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Cosmological aspects of Mimetic Gravity

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Declaration of authorship

I hereby certify that the thesis I am submitting is my own original work: it has been composed by me and is based on my own work, unless stated otherwise. Any use of the works of any other author is properly stated when needed: all references have been quoted, and all sources of information have been acknowledged.

Date: Signature:

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Abstract

The thesis reviews the original proposal of Mimetic Dark Matter, a reformulation of General Relativity, in which the physical metric is parametrized in terms of an auxiliary metric and a scalar field. The equations of motion result to be a modified Einstein equation and a continuity equation for the scalar field, whose kinematical constraint leads to its identification with cosmological time in synchronous reference frame.
From the solution of the additional equation it is seen that the scalar field can mimic Dark Matter also in absence of ordinary matter, from which the adjective *mimetic*.
This extra degree of freedom is shown to be due to the singularity ot the disformal transformation linking the physical and auxiliary metrics.
Then, a straightforward generalization of the original model is reviewed: a potential is added to the Lagrangian and different cosmological scenarios such as Inflation, Bouncing Universe and Quintessence are reproduced in this context.
Further developments on mimetic cosmological phenomenology are also studied and an interlude on static spherically symmetric solutions in Mimetic Gravity, permits to understand how the rotational curves of galaxies, one of the most important evidences for Dark Matter, can be described through the mimetic scalar field subject to potential.
As a next step, the Hamiltonian analysis of the theory is pursued proving in a more formal way the presence of one more degree of freedom and showing the absence of Ostrogradski instability under certain conditions. Finally, Mimetic Dark Matter is identified with the Dust Field formalism at the classical level and the quantization of the latter is pursued and the results are compared with the well known Gauge-fixed picture.

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Chapter 1

Introduction

General Relativity (GR) is a well established theory which describes well our Solar System and all low energy gravitational phenomena. Despite of its successes in experimental predictions such as the perihelion precession of Mercury and gravitational waves, singularities point out regimes in which it fails, inside black holes and at the beginning of the Universe.
Furthermore, the current phase of accelerated expansion of the Universe can be obtained only by putting an extra term, the cosmological constant Λ or Dark Energy, into the Einstein equations for a Universe described by the the FLRW metric.
Besides to this component in the widely accepted model describing our Universe, the so called Λ-CDM, there is also Cold Dark Matter introduced to explain some well established astrophysical evidences such as the flatness of galaxies rotation curves, bullet clusters and gravitational lensing, whose direct detection is still missing.
Still, in this model there are some problems such as the homogeneity problem and the flatness problem, which can be solved by conjecturing the existence of a phase in which the Universe goes through an accelerated expansion, called Inflation.
Finally, more importantly GR describes gravity at the classical level and the attempts to find a consistent theory of Quantum Gravity valid at Planck scales have led to, among others, two theories, Loop Quantum Gravity and String Theory which, so far, have not produced any experimental prediction at energies currently available in particle accelerators. Because of this unhappy situation, in the last decades many physicists start to study theories generalizing the Einstein-Hilbert action at the classical level, with different approaches:

1. by considering functions of the Ricci scalar as in Starobinsky R^2 theory, the ancestor of $F(R)$ theories;

2. by adding scalars as in Brans-Dicke theory, which is the protoype of the more general Horndeski theory, the most general theory of gravity in

four dimensions whose Lagrangian contains as fields the metric tensor and a scalar field and leads to second order equations of motion;

3. by considering large extra dimensions as in Randall-Sundrum, ADD and DGP models.

Among the many ensembles of theories of modified gravity appeared so far, a very promising one is represented by Mimetic Gravity.
The first proposal of a model in Mimetic Gravity is due to V. Mukhanov and A. Chamseddine in 2013 [1].
The original paper starts with the Einstein-Hilbert action with the physical metric reparametrized through a singular disformal transformation in terms of an auxiliary metric and a scalar field, called mimetic, providing a theory in which a new degree of freedom can mimic the Cold Dark Matter component even in the absence of ordinary matter. Since the first work, many papers have appeared expanding the new idea [2] and showing an increasing interest in the field: a list of the many different mimetic modified theories of gravity that have appeared so far includes

1. mimetic $F(R)$ gravity [3];
2. Lorentz violating Galileon theory [4];
3. mimetic Randall-Sundrum II model [5];
4. mimetic Horndeski gravity [6,7];
5. modified Gauss-Bonnet gravity [8].

The aim of this work is to present a review of some important cosmological results in Mimetic Gravity.
Two of the most relevant things studied in the thesis regard the Hamiltonian analysis of the original model, showing in a formal way that there is one more degree of freedom than in General Relativity and solving under certain conditions the problem of the Ostrogradsky instability, and the identification at the classical level of Mimetic Dark Matter (MDM) with the Dust Field formalism, with its quantization in Mini-superspace. The use of Mini-superspace formalism in Quantum Cosmology, in which the Ricci scalar of the usual Einstein gravity is written in terms of the scale factor of a FLRW metric and the lapse function, before fixing the gauge, is justified by the fact that there are finite degrees of freedom and other mathematical and physical complications arising from considering the full gravitational Hamiltonian are removed in the attempt to apply quantum physics to the whole universe. The Dust Field formalism offers a way to solve the problem of time in doing Quantum Cosmology due to its different roles in Quantum Mechanics (a parameter) and in General Relativity (a coordinate on the same foot of the spatial coordinates): it is precisely the mimetic scalar field

to supply time as a parameter for the wave function of the whole Universe. The results for the expectation values for the relevant physical quantities such as scale factor, Hubble constant and energy density in Many Worlds Interpretation and in Bohmian Mechanics obtained in Dust Field formalism are then compared with those of Gauge-fixed picture, where the Hamiltonian in Mini-superspace, after the gauge-fixing procedure, evolves in time. The comparison reveals that, modulo numerical factors with some arbitrariness entailed, the Dust Field formalism and hence, MDM represents, as expected, a particular case of the other approach.
Future attempts will be done in quantization of the MDM model supplemented by a potential for the mimetic scalar field with the same procedure.
This work is structured in this way:

- in Chapter 2, some basics of Cosmology, the history of the Universe and some aspects of Inflation, Dark Energy and Dark Matter are reviewed;

- in Chapter 3, the original MDM model is reviewed, pointing out the reason of its difference with GR through a discussion of general disformal transformations and describing the Lagrange multiplier equivalent formulation;

- in Chapter 4, a straightforward generalization of MDM with a non-vanishing potential for the mimetic scalar field is reviewed, reproducing different cosmological scenarios, such as Inflation, Quintessence and Bouncing Universe. Cosmological perturbations in this setting and direct coupling of the mimetic field with ordinary matter, giving a model of gravitational baryogenesis, are also studied;

- in Chapter 5, static spherically symmetric solutions in Mimetic Gravity are reviewed and the rotational curves of spiral galaxies, which is one of the most important astrophysical evidences for Dark Matter, are obtained.

- in Chapter 6, the Hamiltonian analysis of the original proposal is reviewed, proving in a formal way the presence of a more degree of freedom than in GR and solving the problem of Ostrogradsky instability inside the theory;

- in Chapter 7, the classical Mini-superspace is introduced and the quantization in Gauge-fixed picture is pursued. Then the identification of the Lagrange multiplier formulation of the MDM model with the Dust Field formalism is made, the quantization of the latter in Mini-superspace is performed and the expectation values of the relevant physical quantities are compared with those of Gauge-fixed picture, showing that they coincide for dust as expected.

Chapter 2

Cosmology

The Big Bang theory is the prevailing cosmological model for the Universe from the earliest periods through its large-scale evolution. The observations, made so far, establish the ingredients of the Big Bang theory, known also as Cosmological Standard Model [9,10]:

1. at sufficiently large scales, the universe is isotropic: its properties are independent of the direction of observation.
2. Copernican principle: our location is not special. As a consequence, if the Universe is observed as isotropic from everywhere, it is also homogeneous;
3. the Universe is composed of radiation and baryonic as well as non-baryonic matter;
4. the Universe expands and at late times, the expansion is accelerated.

The first two points are summarized in the Cosmological Principle.
The latest version of the Cosmological Standard Model, called the Λ-CDM model, where Λ is the cosmological constant and CDM stands for Cold Dark Matter, accounts for the last two points, describing a Universe that contains usual luminous matter of the Standard Model of Particle Physics, Dark Matter (DM) and Dark Energy (DE), as required from cosmological and astrophysical data available [11].

2.1 General Relativity and Cosmology

The above ingredients can be formalized in GR: the spacetime is assumed to be a globally hyperbolic Lorentzian manifold endowed with a metric $g_{\mu\nu}$ [12].
The action, which describes the dynamics of a generic spacetime with the metric $g_{\mu\nu}$, is a sum of the Einstein-Hilbert action S_{EH}, describing gravity,

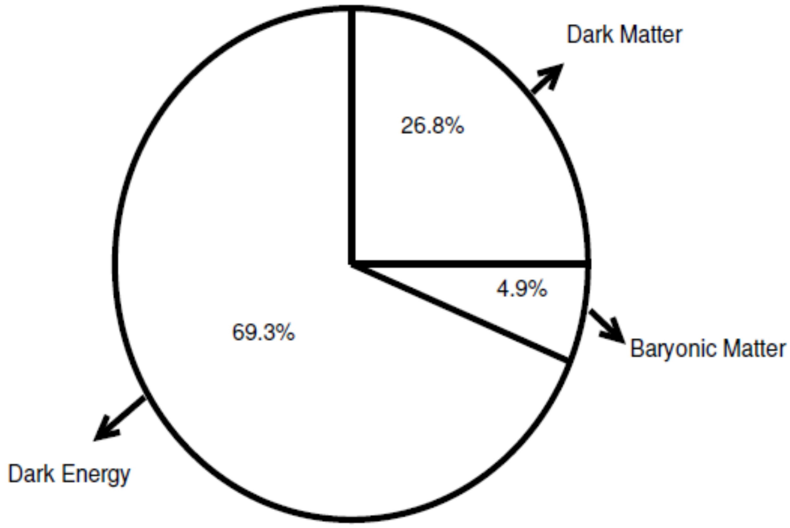

Figure 2.1: Relative abundacies of the matter components of the Universe at the present [11]

and of the matter component action S_M with a generic Lagrangian $\mathcal{L}_M$, given by

$$S = S_{EH} + S_M = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\mathcal{L}_M, \tag{2.1}$$

where G is the Newton gravitational constant, $R = g_{\mu\nu}R^{\mu\nu}$ is the Ricci scalar with $R_{\mu\nu}$, the Ricci tensor, g is the determinant of the metric and the speed of light $c = 1$.
From the variation with respect to the metric $g_{\mu\nu}$, the Einstein field equations yield, which relate the spacetime evolution to the energy of the matter content

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.2}$$

with

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{2.3}$$

and

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \tag{2.4}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of the matter component.
A homogeneous and isotropic Universe is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{2.5}$$

where $a(t)$ is the scale factor with t being the cosmic time and k is the spatial curvature constant, whose value can be $+1$, 0 or -1 describing closed, flat, and open Universes, respectively.
The dynamics of the FLRW Universe is described by the first Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho + \frac{k}{a^2}, \tag{2.6}$$

being the tt component of Einstein equation (2.2), and by the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{2.7}$$

being the rr component of Einstein equation (2.2). The Hubble constant H is defined as

$$H \equiv \frac{\dot{a}}{a}. \tag{2.8}$$

and the matter content of the Universe is described by the energy-momentum tensor of a perfect fluid given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{2.9}$$

where $\rho(t)$ and $p(t)$ are respectively its energy density and its isotropic pressure.
Furthermore, the conservation of the energy-momentum tensor $\nabla_\nu T^{\mu\nu} = 0$ leads to the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.10}$$

Since the Friedmann equations (2.6) and (2.7) and the continuity equation (2.10) are not independent, an extra equation is needed to complete the set of equations to specify all the three unknown variables ρ, p and a.
For the matter described by a perfect fluid, the extra equation is given by the equation of state

$$p = \omega\rho \tag{2.11}$$

where ω is a constant.
From the continuity equation (2.10), the relation between ρ and the scale factor a follows

$$\rho \propto a^{-3(1+\omega)}. \tag{2.12}$$

The most important examples of fluids are:

- dust matter, $\omega = 0$

$$a \propto t^{2/3} \quad \rho \propto a^{-3} \tag{2.13}$$

- radiation, $\omega = 1/3$

$$a \propto t^{1/2} \quad \rho \propto a^{-4} \tag{2.14}$$

- dark energy[1], $\omega = -1$

$$a \propto e^{\sqrt{\frac{8\pi G}{3}\rho} t} \qquad \rho \propto \text{const.} \tag{2.15}$$

The first Friedmann equation (2.6) can also be rewritten as

$$\Omega - 1 = \frac{k}{a^2 H^2} \tag{2.16}$$

where

$$\Omega \equiv \frac{\rho}{\rho_c}, \quad \text{with} \quad \rho_c \equiv \frac{3H^2}{8\pi G}, \tag{2.17}$$

where the density parameter Ω is the ratio of the energy density to the critical density ρ_c, for which the Universe results to be flat ($k = 0$).
The data available today show that the present value of the density parameter is $\Omega = 1$ and the Universe is flat: the presence of DM and DE is also needed for this reason due to the fact that ordinary matter represents only a small percentage of the critical density ρ_c (Figure 2.1) [11].

