamentals of classical solutions in quantum field theory (QFT) are reviewed. In Chapter 2, the discoveries of [1–3] are explained, with focus on its implications on confinement. In Chapter 3, the original part of the work is shown, where different spontaneous symmetry breaking patterns via the introduction of Higgs fields are investigated. Finally, the conclusions are summarized in the last section.

Chapter 1

Classical solutions in quantum field theory

1.1 Kinks and vortices

A classical solution to a quantum field theory is a solution to the classical equations of motion, i.e. Euler-Lagrange (of the original action). Such a solution is called a soliton if it has a profile with finite spatial size, in other words, it is localized within a region of space.

The lowest-dimensional and therefore easiest kind of soliton is the kink, which arises in 1+1-dimensional scalar field theory. Following [6, 7], consider the action:

$$S = \int d^2x \left[\frac{1}{2}\left(\partial_\nu \varphi\right)\left(\partial^\nu \varphi\right) - V(\varphi)\right]; \quad V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - v^2\right)^2 \tag{1.1}$$

It has the trivial minima $\varphi = \pm v$ (solutions of EOM), which obviously is not a soliton. But we can find a static solution of the field equation that interpolates between both minima, e.g. $\varphi(\pm\infty) = \pm v$. For a static field ($\dot{\varphi} = 0$), the field equations read:

$$\varphi'' = \frac{\partial V}{\partial \varphi} \tag{1.2}$$

This immediately reminds us of the equation of motion in classical mechanics for a potential -V. This is a very useful picture in the one dimensional case in order to inspect for possible non-trivial solutions (see later). In this case, the equation can be rewritten by means of multiplying by φ', which leads to:

$$\varphi' = \pm\sqrt{2V} = \pm\sqrt{\frac{\lambda}{2}}\left(v^2 - \varphi^2\right)$$

The previous equation can be directly integrated to give the solution:

$$\varphi_k(x) = \pm v \tanh\left[\frac{\lambda}{\sqrt{2}}(x - x_0)\right] \tag{1.3}$$

This solution is called a kink or antikink depending on which sign we take, which has to be consistent with our boundary conditions. It is localized in the sense that the region where it deviates from its vacuum values is finite. We can see this via the energy density:

$$a \equiv v\frac{\lambda}{\sqrt{2}}(x-x_0); \quad \varepsilon(x) = \frac{1}{2}(\varphi_k')^2 + V(\varphi_k) = \frac{\lambda v^4}{2}(1-\tanh^2 a)^2$$

It reaches zero (vacuum) when $a \sim 1$, i.e. $|x| \sim \sqrt{\frac{2}{\lambda}}\frac{1}{v}$. This is the "size" of the soliton.

A time-dependent solution can be immediately obtained by applying a Lorentz boost ($x \to x - ut$) to the static solution. We can then interpret it as a moving object and we can compute its (classical) mass by plugging φ_k in the energy functional.

The appearance of a soliton is related to the existence of two degenerate vacua, which constitute the values at infinity of the field configuration. There is also a conserved charge which is not directly related to any symmetry of the Lagrangian (therefore, it can't be derived via Noether theorem). If we define:

$$J_t^\mu = \frac{1}{2v}\epsilon^{\mu\nu}\partial_\nu\varphi$$

We see that it is conserved ($\partial^\mu J_\mu = 0$) by commuting partial derivatives and defines a conserved charge,

$$Q = \int_{-\infty}^{+\infty} dx \; J_t^0 = \frac{1}{2v}\left[\varphi(+\infty) - \varphi(-\infty)\right] = \pm 1 \tag{1.4}$$

which takes positive or negative value for a kink or antikink, respectively. Whenever there is a conserved charge that involves different vacuum values at infinity, there is a connection with topology that will become apparent later and thus we speak of topological solitons, with conserved topological charge. We can expand pertubatively the quantum field around the kink solution $\varphi = \varphi_k + \eta$ and then split the Lagrangian:

$$\mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{quad} + \mathcal{L}_{int} + \mathcal{L}_{ct}$$

where the terms stand, respectively, for the classical Lagrangian, the quadratic part in the perturbation, higher order interactions and the counterterms which will be determined from the trivial vacuum sector.

We then would proceed by diagonalizing the quadratic part of the Lagrangian, just as in regular perturbation theory (where normal modes in Fourier space $f_k(x)$ are plane waves and we obtain the dispersion relation $\omega(k) = \sqrt{k^2+m^2}$):

$$\eta(x,t) = \sum_j f_j(x)e^{-i\omega_j t} \tag{1.5}$$

$$\left[-\frac{d}{dx^2} + V''(\varphi_k(x))\right] f_j(x) = \omega_j^2 f_j(x) \tag{1.6}$$

In the path integral formulation, which will be introduced later, these eigenvalues will appear in the form of a determinant. Let us now consider again the equation of motion (1.2) for φ_k and take the spatial derivative:

$$0 = \frac{d}{dx}\left[-\varphi_k'' + V'(\varphi_k)\right] = \left[-\frac{d^2}{dx^2} + V''(\varphi_k)\right]\varphi_k'$$

Therefore, the spatial derivative of the kink fulfills the eigenvalue equation with $\omega_j = 0$, therefore it is a zero mode.[1] Zero modes are a consequence of a symmetry of the Lagrangian being broken by the soliton and lead to infrared divergences (the determinant as product of eigenvalues is trivially zero) and must be handled separately.

In order to do this, instead of using the corresponding coefficient $c_0(t)$ in the expansion (1.5), we introduce a collective coordinate [8] $z(t)$ in the following way:

$$\varphi(x,t) = \varphi_k(x - z(t)) + 2\,\mathrm{Re}\sum_{j\neq 0} c_j(t) f_j(x - z(t))$$

We'll see how this works in the path integral formalism, where a Jacobian factor will appear as a consequence of this change, but the idea is the same. Let us now turn to what happens when we add fermions to the theory. New terms are added to the Lagrangian in (1.1),

$$\mathcal{L} = \frac{1}{2}\left(\partial_\nu\varphi\right)\left(\partial^\nu\varphi\right) - \frac{\lambda}{4}\left(\varphi^2 - v^2\right)^2 + i\overline{\psi}\partial\!\!\!/\psi - g\overline{\psi}\varphi\psi \tag{1.7}$$

where we used the conventional notations for the Dirac adjoint ($\overline{\psi} \equiv \psi^\dagger\gamma^0$) and the Feynman slash ($\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$). The fermion field equation is then the following Dirac equation:

$$0 = i\partial\!\!\!/\psi - g\varphi\psi$$

If we expand the fermion field as $\psi(x,t) = \sum_j \chi_j(x)e^{-i\omega_j t}$ and set the bosonic field to its background value $\varphi_k(x)$, we obtain an eigenvalue equation for the spinor $\chi_j(x)$:

$$\left[-i\gamma^0\gamma^1 + g\gamma^0\varphi_k(x)\right]\chi_j = \omega_j\chi_j \tag{1.8}$$

An important feature of this equation that will appear also in higher dimensions is the fact that non zero eigenvalues come in pairs (the spinor $-i\gamma_1\chi_j$ has eigenvalue $-\omega_j$). For the same reason, zero modes can be chosen as eigenvalues of $-i\gamma_1$.[2] The equation above can then be integrated:

$$\chi_0 = \exp\left(\mp g\int_0^x \varphi_k(r)dr\right)s_\pm \tag{1.9}$$

[1] Note that this does not depend on the specific form of the potential

[2] Note that this is the 2-dimensional chirality operator, which is important for cosmic strings - kinks in 2+1 dimensions

$s_\pm$ is a constant spinor satisfying $(-i\gamma_1)s_\pm = \pm s_\pm$. We see that only the upper sign produces a normalizable zero mode for the kink (and viceversa for the antikink), therefore we can interpret this as a degeneracy in the fermionic spectrum (the occupation number of this mode can be either 0 or 1, giving a state with the same energy). One way to express this behaviour is to assign a fractional fermion number $\pm 1/2$. This will generalize in higher dimensions as having a 2^N degeneracy of the spectrum, with N being the number of fermion zero modes.

Another relevant case of solitons are the vortices, which occur in 2+1-dimensional spacetime. In this setting the spatial infinity is now connected, therefore if we want different vacuum values at each point (in the same spirit as before, where the soliton interpolated between two distinct vacuum values at infinity), we need a continous family of vacuum states. Thus, consider a U(1) Higgs Lagrangian[3] :

$$\phi(x) = \rho(x)e^{i\alpha(x)} \tag{1.10}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^\dagger\partial_\mu\phi - \frac{\lambda}{4}\left(|\phi|^2 - \frac{\mu^2}{\lambda}\right)^2 \tag{1.11}$$

The action is minimised by $|\phi| = \rho = v \equiv \sqrt{\frac{\mu^2}{\lambda}}$, with arbitrary uniform α, giving a continuous 1-parameter family of vacuum states.

When we construct a soliton solution, i.e. one that interpolates between different vacua at infinity, α will take different values at each radial direction for $r \to \infty$. Therefore, each solution will have a distinctive quantity, namely the winding number:

$$n = \frac{1}{2\pi}\oint_{r\to\infty}(d\vec{l}\cdot\vec{\nabla})\alpha(x) = \lim_{r\to\infty}\frac{1}{2\pi}\int_0^{2\pi} d\theta\frac{d\alpha}{d\theta} \tag{1.12}$$

This number counts how many 2π rotations are completed by the phase $\alpha(x)$ along the circle at spatial infinity. By construction, it is an integer. It will be important as it distinguishes between topologically inequivalent solutions, as we will see later.

Nonetheless, there is a problem with this model. Derrick's theorem[4] implies that we need a gauged symmetry in order for these non-trivial solutions to exist. Therefore to our Lagrangian we must add a gauge kinetic term and a covariant derivative:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu; \quad D_\mu\phi \equiv (\partial_\mu - ieA_\mu)\phi \tag{1.13}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^\dagger D_\mu\phi - \frac{\lambda}{4}\left(|\phi|^2 - \frac{\mu^2}{\lambda}\right)^2 \tag{1.14}$$

[3] We follow again the presentation in [7]

[4] Basically, this theorem demands for the solution to be have minimal energy under coordinate rescaling, see [7] for details.

This Lagrangian is invariant under the gauge transformations:

$$\phi(x) \to e^{ie\Lambda(x)}\phi(x); \quad A_\mu(x) \to A_\mu(x) - \partial_\mu \Lambda(x)$$

Nonetheless, for a solution of certain n any gauge transformation will not change it, since $\oint (\vec{dl} \cdot \vec{\nabla})\Lambda(x) = 0$ for any non-singular $\Lambda(x)$. Therefore it is an invariant that takes integer numbers. Also, we can express it as a magnetic flux which, therefore, is quantized. This feature will also appear in the U(1) monopole.

We will now briefly describe the process of finding an ansatz, leaving the actual calculation for latter and more relevant cases. We want to exploit the symmetries of the theory, but we know that a rotationally invariant solution would have zero winding number. Therefore, non-trivial solutions are "rotationally invariant" in a special sense: any rotation can be countered by an appropiate gauge transformation. In this case: $\phi(r,\theta) = f(r)\exp(i\theta)$ and $A_j = \epsilon_{jk}\hat{x}^k a(r)$, but this reasoning applies in general (monopoles, for example). To find the solution, either we plug our ansatz into the field EOM or we vary the action with respect to our unknowns. The latter is usually the simplest procedure.

Now, we have two zero modes associated to translations, but also gauge zero modes corresponding to gauge transformations. Imposing orthogonality of generic transformations to this physically irrelevant modes leads to the so-called background gauge condition:

$$\partial_j(\delta A_j) - \frac{1}{2}ie(\phi^*\delta\phi - \phi\delta\phi^*) = 0$$

The lessons we have learned from the kink and vortex will be useful for more complicated cases, such as monopoles and instantons.

1.2 Brief recapitulation of topology

Both the kink and vortex were solutions that interpolated between multiple vacua at infinity, which can't be continously deformed into a uniform vacuum solution. This has a natural explanation in the language of homotopy classes. In general, we want to consider a scalar field potential V with degenerate minima[5] which spontaneously break the symmetry group of the Lagrangian G into a subgroup H.
Since the action of $h \in H$ leaves the vacuum invariant, but $g \in G$ does not, we can define equivalence classes of elements in G as $g \sim gh$. They form a coset space G/H which defines the vacuum manifold M (also called moduli space).

The concept of homotopy can be understood as follows: two closed paths $f(t), g(t)$ in

[5] Cfr. the case of the kink, with two degenerate minima $\pm v$ that break the Z_2 symmetry.

a manifold are continously deformable into each other if there exists a continous function $k(s,t)$ such that:

$$k(0,t) = f(t); \quad k(1,t) = g(t)$$
$$k(s,0) = k(s,1) = x_0$$

Then, f and g are homotopic at the point x_0, with k being an homotopy. We can then classify all loops in a manifold in homotopy classes, where two paths are equivalent if they are homotopic to each other. By defining a product of homotopy classes via concatenation of paths, a group structure arises which is named the first homotopy group of the manifold $\pi_1(M, x_0)$. For a connected manifold, it doesn't depend on the point x_0.

There is a natural generalization $\pi_n(M)$ to higher dimensions, which classify maps from S^n to the manifold instead of maps from the circle S^1 (continously deformable to any closed loop) to M. Let us summarize some important results without delving into details, which can be found at the same reference [7]:

$$\pi_1(S^n) = 0; \quad \pi_n(S^n) = \mathbb{Z} \tag{1.15}$$

$$\pi_1(G/H) = \begin{cases} 0 & H \text{ continous} \\ H & H \text{ discrete} \end{cases} \tag{1.16}$$

$$\pi_2(G/H) = \pi_1(H) \qquad \text{G compact and simply connected} \tag{1.17}$$

$$\pi_3(G) = \mathbb{Z} \qquad \text{G compact, simple and connected} \tag{1.18}$$

The existence of topologically stable solitons in $D-1$ spatial dimensions is subject to the condition $\pi_{D-2}(M) \neq \varnothing$, where M is the vacuum manifold. The number of independent solutions will then be given by the dimension of the group.

Let's see how this works for the $U(1)$ vortex in 2 spatial dimensions. $U(1)$ can be written as the quotient group $\mathbb{R}/\mathbb{Z}$, whose first homotopy group is $\mathbb{Z}$ (cfr. eq. (1.16)).

Finally, note that the winding number (1.12) classifies the maps $S^1 \to S^1$. That quantity can be easily generalized to higher dimensions and we can interpret it as a topological charge, since it will be stable against continous deformations of the field configuration (see above).

1.3 't Hooft-Polyakov monopole

We explore now topological soliton configurations in 3 space dimensions. Owing to the previous discussion, these will arise in theories where the moduli space M has non-trivial $\pi_2(M)$. The spatial infinity is now a 2-sphere, characterized by the polar and azimuthal angles (θ, φ). The simplest framework of spontaneous symmetry breaking (SSB) in which these conditions are met is the Georgi-Glashow model, where the symmetry of the Lagrangian $SO(3)$ is broken down to $SO(2) \cong U(1)$ by the vacuum expectation value (VEV) of a triplet scalar field ϕ (i.e. a Lorentz scalar transforming in the fundamental representation of $SO(3)$). Since the vacuum manifold is S^2, the relevant homotopy group is $\pi_2(S^2) = \mathbb{Z}$.

As we said before, there is a topological charge classifying different field configurations:

$$N = \frac{1}{8\pi}\epsilon^{ijk}\int dS^i \hat{\phi}\cdot\partial_j\hat{\phi}\times\partial_k\hat{\phi} \qquad \hat{\phi}\equiv\frac{\phi}{|\phi|}$$

We use the usual inner and vector product notation for the triplet ϕ. The Lagrangian density is the same as (1.14), with the following modifications:

$$D_\mu\phi \equiv \partial_\mu\phi + eA_\mu\times\phi \tag{1.19}$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + eA_\mu\times A_\nu \tag{1.20}$$

The potential (and therefore the action) is minimised when $|\phi|^2 = v^2 \equiv \mu^2/\lambda$, a convenient parametrization is $\phi = (0, 0, v)$. Any finite energy configuration fulfills the well-known quantisation condition of the magnetic charge associated to the magnetic field $B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}$:

$$Q_M = \int dS^i\hat{\phi}\cdot B_i = N\left(\frac{4\pi}{e}\right)$$

When trying to find a solution, one must take in account the “spherical symmetry”, i.e. any spatial rotation should potentially be undone by an appropriate $SU(2)$ gauge transformation. A convenient gauge choice (hedgehog gauge) for this solution of the form [9, 10]:

$$\phi^a = \hat{r}^a h(r); \quad A_0^a = 0; \quad A_i^a = \epsilon^{aim}\hat{r}^m\left[\frac{1-u(r)}{er}\right] \tag{1.21}$$

Whereas we cannot solve for the unknown functions (proceeding identically as in the vortex case), we can learn their assymptotic behaviour, knowing that they approach the vacuum solution at infinity. In particular, we see that $u(r) \sim \exp(evr)$, so that the radius of the monopole is $1/ev$.