2.2 Brief thermal history of the Universe

After having discussed some elements of Cosmology, here a brief description of the thermal history of the Universe is given [9].

- **Big Bang**
  The starting point of the Universe is assumed to be the Big Bang which occurs at zero time, but not much is known about this state.

- **Planck scale** $\sim 10^{-43}$ s (10^{19} GeV)
  After 10^{-43} s, the radius of the Universe is 10^{-35} m, called the Planck length and the Universe has cooled to a temperature of about 10^{32} K. Near the Planckian scale, nonperturbative quantum gravity dominates and GR fails: gravity is unified with the other three forces.

- **Grand Unification scale** $\sim 10^{-14} - 10^{-43}$ s (10 TeV $\lesssim E \lesssim 10^{19}$ GeV)
  There is no reason to expect that non-perturbative Quantum Gravity plays any significant role below 10^{19} GeV.
  After 10^{-38} s, in fact, gravity has decoupled from the other forces and classical spacetime makes sense to describe the dynamics of the Universe but electromagnetic, weak and strong forces remain unified as one single force with a single coupling constant, up to $T \simeq 10^{29}$ K. Quarks and leptons exist in a plasma state at energy scale $E \sim 10^{16}$ GeV. At these energy scales, there is some uncertainty regarding the

[1] A more detailed discussion in a later section.

matter content.
It might be that there are many more particle species than those found in the particle accelerators so far and many theories of Physics Beyond the Standard Model are supposed to try to describe the matter content in this regime.

- **Electroweak scale** $\sim 10^{-10} - 10^{-14}$ s ($E \sim 100$ GeV - 10 TeV)
  In this time interval, strong interactions decouple from electromagnetic and weak interactions.
  The Higgs field condenses and acquires a spontaneous vacuum expectation value, endowing masses to the quarks, leptons and weak bosons. The typical energy of particles in the Universe is given by the electroweak scale, $E_{EW} \sim 1$ TeV, with the Higgs phase transition temperature given by $T_H = 10^{16}$ K.

- **Quark-Gluon transition, Baryogenesis** $\sim 10^{-6}$ s ($E \sim 1$ GeV)
  The quark-gluon transition takes place: free quarks and gluons become confined within baryons and mesons.
  The matter/antimatter symmetry is also broken and matter becomes dominant. This process is called Baryogenesis.

- **Neutrino decoupling** ~ 0.2 s ($E \sim 1 - 2$ MeV)
  The primordial neutrinos decouple from other particles and propagate without scatterings and the ratio of neutrons to protons freezes out because the interactions that keep neutrons and protons in chemical equilibrium become inefficient.

- **Electron-positron asymmetry** ~ 1 s ($E \sim 0.5$ MeV)
  The typical energy at this time is of the order ofthe electron mass. Electron- positron pairs begin to annihilate when the temperature drops below their rest mass and only a small excess of electrons over positrons, roughly one per billion photons, survives after annihilation.

- **Nucleosynthesis** 10^2 s $\simeq 3$ min ($E \sim 0.05$ MeV)
  Nuclear reactions become efficient at this temperature: helium and other light elements start to form in the process of Nucleosynthesis.

- **Cosmic Microwave Background** $\sim 10^{13}$ s $\simeq 380000$ years ($E \sim$ eV)
  Matter-radiation equality occurs: the radiation-dominated epoch ends and the matter-dominated epoch starts, with the Universe cooling to the temperature $T \simeq 3000$ K.
  At this temperature, atoms form, and the absence of free charges makes the Universe transparent to radiation which decouples from matter.

The Cosmic Microwave Backgroung (CMB) radiation is the relic radiation, travelling freely after this Recombination epoch.
The CMB temperature fluctuations, induced by the slightly inhomogeneous matter distribution in this epoch, survive to the present and carry information about the state of the Universe at the last scattering surface.

- **Large structures** $\sim 10^{16} - 10^{17}$ s
Galaxies and their clusters have formed from small initial inhomogeneities due to gravitational instability.
The CMB radiation has further cooled to the temperature of 2.7 K.
The main unresolved fundamental issue regarding this period is the nature of DM and DE.

2.3 Problems of the Standard Big-Bang model

Here, are briefly summarized the main issues that affect the Standard Big-Bang model, which lead to the need to introduce the inflationary scenario [9,10,13,14].

1. **Flatness problem**
In the Cosmological Standard Model, for the Universe it is valid that $\ddot{a} < 0$ for all time, i.e. it is decelerating, so a^2H^2 decreases: this indicates that Ω tends to shift away from unity with the expansion of the Universe.
However, present observations indicate that Ω is very close to one. From this follows that in the past it should have been closer to one. For instance, it is required $|\Omega - 1| < \mathcal{O}(10^{-20})$ at the epoch of Nucleosynthesis, leading to a huge fine-tuning.

2. **Horizon problem**
CMB photons, which are propagating freely since they decoupled from matter at the epoch of last scattering, appear to be in thermal equilibrium at almost the same temperature. To explain this, it is supposed that the Universe has reached a state of thermal equilibrium through interactions among different regions before the last scattering: the cosmological scales that can be seen must have been casually connected before the decoupling of radiation from matter. But this is not possible for the regions that became casually connected recently to interact before this event, because of the finite speed of light.
The new regions of the Universe that appear from the cosmological horizon scale should not be in causal connection if their angular distance is of order 1°. However, photons are seen to have almost the same temperature with anisotropies of the order $\Delta T/T \simeq 10^{-5}$ in all the CMB sky.

3. **Monopole problem**
   Particle physics predicts that spontaneous symmetry breakings occur in the primordial Universe, with high temperatures and high densities. These events could have produced many unwanted relics such as monopoles, cosmic strings, and topological defects. If these particles existed in the early stage of the Universe, their energy densities would decrease as a matter component and these massive relics would be the dominant matter, differently from what is measured today.

2.4 Inflation

To solve these problems, it is claimed that, besides the series of the events described above, the Universe underwent a period of exponential expansion lasted from 10^{-36} s to 10^{-32} s, after the Big Bang. To have such exponential expansion, a necessary condition to occur is the violation of the Strong Energy Condition (SEC)

$$\rho + 3p \geq 0, \tag{2.18}$$

for whatever matter component of the Universe present during this phase, as can be seen from the second Friedmann equation (2.7).
In one of the simplest models of Inflation, the dominating matter component is a minimally coupled scalar field φ, called the inflaton, to drive the accelerated expansion in a FLRW Universe

$$S = \frac{1}{2} \int d^4x \sqrt{-g}[g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi)]. \tag{2.19}$$

The energy-momentum tensor is obtained by using the definition (2.4) and has the form

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \Big(\frac{1}{2} g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi - V(\varphi) \Big). \tag{2.20}$$

By comparing this expression with the energy-momentum tensor of a perfect fluid (2.9) and taking into account that the scalar field must respect the symmetries of the FLRW Universe, the following expression for the energy density and the pressure yield

$$\rho = \frac{1}{2} \dot{\varphi}^2 + V(\varphi), \tag{2.21}$$

$$p = \frac{1}{2} \dot{\varphi}^2 - V(\varphi). \tag{2.22}$$

The Friedmann equations (2.6) and (2.7) take the form

$$3H^2 = V(\varphi) + \frac{1}{2} \dot{\varphi}^2, \tag{2.23}$$

$$\ddot{\varphi} + 3H\dot{\varphi} = -\partial_\varphi V(\varphi). \tag{2.24}$$

where the second equation coincides with the equation of motion for φ.
To violate the SEC (2.18) it must be valid that the kinetic energy is dominated by the potential energy

$$\dot{\varphi}^2 < V(\varphi). \tag{2.25}$$

In order for Inflation to end, it is required that a stable point where such condition is violated, must exist. Furthermore, the model must provide a graceful exit to a radiation dominated epoch.
To solve the equations (2.23) and (2.24), one usually uses the so-called slow-roll approximation

$$\dot{\varphi}^2 << V(\varphi). \tag{2.26}$$

In this approximation the equations (2.23) and (2.24) become

$$3H^2 \simeq V(\varphi), \tag{2.27}$$

$$3H\dot{\varphi} \simeq -\partial_\varphi V(\varphi). \tag{2.28}$$

If the scalar field satisfies the slow-roll conditions, then Inflation is guaranteed.
The shape of the potential is often characterized by slow-roll parameters defined as

$$\epsilon(\varphi) \equiv \frac{1}{2}\Big(\frac{\partial_\varphi V}{V}\Big)^2, \quad \eta(\varphi) \equiv \frac{\partial^2_\varphi V}{V}, \tag{2.29}$$

where ϵ measures the slope of the potential, and η measures the curvature.
Necessary conditions for the slow-roll approximation are

$$\epsilon << 1, \quad |\eta| << 1. \tag{2.30}$$

Inflation has to last long enough in order to bring Ω sufficiently close to 1.
A standard measure of the expansion during Inflation is the number of e-foldings, which is given by

$$N = \log\frac{a(t_{end})}{a(t_{in})}, \tag{2.31}$$

where $a(t_{in})$ and $a(t_{end})$ are the values of the scale factor at the initial and at the end time of the inflationary epoch.
By using the slow-roll approximation, this number can be expressed without solving the equations of motion

$$N = -\int_{\varphi_{in}}^{\varphi_{end}} \frac{V}{\partial_\varphi V} d\varphi \tag{2.32}$$

Thanks to the features of Inflation, the problems discussed above result to be solved in inflationary scenario.

For the flatness problem, since the a^2H^2 term in (2.16) decreases during inflation, Ω rapidly approaches unity. When the inflationary period ends, the evolution of the Universe is followed by the decelerated expansion and $|\Omega - 1|$ begins to increase in the radiation and matter dominated epochs. However, if the inflationary expansion occurs for a sufficiently long period, Ω remains of order unity even at the present epoch.
Regarding the horizon problem, during Inflation one has $\ddot{a} > 0$ and this implies that the comoving Hubble radius r_H is decreasing. So the physical wavelength $\sim a$ grows faster than the Hubble radius during the inflationary epoch. This means that regions that are causally connected were stretched on scales larger than the Hubble radius.
In order to solve the horizon problem, it is thus required that

$$r_H(t_0) < r_H(t_i). \tag{2.33}$$

The horizon and flatness problems can be solved if the Universe expands about e^{70} times during the inflationary period, thus $N \simeq 70$.
For the last problem, the accelerated expansion dilutes unwanted relics density since it is proportional to a^{-3} and this fits the experimantal observations that show that the contribution of these particles to the Universe density is completely negligible today.
For further discussions on models of Inflation, see [10].

2.5 Dark Energy and Quintessence

Besides to adding a perfect fluid with equation of state $p = -\rho$ as done in section 2.1, a way to model DE is adding a positive cosmological constant Λ to the Einstein-Hilbert action (2.1)

$$S_\Lambda = -\int d^4x\sqrt{-g}\Lambda. \tag{2.34}$$

Variation with respect to the metric gives the Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{2.35}$$

which has one more term with respect to equation (2.2).
The associated Friedmann equations read

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} + \frac{k}{a^2} \tag{2.36}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.37}$$

So, from equation (2.36) for a flat Universe dominated by the cosmological constant, the scale factor

$$a(t) \propto \exp\Big(\sqrt{\frac{\Lambda}{3}}t\Big). \tag{2.38}$$

This solution gives rise to a late-time accelerated expansion because Λ remains constant for all time, while dust decays as seen in (2.13) and as time grows, Λ starts to compete and then overcomes dust.
An alternative way to model the late-time acceleration is to consider DE varying with time, through an additional scalar field.
As seen above, a scalar field can naturally provide a cosmic fluid with negative pressure that can drive accelerated expansion.
One can consider the same mechanism to give account for the late-time observed phenomenology.
Such models are usually called Quintessence. Many models with non-trivial kinetic terms have been proposed so far but in this section a basic one with standard kinetic term is discussed. For a review of these models, see [15].
The scalar field χ is described by the following action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\chi)\right]. \tag{2.39}$$

Similarly to the Inflation the non-trivial behavior stems from the choice of the shape of the potential.
In a flat FLRW Universe the equation of motion for the field χ is

$$\ddot{\chi} + 3H\dot{\chi} = -\partial_\chi V(\chi), \tag{2.40}$$

where the spatial derivatives as in the case of the inflaton, are not taken into account, since χ is assumed to be homogeneous because it has to respect the symmetries of the metric describing the Universe.
The energy-momentum tensor takes the same form as for the inflaton (2.20) and the energy density and the pressure for this field is the same as (2.21) and (2.22).
The ratio of the pressure over the energy density is given by

$$\omega_\chi = \frac{p}{\rho} = \frac{\dot{\chi}^2 - 2V(\chi)}{\dot{\chi}^2 + 2V(\chi)}, \tag{2.41}$$

with $\omega_\chi \in [-1, 1]$. The lower bound corresponds to the slow-roll approximation. The solution of the continuity equation (2.10) for time-dependent equation of state reads

$$\rho = \rho_0 \exp\Big(\int 3(1+\omega_\chi)\frac{da}{a}\Big), \tag{2.42}$$

where ρ_0 is a constant of integration.
By using the last expression for the scale factor a, the condition that the potential has to satisfy such that accelerated expansion can be produced, can be derived.
The border between accelerated and decelerated expansion is

$$a(t) \propto t. \tag{2.43}$$

The potential can be found with no extra effort if the scale factor obeys a power law

$$a(t) \propto t^p. \tag{2.44}$$

From Friedmann equations (2.6) and (2.7), it is obtained the following relation

$$\dot{H} = -4\pi G \dot{\chi}^2. \tag{2.45}$$

By using the last equation and the expressions for ρ and p (2.21) and (2.22) in the second Friedmann equation (2.7), the potential V and the field χ are expressed in terms of H and $\dot{H}$

$$V = \frac{3H^2}{8\pi G}\Big(1 + \frac{\dot{H}}{3H^2}\Big), \tag{2.46}$$

$$\chi = \int dt(-2\dot{H})^{1/2}. \tag{2.47}$$

By substituting into these formulas the scale factor (2.44), and by using the definition of the Hubble constant (2.8), equations (2.46) and (2.47) become

$$V = \frac{p(3p-1)}{8\pi G t^2}, \tag{2.48}$$

$$\chi = \pm\Big(\frac{p}{4\pi G}\Big)^{1/2} \log(t). \tag{2.49}$$

By expressing t from the positive branch of the last equation and plugging it in (2.48), it is obtained

$$V(\chi) = \frac{p(3p-1)}{8\pi G} \exp\Big(-\Big(\frac{16\pi G}{p}\Big)^{1/2}\chi\Big). \tag{2.50}$$

The major difference with the Inflation is that the potential is chosen to achieve accelerated expansion at late time, instead of the early Universe.