The zero modes of the solution (which are called bosonic, in opposition to those arising when fermions are included on the theory) proceed very similarly to the vortex case. We will study bosonic zero modes in more detail when considering the BPST instanton, which is more relevant for our purposes.

Finally, let’s see how fermionic zero modes arise in the monopole background, originally found by Jackiw and Rebbi [11]. Fermions can be added transforming in some representation of an internal “flavor” group, in this case one may choose $SU(2)$ so that it coincides with the original symmetry of the theory.[6] For such a fermion multiplet ψ_n, the fermion part of the Lagrangian reads -cfr. (1.7)-:

$$\mathcal{L}_f = i\overline{\psi_n}\gamma^\mu (D_\mu \psi)_n - g\overline{\psi_m} T^a_{mn} \psi_n \phi^a \tag{1.22}$$

T^a_{mn} are the generators in the appropiate representation of the theory. Note that the coupling to the scalar field can be realized in this way since $N = N_f$. In hedgehog gauge (1.21), $\psi_n = (\psi_n^+, \psi_n^-)$ and using as basis for the Dirac matrices ($S^j = \sigma^j/2$):

$$\gamma^0 = \begin{pmatrix} 0 & -i\mathbb{I} \\ +i\mathbb{I} & 0 \end{pmatrix} \quad \gamma^j = \begin{pmatrix} -2iS^j & 0 \\ 0 & 2iS^j \end{pmatrix}$$

the static Dirac equation can be written as:

$$0 = \left(\delta_{mn} \vec{S} \cdot \vec{\nabla} + i \left[\frac{1-u(r)}{r} \right] T^a_{mn} (S \times \hat{r})^a \mp \frac{g}{2} h(r) \hat{r}^a T^a_{mn} \right) \psi_n^\pm \tag{1.23}$$

In the fixed monopole background, the conserved fermion angular momentum is $\vec{J} = \vec{L} + \vec{S} + \vec{T}$. For an isodoublet spinor ($T = 1/2$) in the channel $J = L = 0$, the wave function is spherically symmetric and the spin and isospin are anticorrelated, this means $\psi^\pm_{\alpha m} = f_\pm(r)\epsilon_{\alpha m}$, where we now introduce the spinor index α, implicit before. Plugging this into (1.23), one obtains a normalizable solution for $f_\pm(r)$.

$$0 = \left[\frac{\partial}{\partial r} - \frac{1-u(r)}{r} \pm \frac{g}{2} h(r) \right] f_\pm$$

This procedure can be also done for other isospin channels, and it was found by Jackiw and Rebbi that there are 2 additional zero modes for the isotriplet channel [11]. The purpose of this calculation was to illustrate how fermionic zero modes are found in a non-trivial background, which will also be important in the instanton case.

1.4 Quantum mechanics and path integral formulation

In this section we will use as main reference [12]. First, recall the path integral formulation for amplitudes in (one-dimensional) quantum mechanics for some boundary conditions $A = (-t_0/2, x_i)$; $B = (t_0/2, x_f)$:

$$\langle x_f | e^{-iHt_0} | x_i \rangle = N \int_A^B \mathcal{D}x \; e^{iS[x(t)]} \tag{1.24}$$

where the measure $\mathcal{D}x$ includes all functions $x(t)$ compatible with the boundary conditions, and H is the Hamiltonian operator.

[6] We say that the scalar field transforms in a $SU(N)$ representation, whereas the fermions do so in a $SU(N_f)$. The situation above can be shortly described as $N = N_f$.

We can directly extract the energy of the ground state from this amplitude via the following procedure: expand in energy eigenstates, perform a Wick rotation and take the limit of Euclidean time to infinity:

$$\langle x_f|e^{-iHt_0}|x_i\rangle = \sum_{n,m} \langle x_f|n\rangle\, e^{-iE_nt_0}\delta_{n,m}\, \langle m|x_i\rangle \xrightarrow[\tau_0\to\infty]{\tau\equiv it} e^{-E_0\tau_0}\psi_0(x_f)\psi_0^*(x_i)[1+\mathcal{O}(e^{-\tau_0})]$$

This means that when we compute the amplitude via the path integral in equation (1.24) in the limit $\tau_0 \to \infty$, the energy of the ground state E_0 will appear in the exponent.

If we now define the euclidean action S_E as

$$S_E = -iS[x(t)] = \int_{-\tau_0/2}^{\tau_0/2} \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right] d\tau \tag{1.25}$$

it coincides with the "energy" of the field configuration in Euclidean time (in the path integral it will appear as a factor e^{-S_E}). From now on, when dealing with an Euclidean signature, we refer to the Euclidean action only as S.

In order to parametrize all possible functions in the measure $\mathcal{D}x$, we consider,

$$x(\tau) = X(\tau) + \sum_n c_n x_n(\tau) \tag{1.26}$$

where $X(\tau)$ is some function fulfilling the boundary conditions (A, B) and $x_n(\tau)$ form a complete set of orthonormal functions that vanish on the endpoints. In this way, the variation of the coefficients as independent variables will enter the measure up to a numerical coefficient, which by convenience is chosen as:

$$\mathcal{D}x \equiv \prod_n \frac{dc_n}{\sqrt{2\pi}} \tag{1.27}$$

Since the path integral can't be computed directly, some method of approximation is needed. For the cases at hand, where the action is large, the method of steepest descent (called quasiclassical approximation in [13]) is an appropiate one. By choosing $X(\tau)$ to extremalize the action, we can write:

$$\left.\frac{\delta S}{\delta x}\right|_{X(\tau)} = 0$$

$$S \simeq S_0 + S_{quad} \equiv S[X(\tau)] + \frac{1}{2}\left.\frac{\delta^2 S}{\delta x^2}\right|_{X(\tau)} (x-X)^2 \tag{1.28}$$

In terms of the potential and kinetic energy, the extremal trajectory $X(\tau)$ satisfies,

$$\frac{d^2X}{d\tau^2} = V'(X) \tag{1.29}$$

which is clearly the Euler-Lagrange equation[7] for a potential $-V$ (note that the definition of the euclidean action (1.25) changes the sign of the kinetic term).

Now, the complete set of functions $\{x_n\}$ in (1.26) can be chosen to be any particular one, namely the set of eigenfunctions of the previous equation analogously as in (1.6):

$$\left[-\frac{d^2}{d\tau^2} + V''(X)\right] x_n(\tau) = \varepsilon_n x_n(\tau) \tag{1.30}$$

Then, taking in account (1.26) and (1.28), S reduces to:

$$S = S_0 + \frac{1}{2}\sum_{n,m} c_n c_m \varepsilon_m \underbrace{\int_{-\tau_0/2}^{\tau_0/2} d\tau\ x_n(\tau) x_m(\tau)}_{\delta_{n,m}} = S_0 + \frac{1}{2}\sum_n \varepsilon_n c_n^2$$

This, together with the definition of $\mathcal{D}x$ (1.27) and the gaussian integral

$$\int_{-\infty}^{\infty} dc \exp\left(-\frac{1}{2}\varepsilon c^2\right) = \frac{\sqrt{2\pi}}{\sqrt{\varepsilon}}$$

leads to the following result for the path integral:

$$\langle x_f|e^{-iHt_0}|x_i\rangle = N\int_A^B \mathcal{D}x\ e^{-S_E} = Ne^{-S_0}\prod_n \epsilon_n^{-1/2} \equiv Ne^{-S_0}\left(\det\left[-\frac{d}{d\tau^2} + V''(X)\right]\right)^{-1/2} \tag{1.31}$$

where we introduced a formal determinant to describe the product of the eigenvalues of the differential operator.

As a first example, we take the quantum mechanical version of the kink, a potential $V(x) = \lambda(x^2 - \eta^2)^2$. This potential[8] has two minima separated by a finite wall at $x = 0$ (Fig. 1.1). This is a well-known problem in quantum mechanics, namely tunneling through a finite barrier. We know its solution from the WKB approximation [14].

We will now apply the machinery described above. First, we need to find the (finite action) solutions $X(\tau)$ to (1.29) such that they contribute to the amplitude $\langle \pm\eta|e^{-H\tau_0}|\mp\eta\rangle$ (this fixes the boundary conditions). By inverting the potential we can intuitively grasp the existence of such a solution: starting from one minimum at $t = -\tau_0/2$, rolling through the

[7] Cfr. (1.2)

[8] We also define $\omega^2 = 8\lambda\eta^2$.

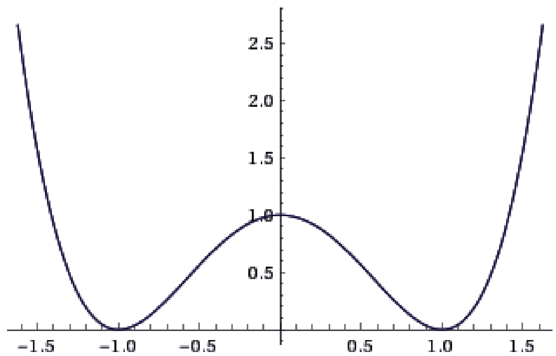

Figure 1.1: Plot of $V(x) = \lambda(x^2 - \eta^2)^2$ with $\lambda = \eta = 1$.

valley to the other one and landing there at $t = \tau_0/2$. It is no surprise that, just as in the case of the kink (1.3), the solution is given by:

$$X(\tau) = \pm\eta \tanh \frac{\omega(\tau - \tau_C)}{2} \tag{1.32}$$

This non-trivial solution to the classical EOM receives the name of (anti-)instanton, depending on the sign fixed by the boundary conditions. Plugging this solution back into the action, one obtains $S_0 = \omega^3/12\lambda$. This will be the exponent in our formula (1.31).

What remains now is to compute the preexponential factor in (1.31). The presence of an arbitrary parameter τ_C is a consequence of spontaneous symmetry breaking of time translations and therefore, there will be a zero mode in the computation in the determinant. We rewrite (1.31) as:

$$\langle -\eta | e^{-H\tau_0} | +\eta \rangle = N e^{-S_0} \left(\frac{\det\left[-\frac{d}{d\tau^2} + V''(X)\right]}{\det\left[-\frac{d^2}{d\tau^2} + \omega^2\right]} \right)^{-1/2} \det\left[-\frac{d^2}{d\tau^2} + \omega^2\right]^{-1/2} \tag{1.33}$$

Plugging (1.32) into the eigenvalue equation (1.30) gives:

$$\left[-\frac{d^2}{d\tau^2} + \left(\omega^2 - \frac{3\omega^2}{2\cosh^2(\omega\tau/2)}\right)\right] x_n(\tau) = \varepsilon_n x_n(\tau) \tag{1.34}$$

Recall that the boundary conditions are $x_0(\pm\tau_0/2) = 0$. This equation turns out to have two discrete levels and a continous region of the spectrum. As said before, there exists a mode x_0 [9] with $\varepsilon_0 = 0$, which will make the gaussian integral $\propto \varepsilon^{-1/2}$ diverge. In order to avoid this divergence, instead of integrating over c_0 (the corresponding coefficient in

[9] It can be easily checked that $x_0 = \sqrt{\frac{3\omega}{8}} \frac{1}{\cosh^2(\frac{\omega(\tau-\tau_C)}{2})}$. Note that $x_0 \propto \frac{dX(\tau)}{d\tau_C}$.

the expansion (1.26)) we change variables to τ_C, which is called the collective coordinate. The latter is defined as the arbitrary parameter that appears in the solution and breaks a symmetry of the Lagrangian (translation, scale,...). The Jacobian can be easily deduced:

$$\begin{cases} \Delta x(\tau) & = x_0(\tau)\Delta c_0 \\ \Delta x(\tau) & = \dfrac{dx}{d\tau_C}\Delta\tau_C = \dfrac{-dx_0}{d\tau}\Delta\tau_C = -\sqrt{S_0}x_0(\tau)\Delta\tau_C \end{cases} \Longrightarrow dc_0 = \sqrt{S_0}d\tau_C$$

The result will then include an integral over τ_C instead of the previous gaussian integral for the zero mode. There is also a second discrete eigenvalue $\varepsilon_1 = 3\omega^2/4$ and a continous region. Since the eigenvalues of the harmonic oscillator that were introduced in (1.33) are $\varepsilon'_n = \omega^2 + p_n$ with $p_n = \pi n/\tau_0$, allowing to the boundary conditions we can rewrite the quotient of determinants as:

$$\frac{\det\left[-\frac{d}{d\tau^2} + V''(X)\right]}{\det\left[-\frac{d^2}{d\tau^2} + \omega^2\right]} = \sqrt{S_0}\; d\tau_C \left(\frac{3\omega^2}{4}\right)^{-1/2} \left(\prod_{n=2}^{\infty} \frac{\omega^2 + \tilde{p}_n^2}{\omega^2 + p_n^2}\right)^{-1/2}$$

$$\tilde{p}_n \equiv \frac{\pi n + \delta_p}{\tau_0} \qquad\qquad e^{i\delta_p} = \frac{1 + ip/\omega}{1 - ip/\omega}\frac{1 + 2ip/\omega}{1 - 2ip/\omega}$$

The dominant contribution comes from $n >> \omega\tau_0$, where $\omega^2 + p_n^2 = \omega^2 + \tilde{p}_n^2$. Rewriting the product as an exponential of a sum of logarithms and going to the continuum limit gives the result:[10]

$$\prod_{n=2}^{\infty} \frac{\omega^2 + \tilde{p}_n^2}{\omega^2 + p_n^2} = \exp\left[\sum_n \ln\frac{\omega^2 + \tilde{p}_n^2}{\omega^2 + p_n^2}\right] \simeq \exp\left[\frac{1}{\pi}\int_0^{\infty} \frac{2pdp}{p^2 + \omega^2}\right] \underset{\text{IBP}}{=}$$
$$\exp\left[\frac{-2}{\pi^2}\int_0^{\infty} dy \left(\frac{1}{1+y^2} + \frac{2}{1+4y^2}\right)\ln(1+y^2)\right] = \frac{1}{9}$$

Only the calculation of the determinant of the harmonic oscillator (including the normalization constant N) remains, however, it is quite straightforward and can be found in the same reference [12]. Plugging everything into (1.33), in the limit of large τ_0 we get:

$$\langle -\eta|e^{-H\tau_0}| + \eta\rangle = \frac{\omega}{\pi}\int_{-\tau_0/2}^{\tau_0/2} d\tau_C \sqrt{6\omega S_0}e^{-S_0 - \omega\tau_0/2} \tag{1.35}$$

This formula contains the contribution of one instanton centered at τ_C to the amplitude connecting $-\eta$ to η. Nonetheless, there could be several other instantons centered at different places, and therefore the total amplitude will have to include the contribution of n instantons. If the distance between the centers is small enough (this assumption is valid since there is a free parameter λ that we can adjust), the total action is simply the sum of

[10] In the calculation, the variable change $y = p/\omega$ is used.

the individual action and for a certain number of instantons n:

$$d \equiv \sqrt{\frac{6S_0}{\pi}} e^{-S_0}$$

$$\frac{\omega}{\pi} \int_{-\tau_0/2}^{\tau_0/2} d\tau_C \sqrt{6\omega S_0} e^{-S_0-\omega\tau_0/2} \longrightarrow \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_0/2} \frac{d^n \omega^n}{n!} \left[\int_{-\tau_0/2}^{\tau_0/2} d\tau_C \right]^n$$

The $n!$ accounts for permutations of identical objects (alternatively, one can consider a particular order in space which fixes the limit of each integral, with identical result). Now, since one instanton represents a solution starting in $-\eta$ and ending in η, only an odd number of solutions will contribute to amplitudes with different endpoints such as (1.33). Therefore, we find:

$$\langle -\eta | e^{-H\tau_0} | +\eta \rangle = \sum_{n \text{ odd}} \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_0/2} \frac{(\omega\tau_0 d)^n}{n!} = \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_0/2} \sinh(\omega\tau_0 d) \tag{1.36}$$

We can finally retrieve the energy of the ground state, looking at the exponent in the limit $\tau_0 \to \infty$:

$$e^{-\omega\tau_0/2} \sinh(\omega\tau_0 d) \longrightarrow e^{-\omega\tau_0/2+\omega\tau_0 d} \equiv e^{-E_0\tau_0} \Rightarrow E_0 = \frac{\omega}{2} \left[1 - \sqrt{\frac{2\omega^3}{\pi\lambda}} e^{-\omega^3/12\lambda} \right] \tag{1.37}$$

This result agrees with the energy of the symmetric ground state for this potential with usual methods in quantum mechanics. Here concludes this section where the instanton calculation is presented in a simple example in quantum mechanics (which is analog to the kink), before shifting to instantons in Yang-Mills theories.

1.5 BPST Instantons

We'll adress now the topic of how instantons arise in Yang-Mills theories [15], still with [12] as main reference. Other useful sources are [16–19]. But first, let us introduce its Euclidean formulation as a convenient way to write the theory in its Wick-rotated version. Minkowski coordinates (x_0, x_1, x_2, x_3) are replaced by Euclidean coordinates $(x_1, x_2, x_3, x_4 \equiv ix_0)$, denoted by an index $\mu = 1, ..., 4$. Correspondingly, we define an Euclidean gauge vector field:

$$\begin{cases} A_4 \equiv -iA_0^{Mink} \\ A_k \equiv -A_k^{Mink} \quad\quad k = 1, 2, 3 \end{cases}$$

The corresponding covariant derivative is analog to its Minkowski version. For completeness, let us note that the gauge field strength remains the same except for $G_{0k}^a = -iG_{4k}^a$. Gamma matrices are also redefined:

$$\begin{cases} \gamma_4 = \gamma_0^{Mink} \\ \gamma_k = -i\gamma_k^{Mink} \quad k = 1, 2, 3 \end{cases}$$

Lastly, new anticommuting fermion fields are defined, such that the Lorentz scalar is $\psi^\dagger\psi$ [11], i.e. $\psi = \psi^{Mink}$ and $\bar{\psi} \equiv i\bar{\psi}^{Mink}$. The Euclidean action (defined in (1.25)) for a Yang-Mills theory with Dirac fermions is:

$$S = \int d^4x \left[\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \bar{\psi}(-i\gamma_\mu D_\mu - iM)\psi\right] \tag{1.38}$$

where a is the color index[12] and a sum over repeated indices is implied, since upper and lower indices are equivalent for the Euclidean flat metric $\delta_{\mu\nu}$.

We consider now pure SU(2) Yang-Mills and, in the spirit of the previous sections, we try to find the condition for having topologically non-trivial field configurations with finite action. By looking at (1.38), it is apparent that $G^a_{\mu\nu}$ must decrease faster than $1/x^2$ at large distances. Thus in the Laurent expansion for the gauge field A^a_μ no $1/x$ term will appear. Therefore, up to terms of inverse quadratic order in x, A^a_μ at large distances will be gauge equivalent to zero, i.e. of pure gauge form:

$$A^a_\mu \sim iS\partial_\mu S^\dagger \tag{1.39}$$

The problem then reduces to classifying the matrices $S(x) \in SU(2)$ (the gauge field belongs to the adjoint representation). The corresponding topological quantity is the Maurer-Cartan form, which integrated over a path gives what is known as the Pontryagin index [17], which is the higher dimensional analog of the winding number, i.e. it counts how many times is the manifold covered by a certain mapping. In terms of the gauge fields, it can be expressed in terms of the Chern-Simons form k_μ or in terms of the product of the gauge field strength and its dual $\tilde{G}_{\mu\nu}$:

$$\tilde{G}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\gamma\delta}G_{\gamma\delta} \qquad\qquad k_\mu \equiv 2\varepsilon_{\mu\nu\gamma\delta}\left(A^a_\nu\partial_\gamma A^a_\delta + \frac{g}{3}\varepsilon^{abc}A^a_\nu A^b_\gamma A^c_\delta\right)$$

$$n = \frac{-1}{24\pi^2}\int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\theta_3\ \varepsilon_{ijk}\mathrm{tr}(S\partial_i S^{-1}S\partial_j S^{-1}S\partial_k S^{-1}) = \tag{1.40}$$

$$= \frac{g^2}{32\pi^2}\int_{S^3_\infty} d^3S\hat{r}_\mu k_\mu = \frac{g^2}{32\pi^2}\int d^4x G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \tag{1.41}$$

Therefore, we are left with configurations of the form (1.39) with matrices $S \in SU(2)$ and classified by (1.40). Recalling that $\exp(i\theta\vec{v}\cdot\vec{\tau}) = \cos\theta\mathbb{I} + i\sin\theta(\vec{v}\cdot\vec{\tau})$, an example of $n = 1$ can be written as:

$$S = \frac{x_4\mathbb{I} + i\vec{x}\cdot\vec{\tau}}{\sqrt{x^2}} \tag{1.42}$$

[11] See [20]; Chapter 5, section 19.

[12] Here, it labels the adjoint representation of the SU(N) group.

How many independent configurations (i.e. matrices) are there? We need to look at the relevant homotopy group, in this case where we are considering 4-dimensional space and $SU(2)$ gauge theory, by formula (1.18):

$$\pi_3(SU(2)) = \mathbb{Z}$$

We then conclude that n uniquely labels the different configurations and any other $n = 1$ configuration can be related via an appropriate gauge transformation. Furthermore, due to the cyclicity properties of the Maurer-Cartan integral $n(SS') = n(S) + n(S')$ the configuration with arbitrary n can be found by simply exponentiating (1.42). Plugging it into (1.39), we recover the known form of the BPST instanton with use of the 't Hooft symbols introduced in [21]:

$$A^a_\mu \xrightarrow{x\to\infty} \frac{2}{g}\eta_{a\mu\nu}\frac{x_v}{x^2} \tag{1.43}$$

$$\overset{(-)}{\eta}_{a\mu\nu} \equiv \begin{cases} \varepsilon_{a\mu\nu} & \mu,\nu \neq 4 \\ (-)\delta_{a\nu} & \mu = 4 \\ -\delta_{a\mu} & \nu = 4 \\ 0 & \mu = \nu = 4 \end{cases}$$

The corresponding anti-instanton solution ($n < 0$) is found by changing η to $\bar{\eta}$. There is an alternative way of deriving the full BPST solution directly from the Yang-Mills action:

$$S = \int d^4x \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} = \int d^4x \left[\pm\frac{1}{4} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} + \frac{1}{8}(G^a_{\mu\nu} \mp \tilde{G^a_{\mu\nu}})^2\right] = \frac{8\pi^2}{g^2}|n| + \frac{1}{8}\int d^4x (G^a_{\mu\nu} \mp \tilde{G^a_{\mu\nu}})^2 \tag{1.44}$$

It is immediately seen that, for a certain value of $n > 0$, the minimum of the action is acquired by a self-dual $G^a_{\mu\nu} = \tilde{G^a_{\mu\nu}}$ field configuration, whereas for $n < 0$ an anti-self-dual $G^a_{\mu\nu} = -\tilde{G^a_{\mu\nu}}$ field configuration is required[13]. Solving the self-duality equations leads to the solution:

$$A^a_\mu = \frac{2}{g}\eta_{a\mu\nu}\frac{(x - x_0)_v}{(x - x_0)^2 + \rho^2} \tag{1.45}$$

where ρ and x_0 appear as arbitrary constants of integration, signaling the breaking of scale and translational invariance, respectively. A gauge equivalent way of writing this solution is the so-called 'singular gauge", in which a solution of n instantons (parallel in $SU(2)$ isospace orientation) can be compactly written as:

$$A^a_\mu = \frac{-1}{g}\bar{\eta}_{a\mu\nu}\partial_\nu \ln W(x); \quad W(x) = 1 + \sum_{i=1}^{n} \frac{\rho_i^2}{(x - x_i)^2} \tag{1.46}$$

A nice explanation of why $SU(2)$ works as a gauge group for these solutions comes in the previous spirit of (1.21), where we tried to find a solution for which a spatial rotation

[13] The Bianchi identity $D_\mu \tilde{G}_{\mu\nu} = 0$ guarantees the fulfillment of the EOM $D_\mu G_{\mu\nu} = 0$.

could be undone by an appropriate constant gauge transformation. In our case, on one hand there is a residual gauge invariance corresponding to constant $SU(2)$ gauge transformations ($S \to SU^\dagger$) plus a global isotopic SU(2) invariance corresponding to transformations of the form $S \to US$. On the other hand, the rotation group of 4-dimensional Euclidean space is also $SU(2) \times SU(2)$, under which the coordinates transform in the representation $(1/2, 1/2)$, i.e. S, given by (1.42), transforms as $USU^\dagger$. Thus, any spatial rotation on the BPST instanton can be undone by an appropriate global transformation.

As in the quantum mechanical example, we want now to examine the vacuum-vacuum transition mediated by the instanton. We then expand the gauge field around the instanton background (1.45) ($A_\mu = A_\mu^{inst} + a_\mu$) and the quadratic expansion for the action reads:

$$S = \underbrace{\frac{8\pi^2}{g^2}}_{S_0} + \frac{1}{2}\int d^4x \ a_\mu^a L_{\mu\nu}^{ab} a_\nu^b; \quad L_{\mu\nu}^{ab} = D^2\delta^{ab}\delta_{\mu\nu} - D_\mu D_\nu \delta^{ab} - 2g\varepsilon^{abc}G_{\mu\nu}^c \tag{1.47}$$

To this we must add a gauge fixing term $(\Delta L)_{\mu\nu}^{ab}$ and a kinetic term L_{gh} for the scalar, anticommuting Faddeev-Popov ghosts. By making use of the Grassmann integration for the ghosts and the usual gaussian integration for the quantum fluctuations, the transition amplitude can be written:

$$\langle 0|e^{-HT}|0\rangle \equiv \langle 0|0_T\rangle = [\det(L + \Delta L)]^{-1/2} \det L_{gh} e^{-S_0} \tag{1.48}$$

A Pauli-Villars (PV) regularisation scheme is usually preferred over dimensional regularisation, since having 4 dimensions is crucial to attain the instanton solution (as seen in our homotopy discussion). This amounts to regularise the determinants introducing a new mass scale M:

$$\langle 0|0_T\rangle = \left[\frac{\det(L + \Delta L + M^2)}{\det(L + \Delta L)}\right]^{1/2} \frac{\det L_{gh}}{\det(L_{gh} + M^2)} e^{-S_0}$$

Furthermore, we divide by the perturbation theory probability to normalize to 1 the vacuum state [21].

The determinant will include both zero modes and positive frequency modes. The first are handled via collective coordinates, which we can identify through the (spontaneously broken) symmetries: translation, scale and (iso)rotation.[14] Therefore, we have 8 collective coordinates: 4 due to translations (x_0), 1 due to scale (ρ) and 3 due to (iso)space orientation (θ, ϕ, ψ for $SU(2)$ orientation). The Jacobian is already known from the QM example ($\sqrt{S_0}$) and we have to take in account the factors of M from PV regularisation, both arise once per collective coordinate:

$$\int d^4x_0 \ \rho^3 \ d\rho \ \sin\theta \ d\theta \ d\phi \ d\psi \ (M\sqrt{S_0})^8$$

[14] Recall that for $SU(2)$ there is an equivalence between the group of spatial rotations and global gauge transformations

The collective coordinate formalism was first introduced in [8] and used by t'Hooft in his celebrated calculation [21]. For the treatment of the gauge zero modes, see [22].

Using this, we can rewrite the amplitude as:

$$\frac{\langle 0|0_T\rangle_{\rm ins}}{\langle 0|0_T\rangle_{\rm pert}} = C\int \frac{d^4x_0\ d\rho}{\rho^5}\left(\frac{8\pi^2}{g^2}\right)^4 \exp\left[-\frac{8\pi^2}{g^2} + 8\ln M\rho + \phi_1\right] \tag{1.49}$$

where C is a numerical constant and ϕ_1 accounts for the contribution of the positive frequency modes (to be dealt with later). Notice that the action S_0 has been replaced by its explicit value (1.47) and the logarithmic term appearing in the exponential comes just from reassembling the powers of M and ρ. The convenience of this form will be seen immediately.

For the positive frequency modes, usual perturbation theory can be used. It is highly instructive to look up t'Hooft's calculation [21], where he obtains the result for an arbitrary number of fields of different isospin by using the special properties of the eigenvalue equation. On the other hand, [12] uses a simpler approach by computing a single Feynman diagram. Both ways lead to the same inevitable result: as a consequence of the renormalizability of the theory, the coupling constant is substituted by the *running* coupling constant. In our case (pure $SU(2)$), at one loop this is determined by the Gell-Mann-Low beta function [23]:

$$\frac{8\pi^2}{g^2} \to \frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g^2(\rho_0)} + \frac{22}{3}\ln\frac{\rho_0}{\rho}$$

The relevant scale is set by the regulator cutoff $\rho_0 = 1/M$ and therefore the substitution in (1.49) is simply:

$$\frac{\langle 0|0_T\rangle_{\rm ins}}{\langle 0|0_T\rangle_{\rm pert}} = C\int \frac{d^4x_0\ d\rho}{\rho^5}\left(\frac{8\pi^2}{g_0^2}\right)^4 \exp\left[-\frac{8\pi^2}{g_0^2} + \frac{22}{3}\ln M\rho\right] \tag{1.50}$$

where $g_0 = g^2(\rho_0)$. In order to replace consistently the coupling constant in the exponential prefactor, we need to resort to two-loop order [12].

In order to work in larger gauge groups, such as in QCD, we can use one of the $SU(2)$ subgroups to embed the BPST solution (1.45) in the following way: $A_\mu^{inst} \equiv A_\mu^a T^a$, where T^a are the generators of the subgroup fulfilling $\mathrm{tr} T^a T^b = c/2\ \delta^{ab}$, $c \geq 1$. The instanton number is then amplified $n_{inst} = cn$. For example, for the case of $SU(3)$ there are two subgroups, one with $\lambda_1/2, \lambda_2/2, \lambda_3/2$ as generators and $c = 1$ and another with $\lambda_2, \lambda_5, \lambda_7$ and $c = 4$.[15] These are not fundamentally different, but the latter one represents a symmetric configuration of multiple (in this case, 4) $c = 1$ instantons[16]. With regards to the previous calculations, two modifications are needed. First, when computing the zero modes there is an additional little group (leaves the instanton solution unchanged) which leads to a total

[15] It can be easily checked that they fulfill the $\mathfrak{su}(2)$ algebra.

[16] Recall that all BPST solutions are uniquely classified by n.

of $4N$ zero modes. A very interesting discussion about this matter in terms of the index theorem can be found in [7, 24]. Accordingly, the positive-frequency modes will have an additional contribution of $4(N-2)$ vector fields corresponding to the new gauge bosons. Finally, when the integral over the gauge orientations is performed, the embedding volume has to be taken in account [22].

Finally, when we include fermions in the theory, they appear as well in the path integral. By virtue of Grassmann integration, it will render the determinant of the quadratic fermion operator, instead of the inverse square root of it. Namely, for Dirac fermions the contribution will amount to the product of eigenvalues of the Dirac operator (shifted by the mass):

$$Z_F = \int D\psi D\bar{\psi} e^{-S_F} = \det(i\not{D} + m)$$

Due to the properties of the Dirac matrices and just as in the case of the kink, non-zero eigenvalues come in pairs of opposite chirality and in the limit of massless fermions, there exist fermionic zero modes. Let us first compute these explicitly in an instanton background. The Dirac equation reads,

$$i\gamma_\mu D_\mu u_0(x) = 0 \xrightarrow{u_0=(\chi_L,\chi_R)^T} \begin{cases} \sigma_\mu^- D_\mu \chi_R = 0 \\ \sigma_\mu^+ D_\mu \chi_L = 0 \end{cases}$$

where $\sigma_\mu^\pm \equiv (\vec{\sigma}, \mp i\mathbb{I})$ and the covariant derivative includes the BPST gauge field (1.45). We can either multiply by $\sigma_\nu^\pm D_\nu$ respectively [20] or solve directly the first degree differential equation [7]. In both cases we need first to infer the index structure, which is fixed by the requirement of invariance under a combined $SU(2)$ gauge transformation and rotation, i.e. $\chi_R \propto \varepsilon_{\alpha k}$. Another way to see this is that by the first mentioned procedure, the equation for χ_R is:

$$\left[-D_\mu^2 + 4\vec{\sigma}\cdot\vec{\tau}\frac{\rho^2}{x^2+\rho^2}\right]\chi_R = 0$$

The positive definiteness of the operator $-D_\mu^2$ implies automatically that the scalar product of generators $\vec{\sigma}\cdot\vec{\tau}$ must be negative. Anticorrelated spin and color fulfill this requirement, since $(\vec{\sigma}+\vec{\tau})\chi = 0$ implies $\vec{\sigma}\cdot\vec{\tau} = -\sum_a \tau_a^2 = -3$.