2.6 Dark Matter

The main features of DM are naively summarized as follows [11]:

1. DM is a nonluminous matter component which has no electromagnetic interaction. So, it should consist of chargeless neutral particles;

2. DM should be composed by stable particles which do not decay to known particles, pervades the Universe and helps the formation of large-scale structure;

3. The interaction of DM with Standard Model particles is very weak and only of gravitational type.

The major evidence for the existence of DM is given by spiral galaxies rotation curves.
The stars in a galaxy orbit around the center of the galaxy. These orbits around the galactic center are roughly circular but oscillate in their bound closed paths due to the gravitational influence of other objects in the galaxy, such as other stars and planets.
So each star has also a back-and-forth motion along the radial direction along its orbit. Studies of the radial motion are performed by the spectroscopic method, measuring the shift of the light coming from the star, due to the Doppler effect. However, the average motion of a star in a spiral galaxy is circular with great approximation. Thus, the velocity of this circular motion is such that it balances the gravitational force on the star toward the galactic center to keep it in the circular motion.
A rotational curve of a galaxy is the orbital speed of the stars in a galaxy as a function of the radial distance of the stars from the galactic center. For a

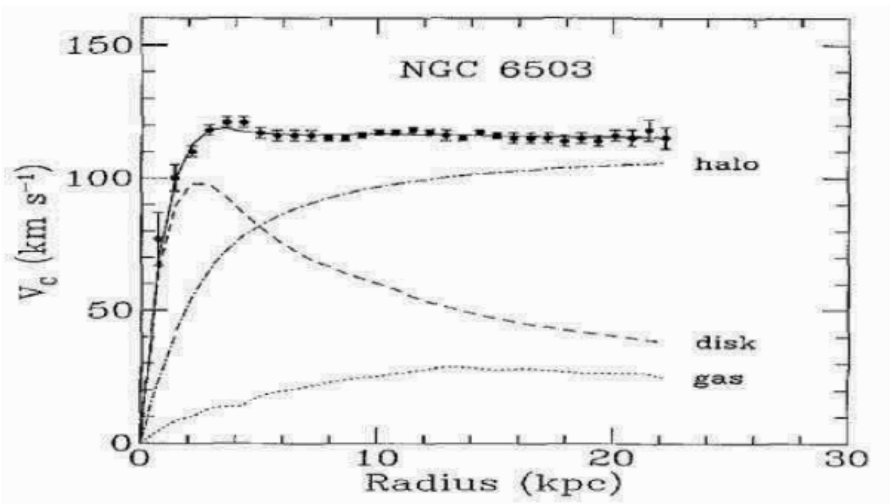

Figure 2.2: An observed rotation curve data of a spiral galaxy [2]

spiral galaxy there is a central bulge where most of the mass is concentrated and the spiral arms are spread over a disk.
For a star in such a galaxy at a distance r from the galactic center moving with a circular velocity $v(r)$, the balance between the gravitational force and

the centrifugal force is given by

$$\frac{mv(r)^2}{r} = \frac{GmM_{<r}}{r^2}, \tag{2.51}$$

where $M_{<r}$ is the mass in the sphere of radius r. If the star is in the bulge of the galaxy with average density ρ, then $M_{<r} = \frac{4}{3}\pi r^3 \rho$. Then, for a star in the central bulge from equation (2.50) it is expected

$$v(r) \sim r. \tag{2.52}$$

But for a star outside this dense center, the mass $M_{<r}$ can be assumed constant and coincides approximately with the mass of the central bulge. Then from equation (2.51) it follows that

$$v(r) \sim \frac{1}{\sqrt{r}}. \tag{2.53}$$

Thus the variation of $v(r)$ with r for a spiral galaxy should initially increase and then decrease.
However, observational data of rotation curves for several spiral galaxies show $v(r) =$ constant for large r, as can be seen in the Figure 2.2. Then one gets from equation (2.51) that $M_{<r} \sim r$, suggesting the presence of a huge quantity of nonluminous only gravitationally interacting mass in the galaxy. This is one of the astrophysical evidence for the DM hypothesis and DM is believed to form a halo in which the galaxy is embedded.
The different forms of DM that have been conjectured to fill the Universe are distinguished on the basis of the mechanism of their production, particle types and their masses and speed.
First of all, DM can be classified on the basis of whether it was produced thermally or non-thermally in the early Universe.
In the case of thermal production, DM is produced via the collision of cosmic plasma in radiation-dominated era, while the non-thermal DM particles are produced by other mechanisms, such as the decay of some massive particles.
The particle nature of DM can be of two types, baryonic or non-baryonic.
DM can not be made of known particles, as already said, but the visible Universe can not account for the baryon density in the Universe: at least some DM must be baryonic in order to account for the astrophysical data available. The baryonic DM can be inside Massive AstrophysiCal Halo Objects (MACHOs). Other suggestions for baryonic DM are, for instance, brown dwarfs and primordial black holes.
The baryonic contribution to DM is negligible, so the DM should be mostly non-baryonic. Non-baryonic DM particles have very weak interactions with ordinary matter and hence they are hard to detect. Some of possible non-baryonic DM candidates can be massive enough to account for DM relic density of the Universe.

The relic particles comoving density becomes constant when the interaction rate of DM particles falls below the expansion rate of the Universe.
The mass of DM particles and the temperature of the Universe at the time of their decoupling determine whether DM particles were relativistic or non-relativistic at that time. One can distinguishes between Hot Dark Matter (HDM) and Cold Dark Matter (CDM).
HDM is characterized by relativistic speeds and at the time of freeze-out, was extremely relativistic with masses less than kinetic energies.
More precisely, given a HDM species, let m be the mass of the particles and T_f its freeze-out temperature, it holds

$$x_f \lesssim 3 \quad \text{with} \quad x_f = \frac{m}{T_f}. \tag{2.54}$$

On the other hand, for CDM it holds

$$x_f \gtrsim 3 \tag{2.55}$$

and at the freeze-out, this type of DM particles were non-relativistic and heavier than HDM particles.
In the Λ-CDM model, HDM is negligible with respect to CDM.
No DM particle candidates have been detected so far. Therefore, many attempts to explain phenomenology of galaxies by modifying Gravity have been made. One is represented by Mimetic Gravity, reviewed in this work.

Chapter 3

Mimetic Dark Matter

In this chapter, the features of the original model of Mimetic Gravity, called Mimetic Dark Matter (MDM) [1], will be presented, pointing out its relation with general disformal transformations.
An equivalent formulation will be introduced and will be used in the next chapters for further developments.

3.1 The original proposal

The physical metric $g_{\mu\nu}$ is reparametrized in terms of a scalar field ϕ and an auxiliary metric $l_{\mu\nu}$

$$g_{\mu\nu} = l_{\mu\nu}(l^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) \equiv w l_{\mu\nu} \tag{3.1}$$

such that the physical metric $g_{\mu\nu}$ results to be invariant under the conformal transformation of the auxiliary metric $l_{\mu\nu} \to \Omega^2 l_{\mu\nu}$.
While the Einstein equations are obtained by varying the Einstein-Hilbert action and the matter action (2.1) with respect to the metric $g_{\mu\nu}$, here the fundamental fields describing gravity are taken to be the auxiliary metric $l_{\mu\nu}$ and the scalar field ϕ and the new equations are found by varying with respect to these two fields.
For this reason, the usual general relativistic Einstein-Hilbert action can be written in terms of the physical metric $g_{\mu\nu}$, considered as a function of the auxiliary metric $l_{\mu\nu}$ and on the scalar field ϕ.
Hence, the full action of the theory is

$$S = \int d^4x \sqrt{-g(l_{\mu\nu}, \phi)} \left[-\frac{1}{2\kappa} R(g_{\mu\nu}(l_{\mu\nu}, \phi)) + \mathcal{L}_M \right] \tag{3.2}$$

where $\kappa \equiv 8\pi G$ and $c = 1$ and the form of the matter Lagrangian $\mathcal{L}_M$ is not specified.
The variation with respect to the physical metric $g_{\mu\nu}$ as usual, leads to

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{-g} (G^{\alpha\beta} - \kappa T^{\alpha\beta}) \delta g_{\alpha\beta}. \tag{3.3}$$

Now, the variation $\delta g_{\alpha\beta}$ can be recasted in terms of the variation of the auxiliary metric $\delta l_{\alpha\beta}$ and the variation of the scalar field $\delta\phi$

$$\begin{aligned}\delta g_{\alpha\beta} &= w\delta l_{\alpha\beta} + l_{\alpha\beta}\delta w = \\ &= w\delta l_{\mu\nu}(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_{\kappa}\phi\partial_{\lambda}\phi) + 2g_{\alpha\beta}g^{\kappa\lambda}\partial_{\kappa}\delta\phi\partial_{\lambda}\phi\end{aligned} \tag{3.4}$$

By plugging this expression in (3.3), the following equations of motion are obtained

$$(G^{\mu\nu} - \kappa T^{\mu\nu}) - (G - \kappa T)g^{\mu\alpha}g^{\nu\beta}\partial_{\alpha}\phi\partial_{\beta}\phi = 0, \tag{3.5}$$

and

$$\frac{1}{\sqrt{-g}}\partial_{\kappa}(\sqrt{-g}(G - \kappa T)g^{\kappa\lambda}\partial_{\lambda}\phi) = \nabla_{\kappa}((G - \kappa T)\partial^{\kappa}\phi) = 0, \tag{3.6}$$

where (3.5) are modified Einstein field equations and (3.6) takes the form of a continuity equation for the scalar field ϕ.
To go further in the analysis of the model, it is worth noticing that from (3.1), the inverse physical metric $g^{\mu\nu}$ can be written in terms of the inverse auxiliary metric $l^{\mu\nu}$ as

$$g^{\mu\nu} = \frac{1}{w}l^{\mu\nu}, \tag{3.7}$$

and for consistency, the scalar field satisfies the constraint equation

$$g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi = 1. \tag{3.8}$$

The trace of (3.5) is given by

$$(G - \kappa T)(1 - g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi) = 0 \tag{3.9}$$

which is identically satisfied even for $G - \kappa T \neq 0$.
In fact, the trace $G - \kappa T$ is determined from (3.6) and (3.8) and even in the absence of matter, the equations for the gravitational field have non-trivial solutions for the conformal mode.
As will be shown explicitly later in this section, solving equation (3.6) for ϕ, it determines $G - \kappa T$. Hence, in addition to two transverse degrees of freedom, already present in GR and describing gravitons, an extra longitudinal degree of freedom shared by the scalar field ϕ and a conformal factor of the physical metric becomes dynamical.
Now it is easy to show what this extra degree of freedom describes, by rewriting (3.5) in the form

$$G^{\mu\nu} = \kappa T^{\mu\nu} + \tilde{T}^{\mu\nu}, \tag{3.10}$$

where

$$\tilde{T}^{\mu\nu} = (G - \kappa T)g^{\mu\alpha}g^{\nu\beta}\partial_{\alpha}\phi\partial_{\beta}\phi, \tag{3.11}$$

By comparison with the energy-momentum tensor of a perfect fluid (2.9), the energy density $\tilde{\varepsilon}$ and the four-velocity u^μ are given by

$$\tilde{\varepsilon} \equiv G - \kappa T, \quad u^\mu \equiv g^{\mu\alpha}\partial_\alpha \phi. \tag{3.12}$$

Hence, the energy-momentum tensor $\tilde{T}^{\mu\nu}$ describes pressureless dust with energy density $\tilde{\varepsilon} = G - \kappa T$ and the scalar field plays the role of a velocity potential, whose normalization condition is given by the kinematical constraint (3.8).
Furthermore, an interesting property of $\tilde{T}^{\mu\nu}$ is that from its conservation law, the equation for the scalar field ϕ (3.6) is recovered.
To find a general explicit solution for equation (3.6), it is convenient to consider a synchronous coordinate system where the metric takes the form

$$ds^2 = dt^2 - \gamma_{ij}dx^i dx^j, \tag{3.13}$$

where γ_{ij} is a 3-dimensional metric.
Moreover, by assuming that the scalar field ϕ is spatially homogeneous, the constraint (3.8) reduces to $g^{00}(\partial_t\phi)^2 = 1$, whose general solution is given by

$$\phi = \pm t + C. \tag{3.14}$$

So the scalar field ϕ can be identified with time in a synchronous background.
By using this identification, equation (3.6) in a synchronous coordinate system, simplifies to

$$\partial_t\Big(\sqrt{-\gamma}(G - \kappa T)\Big) = 0, \tag{3.15}$$

with $\gamma = \det \gamma_{ij}$.
The solution of this equation is given by

$$G - \kappa T = \frac{C(x^i)}{\sqrt{-\gamma}}, \tag{3.16}$$

where $C(x^i)$ is a constant of integration depending only on spatial coordinates.
In flat FLRW universe, where $\gamma_{ij} = a^2(t)\delta_{ij}$, the solution can be rewritten as

$$G - \kappa T = \frac{C(x^i)}{a^3}, \tag{3.17}$$

that is, the energy density of the extra scalar degree of freedom is proportional to a^{-3} and imitates pressureless dust, whose amount is determined by the time-independent constant of integration $C(x^i)$.
The presence of ordinary matter matter modifies only the amount of energy density but not how it scales with the scale factor.
This is the reason why this model has been called Mimetic Dark Matter (MDM) and to the scalar field is added the adjective *mimetic*.