The coordinate dependence now can be solved; in the second case (the simplest one to solve explicitly), the first order differential equation is:

$$h'(x^2) = -\frac{3}{2}\frac{h}{x^2+\rho^2} \rightarrow h(x^2) \propto (x^2+\rho^2)^{-3/2}$$

which leads to the (normalised) zero mode of the form:

$$u_0(x) = \frac{1}{\pi}\frac{\rho}{[(x-x_0)^2+\rho^2]^{3/2}}\begin{pmatrix} 0 \\ \varepsilon_{\alpha k}\end{pmatrix} \tag{1.51}$$

From this, we see that the number of zero modes will scale as the number of flavors, since each of them will possess such a solution. Also note that these zero modes are localized, just as the instanton itself.

To conclude this section, let us briefly discuss the physical consequences of these fermionic zero modes. Consider the familiar setting of $SU(N_C)$ gauge theory with fermions transforming in the fundamental representation of this color group and additionally related by a global flavor symmetry $SU(N_F)$, under which they also transform in the fundamental representation. The well-known chiral anomaly leads to a conserved current being anomalous at the quantum level. In particular, the chiral current $j_5^\mu \equiv \bar{\psi}\gamma^5\psi$ fulfills the equation:

$$\partial_\mu j_5^\mu = \frac{g^2}{16\pi^2} N_F \partial_\mu k^\mu$$

where k^μ is the Chern-Simons form defined above (1.40). Not surprisingly, at the level of conserved charges, j_5^μ leads to the difference of right- and left-handed fermions, whereas the right-hand side leads to the instanton (winding) number. This implies that chiral symmetry is explicitly broken by instantons even in the massless limit for fermions, leading to a non-conservation of the corresponding $U(1)_A$ symmetry,[17] which provides a solution of the famous $U(1)$ problem (large mass of η' meson).

[17] Non-conservation of chirality in the presence of fermionic zero modes is quite a general result (in even spacetime dimensions), for example in cosmic strings or domain walls [7].

Chapter 2

The role of the entropy bound

Now that we acquired an understanding for these essentially non-perturbative objects that are classical solutions in QFT, we want to examine their properties in a closer way. In particular, we will review [1–3], where it is argued that at the point where perturbative unitarity is saturated, three entropy bounds are fulfilled simultaneously.

First, we introduce the Bekenstein entropy bound [25]. In three spatial dimensions this bound takes the form: [1]

$$S_{\max} = 2\pi MR \sim MR \tag{2.1}$$

Originally motivated by black hole thermodynamics, this bound relates the maximal entropy of a physical system to its energy and radius. This bound has been studied extensively in this context (see for example [26], where (2.1) is generalized for n dimensions). Although black holes do saturate this bound, it is independent of the presence of gravity.

On the other hand, the celebrated Bekenstein-Hawking black hole entropy [27] is proportional to the area (in d spatial dimensions) of the enclosing sphere:

$$S_{\max} \sim (Rf)^{d-1} \tag{2.2}$$

In this relation f is the relevant energy (inverse length) scale, in gravity being $f = M_P$. The origin of this area law has been also object of intense study, one of the most popular approaches being AdS/CFT correspondence [28]. With regards to the microscopic origin of black hole entropy, the string-theoretic construction of Strominger and Vafa [29] stands out.

Recently [1–3], it has been shown that both (2.1) and (2.2) are not exclusive to black holes, but arise as entropy bounds in other contexts. In particular, different settings provide a different scale f (2.2). On one hand, this scale works as symmetry breaking order parameter and on the other hand, it determines the couplings of the Goldstone (gapless) modes ocurring as a result of sponteanous breaking of internal symmetries. It is the presence

[1] In the remaining of this section, coefficients of order one will be ubiquitously dropped in order to keep track of the scaling of the relevant quantities more easily.

of these modes that endow the object with a large entropy, as we will see later. In the monopole case, for example, it will be the Higgs VEV ($f = v$).

Furthermore, an independent third entropy bound is proposed, namely the inverse coupling bound, which relates the maximal entropy to the running coupling constant:

$$S_{\max} \sim \left.\frac{1}{\alpha(q)}\right|_{q=1/R} \tag{2.3}$$

Here, α is the coupling of the relevant long range interaction, which typically will be the coupling among the Goldstones, and it is evaluated at the the momentum transfer $q = 1/R$. These systems at the critical point present a particular feature in the sense that they have a single fundamental scale that defines their properties, which has a resemblance to the quantum portrait of black holes in [30].

The observation made in [1, 2] is that self-sustained objects -such as solitons- saturate all three bounds simultaneously at the critical point:

$$S_{\max} \sim MR \sim \frac{1}{\alpha} \sim (Rf)^{d-1} \tag{2.4}$$

Furthermore, it is shown in [3] that the saturation of the bound is intrinsically linked to saturation of unitarity of certain processes. It is also discovered that all three bounds do not stand in equal footing, since by imposing unitarity in $2 \to n$ particle scattering amplitudes the Bekenstein bound is found to be less stringent than the rest.

Our first step is then to see how these entropy bounds are saturated. We will not review the 't Hooft-Polyakov monopole [1], since the soliton/instanton case [2] is perhaps more interesting in the sense that it generalizes the concept of entropy to an instanton (a virtual process, not an object). In both cases, the arguments run along similar lines.

2.1 Monopoles and solitons

We start considering the following Lagrangian in 5 dimensions with Minkowski signature (the usual Einstein convention applies) [2]:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{g^2}{4}\sigma^4$$

Because we are in 5-dimensional spacetime, the coupling g^2 is dimensionful. The corresponding dimensionless four-point coupling will scale with energy $\alpha = g^2 E$. Therefore we see that the unitarity of the amplitude will inevitably be lost at energies $\Lambda \sim 1/g^2$. This is a characteristic feature of non-renormalizable theories, such as the Proca Lagrangian.

The corresponding equation of motion admits a static, spherically symmetric solution [31], namely:

$$\sigma(x) = \frac{\sqrt{8}}{g}\frac{R}{x^2 + R^2} \tag{2.5}$$

It can be easily checked that $f(r) = R/(r^2 + R^2)$, with R an arbitrary integration constant, fulfills the radial EOM $(r^3 f')' = 8r^3 f^3$. The mass of this configuration (soliton) is given by:

$$M_{\text{sol}} = \int d^4x \left[\frac{1}{2}(\nabla\sigma)^2 - \frac{g^2}{4}\sigma^4\right] = \frac{8\pi^2}{3g^2} \tag{2.6}$$

Note that (2.6) does not depend on R, in accordance with the scale invariance of the theory (in fact, it is conformally invariant [31]). In fact, this solution breaks both translation and dilatation invariance, which will have associated their corresponding Goldstone bosons. Therefore, they produce a family of degenerate solutions, which, if they were to be indistinguishable to an observer, would endow this object with an entropy. However, since we want to study its upper bound for the entropy, we need to endow this solution with a parametrically large[2] entropy. One way to achieve this is to introduce a "flavor" group $SO(N)$, under which the fields σ_β transform in the fundamental representation. The Lagrangian now becomes:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma_\beta)(\partial^\mu\sigma_\beta) + \frac{g^2}{4}(\sigma_\beta\sigma_\beta)^2 \tag{2.7}$$

Now, an important consideration that we will need to keep in mind for the rest of this work: the introduction of N will impose a unitarity constraint on the coupling, namely $\alpha \lesssim 1/N$. For any amplitude, the N fields will run in the loop times the dimensionless coupling have to be less than one if unitarity is to be conserved. This, in turn, defines a new cutoff $\Lambda \sim (g^2 N)^{-1}$ above which processes will violate unitarity.

Now the previous solution is trivially generalized to the N components in σ_β, i.e.:

$$\sigma_\beta(x) = \frac{\sqrt{8}}{g}\frac{R}{x^2 + R^2}\frac{a_\beta}{\sqrt{N_\sigma}} \tag{2.8}$$

with the constraint $\sum_{\beta=1}^{N} |a_\beta|^2 = N_\sigma$. This normalization constant N_σ is irrelevant in the classical theory, but as we will see later, is crucially important in the quantum theory.

As mentioned before, this produces a huge degeneracy of the classical solution, which we can understand in two ways. First, the number of vacua has been increased by means of embedding the soliton in $SO(N)$, i.e. there is now a moduli space (vacuum manifold) parametrized by $SO(N)$ transformations which do not leave this configuration invariant.

[2] It is then natural that, in the following, we work in the large N regime.

It is easy to see that this moduli space will be $SO(N)/SO(N-1) = S^{N-1}$, for instance, by choosing a convenient parametrization in which only one component is non-zero.

An alternative language to express this degeneracy is the language of Goldstone modes: the VEV of σ_α breaks the flavor group $SO(N)$ down to $SO(N-1)$ and, as a consequence of Goldstone theorem, produces the corresponding massless bosons. Since the solution is localized (i.e. tends to the trivial vacuum at large distances), these gapless modes are localized at the core of the symmetry-breaking configuration. Thus, the excitations of these modes contribute to the micro-state degeneracy of the object (and not free particles over the whole spacetime), i.e. its entropy.

Classically, this degeneracy is infinite (the number of solutions is in one-to-one correspondence with the number of points in the vacuum manifold). If we regard a_α as the VEV of quantum creation and annihilation operators $[\hat{a}_\beta, \hat{a}^\dagger_\gamma] = \delta_{\beta\gamma}$, its effective Hamiltonian just enforces the constraint given above via a Lagrange multiplier X:

$$H = X\left(\sum_{\beta=1}^{N} \hat{a}^\dagger_\beta \hat{a}_\beta - N_\sigma\right) \tag{2.9}$$

The degenerate eigenstates of the Hamiltonian can be expressed as $|n_1...n_N\rangle$, with each slot describing the occupation number of each creation/annihilation operator. The total number of states is equivalent to the classic problem of distributing N_σ balls in $N-1$ lines and thus, given by the combinatoric coefficient:

$$n_{\rm st} = \binom{N_\sigma + N - 1}{N_\sigma} \tag{2.10}$$

It is quite clear that the degeneracy grows with N, and therefore the associated entropy. If we assume that $N_\sigma \sim N$, using Stirling approximation $n! \sim \sqrt{2\pi n}\; n^n e^{-n}$ [32, 33], the entropy then scales as:

$$S \sim \ln n_{\rm st} \sim \underbrace{\frac{64\pi^2}{g^2} R \ln 2}_{\propto\, N} \approx 1.3\frac{4\pi}{3}(MR) \tag{2.11}$$

We can now determine N_σ scaling from the bound state description, i.e. describing the solitonic solution as a composite state made up of several (soft) quanta in a mean field approximation [30, 34]. Each quantum is an excitation of the gapless modes ($\hat{a}^\dagger_\beta$, $\hat{a}_\beta$), in total we have N_σ. This bound state has a size R and an attractive interaction dictated by the coupling g^2. The kinetic energy of each quantum scales as $1/R$,[3] which has to be balanced against the potential energy well created by the rest of the quanta[4] $\sim N_\sigma g^2/R^2$, i.e.:

$$N_\sigma \sim \frac{R}{g^2} \qquad N \sim \frac{\Lambda^{-1}}{g^2} \tag{2.12}$$

[3] cfr. (2.3) and comment below.

[4] Recall that Coulomb interaction in five dimensions is proportional to R^{-2}

Since the coupling g^2 is the same, the unitarity bound of N applies for N_σ as well, thus $N \sim N_\sigma$. We then see that this imposes a minimal size R for the soliton in terms of the cutoff of the theory:

$$R = \Lambda^{-1} \longleftrightarrow E = \Lambda = R^{-1} \tag{2.13}$$

It only remains to find the scale f, the order parameter of the SSB. This breaking is dictated by the VEV of the sigma field, which at the origin (maximum) is $\sigma(0) \propto (gR)^{-1}$. From here[5] we can extract an inverse length scale $f = (gR)^{-2/3}$. Now we are equipped to compare the different entropy bounds at the unitarity limit (2.13):

$$S_{\text{sol}} \sim N \sim (\Lambda g^2)^{-1} \tag{2.14}$$

$$S_{\text{Bekenstein}} \sim M_{\text{sol}} R \sim (\Lambda g^2)^{-1} \tag{2.15}$$

$$S_{\text{coupling}} \sim \left.\frac{1}{\alpha(q)}\right|_{q=R^{-1}=\Lambda} \sim (\Lambda g^2)^{-1} \tag{2.16}$$

$$S_{\text{area}} \sim (Rf)^3 \sim (\Lambda g^2)^{-1} \tag{2.17}$$

Therefore, we see that in the limit imposed by unitarity, the entropy reaches its maximal value, which in turn coincides with all three bounds (2.1), (2.2), (2.3).

If we consider now a gauge soliton in pure Yang-Mills, we retrieve the BPST solution (1.45), now in 5 dimensions. Here, a correspondence between a 5-dimensional soliton and a 4-dimensional instanton is established. Sufficient is for us the intuitive idea: a soliton tunneling through the 4-dimensional brane is seen by a 4 dimensional observer as an instanton [35, 36]. Thus, the dimensionless coupling is still $\alpha = g^2 E$ and by a similar calculation as before, its mass is independent of the scale, position and orientation $M \sim 1/g^2$ [2]. Similarly as before, we can enlarge the moduli space by embedding this configuration (originally in $SU(2)$) into $SU(N)$. We can define the analog of the t'Hooft coupling [37]:

$$\lambda_t \equiv E\frac{g^2 N}{8\pi^2} \lesssim 1$$

As discussed in the previous chapter, this embedding leads to a total of $4N$ (bosonic) zero modes. The previous discussion follows through, just replacing N_{mon} by N_σ, with the same scaling R/g^2. We can consrruct an ansatz from the topological invariant (1.40) that has the same scaling (here we note the field strength as $F_{\mu\nu}$):

$$N_{\text{mon}} = R\int d^4x\; F\tilde{F} = \frac{32\pi^2 R}{g^2} = \frac{4N}{\lambda_t} \tag{2.18}$$

Similarly as before, we obtain the number of states as combinatoric coefficient:

$$n_{\text{st}} = \binom{N_{\text{mon}} + 4N}{N_{\text{mon}}} \tag{2.19}$$

[5] Recall that we are in 5 spacetime dimensions, therefore the mass (inverse length) dimension of σ is deduced from the kinetic term $5 = 2d + 2$.

Going to the unitarity limit $\lambda_t = 1$ (with (2.13)), we obtain this expression for the entropy [2]:

$$S \sim 8 \ln 2 \; (MR) \simeq 1.3 \left(\frac{4\pi}{3} MR \right) \tag{2.20}$$

We see that this expression saturates the Bekenstein bound. Similarly, one can deduce the area law form [2]. If we instead populated this solution with fermion zero modes, as in (1.51), all the previous arguments exactly follow through: these are localized gapless solutions that scale with the (arbitrary large) number of fermions. Therefore we conclude that this entropy bound saturation happens independently of the statistics of the zero modes.

Now, as mentioned before, we can connect this 5-dimensional soliton with its 4-dimensional instanton counterpart [35]. The solution is unchanged apart from the trivial modifications in dimensionality. As a consequence, the dimensionless t'Hooft coupling is now:

$$\lambda_t = \frac{g^2 N}{8\pi^2}$$

We can directly assign the "entropy" of this instanton by using formula (2.19) and, due to the change in dimensionality, substituting N_{mon} by $N_{\text{inst}} = 32\pi^2/g^2$.

One must now identify the physical meaning of the "entropy bound violation" by an instanton. First, as this solution lives in Euclidean space (see section 1.5), there is no notion of energy. Therefore, the Bekenstein bound is ill-defined. Also, if we perform the same calculation as in (2.20), due to the absence of the E factor, we now get,

$$S \sim \frac{64\pi^2 \ln 2}{g^2} \tag{2.21}$$

where the coupling is evaluated at the scale $\Lambda = R^{-1}$. We then retrieve the inverse coupling bound in 4 dimensions, while the Bekenstein bound is unapplicable. This fact hints to the hierarchy between the bounds that is discovered in [3]. The area law is also retrieved (notice that the only available scale of inverse length is $1/R$ and therefore R will not appear). In both cases (soliton and instanton), saturation of the entropy bounds occur when their respective t'Hooft couplings are of order one.

It is inevitable to establish a connection to black holes. It is well known that they fulfill both the area law and the Bekenstein bound and, as shown in e.g. [30], its entropy also can be written as an inverse coupling bound with $\alpha_{\text{gr}} = (RM_P)^{2-d}$, d being the number of spacetime dimensions ($d-1$ spatial dimensions).

As a conclusion of this part, we see that on the one hand non-perturbative objects at the critical scale such as solitons or instantons (also monopoles, baryons $[1, 2]$) and on the other hand black holes share a number of properties, which therefore give a strong hint that these are not specific to gravity or black holes themselves, but must be connected to more fundamental aspects of QFT such as unitarity and assymptotic freedom.[6]

2.1.1 Application of the entropy bounds to $SU(N_C)$

Consider a $SU(N_C)$ theory, with coupling g and N flavors, with each flavor consisting of a pair of left- and right-handed Weyl fermions which transform in the fundamental representation (quarks). In the massless limit, this theory possesses an additional global (chiral) symmetry $U(N)_L \otimes U(N)_R = SU(N)_L \otimes SU(N)_R \otimes U(1)_V \otimes U(1)_A$, which includes the anomalous $U(1)_A$ mentioned in Sec. 1.5. To make the connection with what we dealt before, we will work in the t'Hooft (planar) limit [37]: $N_C \to \infty$, $g \to 0$, $g^2 N_C$= finite, whilst keeping the QCD scale Λ fixed.

Furthermore, it is of common acceptance that the chiral symmetry is broken: $SU(N)_L \otimes SU(N)_R \to SU(N)_F$ and the relevant degrees of freedom are color singlets. In particular, after the quark condensate $\langle\bar{\psi}\psi\rangle$ acquires a VEV, the Goldstone bosons of this breaking (pions) conform the low energy EFT, with decay constant $f_\pi \sim \sqrt{N}\Lambda$. This plays the role of the scale in (2.2), very much like the Higgs VEV.

T'Hooft [37] pioneered in this regime the $1/N_C$ expansion, in which diagrams can be classified by powers of $1/N_C$. This alternative pertubative series gives an insight on the non-perturbative effects on the usual coupling constant expansion. Later, Witten [34] proposed a model of baryons described as a bound state of N_C heavy quarks in the mean field approximation. Similarly to the reasoning of (2.12), the mass and radius scalings of such a state are deduced:

$$\begin{aligned} M_B &= N_C(m+T) + \frac{1}{2}N_C^2\,\frac{V}{N_C} \sim N_C\Lambda \\ R_B &\sim \Lambda^{-1} \end{aligned}$$

where m is the rest mass of a quark, T its average kinetic energy and V the strength of the potential (which is given by interactions of order $1/N_C$). The important fact is that the mass of the baryon scales with N_C, whereas the radius does not. Alternatively, he shows that in this limit this description is totally equivalent as the skyrmion one [38], which is a soliton of pions. Later, when we deal with scattering amplitudes, we will encounter again this equivalence of descriptions between a classical solution and a multi-particle bound state.

Consider now a baryon of spin s. By the same logic as in (2.10) and taking in account the additional spin degeneracy, it is inmediate to identify the number of degenerate microstates:

[6] Renormalizability is discarded since neither gravity nor the 5 dimensional soliton Lagrangian are renormalizable.

$$n_{\rm st} = (2s+1)\binom{N_C+N-1}{N_C} \tag{2.22}$$

Again, the bound for the entropy can be saturated for $N \sim N_C$ in the unitarity limit $g^2 N_C \sim 1$ and it acquires yet again the Bekenstein and area law form:

$$n_{\rm st} \sim 2^{2N_C} \to S_{\rm max} \sim N_C \sim M_B R_B \sim (R_B f_\pi)^2 \tag{2.23}$$

The question that arises now is, can confinement be a succesful mechanism to avoid the violation of this bound? Following [1], consider only 1 flavor. The combinatoric coefficient then provides no degeneracy and the entropy is solely given by the spin degeneracy, for spin $s = N_C/2$ it is:

$$S_B \sim \ln N_C \tag{2.24}$$

Comparing with the maximum allowed by the bound (2.23), as long as N is below the threshold of perturbative unitarity ($g^2 N \lesssim 1$, see before), the production of color singlets as assymptotic states effectively prevents the violation of the entropy bound.

Conversely, if we allowed for assymptotic colored states, both quantities change. In order to keep assymptotic freedom safe, we assume $N_C >> N >> 1$.[7] Now, the bound will be fixed by the mass relation (now a state is made up of $N \neq N_C$ quarks) $M = N/R$. Correspondingly, the Bekenstein bound on its entropy will be:

$$S_{\rm max} \sim MR \sim N \tag{2.25}$$

The actual entropy of this bound state is calculated analogously as before, distributing N quarks over N_C colors:

$$n_{\rm st} \sim (2s+1)\binom{N_C+N-1}{N_C} \sim \binom{N_C}{N} \sim (N_C/N)^N \to S_B \sim N \ln\frac{N_C}{N} \tag{2.26}$$

We are led to a situation in which $S_B/S_{\rm max} >> 1$. Thus, we conclude that allowing for assymptotic colored states will eventually lead to a violation of the entropy bound and confinement can act as a mechanism to avoid this violation [1].

2.2 Scattering amplitudes

In this section we want to review [3]. Due to its extension, we will not be able to go through all the results, especially those related to black holes. However, it is highly instructive to read the article in full, since all the ramifications stem from the same fundamental concepts.

[7] Antisymmetry in color still holds, only that it will not render a singlet state since $N \neq N_C$.

In this article, scattering processes of the form $2 \to n$ are analyzed. A correspondence between the saturation of the entropy bounds (2.2), (2.3) and the saturation of unitarity of the aforementioned scattering processes with $n = \alpha^{-1}$ and momentum transfer $q = R^{-1}$. A hierarchy between the bounds is found, since the Bekenstein bound (2.1) is respected in some examples where (2.2) and (2.3) are violated. However, these imply a violation of unitarity and in a consistent theory all three are simultaneously satisfied.

At the saturation point, these amplitudes can be thought of describing a classical object, here named saturon. Normally, the associated cross-section is exponentially supressed [39], but here this effect is compensated by the huge degeneracy of classically equivalent states. This provides the connection between both approaches.

The consequence of this on $SU(N)$ gauge theories is that, through this relation to the corresponding scattering amplitude (here, gluons), it is inferred that avoiding entropy bound violation at the IR scale requires some mechanism, which in the case of pure glue could only be confinement.

Other results include the difficulty of actually producing a saturon in a collider ("kinematic window of opportunity") and more on the connection with black holes and their interpretation of a saturated bound state of soft gravitons, which as said before, we will not cover.

With respect to the previous section, there is a slight change of notation. For example, comparing with (2.26) the following replacement is performed: $(N_C, N) \to (N, n)$. Furthermore, two additional couplings are defined:

$$\lambda_t \equiv \alpha N \qquad \text{t'Hooft coupling} \tag{2.27}$$

$$\lambda_c \equiv \alpha n \qquad \text{collective coupling} \tag{2.28}$$

where α is the coupling of the interaction term in the Lagrangian $\alpha = g^2/4\pi$. From the usual number eigenstates (coherent states) of a certain quantum field, which have a definite momentum, we can construct a superposition of states with different momentum which is sharply centered around $q = R^{-1}$. At weak coupling (t'Hooft limit), the classical evolution of such a state is valid for a sufficiently long time[8] and thus, we can consider this state as an n-particle state, each with momentum $q = R^{-1}$. The total energy is then:

$$E \sim nq = \frac{n}{R} \tag{2.29}$$

Now, when will be this configuration self-sustained? As in (2.12), we can perform the energy balance between the kinetic energy of each quark and the potential energy created by the

[8]Very much like the wave packet in quantum mechanics, where phase and group velocity differ and this leads to the spread of a localised configuration. If they are sufficiently similar, the classical description (particle) is valid.

rest. This leads to the condition:

$$\lambda_c = \alpha n \sim 1 \to n \sim \alpha^{-1} \tag{2.30}$$

At this point, the energy (2.29) of the configuration (soliton) will be:

$$E \sim \frac{q}{\alpha} = (\alpha R)^{-1} \tag{2.31}$$

This new notation allows us to rewrite the number of (classically indistinguishable) microstates (2.10):

$$n_{\rm st} = \binom{n+N-1}{N} \sim \left[\left(1+\frac{\lambda_t}{\lambda_c}\right)^{\lambda_c}\left(1+\frac{\lambda_c}{\lambda_t}\right)^{\lambda_t}\right]^{1/\alpha} \tag{2.32}$$

where again we retain only the exponential terms, since the parameters n, N are arbitrarily large.[9] At the critical value of the collective coupling $\lambda_c = 1$, we see that this will saturate the inverse coupling entropy bound (2.3) when:

$$\left[(1+\lambda_t)\left(1+\frac{1}{\lambda_t}\right)^{\lambda_t}\right]^{1/\alpha} \sim e \Longleftrightarrow \lambda_t \approx 0.54 \sim 1 \tag{2.33}$$

We recover again the relation of [1,2] in which the entropy bound and unitarity are simultaneously saturated (couplings are of order 1). Similarly, the area-law (2.2) and Bekenstein (2.1) are recovered [3].

We want to examine now the mentioned $2 \to n$ processes, where the saturation of unitarity can occur. One must clearly discriminate between unphysical saturation, which can be eliminated by the resummation of different diagrams eliminates and physical, which implies a sum over states in the cross-section.

Our final state in its multiparticle description is made up of a high number of consituents n with characteristic momentum $q = R^{-1}$. The perturbative expansion has a naive factorial growth, but this expansion cannot be trusted for $\lambda_c \gtrsim 1$ (where terms start to increase order by order in α). Non-perturbatively, the production of such a state is exponentially suppressed (see Appendix A of [3]):

$$\sigma_{2\to n} \lesssim n!\, n^{-n} \sim e^{-n} \tag{2.34}$$

This cross-section is associated to the production of a single microstate (in our language). Now, when we introduce the large internal group, a lot of different microstates are classically equivalent. Therefore, the production of this classical object is obtained by summing over all microstates, or equivalently, multiplying (2.34) by the number of states. At the critical

[9] Note that the corrections are $\mathcal{O}\left(\frac{\ln N}{N}\right)$, since $\lambda' \sim (N^d \lambda^N)^{1/N} = \lambda e^{d \ln N/N} \approx \lambda\left(1 + d\frac{\ln N}{N}\right)$

point, $n = \alpha^{-1}$, the total cross-section is:

$$\sigma = \sigma_{2\to n} e^{S} \sim e^{S-1/\alpha} \tag{2.35}$$

This formula shows the following statement: when the entropy saturates the bound (2.3), the cross-section saturates unitarity (becomes order 1). The (running) t'Hooft coupling fulfills the same relation (2.33).

This object which saturates unitarity of production cross-section and entropy bound simultaneously here receives the name of saturon. At the scale q which is fixed by $\lambda_t(q) \sim 1$, it has mass and radius:

$$R \sim q^{-1} \tag{2.36}$$

$$M \sim nq \sim (\alpha R)^{-1} \tag{2.37}$$

Again, it can be shown that the area-law (2.2) and Bekenstein bound (2.1) are saturated. The corresponding scale f is given by the maximal field value, which can be deduced from the gradient:

$$E \sim \int d^4x\ (\nabla\phi)^2 \sim R^3\ (\nabla\phi)^2 = (\alpha R)^{-1} \Longleftrightarrow f \sim R\ \nabla\phi \sim \frac{1}{R\sqrt{\alpha}}$$

Therefore, since a classical observer cannot resolve the different degenerate microstates due to the internal symmetry (the coupling is arbitrarily small), the production of any of these will be interpreted as the production of the same classical state. This lifts the exponential suppression of each microstate and at the critical point, the cross-section is saturated.

We learned that the saturation of the entropy bound is controlled by the 't Hooft and collective couplings λ_c, λ_t. The next question that can be studied is what happens if the theory is deformed beyond the saturation point. Since λ_c is a parameter associated only to the state (and not the theory), we can fix it to a certain value (in particular, to 1, since we are interested in the regime where this classical state is a saturon) and study the limit $\lambda \to \infty$. From (2.32), the entropy of the classical state behaves as:

$$S \sim \frac{1}{\alpha}(1 + \ln \lambda_t) \tag{2.38}$$

and therefore, the entropy bound (2.3) is violated for sufficiently large λ_t. Consequently, the associated cross-section violates unitarity. This cannot happen in a consistent theory and therefore we would like to find a dynamical mechanism that forbids the theory to enter in this dangerous domain of parameter space. As mentioned before, confinement in $SU(N)$ gauge theories plays such a role [2].

2.3 Implications on confinement

We consider pure $SU(N)$ Yang Mills (no fermions), which is asymptotically free [23] (running coupling vanishes at infinite energies). As before, we work in the 't Hooft limit [37].

Let us first briefly recall the problem of confinement. Quantum chromodynamics is known to become strongly interacting towards the IR and behave as a free theory at assymptotically high energies [4,5]. This knowledge, together with experimental observation, leads to the common acceptance that at low energies, color charges are confined (i.e. all observable particles have zero color charge). This extrapolates as well to any $SU(N_C)$ theory, where the behaviour of the coupling is determined by the beta function. Let us for reference display the one-loop beta function for complex scalars (Higgs) and Weyl fermions in the fundamental representation [40]:

$$\beta(g) = \frac{\partial g}{\partial(\log q)} = \left(-\frac{11}{3}N_C + \frac{1}{6}N_{\text{Higgs}} + \frac{2}{3}N_F\right)\frac{g^3}{16\pi^2} \tag{2.39}$$

Now, in [1] it is suggested that this mechanism could be understood as a preventive mechanism against the violation of the entropy and unitarity bounds. It has been previously shown that at the critical point the entropy of a baryon of $N \sim N_C$ quarks saturated the entropy bound. The same will happen when we consider glueballs (colorless bound states of gluons), since now the role of N is played by N_C itself. However, we want to discuss what happens if the theory enters the regime $\lambda_t \gtrsim 1$.

By following the previous reasoning, increasing λ_t will lead to violation of unitarity in $2 \to n$ gluon scattering amplitudes (2.35), due to the enhancement of classically equivalent microstates. We can regard the increase of λ_t in two ways: first, if we fix the scale q, this amounts to changing the relation between $\alpha(q)$ and N, i.e. changing the theory. On the other hand, we can fix N and therefore this amounts to running the scale towards the infrared (what increases now is $\alpha(q)$, recall that the beta function is negative), so that we compare processes at different energy scales. Note that, since we compare them at the critical point $\lambda_c = \alpha n = 1$, they will have different number of gluons in the final state.

Therefore, it is inevitable that these scattering amplitudes will violate unitarity and the entropy bound at some scale in the IR if the gluons remain to be the relevant degrees of freedom, rendering the theory inconsistent. Then, the key point is that we need to lift the degeneracy of microstates in order to stop the growth of the coupling constant towards IR. This can be done in two ways: by hitting a fixed point (λ_t stops growing and entropy bound/unitarity is respected at all scales) or generating a mass gap (so that there are no gapless degrees of freedom). From the soliton/instanton perspective, we have seen that confinement effectively eliminates this huge degeneracy and avoids the violation of entropy/unitarity. Alternatively, if the theory is higgsed the coupling constant will have a zero or positive beta function below the Higgs scale and the violation of the bounds will be avoided.

In short, the IR running of the 't Hooft coupling will eventually lead to a violation of both the entropy bound by a saturon and the violation of unitarity of the corresponding $2 \to n$ scattering. In order to avoid this and keep the theory consistent and assymptotically free without introducing additional matter fields confinement is the only mechanism available.

The introduction of fermions in the theory [3] does not modify the argument significantly. The saturation of the entropy enforces the condition $N_F \sim N_C$ whereas the asymptotical freedom keeps N_F from being too large. Furthermore, adjusting the number of fermions allows for the theory to develop an infrared fixed point (conformal window, [41])[10].