3.2 Disformal transformations

3.2.1 General disformal transformations

Now the question regarding MDM model introduced in the previous section, is: why the seemingly innocous reparametrization of the physical metric described above leads to such dramatic change for the GR equations of motion. The answer is that this is a particular singular disformal transformation [16]. To analyse this important issue, instead of the physical metric reparametrization adopted in (3.1), here a more general one, called disformation or disformal transformation is taken into account

$$g_{\mu\nu} = F(\phi, w) l_{\mu\nu} + H(\phi, w)\partial_\mu\phi\partial_\nu\phi, \tag{3.18}$$

where F and H are a priori arbitrary functions of the field ϕ and of the conformal factor w, defined above.
It can be easily noticed that the reparametrization leading to the MDM model is just a special case: $F = w$ and $H = 0$.
By following the same steps as in the previous section, the following equations of motion are obtained

$$F(G^{\mu\nu} - \kappa T^{\mu\nu}) = \left(A\frac{\partial F}{\partial w} + B\frac{\partial H}{\partial w}\right)(l^{\mu\rho}\partial_\rho\phi)(l^{\nu\sigma}\partial_\sigma\phi), \tag{3.19}$$

$$\frac{2}{\sqrt{-g}}\partial_\rho\left\{\sqrt{-g}\partial_\sigma\phi\left[H(G^{\rho\sigma} - \kappa T^{\rho\sigma}) + \left(A\frac{\partial F}{\partial w} + B\frac{\partial H}{\partial w}\right)l^{\rho\sigma}\right]\right\} = A\frac{\partial F}{\partial\phi} + B\frac{\partial H}{\partial\phi}. \tag{3.20}$$

The first equation is the modified Einstein equation and the second equation is the continuity equation for the mimetic scalar field ϕ, in which the following definitions are used

$$A \equiv (G^{\rho\sigma} - \kappa T^{\rho\sigma})l_{\rho\sigma}, \quad B \equiv (G^{\rho\sigma} - \kappa T^{\rho\sigma})\partial_\rho\phi\partial_\sigma\phi. \tag{3.21}$$

3.2.2 Veiled General Relativity

Contractions of the modified Einstein equation (3.19) with $l_{\mu\nu}$ and $\partial_\mu\phi\partial_\nu\phi$ yield

$$A\left(F - w\frac{\partial F}{\partial w}\right) - Bw\frac{\partial H}{\partial w} = 0,$$

and

$$Aw^2\frac{\partial F}{\partial w} - B\left(F - w^2\frac{\partial H}{\partial w}\right) = 0, \tag{3.22}$$

respectively. This is a system of two algebraic equations in the variables A and B and its determinant is given by

$$\det = w^2 F\frac{\partial}{\partial w}\left(H + \frac{F}{w}\right). \tag{3.23}$$

For det $\neq 0$, the unique solution of the system (3.22) is $A = B = 0$ and the equations (3.19) and (3.20) reduce to

$$F(G^{\mu\nu} - \kappa T^{\mu\nu}) = 0, \quad \partial_\rho[\sqrt{-g}\partial_\sigma\phi H(G^{\rho\sigma} - \kappa T^{\rho\sigma})] = 0 \tag{3.24}$$

where $F \neq 0$.
The first equation is the usual Einstein equation and, as a consequence, the second one is identically zero.
Thus, in the generic case, the variation of the Einstein-Hilbert action with respect to the disformed metric $g_{\mu\nu}$ or with respect to $l_{\mu\nu}$ and ϕ, are completely equivalent.
An easy proof of a general theorem in classical field theory stating that, for a general non-singular transformation, connecting two different sets of fields describing the same theory, the number of degrees of freedom in the theory, i.e. the number of initial configurations needed to specify the time evolution of the fields, remains the same, is given in [17].
This is valid in particular for non-singular disformal transformations.

3.2.3 Mimetic Gravity

The theorem in [17] is not in general true for a singular disformal transformation: in this case the number of degrees of freedom can decrease or increase.
A singular disformal transformation (3.1) connects the physical metric $g_{\mu\nu}$ with the auxiliary metric $l_{\mu\nu}$ and the mimetic scalar field ϕ: when GR is rewritten in terms of this new variables is not guaranteed that the number of degrees of freedom remains the same as when it is written in terms of the physical metric and, indeed, there is one more degree of freedom.
The Hamiltonian analysis in Chapter 6 will be crucial for convincing oneself of the truth of the last statement.
Hovever, here is interesting to see what happens in a general situation, of which (3.1) is a particular case.
By turning to the case when the determinant (3.23) is zero, a very different situation is found.
For $F \neq 0$, the function $H(w, \phi)$ takes the form

$$H(w, \phi) = -\frac{F(w, \phi)}{w} + h(\phi). \tag{3.25}$$

The solution of the system (3.22) is $B = wA$ and the equations of motion (3.19) and (3.20) become

$$G^{\mu\nu} - \kappa T^{\mu\nu} = \frac{A}{w}(l^{\mu\rho}\partial_\rho\phi)(l^{\nu\sigma}\partial_\sigma\phi), \quad \frac{2}{\sqrt{-g}}\partial_\rho(\sqrt{-g}hAl^{\rho\sigma}\partial_\sigma\Psi) = Aw\frac{dh}{d\phi} \tag{3.26}$$

For $h(\phi) \neq 0$, these equations of motion can be written in terms of the disformed metric $g_{\mu\nu}$ only

$$G_{\mu\nu} - \kappa T_{\mu\nu} = (G - \kappa T)h\partial_\mu\phi\partial_\nu\phi, \quad 2\nabla_\rho[(G - \kappa T)h\partial^\rho\phi] = (G - \kappa T)\frac{1}{h}\frac{dh}{d\phi}, \tag{3.27}$$

where it is used that the disformed metric (3.18) can be expressed, by using equation (3.25), as

$$g_{\mu\nu} = F(w, \phi)l_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi\Big(- \frac{F(w, \phi)}{w} + h(\phi)\Big). \tag{3.28}$$

and the inverse metric $g^{\mu\nu}$ is given by[1]

$$g^{\mu\nu} = \frac{l^{\mu\nu}}{F} + \frac{F - wh}{Fhw^2}(l^{\mu\rho}\partial_\rho\phi)(l^{\nu\sigma}\partial_\sigma\phi). \tag{3.29}$$

By using the last equation, one has that $A = (G - \kappa T)/(hw)$ and $l^{\mu\rho}\partial_\rho\phi = hw\partial^\mu\phi$, where $G - \kappa T \equiv g_{\rho\sigma}(G^{\rho\sigma} - \kappa T^{\rho\sigma})$ and $\partial^\mu\phi \equiv g^{\mu\rho}\partial_\rho\phi$.
Finally, by means of a field redefinition, the function $h(\phi)$ can be eliminated, by introducing the field Φ such that $\frac{d\Phi}{d\phi} = \sqrt{|h|}$, yielding

$$G_{\mu\nu} - \kappa T_{\mu\nu} = \epsilon(G - \kappa T)\partial_\mu\Phi\partial_\nu\Phi, \quad 2\nabla_\rho[(G - \kappa T)\partial^\rho\Phi] = 0, \tag{3.30}$$

where $\epsilon = \pm 1$ depending on the sign of the norm of $\partial_\mu\phi$: $g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = \epsilon$, which depends on the choice of the metric signature.
For $\epsilon = +1$, these equations are exactly the same as the original equations of motion for MDM (3.5) and (3.6).
So, for a general non-singular disformal transformation, MDM is recovered by means of a field redefinition.

3.3 Lagrange multiplier formulation

Besides of the original formulation of MDM model, an equivalent formulation by using a Lagrange multiplier can be derived [18]. For an alternative derivation, see [19].
For this purpose, a set of Lagrange multipliers $\lambda^{\mu\nu}$ is introduced to impose the reparametrization of the physical metric $g_{\mu\nu}$ in terms of the auxiliary one $l_{\mu\nu}$ and the mimetic field ϕ. The action, ignoring the matter Lagrangian, reads

$$S = \int d^4x\sqrt{-g}\Big[- \frac{1}{2\kappa}R(g_{\mu\nu}) + \lambda^{\mu\nu}\Big(g_{\mu\nu} - l_{\mu\nu}(l^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)\Big)\Big] \tag{3.31}$$

By varying with respect to ϕ one has

$$\nabla_\mu(\lambda\partial^\mu\phi) = 0, \tag{3.32}$$

[1]The assumption $h \neq 0$ is crucial for the invertibility of the metric.

where $\lambda \equiv \lambda^{\mu}_{\mu} = g_{\mu\nu}\lambda^{\mu\nu}$.
By comparing this equation with (3.6), it is obtained that $\lambda = G - \kappa T$ or $\lambda = G$ in absence of ordinary matter.
Variation with respect to the physical metric $g_{\mu\nu}$ yields the Einstein equation

$$G_{\mu\nu} - \lambda_{\mu\nu} = 0. \tag{3.33}$$

Finally, by varying with respect to $l_{\mu\nu}$ one obtains

$$\lambda^{\mu\nu}(l^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \lambda^{\rho\sigma}l_{\rho\sigma}l^{\mu\alpha}\partial_\alpha\phi l^{\nu\beta}\partial_\beta\phi = 0, \tag{3.34}$$

which, by using equation (3.1), gives

$$\lambda_{\mu\nu} = \lambda\partial_\mu\phi\partial_\nu\phi. \tag{3.35}$$

By using $\lambda = G$ and substituting the last equation in (3.33), the generalized Einstein equation of mimetic DM model above with vanishing energy-momentum tensor is recovered.
So, the theory described by the action (3.31) is equivalent to the original action (3.2).
Moreover, the new action can be further simplified: $\lambda_{\mu\nu}$ is fully determined by its trace and only the trace part of the constraint-fixing term can be left in the action, leading to

$$S = \int d^4x\sqrt{-g}\Big(-\frac{1}{2\kappa}R(g_{\mu\nu}) + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1)\Big) \tag{3.36}$$

This Langrange multiplier formulation of the MDM action will be used in the following chapters for reviewing further developments in Mimetic Gravity.

Chapter 4

Mimetic Gravity Cosmology

After having introduced the model and discussed some issues regarding it in the previous chapter, it is particularly interesting to consider some extensions.
In this chapter a potential for the mimetic scalar field is introduced, through which it is possible to mimic many cosmological scenarios: this generalized MDM model can provide Inflation and Quintessence and can also lead to a bouncing non-singular Universe [20]. The role of the potential will be similar to that seen for the models of Quintessence and Inflation discussed in Chapter 2.
Motivated by the discussion regarding the known facts about the Universe in Chapter 2, only the flat FLRW metric will be considered.
Then, a general method for resolving cosmological singularities [21], a direct coupling between MDM and ordinary matter [22] and cosmological perturbations in Mimetic Gravity setting [20] are discussed.