[10]This is related to black holes in gravity in the sense that it provides saturons for different scales $\lambda_t(q) \sim 1$.

Chapter 3

Higgs mechanism to avoid entropy bound violation

3.1 Adjoint representation

In this section, the original part of the work is displayed. Confinement has been proposed as a mechanism to avoid the entropy bound violation in renormalizable gauge theories [3], in particular with an $SU(N)$ group in the large N regime. The underlying mechanism consists in developing a mass gap that lifts the degeneracy of a certain classical state and prevents it from reaching large entropy. To achieve this, another plausible mechanism would be to introduce one or several Higgs fields that would spontaneously break the symmetry and stop the unlimited growth of the coupling constant. We will consider now two such scenarios and study the behaviour of the theory after the SSB.

First, we will deal with one Higgs field Σ in the adjoint representation breaking the $SU(N)$ symmetry down to $U(1)^{N-1}$. Since this gauge group is abelian, there is no anti-screening (i.e. the effective charge does not decrease when the distance increases) and the beta function[1] is non-negative, therefore the IR growth of the coupling is stopped.

If we represent Σ as an $N \times N$ hermitian traceless matrix in the basis in which it is diagonal[2] $\Sigma_\alpha^\beta = \text{diag}(a_1, ..., a_N)$, we see that in order to get this breaking pattern we need all the eigenvalues to be different (if not, any transformation interchanging any two equal eigenvalues would remain a symmetry of the ground state). This can be achieved via a non-renormalizable potential[3] (we need a polynomic equation of order $N-1$ to obtain $N-1$ different roots and the last one is fixed by the tracelessness) or no potential at all, in which case any state would be the ground state up to the quantum corrections arising

[1] The one-loop formula for a $U(1)$ theory can be found, for instance, in [42].

[2] Clearly by hermiticity, $a_k \in \mathcal{R}\ \forall k = 1, ..., N$.

[3] In [43] it is explicitly shown that the most general renormalizable Higgs potential for the adjoint representation of SU(N) has extrema only if at most two eigenvalues are different.

from the Coleman-Weinberg potential. In Appendix B this issue is adressed in more detail, nonetheless, let us first assume this breaking and study how the theory behaves. The Lagrangian is:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\mathrm{Tr}((D_\mu\Sigma)^\dagger(D^\mu\Sigma)) + V(\Sigma^\dagger\Sigma) \tag{3.1}$$

where in the matrix representation for the gauge fields ($A_\mu \equiv A_\mu^a T^a$, $F_{\mu\nu} \equiv \partial_{[\mu}A_{\nu]} - ig[A_\mu, A_\nu]$) the covariant derivarive acts as the commutator: $D_\mu\Sigma = \partial_\mu\Sigma - ig[A_\mu, \Sigma]$.

Let us recall the familiar case $N = 2$. We parametrize such a vacuum solution and the gauge fields as

$$\Sigma = v\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad\qquad A_\mu = \begin{pmatrix}A_\mu^3 & A_\mu^+\\ A_\mu^- & -A_\mu^3\end{pmatrix} \tag{3.2}$$

The mass terms come from the quadratic part in the gauge bosons of the Higgs kinetic term:

$$\frac{1}{2}\mathrm{Tr}(-g^2([A_\mu, \Sigma])^2) = g^2v^2A_\mu^+A_\mu^- \tag{3.3}$$

We learn from here that the diagonal boson does not get a mass, whereas the complex linear combination of the off-diagonal ones gets a mass proportional to the difference of the eigenvalues.

Going back to the general case, we can write the commutator using the (hermitian) matrix elements $(A_\mu)_{ij}$, $\Sigma_{ij} = a_j\delta_{ij}$:

$$([A_\mu, \Sigma])_{ij} = \sum_k (A_\mu)_{ik}a_j\delta_{kj} - a_i\delta_{ik}(A_\mu)_{kj} = (a_j - a_i)(A_\mu)_{ij}$$
$$([A_\mu, \Sigma])_{ji} = (a_i - a_j)(A_\mu)_{ji} = -(a_j - a_i)(A_\mu^\dagger)_{ij}$$

where for clarity we explicitly write the sums over repeated indices. Then, the following mass terms arise:

$$\frac{1}{2}\mathrm{Tr}_{i,j}(-g^2([A_\mu, \Sigma])^2) = -\sum_k \mathrm{Tr}_{i,j}\frac{g^2}{2}[A_\mu, \Sigma]_{ik}[A_\mu, \Sigma]_{kj} =$$
$$= \sum_k \mathrm{Tr}_{i,j}\frac{g^2}{2}(a_k - a_i)(a_k - a_j)(A_\mu)_{ik}(A_\mu^\dagger)_{jk} = \sum_{i,k}\frac{g^2}{2}(a_i - a_k)^2|A_\mu|^2_{ik} \tag{3.4}$$

As expected, the $N - 1$ diagonal bosons remain massless, whereas the $N^2 - N$ off-diagonal ones get a non-zero mass:

$$m_{ij}^2 = g^2(a_i - a_j)^2 \tag{3.5}$$

Now, since we are working in the t'Hooft limit $g^2 \sim 1/N$, we have to establish the scaling of the boson masses. If we let m^2 scale with some power of N, eventually either the masses will become too big (so that there is no original symmetry in the UV) or increasingly small (so that the original symmetry will be recovered in the IR). Therefore we set the gauge boson masses to be independent of N, which implies $|a_i - a_j| \sim \sqrt{N}$. But this now leads to a contradiction, since we have to fit N eigenvalues and the difference between any pair has to be of order $\mathcal{O}(\sqrt{N})$, which is clearly unfeasible.

Thus, a simple counting argument discards this particular scenario (having $U(1)^{N-1}$ in the large N limit) as a viable possibility to avoid the entropy bound violation. However, there are other possibilities.

It is impossible to have all gauge boson masses without N scaling, but for example we can require the condition of not having "heavy" gauge bosons, i.e. with masses scaling with some positive power of N. This fixes the scaling of the minimum and the maximum of all eigenvalues, which have to sum to zero due to the tracelessness requirement (remember that $\Sigma_{ij} = a_j \delta_{ij}$ belongs to the adjoint representation).

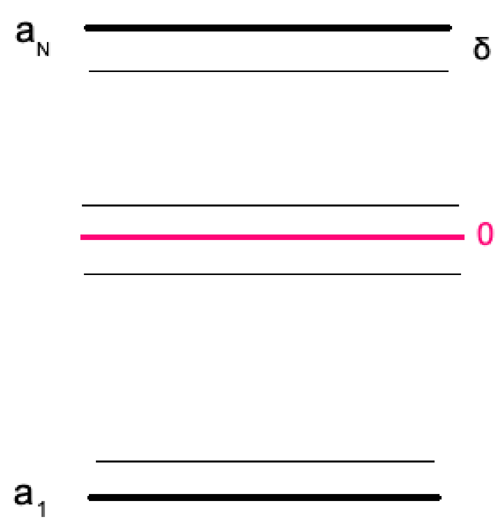

Figure 3.1: Distribution of eigenvalues with the requirement of not producing "heavy" gauge bosons. This implies that the maximum mass is order 1, therefore according to (3.5), $|a_N - a_1| \sim \sqrt{N}$. In this symmetric distribution $a_N = -a_1 = c\sqrt{N}$, where all the eigenvalues are separated by a constant distance $\delta \sim 1/\sqrt{N}$.

In this way, a "continuum" of masses according to (3.5) is generated, from $m^2_{i+1,i} \sim 1/N^2$ to $m^2_{N1} \sim 1$, since $a_{i+K} - a_i \sim K\delta \sim K/\sqrt{N}$. The mass matrix scales then as:

$$M^2 \sim \begin{pmatrix} 0 & 1/N^2 & 4/N^2 & ... & 1 \\ 1/N^2 & 0 & 1/N^2 & ... & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & ... & ... & 1/N^2 & 0 \end{pmatrix}$$

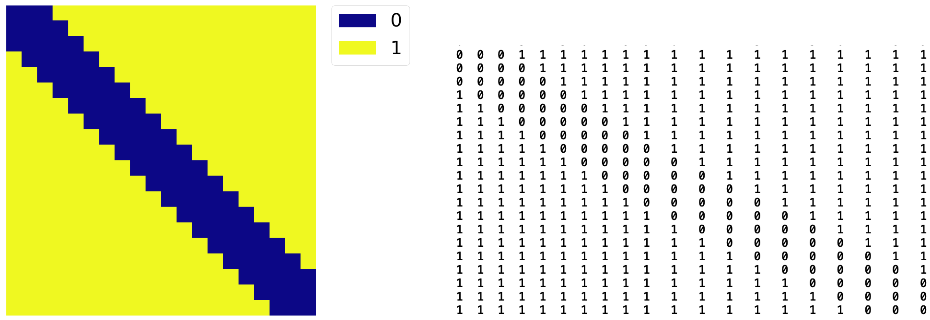

Figure 3.2: Left, graphical depiction of the shape of the gauge boson mass matrix, where vanishing values are represented by blue areas and non-vanishing ones by yellow zones. Right, example of such a matrix with $N = 20$ and a cutoff of 3 adjacent values (i.e., $m^2_{i+4,i} \nrightarrow 0$).

For finite N, this signals in principle a standard breaking $SU(N) \to U(1)^{N-1}$. However, in the large N regime, the shape of the mass matrix can not be identified with a standard Lie group. First, one would want to distinguish which values will tend to zero and which will remain finite, but in the continuum limit there is no such possibility. One alternative route to try and gain some insight is to define a certain 'cutoff" under which one considers that the masses tend to zero and all the masses above this are considered to remain finite. For example, one could set as limit the scaling $m^2_{ij} \sim 1/\log N$. This means that all the masses contained in a separation of $N/\sqrt{\log N}$ vanish, where as the rest do not:[4]

$$m^2_{1,N/\sqrt{\log N}} \sim \frac{1}{N}\left(\frac{N}{\sqrt{\log N}}\frac{1}{\sqrt{N}}\right)^2 = \frac{1}{\log N}$$

Now, in the large N limit, the mass matrix takes the form displayed in Fig. 3.2 (we refer to any non-zero values by 1 for simplicity). First, notice that this structure does not correspond to some $SU(N')$, since this would arise as a $N' \times N'$ block of zeros in the mass matrix. Therefore we suspect that this distribution of eigenvalues leads to some unusual arrangement of gauge bosons.

We make now an ansatz to try to identify this underlying "gauge group". If we look closely at Fig. 3.2, we see that there are actually small squares of the form that would correspond to $SU(c)$, where c is what we called before the "cutoff"[5] (number of nearest neighbour eigenvalues which are sufficiently close to produce a vanishing mass), but they overlap with the adjacent ones. If they didn't, we could identify this with a product group with independent generators. Due to the overlapping, we can then deduce that some generators will be shared between two blocks and therefore they have to be counted out. This leads to the idea of a quotient between these small groups corresponding to the square blocks

[4] In general, one can explore a separation αN, which does not change the consequences of the analysis performed here.

[5] In our example $c = N/\sqrt{\log N}$.

and the shared zeros will always form a sub-block with one row/column less. Therefore, we propose the following ansatz for the SSB:

$$SU(N) \to \; SU(c)^{N-c+1}/SU(c-1)^{N-c} \tag{3.6}$$

We now want to check if this ansatz has the same dimension. A way to count how many zeros (correspondingly, surviving generators) are there in the matrix is to count the number ε of non-zero entries in the upper diagonal and using the symmetric nature of M^2, subtract it twice from the total number of generators. Looking at Fig. 3.2, it is immediate to find this ε:

$$\varepsilon = \sum_{i=1}^{N-c} i = \frac{(N-c)(N-c+1)}{2} \tag{3.7}$$

The number of surviving generators, i.e. massless gauge bosons g is then:

$$g = N^2 - 1 - 2\varepsilon = -c^2 + 2cN + c - N - 1 \tag{3.8}$$

On the other hand, the number of generators corresponding to independent transformations of the quotient group (3.6) is:

$$g = (N-c+1)(c^2-1) - (N-c)((c-1)^2-1) = -c^2 + 2cN + c - N - 1 \tag{3.9}$$

Therefore, the number of generators of the ansatz is the same as in the actual SSB.

Now, let us look at the large N limit of (3.6). From (3.9), the number of generators in this limit is:

$$g = (2c-1)N \tag{3.10}$$

This corresponds to a quotient group:

$$SU(c)^N/SU(c-1)^N \cong (S^{2c-1})^N \tag{3.11}$$

We can derive this also from (3.6) by identifying which variation from N has a greater contribution to the total number of generators. Consider a quotient group of the form:

$$SU(c)^{d+t}/SU(c-s)^d$$

where $d >> c, s, t$. The total number of independent generators (i.e. the dimension of the quotient algebra) is:

$$g = (d+t)(c^2-1) - d((c-s)^2-1) \sim c^2t - cs^2 + 2cds$$

where we selected the leading s, t-dependent contributions. It is clear that the last term, proportional to d, makes the contribution of s is bigger than the one of t, therefore applying this argument to equation (3.6), we recover (3.11).

To sum up, we found that by imposing that all the masses acquired by the gauge bosons after the SSB are bounded by a fixed scale that does not grow with N, this does not lead to a IR Yang-Mills theory in the large N limit. Since in order to potentially violate the entropy bound we need a parametrically large N, this mechanism can't be considered a viable alternative to confinement.

There is an analogy to another physical situation that is worth noting. In the Kaluza-Klein theory, where a fifth dimension is compactified and a four-dimensional effective theory arises, a massless 5-dimensional scalar field leads to a tower of massive scalar fields from the 4-dimensional point of view. This is easy to see, using the Klein-Gordon equation and the periodicity of the field in the compact coordinate:

$$\Box_5 \Phi(x, x^5) = 0 \qquad \Phi(x, x^5) = \sum_{n=0}^{\infty} \Psi_n(x) e^{i\frac{n}{R}x^5}$$

$$\Box_5 \Phi(x, x^5) = \sum_{n=0}^{\infty} \left[\Box \Psi_n(x) + \frac{n^2}{R^2} \Psi_n(x) \right] e^{i\frac{n}{R}x^5}$$

$$\left(\Box + \frac{n^2}{R^2} \right) \Psi_n(x) \equiv \left(\Box + m_n^2 \right) \Psi_n(x) = 0$$

This happens as well for the gauge bosons (see for example [44]). Now, for a similar SSB as our scenario but in this extra-dimensional context, those fields which at finite R acquire a non-vanishing mass will, in the limit $R \to \infty$, become massless. This is quite similar to what we found, with N playing the role of R.

In the extra-dimensional case, any such SSB pattern will then not possibly be understood from the low-dimensional point of view (the number and distribution of massless fields would not match the regime of finite and infinite R). It is intuitively obvious that when $R \to \infty$ and the extra dimension is no more compactified, it would be impossible to describe it as a 4-dimensional effective theory, since the 5th one would stand in equal footing as the rest. This striking resemblance could suggests that the strange pattern of SSB that we encountered might correspond to an insufficient knowledge about the degrees of freedom of the theory (needing some extra "dimension"), although it is unclear how exactly one could proceed.

This connection between the number of massless fields and the compactification radius has a natural explanation in the context of large extra dimensions [45]. In this article, it is shown that the four-dimensional coupling and the fundamental ($4+d$-dimensional) one are related by the compactification radius R and the fundamental scale M_*:

$$\alpha_{4+d} = (RM_*)^d \alpha_4$$

Additionally, in [46] a similar relation is deduced, where unitarity determines the number of Kaluza-Klein fields N_{KK}:

$$\alpha_{4+d} = N_{KK} \alpha_4$$

Figure 3.3: Left, a distribution of eigenvalues producing the breaking $SU(N) \to SU(N')$. The first N' eigenvalues are all of order 1 and therefore the associated masses vanish in the large N limit. The distance Λ will have to scale at least as $\sqrt{N}$ in order to have non-vanishing masses. This applies also the individual difference of the rest of the eigenvalues, mimicking the case of Fig. 3.2. Finally, the tracelessness fixes the value of a_N Right, the gauge boson mass matrix for such a distribution exemplified for $N = 20$ and $N' = 3$.

This establishes a relation between N_{KK} and R, with the following consequence. In the limit $R \to +\infty$, with M_* fixed, the number of Kaluza-Klein fields also tends to infinity. Now, the fundamental scale plays the role of a UV cutoff, and since it is fixed this means that in the above limit, the number of massless fields grows with the compactification radius as well.

Finally, we want to explore an alternative setup. In particular, we demand that in the IR the remaining gauge group is $SU(N')$, with $N' << N$. Note that this would stop as well the growth of the coupling constant, since now N' fields would contribute in (2.39) and due to the unitarity limit $g^2 \lesssim 1/N$, this leads to a vanishing beta function in the large N limit.

In Fig. 3.3 such a distribution of the eigenvalues is displayed. The key point is that, since $N' << N$, there will inevitably be some gauge boson masses that scale with N. For instance:

$$
\begin{aligned}
&|a_{N-1} - a_{N'+1}| \sim (N - 1 - (N' + 1))\sqrt{N} \sim (N - N')\sqrt{N} \\
&m^2_{N-1,N'+1} \sim \frac{1}{N}(a_{N-1} - a_{N'+1})^2 \sim (N - N')^2
\end{aligned}
$$

As explained before, if the limit $N'/N \to 0$ applies then we could avoid the entropy bound violation. The caveat is that if masses grow with N, the UV symmetry will effectively be reduced, since the now the infinitely massive gauge fields will not be at any scale obey the corresponding Yang-Mills Lagrangian. In other words, there is no finite UV scale above which the original $SU(N)$ theory is still intact. We can estimate the number of the heavy fields from the previous calculation (3.7), but with the substitution $N \to \tilde{N} \equiv N - N'$.

Using our example $c = N/\sqrt{\log N}$, this leads to a number of heavy fields of the order:

$$n_{\text{heavy}} \sim \tilde{N}^2 \left(1 - \frac{1}{\sqrt{\log \tilde{N}}}\right) \tag{3.12}$$

N	dim $SU(N) = N^2 - 1$	n_{heavy}	% heavy fields
100	9.999000e+03	5009	50.0
1000	9.999990e+05	615726	62.0
10^6	1.000000e+12	730955785654	73.0
10^9	1.000000e+18	780329918974699904	78.0

Table 3.1: Number and proportion of heavy fields for different values of N, with $N' = 3$.

As a consequence, the UV symmetry is no longer $SU(N)$ with large N, which invalidates our starting hypothesis.

3.2 Fundamental representation

Let us now consider a different set up, in which N Higgs fields transforming in the fundamental representation are introduced. We can specifically devise a hedgehog potential which completely breaks the $SU(N)$ symmetry:

$$V(\phi) = -\frac{m^2}{2}\phi^{\dagger j}_{\ A}\phi_{Aj} + \alpha(\phi^{\dagger j}_{\ A}\phi_{Bj})(\phi^{\dagger l}_{\ B}\phi_{Al}) + \frac{\tilde{\alpha}}{4}\underbrace{(\phi^{\dagger j}_{\ A}\phi_{Aj})^2}_{(\phi^{\dagger j}_{\ A}\phi_{Aj})(\phi^{\dagger k}_{\ A}\phi_{Ak})} \tag{3.13}$$

where the sum over repeated indices is implied and all constants are positive. The Higgs field ϕ^j_A has two indices: $j, l = 1, ..., N$ account for the $SU(N)$ gauge (color), whereas $A, B = 1, ..., N$ account for the $SU(N)$ global symmetry (flavor) arising from the fact that all constants are flavor-blind. We can write it as a matrix transforming in the (anti)fundamental representation of the color(flavor) group $\phi \leftrightarrow U_C \phi U_F^\dagger$. Notice the somewhat peculiar structure of the third term in (3.13), which can not be identified with the usual $(\text{Tr}\,|\phi|^2)^2$. Due to the hedgehog term[6] (proportional to α), the potential is minimized by:

$$\langle \phi_{Aj} \rangle = \frac{m}{\sqrt{\tilde{\alpha}}}\delta_{Aj} \equiv v \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{3.13b}$$

[6] It can be compared to the scalar product term in the Ising model, which minimizes the energy for orthogonal spins. Without it the EOM reduces to $\sum_k \phi^{\dagger k}_{\ A}\phi_{Ak} = v^2$, which is satisfied for example by $\phi_{Ak} = v\ \delta_{kc}$, with c a particular color. We will consider this case in more detail afterwards.

The masses of the gauge bosons after the symmetry breaking are again given by the quadratic term of the kinetic part of the lagrangian (the covariant derivative acts as $D_\mu \phi_{Aj} = \partial_\mu \phi_{Aj} - ig(A_\mu)_{jl}\phi^l_A$):

$$m_{ij}^2 = g^2(|\phi|^2_{i,A=i} + |\phi|^2_{j,A=j}) = 2g^2\frac{m^2}{\tilde{\alpha}} \tag{3.14}$$

Using the definition of covariant derivative above, this result is straightforward (we use the Einstein summation convention):

$$\mathcal{L} \supset D_\mu\phi_{Aj}(D^\mu\phi^j_A)^\dagger = g^2(A_\mu)_{jl}\phi^l_A(A^\mu)^\dagger_{jk}\phi^{k\,\dagger}_A \supset g^2(A_\mu)_{ji}(A^\mu)^\dagger_{ji}|\phi^i_A|^2 + g^2(A_\mu)_{ij}(A^\mu)^\dagger_{ij}|\phi^j_A|^2$$

Since the A_μ matrix is hermitian $(A_\mu)_{ij} = (A^\dagger_\mu)_{ji}$, i.e. transposed components are complex conjugate to each other and ϕ_{Aj} acquires the diagonal form (3.13b), we retrieve the mass (3.14).

In order to establish the different N scalings, we need to look at the constraints imposed by unitarity. The diagrams in Fig. 3.4 and Fig. 3.5 impose the following limits:

$$\alpha N \lesssim 1 \leftrightarrow \alpha \lesssim 1/N \qquad\qquad \tilde{\alpha} N \lesssim 1 \leftrightarrow \tilde{\alpha} \lesssim 1/N$$

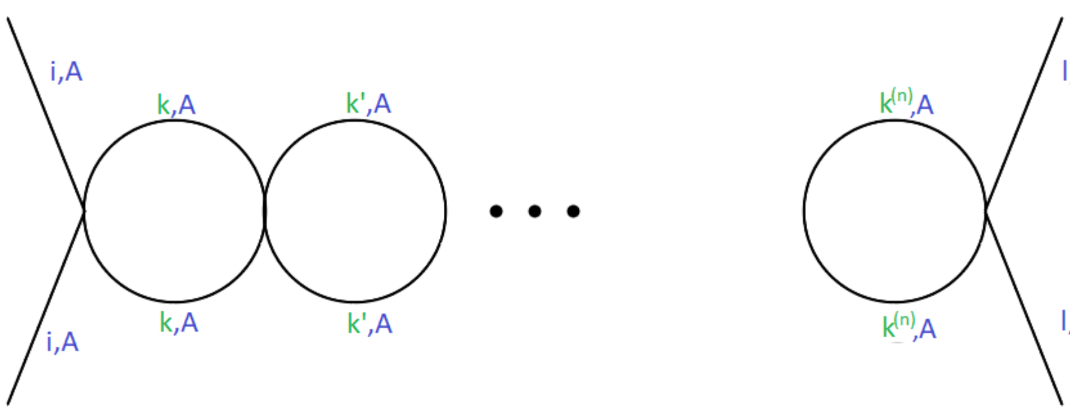

Figure 3.4: The $\tilde{\alpha}$ term in the Lagrangian generates an n-loop amplitude which will be proportional to $(\tilde{\alpha}N)^n$. Each different index in green is summed over, therefore they carry each a factor of N, whereas blue indices are fixed by the external fields.

Then, as before, the scale we want to fix is given by the gauge boson masses. Combining both t'Hooft and unitarity saturation limits in (3.14), the scalings of all quantities are then:

$$g^2 \sim \alpha \sim \tilde{\alpha} \sim 1/N \Longleftrightarrow m_{ij}^2 \sim m^2 \sim 1 \tag{3.15}$$

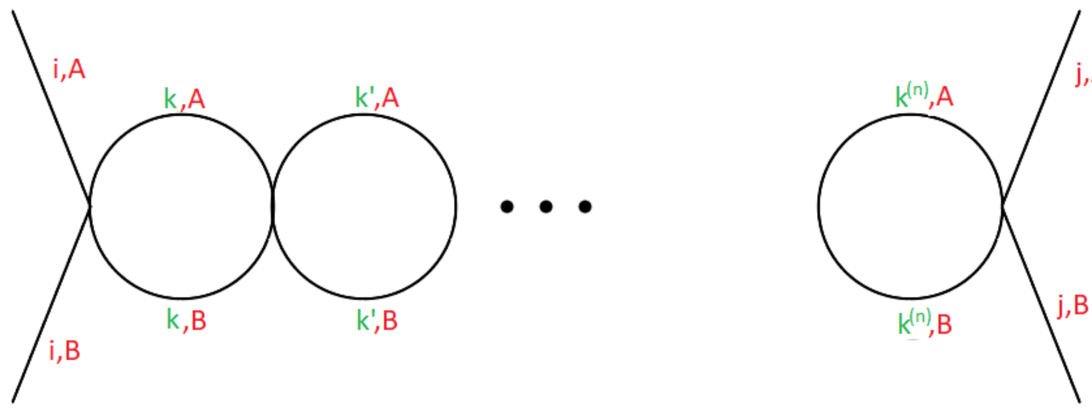

Figure 3.5: The α term in the Lagrangian generates an n-loop amplitude which will be proportional to $(\alpha N)^n$. Each different index in green is summed over, therefore they carry each a factor of N, whereas red indices are fixed by the external fields.

Note that a term proportional to $(\text{Tr } |\phi|^2)^2$ would have a $1/N^2$ scaling, due to an additional summation over the flavors with respect to Fig. 3.4 and Fig. 3.5 and thus would be subleading in the large N limit. Since we are studying this mechanism as an alternative to confinement to violate the entropy bound and m sets the scale at which the gauge symmetry is broken, this implies $m >> \Lambda_{QCD}$. An interesting fact to keep in mind is that, in the limit of large N, the Higgs VEV v is arbitrarily large, whereas the gauge boson masses $\sim m$ stay fixed. So there is an increasingly large range of energies in which the Higgs field is non-dynamical (up to quantum fluctuations, obviously) but the gauge symmetry effectively remains intact. Of course, this has an explanation in that the coupling between the Higgs and the gauge bosons $g^2 \sim 1/N$ is increasingly small. Conversely, if the breaking of the symmetry was only dictated by the Higgs VEV, this would ultimately destroy the original gauge symmetry in the UV. Therefore, we learn that the scale that really matters for the SSB is that of the masses of the gauge bosons.

Let us now examine the symmetries of the theory in Tab. 3.2. In the limit[7] of vanishing α flavor does not exist as a symmetry, since all the terms in the Lagrangian can be expressed as a sum over the "flavor" index and therefore this index only numerates N independent fields. Thus, each field can undergo its own $SU(N)$ global transformation without changing the Lagrangian, with the particularity that the diagonal combination (in which all fields transform the same) is gauged. The Higgs fields acquire a VEV which is determined by the EOM:

$$\sum_k {\phi^\dagger}^k_A \phi_{Ak} = v^2$$

[7]This limit would be more accurate if unitarity imposed something like $\alpha \sim 1/N^2$, but as we will now check, we want to avoid this limit for our purposes.

N_F Higgs fields in fundamental representation of $SU(N_C)$, $N_C = N_F = N$			
$\alpha \to 0$	before SSB:	$SU(N_C)_{\text{gauge}} \times [SU(N_C)]^{N_F-1}_{\text{global}}$	$2N-1$ would-be Goldstones
	after SSB:	$SU(N_C-1)_{\text{gauge}} \times [SU(N_C-1)]^{N_F-1}_{\text{global}}$	$(N-1)(2N-1)$ Goldstones
$\alpha \neq 0$	before SSB:	$SU(N_C)_{\text{gauge}} \times SU(N_F)_{\text{global}}$	N^2-1 would-be Goldstones
	after SSB:	$SU(N)_{\text{diag}}$	0 Goldstones

Table 3.2: Symmetries of the theory before and after spontaneous symmetry breaking.

Therefore, each flavor acquires a vacuum value which is defined by the previous equation up to a unitary transformation in color space, we then expect a huge number of Goldstone bosons due to this high degeneracy of the vacuum. One such possibility is having all the fields aligned in a certain color:

$$\langle \phi_{Aj} \rangle = v \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}}_{A \longrightarrow} \Big\downarrow j$$

This configuration clearly breaks each $SU(N)$ symmetry down to $SU(N-1)$, since only the rotations over the zero values will leave the vacuum invariant. We can calculate the gauge boson masses derived from this configuration ($\phi_{Aj} = v\delta_{jc}$, where c is a particular color).

$$\mathcal{L} \supset \sum_A \sum_j D_\mu \phi_{Aj} (D^\mu \phi_{Aj})^\dagger \supset g^2 v^2 \sum_A \sum_{j,l,k} (A_\mu)_{jl} (A^\mu)^\dagger_{jk} \delta_{lc} \delta_{kc} = N g^2 v^2 \sum_j |A_\mu|^2_{jc} \quad (3.16)$$

We see that only $2N-1$ gauge bosons get a mass[8], which precisely matches the broken generators from $SU(N) \to SU(N-1)$. Note that these masses would render an inconsistent alternative to confinement in the large N limit, since they would scale as $Ng^2v^2 \sim N$, with the aforementioned consequences.

When we restore α, all Higgs fields are now related by the $SU(N)$ global symmmetry (see Tab. 3.2). The hedgehog term lifts the degeneracy between all the vacua satisfying the previous EOM[9] and the vacuum is now uniquely determined by (3.13b). The little group is the one that preserves the identity in the transformation $\phi \leftrightarrow U_C \phi U_F^\dagger$, i.e. the diagonal

[8] The counting is as follows, 1 real field from the diagonal + $(N-1)$ complex fields from the off-diagonal: $1 + 2(N-1) = 2N-1$.

[9] Since all terms in the potential are positive semidefinite, the minimum is achieved by setting to zero the hedgehog term and minimizing the rest.

subgroup $U_C = U_F$[10].

The counting now is simple, from $2(N^2-1)$ generators, N^2-1 become longitudinal components for the gauge bosons and the rest are preserved as the $SU(N)$ diagonal subgroup. In order to avoid the entropy bound violation and according to the discussion in the previous chapter, we have to make sure that no gapless degrees of freedom are left in the theory. Looking at Table 3.2, in the case $\alpha = 0$ we have $N(2N-1)$ scalar excitations, from which $2N-1$ are absorbed as longitudinal components by the gauge bosons and the rest are physical (massless) Goldstone fields. This counting must match the $\alpha \neq 0$ spectrum. From the SSB pattern, we deduce that N^2-1 of these excitations are absorbed by the gauge bosons and no physical Goldstones should appear. Therefore, the remaining N^2-N+1 of these scalars must appear as massive particles, i.e. pseudo-Goldstones.

The pseudo-Goldstones are related to the original symmetry which is explicitly broken by the α term, i.e. they arise from performing a color rotation independently for each flavor (see Tab. 3.2). We can parametrize them in the usual way -repeated color indices are implicitly summed over:

$$\phi_{Ak} = vU^{(A)}_{kl}\delta_{Al} \qquad\qquad U^{(A)}_{kl} = \exp i\theta^{(A)}_a T^{(A)}_a \tag{3.17}$$

Now, plugging (3.17) into the Higgs Lagrangian with the potential (3.13), we can read off the kinetic term and the mass term for these pseudo-Goldstones $\theta^{(A)}_j$. The kinetic term is (upon normalization):

$$\mathcal{L} \supset (\partial_\mu\phi_{Aj})^2 = \partial_\mu\theta^{(A)}_a\partial_\mu\theta^{(A)}_b(T^{(A)}_a)_{jl}(T^{(A)}_b)_{jk}\delta_{Al}\delta_{Ak} = \partial_\mu\theta^{(A)}_a\partial_\mu\theta^{(A)}_b(T^{(A)}_aT^{(A)}_b)_{AA} \tag{3.18}$$

We can learn several things from this result. First, the (pion) decay constant [47] for the pseudo-Goldstones is $f = v$, since the kinetic term is normalized by redefining $\theta^{(A)}_j \to v^{-1}\theta^{(A)}_j$. Secondly, the color structure $(T^{(A)}_aT^{(A)}_b)_{AA}$ makes sure that the counting matches with the breaking $SU(N) \to SU(N-1)$ and this is repeated for each flavor A.