4.1 Potential for Mimetic Matter

The following action is considered

$$S = \int d^4x\sqrt{-g}\Big[-\frac{1}{2\kappa}R(g_{\mu\nu}) + \mathcal{L}_M + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - V(\phi)\Big], \quad (4.1)$$

where a potential term for the mimetic scalar field ϕ is added to the original Lagrangian, and its form will be specified in the following and will depend on which cosmological scenario has to be reproduced.
Variation with respect to λ gives obviously the kinematical constraint (3.8), while varying with respect to $g^{\mu\nu}$ yields the modified Einstein equation

$$G_{\mu\nu} - 2\lambda\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\kappa V(\phi) = \kappa T_{\mu\nu}. \quad (4.2)$$

By taking the trace of the last equation, the Lagrange multiplier can be recasted as

$$\lambda = \frac{1}{2}(G - \kappa T - 4\kappa V), \quad (4.3)$$

where the constraint (3.8) is already used.
Therefore, the equation (4.2) can be rewritten as

$$G_{\mu\nu} = (G - \kappa T - 4\kappa V)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\kappa V(\phi) + \kappa T_{\mu\nu}. \tag{4.4}$$

The equation of motion for the mimetic field ϕ takes the form

$$\nabla^\nu((G - \kappa T - 4\kappa V)\partial_\nu\phi) = -\kappa V'(\phi), \tag{4.5}$$

where the prime denotes derivative with respect to ϕ and it is the generalization of (3.6) in the case of non-vanishing potential.
By comparing the extra contribution to the modified Einstein equation (4.4) with the perfect fluid energy-momentum tensor (2.9), the extra ideal fluid is seen to have a pressure and an energy density given by

$$\tilde{p} = -\kappa V, \quad \text{and} \quad \tilde{\varepsilon} = G - \kappa T - 3\kappa V \tag{4.6}$$

respectively, where the mimetic scalar field ϕ plays the same role as in Chapter 3.
Now, it is interesting to see how the presence of the potential $V(\phi)$ modifies the solution for the MDM model found in Chapter 3.
By considering a flat FLRW Universe, the potential $V(\phi)$ does not spoil the mimetic constraint (3.8) and as a consequence, the field ϕ can be identified with cosmological time, as done in Chapter 3. Hence, the pressure and the energy density associated with the scalar field ϕ, depend only on time and consequently, equation (4.5) reads

$$\frac{1}{a^3}\frac{d}{dt}(a^3(\tilde{\varepsilon} - \kappa V)) = -\kappa\dot{V}. \tag{4.7}$$

This equation, with the pressure given in (4.6), is the usual continuity equation

$$\dot{\tilde{\varepsilon}} = -3H(\tilde{\varepsilon} + \tilde{p}), \tag{4.8}$$

where H is the Hubble constant (2.8).
Upon integration, equation (4.7) gives the energy density in terms of the potential V

$$\tilde{\varepsilon} = \kappa V - \frac{\kappa}{a^3}\int a^3\dot{V}dt = \frac{3\kappa}{a^3}\int a^2 V da. \tag{4.9}$$

A constant of integration in the last equation determines the amount of MDM, which decays as a^{-3} as found in Chapter 3, but now there is an extra contribution besides to that of MDM due to the non-vanishing potential V.
The first Friedmann equation, corrisponding to the tt component of the Einstein equations (4.4), for vanishing or negligible ordinary matter, by using the last equation, takes the form

$$H^2 = \frac{\kappa}{3}\tilde{\varepsilon} = \frac{\kappa^2}{a^3}\int a^2 V da, \tag{4.10}$$

By multiplying the last equation by a^3 and differentiating it with respect to time one obtains the second Friedmann equation

$$2\dot{H} + 3H^2 = V(t), \tag{4.11}$$

where, for simplicity, $\kappa = 1$.
The last equation can be simplified by introducing the new variable

$$y = a^{\frac{3}{2}}. \tag{4.12}$$

With this new variable, the Hubble constant and its first time derivative become

$$H = \frac{2}{3}\frac{\dot{y}}{y},\ \dot{H} = \frac{2}{3}\left(\frac{\ddot{y}}{y} - \left(\frac{\ddot{y}}{y}\right)^2\right), \tag{4.13}$$

and equation (4.11) becomes a linear differential equation

$$\ddot{y} - \frac{3}{4}V(t)y = 0 \tag{4.14}$$

This is the equation of a harmonic oscillator with sign reversed angular frequency which depends on time. For a general procedure for solving this type of differential equations, see [23].
Here, only some interesting potentials by the cosmological point of view and the corresponding solutions for the equation (4.14) are considered.

4.1.1 Cosmological solutions

As a first example, the following potential is studied

$$V(t) = \frac{\alpha}{t^2}, \tag{4.15}$$

where α is a constant.
The general solution of the equation

$$\ddot{y} - \frac{3\alpha}{4t^2}y = 0, \tag{4.16}$$

is given by

$$y = \begin{cases} C_1 t^{\frac{1}{2}} \cos\left(\frac{1}{2}\sqrt{|1+3\alpha|}\ln t + C_2\right), \text{for } \alpha < -1/3, \\ C_1 t^{\frac{1}{2}(1+\sqrt{1+3\alpha})} + C_2 t^{\frac{1}{2}(1-\sqrt{1+3\alpha})}, \text{for } \alpha \geq -1/3, \end{cases}$$

where C_1 and C_2 are constants of integration.
The solution for $\alpha < -\frac{1}{3}$ becomes negative in certain intervals of time, so it

is unphysical.
Instead, the general solution for $\alpha \geq -1/3$ is physical and can be written as

$$a(t) = t^{\frac{1}{2}(1+\sqrt{1+3\alpha})}\Big(1 + At^{-\sqrt{1+3\alpha}}\Big)^{2/3}, \tag{4.17}$$

where $A = C_2/C_1$, assuming $C_1 \neq 0$.
By substituting this solution in the first Friedmann equation (4.10), one finds the energy density

$$\tilde{\varepsilon} = 3H^2 = \frac{1}{3t^2}\Big(1 + \sqrt{1+3\alpha}\frac{1 - At^{-\sqrt{1+3\alpha}}}{1 + At^{-\sqrt{1+3\alpha}}}\Big)^2 \tag{4.18}$$

and, taking into account that

$$\tilde{p} = -\frac{\kappa\alpha}{t^2}, \tag{4.19}$$

for the fluid associated with mimetic scalar field ϕ with equation of state $\tilde{p} = \omega\tilde{\varepsilon}$ one finds

$$\omega = \frac{\tilde{p}}{\tilde{\varepsilon}} = -3\alpha\Big(1 + \sqrt{1+3\alpha}\frac{1 - At^{-\sqrt{1+3\alpha}}}{1 + At^{-\sqrt{1+3\alpha}}}\Big)^{-2} \tag{4.20}$$

In general, this equation of state depends on time but in the limit of small and large t tends to a constant and, for different values of the parameter α, one gets different behaviors for MDM

- $\alpha = -1/3$: ultra-hard equation of state with $\tilde{p} = \tilde{\varepsilon}$ and $a \propto t^{1/2}$;
- $\alpha = -1/4$: ultra-relativistic fluid with $\tilde{p} = \frac{1}{3}\tilde{\varepsilon}$ at large time and $\tilde{p} = 3\tilde{\varepsilon}$ when $t \to 0$ if $A \neq 0$;
- positive α: the pressure is negative and if $\alpha >> 1$ the equation of state approaches the cosmological constant, $\tilde{p} = -\tilde{\varepsilon}$.

4.1.2 Quintessence

If the same potential of the previous section is considered but the Universe is dominated by ordinary matter with the equation of state (2.11), the scale factor reads as $a \propto t^{\frac{2}{3(1+\omega)}}$ and the energy density of MDM (4.9) takes the form

$$\tilde{\varepsilon} = -\frac{\alpha}{\omega t^2}. \tag{4.21}$$

Because the pressure is $\tilde{p} = -\kappa\alpha/t^2$, MDM imitates the equation of state of the dominant matter. However, since the total energy density from (2.6) with $\kappa = 1$ and $k = 0$, is equal to

$$\rho = 3H^2 = \frac{4}{3(1+\omega)^2 t^2}, \tag{4.22}$$

MDM can be subdominant only if $\alpha/\omega << 1$. The more general solution for subdominant MDM, $\phi = t + t_0$, first corresponds to a cosmological constant for $t < t_0$ and only at $t > t_0$ starts to behave similar to a dominant matter.

4.1.3 Inflation

Inflationary solutions can also be constructed in the framework of Mimetic Gravity, provided that the potential satisfies the following limits:

- for $t \to 0$, $V(t) \sim t^2$
- for $t \to \infty$, $V(t) \sim t^2 e^{-t}$

By interpolation, one finds the potential also for intermediate values of t

$$V(t) = \frac{\alpha t^2}{\exp(t) + 1}. \tag{4.23}$$

With $\alpha > 0$, this potential describes inflation with graceful exit to matter dominating universe. In fact, the scale factor grows as

- $a \propto \exp\left(\sqrt{\frac{\alpha}{12}} t\right)$ as $t \to 0$;
- $a \propto t^{2/3}$ for positive t.

For $t \to 0$ it can be easily shown that the Strong Energy Condition (SEC) $\tilde{\varepsilon} + 3\tilde{p} \geq 0$ is violated.

4.1.4 Bouncing Universe

A potential which provides a non-singular bounce in a contracting flat Universe is

$$V(t) = \frac{4}{3} \frac{1}{(1+t^2)^2}. \tag{4.24}$$

The general exact solution of the equation (4.14) with this potential is given by

$$y(t) = \sqrt{t^2+1}(C_1 + C_2 \arctan t), \tag{4.25}$$

and correspondingly the scale factor is

$$a(t) = \left(\sqrt{t^2+1}(1 + A \arctan t)\right)^{2/3}, \tag{4.26}$$

where $A = C_2/C_1$, assuming $C_1 \neq 0$. The constant of integration A can be put equal to zero, and the scale factor

$$a(t) = (t^2+1)^{1/3} \tag{4.27}$$

yields the following energy density and pressure

$$\tilde{\varepsilon} = 3H^2 = \frac{4}{3} \frac{t^2}{(1+t^2)^2}, \quad \tilde{p} = -\frac{4}{3} \frac{1}{(1+t^2)^2}. \tag{4.28}$$

The interesting property of this solution is that for $|t| < 1$

$$\tilde{\varepsilon} + \tilde{p} = \frac{4}{3} \frac{t^2 - 1}{(1 + t^2)^2} \tag{4.29}$$

becomes negative, violating the Null Energy Condition (NEC)

$$\tilde{\varepsilon} + \tilde{p} \geq 0 \tag{4.30}$$

Therefore, the null version of the Penrose-Hawking singularity theorem does not apply and a non-singular bounce is possible [24].
This process is the so-called Big Bounce, as an alternative to the Big Bang: for a review of different models, see [25].
For large negative t the universe is dominated by dust with negligible pressure and it contracts, the energy density first growing as a^{-3}. Then, during the time interval $|t| < 1$ the NEC (4.30) is violated, the energy density first increases reaching the Planck energy at time $t = 0$, and finally, starts to decrease.
Correspondigly, the Universe stops its contraction and starts to expand. After the Planck time, the expansion proceeds as in dust dominated Universe.
In the model considered so far, the bounce happens at Planck scales, where the quantum effects should be relevant but the potential V can be modified in such a way that the bounce will be longer than the Planck time scale.
For this purpose, the potential can be chosen as

$$V(\phi) = \frac{4}{3} \frac{\alpha}{(t_0^2 + t^2)^2} \tag{4.31}$$

For this potential, the equation (4.19) becomes

$$\frac{d^2 y}{d\tilde{t}^2} - \frac{\alpha t_0^{-2}}{(1 + \tilde{t}^2)^2} y = 0, \tag{4.32}$$

where $\tilde{t} = t/t_0$, and its general solution is

$$a(t) = \left[\sqrt{\tilde{t}^2 + 1}\Big(\cos(\beta \arctan(\tilde{t})) + A \sin(\beta \arctan(\tilde{t}))\Big)\right]^{\frac{2}{3}}, \tag{4.33}$$

where $\beta = \sqrt{1 - \alpha t_0^{-2}}$. In this case the bounce happens at scales about αt_0^{-2} during the time interval t_0.

4.2 Resolving Cosmological Singularities

A more general method than that presented in the previous section for resolving cosmological singularities, is presented in [21]. For this purpose, the following action is considered

$$S = -\int d^4x\sqrt{-g}\Big(-\frac{1}{2\kappa}R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + f(\chi) + \mathcal{L}_M\Big) \tag{4.34}$$

where $\chi = \Box\phi$ and f is an arbitrary function of χ for the moment.
The mimetic scalar field ϕ satisfies the constraint (3.8) and therefore the term $f(\chi)$ does not lead to the appearance of higher derivatives.
Variation of the above action with respect to the metric gives the following equations

$$G_{\mu\nu} = \tilde{T}_{\mu\nu} + \kappa T_{\mu\nu} \tag{4.35}$$

as in the previously considered models, but with a more complex energy-momentum tensor associated with the mimetic scalar field ϕ, given by

$$\tilde{T}_{\mu\nu} = 2\lambda\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}(\chi f' - f + g^{\rho\sigma}\partial_\rho f'\partial_\sigma\phi) - \partial_{(\mu}f'\partial_{\nu)}\phi, \tag{4.36}$$

with $f' = df/d\chi$. It is easy to notice that for $f = 0$ this energy-momentum tensor reduces to the one of the original proposal, discussed in Chapter 3.
By considering the synchronous frame, as in Chapter 3, the mimetic scalar field can be identified with the time and then $\chi = \frac{\gamma}{2\dot{\gamma}}$.
In this discussion, the components of the metric γ_{ik} will be considered only function of time and in this case the components of the Ricci tensor are given by

$$R^0_0 = -\frac{1}{2}\dot{\varkappa} - \frac{1}{4}\varkappa^k_i\varkappa^i_k, \quad R^i_k = -\frac{1}{2\sqrt{\gamma}}\frac{d(\sqrt{\gamma}\varkappa^i_k)}{dt}, \tag{4.37}$$

where $\varkappa^i_k = \gamma^{im}\dot{\gamma}_{mk}$, $\varkappa = \varkappa^i_i = \frac{\dot{\gamma}}{\gamma}$.
The components of the mimetic energy-momentum tensor (4.36) are

$$\tilde{T}^0_0 = 2\lambda + \chi f' - f - \dot{\chi}f'', \tag{4.38}$$

and

$$\tilde{T}^i_k = (\chi f' - f + \dot{\chi}f'')\delta^i_k. \tag{4.39}$$

respectively.
Hence, the tt and the $i-j$ components of the Einstein equations read

$$\frac{1}{8}(\varkappa^2 - \varkappa^k_i\varkappa^i_k) = 2\lambda + \chi f' - f - \dot{\chi}f'' + T^0_0, \tag{4.40}$$

and

$$\frac{1}{2\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\kappa^i_k)}{\partial t} = (\lambda + \chi f' - f)\delta^i_k - T^i_k + \frac{1}{2}T\delta^i_k. \tag{4.41}$$

From the variation of the action (4.34) with respect to ϕ, the equation for the mimetic scalar field ϕ reads

$$\frac{1}{\sqrt{\gamma}}\partial_t(2\sqrt{\gamma}\lambda) = \Box f' = \frac{1}{\sqrt{\gamma}}\partial_t(\sqrt{\gamma}f''\dot{\chi}), \tag{4.42}$$

where in the last identity the action of the Laplacian on a function f' depending implicitly on time t through χ, becomes evident.
From the integration of equation (4.42), the expression for the Lagrange multiplier λ is found

$$\lambda = \frac{C}{2\sqrt{\gamma}} + \frac{1}{2}f''\dot{\chi}, \tag{4.43}$$

where C is a constant of integration corresponding to mimetic cold matter. For the usual matter, $T^i_k = -p\delta^i_k$ is valid. With this assumption, by subtracting from equation (4.41) one third of its trace, the equation

$$\partial_t\Big(\sqrt{\gamma}\Big(\varkappa^i_k - \frac{1}{3}\varkappa\delta^i_k\Big)\Big) = 0, \tag{4.44}$$

follows and its solution is given by

$$\varkappa^i_k = \frac{1}{3}\varkappa\delta^i_k + \frac{\lambda^i_k}{\sqrt{\gamma}} \tag{4.45}$$

where λ^i_k are traceless constants of integration.
By taking into account that $\varkappa = 2\chi = \dot{\gamma}/\gamma$ and substituting the last equation together with the expression for the Lagrange multiplier (4.43) into the tt component of the Einstein equations (4.40) one gets

$$\frac{1}{3}\chi^2 + f - \chi f' = \frac{\lambda^i_k\lambda^k_i}{8\gamma} + \frac{C}{\sqrt{\gamma}} + T^0_0. \tag{4.46}$$

One can solve this equation for γ and substitute the result in (4.45) in order to determine all components of the metric, given the function f. The classes of functions f of interest are those which lead to singularity free solutions. One requires curvature invariants to be bounded by some limiting maximal values determined by χ_m which is smaller than the Planck value in order to ignore any quantum effect.
A suitable function for this purpose is of the Born-Infeld type

$$f(\chi) = 1 + \frac{1}{2}\chi^2 - \chi\arcsin\chi - \sqrt{1-\chi^2}. \tag{4.47}$$

This function has the desirable property that at $\chi^2 << \chi_m^2$ the corrections to GR are negligible because its expansion at small χ starts at order χ^4.
After scaling $\chi \to \sqrt{\frac{2}{3}}\frac{\chi}{\chi_m}$ and $f \to \chi_m^2 f$, the function f takes the form

$$f(\chi) = \chi_m^2\left[1 + \frac{1}{3}\frac{\chi^2}{\chi_m^2} - \sqrt{\frac{2}{3}}\frac{\chi}{\chi_m}\arcsin\left(\sqrt{\frac{2}{3}}\frac{\chi}{\chi_m}\right) - \sqrt{1 - \frac{2}{3}\frac{\chi^2}{\chi_m^2}}\right], \tag{4.48}$$

and substituting in (4.46), leads to the equation

$$\varepsilon = \chi_m^2\left(1 - \sqrt{1 - \frac{2}{3}\frac{\chi^2}{\chi_m^2}}\right), \tag{4.49}$$

where $\varepsilon = \frac{\lambda^i_k \lambda^k_i}{8\gamma} + \frac{C}{\sqrt{\gamma}} + T^0_0$, i.e. it is the sum of all the contributions to the energy density present in this setting.
By squaring the last equation and recalling that $\chi = \frac{\dot{\gamma}}{2\gamma}$, the following equation is obtained

$$\frac{1}{12}\Big(\frac{\dot{\gamma}}{\gamma}\Big)^2 = \varepsilon\Big(1 - \frac{\varepsilon}{\varepsilon_m}\Big), \tag{4.50}$$

where $\varepsilon_m = 2\chi_m^2$.
In the FLRW Universe this equation becomes

$$3\Big(\frac{\dot{a}}{a}\Big)^2 = \frac{\varepsilon_m}{a^3}\Big(1 - \frac{1}{a^3}\Big), \tag{4.51}$$

where only the contribution to ε due to MDM is considered and the scale factor a is normalized in order to have $\varepsilon = \varepsilon_m$ at $a = 1$.
The solution of the modified first Friedmann equation reads

$$a(t) = \Big(1 + \frac{3}{4}\varepsilon_m t^2\Big)^{1/3}, \tag{4.52}$$

For $t < -1/\sqrt{\varepsilon_m}$, it describes a cold matter dominated contracting Universe, then it passes through the regular bounce during time interval $-1/\sqrt{\varepsilon_m} < t < 1/\sqrt{\varepsilon_m}$, and after the bounce for $t > 1/\sqrt{\varepsilon_m}$ the universe is expanding as the normal dust dominated FLRW Universe.
For the same procedure in the case of Kasner Universe, see [21]. For black holes, the same analysis can be repeated as done in [26].

4.3 Direct coupling of Mimetic Dark Matter with matter

In this section a direct interaction of MDM with ordinary matter at a relevant energy scale, is considered [22].
To study what happens in this situation, the physical metric $g_{\mu\nu}$ in terms of the auxiliary metric $l_{\mu\nu}$ and the mimetic scalar field ϕ is given by

$$g_{\mu\nu} = l_{\mu\nu} l^{\alpha\beta} \frac{\partial_\alpha\phi\partial_\beta\phi}{M_1^4} \tag{4.53}$$

where a mass scale M_1 is introduced to give the correct mass dimension to the mimetic scalar field.
A requirement that will be clear later is that the action should be invariant under the shift symmetry $\phi \to \phi + C$, where C is a constant, so a non-vanishing potential $V(\phi)$ is excluded.
Hence, the action is given by

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_p^2}{2}R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - M_1^4)\Big] + S_m, \tag{4.54}$$

where for the importance in the following discussion, the reduced Planck mass $M_p^2 = 1/(8\pi G) = 1/\kappa$ is taken into account.
Now, the interaction term that accounts for coupling of MDM with the ordinary matter is added to the above action. The simplest form for the interacting action is

$$S_{int} = \int d^4x\sqrt{-g}\frac{1}{M_2}\nabla_\mu\phi J^\mu, \tag{4.55}$$

which is invariant under the shift symmetry and J^μ is the matter current associated with a certain quantum number, and M_2 is a cutoff scale.
For a system of particles, the current is defined as

$$J^\mu = \frac{1}{\sqrt{-g}}\sum_n q_n \int dx_n^\mu \delta^4(x - x_n), \tag{4.56}$$

where q_n is the charge associated with the n-th particle, x_μ^n is its coordinate, and $\delta^4(x - x_n)$ is the Dirac delta function.
The coupling term in the action is independent of the metric, so the total energy-momentum tensor will be the same as that of the original MDM model,

$$\hat{T}_{\mu\nu} = 2\lambda\partial_\mu\phi\partial_\nu\phi + T_{\mu\nu}, \tag{4.57}$$

so the gravitational field equation is

$$G_{\mu\nu} = \frac{1}{M_p^2}\hat{T}_{\mu\nu} \tag{4.58}$$

with the Lagrange multiplier given by

$$\lambda = \frac{1}{2}\frac{M_p^2 G + T}{M_1^4}, \tag{4.59}$$

where M_1 represents a cut off scale for the contribution of the mimetic scalar field to the Einstein equations. By combining equations (4.57), (4.58) and (4.59), one can easily see that the energy-momentum tensor associated with the mimetic field is

$$\tilde{T}_{\mu\nu} = (G + \frac{T}{M_p^2})u_\mu u_\nu \tag{4.60}$$

which describes a dust-like component with vanishing pressure, energy density $\tilde{\varepsilon} = G + \frac{T}{M_p^2}$ and 4-velocity $u^\mu = \frac{\partial^\mu\phi}{M_1^2}$ which is normalized thanks to the mimetic constraint that can be derived by the action (4.54) by varying with respect to the Lagrange multiplier λ, as usual.
The presence of the derivative coupling term (4.55) modifies the equation of motion of the mimetic field

$$2\lambda\Box\phi + 2\nabla_\mu\lambda\nabla^\mu\phi + \frac{1}{M_2}\nabla_\mu J^\mu = 0. \tag{4.61}$$

If the current is conserved, the last term vanishes and the original equation for the mimetic field (3.6) is recovered. However, when coupled to a non-conserved current, the above equation is equivalent to

$$\nabla_\mu T^{\mu\nu} = \frac{1}{M_2}\nabla_\rho J^\rho \nabla^\nu \phi. \tag{4.62}$$

Hence, in this case there is exchange of energy and momenta between MDM and ordinary matter and this effect is suppressed by the cut off M_2.
As an example of the physical consequences of the coupling between these different matter components, the baryon current, J_B^μ current can be considered.
In the Standard Model of Particle Physics, the baryon number is conserved at low energy scales, but it is violated at high energy scales achieved in the early Universe.
In a FLRW Universe, the coupling $\nabla_\mu \phi J_B^\mu / M_2$ reduces to

$$\frac{\dot{\phi}}{M_2} J_B^0 = \frac{M_1^2}{M_2} n_B, \tag{4.63}$$

here $n_B = n_b - n_{\bar{b}}$ is the net baryon number density.
In such a background with non-vanishing $\dot{\phi}$, the Lorentz symmetry is broken and correspondingly the CPT symmetry in the baryon sector is violated and so a difference between baryons and anti-baryons is introduced.
This is an example of gravitational baryogenesis. Another model of gravitational baryogenesis is studied in [27] and for a more usual approach to this cosmological event, see [28].
In the early Universe when the baryon number violating processes were in thermal equilibrium, the derivative coupling can have induced an effective chemical potential for baryons and an opposite one for anti-baryon

$$\mu_b = \frac{\dot{\phi}}{M_2} = \frac{M_1^2}{M_2} = -\mu_{\bar{b}}. \tag{4.64}$$

This implies that baryons and antibaryons had different thermal distributions and so there was a temperature dependent baryon number density,

$$n_B = n_b - n_{\bar{b}} = \frac{g_b \mu_b T^2}{6}, \tag{4.65}$$

where $g_b = 2$ is the number of degree of freedom of the baryon and T is the temperature. On the other hand, the entropy density of the universe is given by [28]

$$s = \frac{2\pi^2}{45} g_{*s} T^3 \tag{4.66}$$

where g_{*s} is the number of the effective degrees of freedom of the species which contribute to the entropy of the universe.

From the equations (4.65) and (4.66) the baryon-to-entropy ratio is obtained

$$\frac{n_B}{s} = \frac{15g_b}{4\pi^2}\frac{\mu_b}{g_{*s}T} \sim 10^{-2}\frac{M_1^2}{M_2 T}, \tag{4.67}$$

where $g_{*s} \sim 100$ during the radiation dominated epoch.
This provides a way of producing the baryon number asymmetry thermally, in which the key point is the CPT violation, due to the presence of the mimetic field.
The quantity $\frac{n_B}{s}$ became larger at later time with the decreasing of the temperature and then this asymmetry froze out at the temperature T_D when the baryon number violating interactions decoupled from the thermal bath,

$$\left(\frac{n_B}{s}\right)_D \sim 10^{-2}\frac{M_1^2}{M_2 T_D}. \tag{4.68}$$

Below the temperature T_D the baryon number is conserved and the direct coupling between MDM and ordinary matter has no effect on baryons.
The baryon number asymmetry in the Universe is $(n_B/s)_D \sim 10^{-10}$, as required by the Big Bang Nucleosynthesis [28] and the observational data from CMB radiation.
The decoupling temperature of the baryon number non-conservation is $T_D \simeq$ 100 GeV known from the SM, so the relation

$$M_1 \sim 10^{-3}\sqrt{M_2}\text{GeV}. \tag{4.69}$$

is found by using equation (4.68). This implies that the scale M_1 can not be too large: even if $M_2 \sim M_p \sim 10^{18}$ GeV, it is obtained $M_1 \sim 10^6$ GeV.