Let us verify this statement. First, consider the off-diagonal generators, which have the entries of the off-diagonal Pauli matrices in different pairs of rows/columns. The AA element of the product of two such generators will be non-empty only if both share a non-empty Ak element, with $k = 2, ..., N$. This leaves only two possibilities: either they are the same ($T_a = T_b$) or they form a conjugate pair (corresponding to σ_1 and σ_2). Note that due to the symmetry in the color indices a, b, we can replace the product for the anticommutator of the generators, i.e.

$$\theta_a\theta_bT_aT_b = \frac{1}{2}\theta_a\theta_b\{T_a, T_b\}$$

[10] This is obviously only the case when $N_C = N_F$, if $N_F < N_C$ there would remain some unbroken gauge group (which would prevent us from achieving our goal of avoiding the entropy bound due to the IR growth of the coupling constant).

Since the Pauli matrices fulfill $\{\sigma_i, \sigma_j\} \propto \delta_{ij}$, this discards the pairing of off-diagonal pseudo-Goldstones, so that there will be no mixing in color space. As a consequence, each physical field corresponds to the associated generator, i.e. the kinetic terms will only be of the form $a = b$. Now, the square of the generator having non-empty AA element, as mentioned before, requires a non-empty Ak element for fixed A. This is equivalent to computing how many products of N elements I can form without including one in particular (e.g. $\{123\}$ has only one such possibility: $\{32\}$), and twice of those (because of the conjugate pairs) will not appear in the Lagrangian. It is easy to see that we are actually asking to form products of $N-1$ elements, therefore to the total number of possible pairs we have to subtract the associated quantity. As it is well known, the number of possible pairs of N elements is given by the combinatoric coefficient:

$$\binom{N}{2} = \frac{N(N-1)}{2}$$

Thus taking in account the duplicity mentioned before, we can deduce how many pseudo-Goldstones will have non-vanishing kinetic terms and therefore appear in the Lagrangian as a result of the SSB:

$$N(N-1) - (N-1)(N-2) = 2N - 2$$

Only the diagonal generators remain. Since they all in general have non-empty diagonal entries, this will lead to a non-trivial mixing of the associated pseudo-Goldstones. From the symmetry breaking pattern, we know that the remaining group has rank $N-1$ and this ensures that only one of them will appear (since the others would correspond to rotations that are part of the symmetry of the theory), but this happens in a non trivial way. In Appendix A this is explicitly shown by computing the case $N = 3$. In total, we have $2N-2+1 = 2N-1$ Goldstones appearing, which match the SSB of the original symmetry (before attaining masses through the α term).

We focus now on the masses of these pseudo-Goldstones. There is a subtlety in the calculation, because the form (3.17) is only valid for one flavor at a time (the matrix only acts on one column). However, since the rotations are independent, there is no problem because the result will be the same for each flavor upon which the rotation acts. Keeping this in mind, we find the mass terms through the α term in (3.13):

$$\mathcal{L} \supset \alpha({\phi^{\dagger}}^{j}_{A}\phi_{Bj})({\phi^{\dagger}}^{l}_{B}\phi_{Al}) = \alpha v^4 (U^{\dagger(A)} U^{(A)})_{AA} \supset \alpha v^2 \theta_a^{(A)} \theta_b^{(A)} (T_a^{(A)} T_b^{(A)})_{AA} \tag{3.19}$$

The first thing to note is that the color structure is the same as the kinetic term, therefore every Goldstone acquires a mass. This is reassuring since we knew that after restoring α no massless Goldstones should survive. Furthermore, we see that each of them acquires a mass $m^2 \propto \alpha v^2 \sim 1$, meaning that these masses will not scale with N and therefore in the large N limit they will not become neither too small nor too large, which would spoil either the UV or the IR effective symmetry (as with the gauge bosons before). This ensures that this mechanism is also consistent in the t'Hooft limit and our analysis of the entropy bound violation remains intact.

To conclude, let us briefly examine the Goldstone coupling. The setting is very similar to the effective theory of pions in QCD [48], where at low energies the dynamical degrees of freedom are the Goldstones of the symmetry breaking. A simple example of a Goldstone exists in the SU(2) Higgs context, where the coupling between the Goldstone and the Higgs fields is given by both the VEV (decay constant) v and the energy (because the Goldstone couples through a derivative):

$$\phi = (v+h)e^{i\theta}$$
$$\mathcal{L} \supset (\partial_\mu \phi)^\dagger (\partial^\mu \phi) = (\partial_\mu h)^2 + (v+h)^2 (\partial_\mu \theta)^2 \supset \frac{h^2}{v^2}(\partial_\mu \theta)^2$$

This fact is not model-dependent (in general, Goldstones couple through a derivative), so that in our case $m^2 \sim \alpha v^2$ the effective coupling is given by:

$$\alpha_{\text{Gold}} = \frac{E^2}{v^2} \sim \frac{E^2}{m^2 N} \tag{3.20}$$

where we used the unitarity bound $\alpha \sim 1/N$. As in [3], this is the relevant coupling for the relevant degrees of freedom at low energies (pseudo-Goldstones). Now, in the large N limit, the relevant collective coupling (analog of t'Hooft coupling) will be given by $\alpha_{\text{Gold}} N$ (see Fig. 3.5). Therefore, we see that for low energies, the analogous bound to (2.33) will not be saturated.

Therefore we conclude that this mechanism presents no apparent obstruction to be an alternative to confinement in preventing the violation of the entropy bound.

Chapter 4

Conclusions

In this work, we explored the Higgs mechanism as an alternative to confinement to avoid the entropy bound violation by generating a mass gap in the theory. In the fundamental representation, we introduce N fundamental fields. All the gauge bosons acquire a mass of order 1 (3.14), therefore in the large N limit the symmetry breaking pattern remains intact. In addition to this, the pseudo-Goldstone bosons corresponding to independent global flavor transformations acquire as well a mass of order 1.

We checked that the relevant collective coupling for low-energy scattering amplitudes does not exceed the unitarity limit and therefore all the requirements are met to consistently replace confinement. As a side note, there might be some zero modes of a classical solution associated to breaking of translation symmetry for example, but as explained in [1, 2] they will not provide a large enough contribution for the entropy.

In the case of one Higgs field in the adjoint representation, it becomes more complicated to achieve this. The first possibility is to perform the maximum symmetry breaking $SU(N) \rightarrow U(1)^{N-1}$. The reason for considering this breaking is that, since for non-abelian theories the gauge boson contribution to the beta function is negative, the coupling strength increases towards the IR and the inverse coupling bound (2.3) becomes lower, leading to potential violations. In contrast, for abelian theories (in particular $U(1)$) the beta function is positive and the growth of the coupling is stopped.

Naively, there exists a configuration which produces such a pattern. Nonetheless, the masses of the gauge bosons cannot be made simultaneously of order 1, some of them will scale positively and others negatively with N. This means that, in the large N limit (where the entropy becomes parametrically large and the bound is approached), some of the gauge bosons will become increasingly heavy and spoil the UV symmetry, and others will decrease its mass, therefore leaving some gapless degrees of freedom in the IR and a different SSB.

If we instead require that no heavy[1] gauge fields arise after the SSB, the remaining gauge structure in the IR does not have the usual $SU(N)$ Lie group form. An analogy was drawn to Kaluza-Klein theories, where such a configuration can arise as a limiting effect of large compactification radius. Another alternative is to demand the gauge boson mass matrix to match $SU(N')$ in the large N limit, but it is discovered that the number of heavy fields arising from that pattern is big enough to significantly spoil the UV symmetry.

Therefore, we conclude that the Higgs mechanism could provide a workaround to confinement in Yang-Mills theories in the fundamental representation. Of course, as explained in [1,2] the distinctive trait of confinement is that it is already present in pure glue theories, without the need of introducing additional matter content. In addition, as we explicitly checked, not every set of Higgs fields can fulfill this role, for example in the adjoint representation we did not find a configuration of eigenvalues that provided such a replacement to confinement in a consistent way.

[1] In the sense of having a mass which does not scale with N

Appendix A

Explicit example of pseudo-Goldstone masses in N=3

We want to illustrate how the would-be and physical Goldstones acquire a mass from the α term in the potential (3.13) and in particular, show that the number of them is precisely $2N-1$ (for each flavor independently). To this end, we consider the case $N=3$.

The generators of $SU(3)$ are given by the Gell-Mann matrices $T_a = \lambda_a/2$, and we choose the flavor $A=1$. Since the kinetic term and mass term share the same color structure, those who do not have mass will not appear in the Lagrangian, i.e. the corresponding generators are not broken. We take in account the following relations:

$$
\begin{aligned}
\theta_a \theta_b T_a T_b &= \frac{1}{2}\theta_a \theta_b \{T_a, T_b\} \\
\{T_a, T_b\} &= \frac{1}{3}\delta_{ab}\mathbb{I} + \frac{1}{2}\sum_c d^{abc}\lambda^c
\end{aligned}
$$

The constants d^{abc} can be found, for example, in [49]. Therefore, the mass terms will have the following form:

$$
\frac{1}{2}\alpha v^2 \theta_a^{(1)}\theta_b^{(1)}\{T_a, T_b\}_{11} \equiv \frac{1}{2}m_{ab}^2 \theta_a^{(1)}\theta_b^{(1)}
$$

We display the non-zero mass terms, with the notation $m_a^2 \equiv m_{aa}^2$:

$$
\begin{aligned}
m_1^2 = m_2^2 = m_3^2 = m_4^2 = m_5^2 = \frac{\alpha v^2}{2} \qquad & m_6^2 = m_7^2 = 0 \\
m_8^2 = \frac{\alpha v^2}{6} \qquad & m_{83}^2 = m_{38}^2 = \frac{\alpha v^2}{2\sqrt{3}}
\end{aligned}
$$

At first glance, we see that θ_6 and θ_7 have no mass and therefore, their generators are not broken. Naively, the counting would not match, since we need $2N-1=5$ massive fields and we still have 6. Nonetheless, note that the mass matrix is diagonal except for the diagonal bosons (3 and 8). If we display them as a 2x2 matrix, we see that it has zero determinant, therefore one of the eigenvalues will have zero mass. In particular, the combination $\theta_+ = -1/2\ \theta_3 + \sqrt{3}/2\ \theta_8$ will appear as massless after diagonalizing the mass matrix, whereas the orthogonal $\theta_- = \sqrt{3}/2\ \theta_3 + 1/2\ \theta_8$ will have non-zero mass[1] ($m^2 = 2/3\ \alpha v^2$).

We then see that, as expected, 5 Goldstone bosons acquire a mass $\sim \alpha v^2$ from the α term in the Lagrangian, signaling a breaking of $SU(3)$ down to $SU(2)$. This will happen once per flavor, giving the total number of $N(2N-1) = 15$. From these would-be particles, 8 are absorbed as the longitudinal components of the gauge bosons after symmetry breaking and the remaining 7 will conform our spectrum of pseudo-Goldstones.

[1]This coincides with the calculation of gauge boson masses performed in eq. 6 of [50], due to the presence of the anticommutator of the generators.

Appendix B

On the Higgs potential in the adjoint representation

In Section 3.1, the VEV of the Higgs field that fulfilled the required SSB pattern[1] was of the form:

$$\Sigma_\alpha^\beta = \mathrm{diag}(a_1, ..., a_N) \tag{B.1}$$

with all the eigenvalues $\{a_i\}$ different and the constraint $\sum_i a_i = 0$. Of course, it should arise as a solution of the equations of motion and therefore a suitable potential is needed.

The first option is to have a non-renormalizable tree-level potential, which has to be minimized by (B.1). From standard linear algebra, we know that in order to have s different roots it is a necessary condition to have a polynomial of at least order s. Therefore, we explore a potential of the form:

$$V(\Sigma^\dagger\Sigma) = \mathrm{Tr}\sum_{n=1}^{N} c_n(\Sigma^\dagger\Sigma)^n \tag{B.2}$$

will both be invariant under the SU(N) symmetry and provide a polynomial EOM. Working in this diagonal basis, we can write the EOM for its eigenvalues:

$$\mathrm{Tr}\sum_{n=1}^{N} c_n(\Sigma^\dagger_{\alpha\beta}\Sigma_{\beta\gamma})^n = \mathrm{Tr}\sum_{n=1}^{N} c_n(a_\alpha\delta_{\alpha\beta}a_\gamma\delta_{\beta\gamma})^n = \sum_{\gamma,n} c_n a_\gamma^{2n} \tag{B.3}$$

$$0 = \frac{\partial V}{\partial a_\rho} = 2\sum_{n=1}^{N} n c_n a_\rho^{2n-1} \longrightarrow d_1 a_\rho + d_3 a_\rho^3 + ... + d_{2N-1}a_\rho^{2N-1} = 0 \tag{B.4}$$

[1] In the literature the case for a renormalizable potential has been extensively treated (see, for instance, [43]), which leads to a breaking of the type $SU(N) \to SU(h) \times SU(N-h) \times U(1)$, but our case $SU(N) \to U(1)^{N-1}$ has not.

We find that each eigenvalue $a_\rho,\ \rho = 1, ..., N$ fulfills the same polynomial EOM, which is of order $2N-1$. Therefore, by a suitable election of the coefficients this equation will have $2N-1$ different solutions. Note that, by hermiticity of (B.1), all the components must be real, although the polynomial might in general have complex conjugate roots. This is no problem, since we can use the $SU(2) \subset SU(N)$ invariance of the potential to transform the complex conjugate pair into its real and imaginary part, respectively.

Under a suitable choice of the coefficients it should be possible for the tracelessness constraint to be fulfilled. The Vieta formulas [51] relate the sum of the roots of a polynomial to its coefficients, in our notation:

$$r_1 + ... + r_{2N-1} = -\frac{d_{2N-2}}{d_{2N-1}} = 0$$

$$(r_1 r_2 + ... + r_1 r_{2N-1}) + (r_2 r_3 + ... + r_2 r_{2N-1}) + ... + (r_{2N-1} r_1 + ... + r_{2N-1} r_{2N-2}) = \frac{d_{2N-3}}{d_{2N-1}}$$

$$\vdots$$

$$r_1 r_2 ... r_{2N-1} = (-1)^{2N-1} \frac{d_0}{d_{2N-1}} = 0$$

where r_i are the $2N-1$ roots of (B.4). Since the even coefficients are zero in our EOM, some of the formulas have zeros in the left hand side. Equipped with these identities, by tuning the coefficients it is a priori possible to have a subset of N roots to sum to zero.

Another option is to not have a tree-level potential, which then allows to choose any ground state up to quantum corrections. At one-loop level, these are given by the Coleman-Weinberg effective potential [52, 53]:

$$V_{\text{eff}} = \frac{1}{64\pi^2} \text{Tr}(V''(\Sigma))^2 \ln \frac{V''(\Sigma)}{\mu^2} \tag{B.5}$$

where $V"(\Sigma)$ is the second derivative of the potential (in this case, the only contribution comes from the gauge boson mass terms), which depends on the VEV of Σ and μ^2 the renormalisation scale. The structure of the mass terms (3.4) implies that each gauge boson A_{ij} pairs up with its complex conjugate, i.e. there are no mixed terms $A_{ij} A^\dagger_{kl}$. Therefore, the trace can be written as a sum:

$$V_{\text{eff}} = \frac{g^4}{64\pi^2} \sum_{i \neq j} (a_i - a_j)^4 \ln \frac{(a_i - a_j)^2}{\mu^2} \tag{B.6}$$

Finding a global minimum seems like a daunting task, let us first try to find the minimum of one individual term. Defining $x_{ij} \equiv a_i - a_j$, the minimum is found at $x_{ij} = \pm\mu e^{-1/4}$. Now, the variables are not independent, for example $x_{31} = x_{32} + x_{21}$. A simultaneous minimization of all the terms is not possible, since the system of equations would be overconstraining (more equations than independent variables). Therefore, if we assign the individual minima to certain differences of eigenvalues, the others will be automatically fixed. In particular, for the masses (3.4) associated to these "minimum" differences we have:

$$m^2 \sim \mu^2/N \tag{B.7}$$

Nevertheless, due to the interdependence of the variables, other masses will arise with the scaling:

$$m^2 \sim (N\mu)^2/N \sim \mu^2 N \tag{B.8}$$

It is then unclear what spectrum of gauge boson masses would a Coleman-Weinberg potential render and therefore what is the remaining gauge group after SSB in the large N limit.

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