4.4 Cosmological perturbations

In this section, cosmological scalar perturbations in the Universe dominated by MDM are studied [20]. For a general introduction to cosmological perturbations, see [9].
The metric of perturbed Universe in Newtonian gauge can be written as

$$ds^2 = (1 + 2\Phi(x^i, t))dt^2 - (1 - 2\Phi(x^i, t))a^2\delta_{ik}dx^i dx^k, \tag{4.70}$$

where Φ is the Newtonian gravitational potential.
By considering perturbations of the mimetic scalar field ϕ,

$$\phi = t + \delta\phi, \tag{4.71}$$

the mimetic constraint (3.8) at the linear order, gives

$$\Phi = \delta\dot{\phi}. \tag{4.72}$$

The equation for perturbations which follows from the linearized $0-i$ components of Einstein equations reads [9]

$$\left(\dot{\Phi}+H\Phi\right)_{,i}=\frac{1}{2}(\tilde{\varepsilon}+\tilde{p})\delta\phi_{,i}. \tag{4.73}$$

The sum of the energy density and pressure of the mimetic field can be recasted as

$$\tilde{\varepsilon}+\tilde{p}=-2\dot{H}, \tag{4.74}$$

by combining the Friedmann equations (4.10) and (4.11), and by substituting the last expression in equation (4.73), the following equation for $\delta\phi$

$$\delta\ddot{\phi}+H\delta\dot{\phi}+\dot{H}\delta\phi=0 \tag{4.75}$$

is obtained whose general solution is given by

$$\delta\phi=A\frac{1}{a}\int a dt, \tag{4.76}$$

where A is a constant of integration depending only on the spatial coordinates.
The corresponding gravitational potential is

$$\Phi=\delta\dot{\phi}=A\frac{d}{dt}\Big(\frac{1}{a}\int a dt\Big)=A\Big(1-\frac{H}{a}\int a dt\Big). \tag{4.77}$$

This solution is valid for all perturbations irrespective of their wavelength. Normally, one can neglect the spatial derivative terms, multiplied by the speed of sound c_s, for hydrodynamical fluid only for high wavelengths. Here, due to the vanishing of the speed of sound even for non-vanishing pressure at all wavelengths, one cannot define quantum fluctuations as usual, so Inflation realized with the mimetic scalar field, discussed in section 4.1, fails in explaining the large scale structure as originated from quantum fluctuations [29].
A way to make mimetic Inflation viable is considering the following action

$$S=\int d^4x\sqrt{-g}\Big[-\frac{1}{2\kappa}R(g_{\mu\nu})+\lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-1)-V(\phi)+\frac{1}{2}\gamma(\Box\phi)^2\Big] \tag{4.78}$$

where γ is a constant and the Lagrangian of ordinary matter is neglected. Because of the mimetic constraint (3.8), by adding this higher derivative term the total number of degrees of freedom does not change.
By varying the action with respect to the metric the Einstein equation (3.10) is found.
However, the contribution to the energy-momentum tensor due to the mimetic field ϕ is different and is given by

$$\tilde{T}_{\mu\nu}=\Big(V+\gamma\Big(\partial_\alpha\phi\partial_\alpha\chi+\frac{1}{2}\chi^2\Big)\Big)g_{\mu\nu}+2\lambda\partial_\mu\phi\partial_\nu\phi-\gamma\partial_{(\mu}\phi\partial_{\nu)}\chi, \tag{4.79}$$

where $\chi \equiv \Box\phi$.
The general solution (3.14) of the mimetic constraint (3.8) in the FLRW Universe remains the same despite the new term in the action (4.78) and hence

$$\chi = \Box\phi = \ddot{\phi} + 3H\dot{\phi} = 3H. \tag{4.80}$$

By taking this into account, the first Friedmann equation (4.10) and the second Friedmann equation (4.11) become

$$H^2 = \frac{1}{3}V + \gamma\Big(\frac{3}{2}H^2 - \dot{H}\Big) + \frac{2}{3}\lambda, \tag{4.81}$$

$$2\dot{H} + 3H^2 = \frac{2}{2-3\gamma}V, \tag{4.82}$$

respectively. The last equation differs from equation (4.11) only for a normalization factor of the potential V.
Therefore, the presence of the extra term in (4.78) does not modify the cosmological solutions derived in section 4.1 for homogeneous universe up to a numerical factor of order unity.
However, this term dramatically changes the behavior of the short wave cosmological perturbations.
The linear perturbation of $0-i$ component of the energy-momentum tensor is [9]

$$\delta T^0_i = 2\lambda\delta\phi_{,i} - 3\gamma\dot{H}\delta\phi_{,i} - \gamma\delta\chi_{,i}. \tag{4.83}$$

By taking into account that, at the linear order, the perturbation of χ is given by

$$\delta\chi = -3\delta\ddot{\phi} - 3H\delta\dot{\phi} - \frac{\Delta}{a^2}\delta\phi \tag{4.84}$$

and from equations (4.81) and (4.82), one has

$$\lambda = (3\gamma - 1)\dot{H}. \tag{4.85}$$

It follows that the perturbed $0-i$ Einstein equation reduces to

$$\delta\ddot{\phi} + H\delta\dot{\phi} - \frac{c_s^2}{a^2}\Delta\delta\phi + \dot{H}\delta\phi = 0 \tag{4.86}$$

This equation differs from equation (4.75) precisely by the presence of the gradient term, with the speed of sound c_s given by

$$c_s^2 = \frac{\gamma}{2-3\gamma}. \tag{4.87}$$

By Fourier transforming $\delta\phi$, the equation (4.86) for its Fourier components becomes

$$\delta\phi_k'' + \left(c_s^2k^2 + \frac{a''}{a} - 2\Big(\frac{a'}{a}\Big)^2\right)\delta\phi_k = 0. \tag{4.88}$$

where the prime denotes the derivative with respect to the conformal time $\eta = \int dt/a$, which is used here for convenience.
For short wavelength perturbations with $c_s k\eta >> 1$ ($\lambda_{ph} = a/k << c_s H^{-1}$) the conformal time derivative terms inside the bracket can be neglected and the solution is

$$\delta\phi_k \propto e^{\pm c_s k\eta}. \tag{4.89}$$

For long wavelength perturbations with $c_s k\eta << 1$ ($\lambda_{ph} = a/k >> c_s H^{-1}$) the $c_s^2 k^2$-term can be neglected and the solution (4.76) is recovered.
For more discussions on the action (4.78) and further modifications to the same action, leading to the so-called Imperfect DM, see [30].

Chapter 5

Spherically symmetric solutions in Mimetic Gravity

After having discussed some cosmological aspects of Mimetic Gravity in the previous chapters, as an interlude this chapter is devoted to solutions in Mimetic Gravity that can address the most important phenomenological evidence for Dark Matter discussed in Chapter 2: the form of the rotation curves of galaxies. In order to do so, pseudo-static spherically symmetric (SSS) spacetimes in Mimetic Gravity are considered and a reconstruction technique to find the forms of thepotential for the mimetic scalar field ϕ that can reproduce such phenomenology, is described in details, following [31].
The metric of these general spacetimes is given by

$$ds^2 = a(r)^2 b(r) dt^2 - \frac{dr^2}{b(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.1}$$

where $a(r)$, $b(r)$ are at the moment arbitrary functions of the radial coordinate r that will be specified later. The action that is considered here is the same as in Chapter 4 (4.1), i.e. the sum of the Einstein-Hilbert action and the mimetic field action subjected to a potential $V(\phi)$ and the Einstein equations are given by (4.2).
The difference with the previous discussion is that the symmetries of the equations of motion derived with the metric (5.1) require the mimetic field to be a function of r, only instead of the time t, and from the kinematical contraint (3.8), by considering the metric (5.1), one has

$$\phi'(r) = \sqrt{-\frac{1}{b(r)}}, \tag{5.2}$$

leading to a pure imaginary expression for the field, which is to be expected from a time-like vector $\partial_\mu \phi$ with temporal component equal to zero.
The tt and rr-components of the Einstein equations (4.2) with the metric

(5.1) read

$$1 - b'(r)r - b(r) = \frac{V(\phi)r^2}{2}, \tag{5.3}$$

$$\Big(b'(r)r + 2rb(r)\frac{a'(r)}{a(r)} + b(r) - 1\Big) = \frac{\lambda}{2}b(r)r^2\phi'(r)^2 - \frac{V(\phi)r^2}{2}, \tag{5.4}$$

where $\kappa = 1$.
By using equations (5.2) and (5.3), the last equation can be rewritten as

$$4a'(r)b(r) = -\lambda a(r)r. \tag{5.5}$$

From the continuity equation for the mimetic field in the presence of a potential (4.5), the following equation is found

$$\frac{d}{dr}(a(r)b(r)\lambda r^2\phi') = a(r)r^2\frac{dV(\phi)}{d\phi}, \tag{5.6}$$

where the prime denotes derivative with respect to r.
By integrating equation (5.2), the mimetic field ϕ is found

$$\phi(r) = \pm i\int\frac{dr}{\sqrt{b(r)}} \tag{5.7}$$

and the continuity equation (5.6) becomes

$$4\frac{d}{dr}(a'(r)b(r)^{3/2}r) = a(r)r^2\sqrt{b(r)}\frac{dV(r)}{dr} \tag{5.8}$$

where λ is given by equation (4.3) and its expression in terms of the functions appearing in the metric (5.1) is obtained from the associated Ricci scalar

$$R = 3\frac{b'(r)a'(r)}{a(r)} + b''(r) + 2\frac{b(r)a''(r)}{a(r)} + 4\frac{b'(r)}{r} + 4\frac{b(r)a'(r)}{a(r)r} + 2\frac{b(r)}{r^2} - \frac{2}{r^2} \tag{5.9}$$

5.1 Solutions

5.1.1 Vacuum solutions

From equation (5.3), it can be immediately seen that for the choice of $b(r)$ as in Schwarzschild metric

$$b(r) = 1 - \frac{r_s}{r}, \tag{5.10}$$

where r_s is the Schwarzschild radius, $V(r) = 0$ while the rr Einstein equation (5.4), by using the expression for the Ricci scalar (5.9) and its relation with the Lagrange multiplier (4.3) leads to:

$$a(r) = a_1 + \frac{a_2}{\sqrt{1-\frac{r_s}{r}}}\left[\Big(\sqrt{1-\frac{r_s}{r}}\Big)\log\left[\sqrt{\frac{r}{r_0}}\Big(1+\sqrt{1-\frac{r_s}{r}}\Big)\right] - 1\right], \tag{5.11}$$

where a_1, a_2 are dimensionless constants and r_0 is a length scale.
If $a_2 = 0$ and $a_1 = 1$, this solution corresponds to the Schwarzschild solution of GR.
When $a_2 \neq 0$, a_1 can be set equal to zero and the metric is

$$ds^2 = -a_2^2\left[\left(\sqrt{1-\frac{r_s}{r}}\right)\log\left[\sqrt{\frac{r}{r_0}}\left(1+\sqrt{1-\frac{r_s}{r}}\right)\right]-1\right]^2 dt^2+\frac{dr^2}{\left(1-\frac{r_s}{r}\right)}+r^2d\Omega^2. \tag{5.12}$$

5.1.2 Non-vacuum solutions

In this section, the case $V(\phi) \neq 0$ will be considered. It is easily seen that for $V(\phi) = 2\Lambda$ with Λ a cosmological constant, one solution of the tt- and rr-components of the Einstein equations (5.3) and (5.4) is the Schwarzschild-de Sitter metric

$$b(r) = 1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}, \quad a(r) = a_1, \tag{5.13}$$

where a_1 is a constant.
If instead, a linear modification to the Schwarzschild metric, is considered

$$b(r) = 1 - \frac{r_s}{r} + \gamma r, \tag{5.14}$$

where γ is a positive constant, from equation (5.3) the potential results to be

$$V(r) = -\frac{4\gamma}{r}. \tag{5.15}$$

The solution for the mimetic field, obtainable from equation (5.7) is an elliptic function. Hence, a closed expression can only be given in limiting cases:

- for $r \approx r_s$, one can neglect the linear correction to recover the vacuum solution (5.10).
  In this case, the mimetic field reads

  $$\phi(r << \sqrt{r_s/\gamma}) \simeq \pm i\left[r\sqrt{1-\frac{r_s}{r}} + \frac{r_s}{2}\log\left[2r\left(1+\sqrt{1-\frac{r_s}{r}}\right) - r_s\right]\right]. \tag{5.16}$$

  and expanding the result around $r = r_s$:

  $$\phi(r \simeq r_s) \simeq \phi_s \pm 2i\sqrt{r_s(r-r_s)}, r \simeq r_s - \frac{(\phi_s - \phi)^2}{4r_s} \tag{5.17}$$

  with $\phi_s = \pm(ir_s/2)\log(r_s)$.
  The form of the potential, in this case, is

  $$V(\phi \simeq \phi_s) \simeq -\frac{4\gamma}{r_s} - \frac{\gamma(\phi_s - \phi)^2}{r_s^3}. \tag{5.18}$$

- For large distances, the Newtonian term in (5.14) can be ignored and from the equation (5.8) one has for the function $a(r)$

$$a(r >> \sqrt{r_s/\gamma}) \simeq \frac{c_1(4+6\gamma r) + 3c_2\sqrt{1+\gamma r} - c_2(2+3\gamma r)\arctan[\sqrt{1+\gamma r}]}{\sqrt{1+\gamma r}} \tag{5.19}$$

with $c_{1,2}$ constants.
By choosing the values $c_1 = 1/4$ and $c_2 = 0$, the metric reads

$$ds^2(r >> \sqrt{r_s/\gamma}) \simeq -\Big(1 + \frac{3\gamma r}{2}\Big)^2 dt^2 + \frac{dr^2}{1+\gamma r} + r^2 d\Omega^2 \tag{5.20}$$

and the corresponding expressions for the field and the potential are given by

$$\phi(r) \simeq \pm\frac{2i\sqrt{1+\gamma r}}{\gamma}, \quad V(\phi) \simeq \frac{16\gamma^2}{4+\gamma^2\phi(r)^2}, \tag{5.21}$$

Here, it is required $4/\gamma^2 < |\phi|^2$ to guarantee $r > 0$.

Hence, the metric reduces to the usual Schwarzschild space-time for short distances, while at large distances its tt-component of the metric (5.20) and the corresponding Newtonian potential

$$\Phi(r) = -\frac{g_{tt}(r)+1}{2} \tag{5.22}$$

acquire linear and quadratic contributions.
The quadratic correction can be viewed as a negative cosmological constant in the background and can be ignored if $\gamma^2 r^2$ is sufficiently small while, the linear term could help to explain the flatness of galactic rotation curves, discussed in Chapter 2.
This issue will be addressed in the next section.

5.2 Rotation curves of galaxies

For convenience, the tt component of the metric (5.1) is redefined as

$$\tilde{a}(r) \equiv a(r)^2 b(r), \tag{5.23}$$

By inverting the last equation, the derivative with respect to r of the function $a(r)$ reads

$$a'(r) = \frac{1}{2\sqrt{\tilde{a}(r)b(r)}}\left(\tilde{a}'(r) - \tilde{a}(r)\frac{b'(r)}{b(r)}\right), \tag{5.24}$$

and by using (5.3), equation (5.8) becomes

$$\frac{d}{dr}\left[(\tilde{a}'(r)b(r) - \tilde{a}(r)b'(r))\frac{r}{\tilde{a}(r)}\right] = \sqrt{\tilde{a}(r)}\left[-b''(r)r - \frac{2}{r}(1-b(r))\right]. \tag{5.25}$$

The following ansatz for $\tilde{a}(r)$ is made

$$\tilde{a}(r) = 1 - \frac{r_s}{r} + \gamma_0 r - \lambda_0 r^2, \tag{5.26}$$

where r_s, λ_0, γ_0 are positive constants.

5.2.1 Qualitative analysis

Given the metric element $g_{tt}(r)$, the Newtonian potential given by equation (5.22) has different behaviors, depending on which scales are considered:

1. at small distances, the metric leads to a classical Newtonian term r_s/r;
2. at very large distances, the cosmological constant term $\lambda_0 r^2$ becomes relevant and has a de Sitter-like form;
3. at intermediate distance the linear term $\gamma_0 r$ dominates.

At intermediate galactic scales, the $\lambda_0 r^2$ term is negligible, hence the Newtonian potential (5.22) reads

$$\Phi(r) \simeq -\frac{r_s}{2r}\left(1 - \frac{\gamma_0 r^2}{r_s}\right) \tag{5.27}$$

and this gives rise to the rotational velocity profile

$$v_{rot}^2 \simeq v_{Newt}^2 + \frac{\gamma_0 c^2 r}{2} \tag{5.28}$$

where v_{Newt} is the contribution expected from Newtonian mechanics. Therefore, on sufficiently large scales, v_{rot} does not fall-off as expected from Kepler laws, i.e. $v_{rot} \propto 1/\sqrt{r}$, but increases as $\sqrt{r}$. This is valid for galaxies where the Newtonian contribution can not compete with the rising γ_0 term, which occurs for small and medium sized low surface brightness (LSB) galaxies.
The situation is different for large high surface brightness (HSB) galaxies. For these galaxies the Newtonian contribution might become equal with the linear term, $\propto \gamma_0 r$ for some values of r. This leads to a region of approximate flatness consistent with the data for such galaxies. However, the radius of these galaxies can be sufficiently large that the de Sitter term $\propto r^2$ should become relevant. Thus the rotational velocity profile for these galaxies reads

$$v^2 \simeq v_{Newt}^2 + \frac{\gamma_0 c^2 r}{2} - \lambda_0 c^2 r^2, \tag{5.29}$$

where sufficiently far from the center of such galaxies, the quadratic term dominates and eliminates the rising behavior due to the linear term. This is in perfect agreement with data from those HSB galaxies that are large enough to feel the effect of the de Sitter term. Moreover, since v^2 can not become negative, bound orbits are no longer possible on scales greater than $R \sim \gamma_0/2\lambda_0$. This could explain the maximum size of galaxies.

5.2.2 Rotation curves in Mimetic Gravity

Finally, after having discussed qualitatively the effect of the linear and quadratic terms added to the Newtonian potential on the galaxies rotation curves, how such behaviors can be reproduced in Mimetic Gravity is discussed.
In order to reconstruct the complete form of the metric (5.1) together with the definition (5.23), by using equation (5.25) and by inserting the ansatz (5.26) for $\tilde{a}(r)$, the following form for $b(r)$ is found

$$b(r) = \frac{\left(1 - \frac{r_s}{r} + \gamma_0 r - \lambda_0 r^2\right)\left(1 - \frac{3r_s}{r} + \frac{\gamma_0 r}{3} + \frac{c_0}{r^2}\right)}{\left(1 - \frac{3r_s}{2r} + \frac{\gamma_0 r}{2}\right)^2} \tag{5.30}$$

with c_0 a constant.
From equation (5.3), as done in the previous section, one finds a quite involved form for the mimetic potential

$$\begin{aligned} V(r) = &-\frac{2}{3r^2(2r - 3r_s + \gamma_0 r^2)^3}[54r_s^2 r - 27r_s^3 + 171\gamma_0 r_s^2 r^2 - 8\gamma_0^2\lambda_0 r^7 \\ &+ r^4(16\gamma_0 + 7r_s\gamma_0^2 + 324r_s\lambda_0) + 4r_s r^3(-17\gamma_0 - 108r_s\lambda_0) \\ &+ r^6(\gamma_0^3 - 44\gamma_0\lambda_0) + 6r^5(\gamma_0^2 - 12\lambda_0 + 12r_s\gamma_0\lambda_0) \\ &- 12c_0[-r_s + 2r(1 + r_s\gamma_0) + 2r^3(\gamma_0^2 + \lambda_0) - \gamma_0\lambda_0 r^4 + 3r^2(\gamma_0 - 3r_s\lambda_0)]]. \end{aligned} \tag{5.31}$$

By using equation (5.7), the mimetic scalar field ϕ can be found. However, the integral in (5.7) can be done analitically only in some limiting cases:

- For small distances $\gamma_0 = \lambda_0 = 0$, one has

$$\tilde{a}(r) \simeq 1 - \frac{r_s}{r}, \quad b(r) \simeq \frac{4(c_0 + r(r - 3r_s))(r - r_s)}{r(2r - 3r_s)^2} \tag{5.32}$$

By setting $c_0 = \frac{9r_s^2}{4}$, the second equation becomes

$$b(r) = 1 - \frac{r_s}{r}, \tag{5.33}$$

recovering the vacuum Schwarzschild solution of GR. Correspondingly, the mimetic field takes the form

$$\phi(r \simeq r_s) \simeq \phi_s \pm 2i\sqrt{r_s(r - r_s)}, \quad r \simeq r_s - \frac{(\phi_s - \phi)^2}{4r_s}, \tag{5.34}$$

and the potential behaves as

$$V(\phi \simeq \phi_s) \simeq -\frac{32\gamma_0}{3r_s} + \frac{13\gamma_0(\phi_s - \phi)^2}{r_s^3}. \tag{5.35}$$

- For cosmological scales $r_s = \gamma_0 = c_0 = 0$,

$$\tilde{a}(r) = b(r) \simeq (1 - \lambda r_0^2), \tag{5.36}$$

corresponding to the static patch of the de Sitter solution.
In this case, the field has the form

$$\phi \simeq \pm i \frac{\arcsin[\sqrt{\lambda_0} r]}{\sqrt{\lambda_0}}, \quad r \simeq \pm \frac{\sin[\sqrt{\lambda_0}|\phi|]}{\sqrt{\lambda_0}}. \tag{5.37}$$

The positive values of r belong to the range $]0, 1/\sqrt{\lambda_0}[$, where $H_0^{-1} = 1/\sqrt{\lambda_0}$ is the cosmological horizon of the de Sitter solution with positive cosmological constant.
The potential reads

$$V(\phi) \simeq 6\lambda_0 \mp \frac{4\gamma}{3}\left(\frac{\sqrt{\lambda_0}}{\sin[\sqrt{\lambda_0}|\phi|]} + 4\sqrt{\lambda_0}\sin[\sqrt{\lambda_0}|\phi|]\right). \tag{5.38}$$

- For galactic scales, $r_s = \lambda_0 = c_0 = 0$ the function $\tilde{a}(r)$ and $b(r)$ are given by

$$\tilde{a}(r) \simeq (1 + \gamma_0 r), \quad \text{and} \quad b(r) \simeq \frac{4(1 + \gamma_0 r)(3 + \gamma r)}{3(2 + \gamma_0 r)^2} \tag{5.39}$$

In this case, the mimetic field is

$$\phi \simeq \pm \frac{i}{2\gamma_0}\sqrt{3(3 + 4\gamma_0 r + \gamma_0^2 r^2)}, \quad r \simeq \frac{-6 \mp \sqrt{9 - 12\gamma_0^2\phi^2}}{3\gamma_0} \tag{5.40}$$

and the potential reads

$$V(r) \simeq -\frac{2\gamma_0(16 + 6\gamma_0 r + \gamma_0^2 r^2)}{3r(2 + \gamma_0 r)^3}. \tag{5.41}$$

The considered solution turns out to be the Schwarzschild solution at small distances, the static patch of the de Sitter space-time at cosmological distance and, most intriguingly, presents a linear term at the galactic scales which can explain the observed galactic rotation curves in Mimetic Gravity.
To complete the discussion, the data from rotation curves of galaxies are needed to fix the values of the two free parameters γ_0 and λ_0. Here, it has reproduced the conformal gravity potential in Mimetic Gravity, so the task is greatly simplified by adopting the results in [32,33], where the same parameters were fitted to rotation curves.
The fit is done for 138 galaxies, including 25 dwarf galaxies and 21 galaxies in which the de Sitter-like term $\lambda_0 r^2$ becomes relevant. For the full list of galaxies considered, including references to the galactic databases, see [32,33] and references therein.

By doing the same analysis of [32,33], the fit to the rotation curves through the potential given by equation (5.29) is excellent, with $\chi^2_{red} \simeq 1$. Therefore, the linear and quadratic corrections to the Newtonian potential capture the features of the rotation curves not only qualitatively, but also quantitatively. By using such analysis, the best-fit values to γ_0 and λ_0 parameters in Mimetic Gravity to be [32,33]:

$$\gamma \simeq 3.06 \times 10^{-30}\,\mathrm{cm}^{-1}, \quad \lambda_0 \simeq 9.54 \times 10^{-54}\,\mathrm{cm}^{-2}. \tag{5.42}$$

The analysis made here confirms the fact that the large-scale behavior of MDM is only a geometrical effect: even the small-scale phenomenology of rotation curves in Mimetic Gravity suggests that particle dark matter halos in conventional Λ-CDM model might only be an attempt to describe such global geometrical effects in local terms.

Chapter 6

Hamiltonian analysis of Mimetic Dark Matter

6.1 Hamiltonian analysis

The purpose of this chapter is to review the Hamiltonian analysis of the original MDM model introduced in Chapter 3.
In this chapter some important features regarding this model such as the number of degrees of freedom and the possible presence of instabilities and ghosts are considered. A preliminary analysis of the problem of ghosts was already pursued in [19] but here it is followed the more complete one given in [34].
For a pedagogical review of constrained Hamiltonian systems, important for the following discussion, see [35].
The Einstein-Hilbert action of Chapter 3 is recalled

$$S[g_{\mu\nu}, \phi] = -\frac{1}{2\kappa} \int d^4x \sqrt{-g(l_{\mu\nu}, \phi)} R(g_{\mu\nu}(l_{\mu\nu}, \phi)). \tag{6.1}$$

The physical metric has the same form as in Chapter 3,

$$g_{\mu\nu} = (l^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi) l_{\mu\nu} \equiv \Phi^2 l_{\mu\nu} \tag{6.2}$$

where the conformal factor is denoted by Φ^2, for later convenience.
As an initial step, the action can be expressed using the metric $l_{\mu\nu}$ instead of the physical metric $g_{\mu\nu}$. This can be easily done through the formula [36]

$$R(g_{\mu\nu}) = \frac{1}{\Phi^2} \Big(R(l_{\mu\nu}) - 6 \frac{l^{\mu\nu} \nabla_\mu \nabla_\nu \Phi}{\Phi} \Big), \tag{6.3}$$

where the covariant derivative ∇_μ is defined using the metric $l_{\mu\nu}$, because the two metrics are related by the conformal factor Φ^2. By inserting (6.3) into the action, the action reads as

$$S[l_{\mu\nu}, \phi] = -\frac{1}{2\kappa} \int d^4x \sqrt{-l} [\Phi^2 R(l_{\mu\nu}) + 6 l^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi]. \tag{6.4}$$

where the transformation of the determinant is taken into account and a partial integration on the second term is performed.
This action contains second order derivatives of ϕ. This implies that an Hamiltonian analysis is needed in order to exclude the presence of ghosts.
In order to obtain an action with first order derivatives only, the auxiliary field λ is introduced and the action can be recasted into the form

$$S[l_{\mu\nu}, \Phi, \lambda, \phi] = -\frac{1}{2\kappa} \int d^4x \sqrt{-l} [\Phi^2 R(l_{\mu\nu}) + 6 l^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \lambda(\Phi^2 - l^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi)], \tag{6.5}$$

where Φ and ϕ are treated as independent fields, and the auxiliary field λ plays the role of a Lagrange multiplier enforcing the the constraint $\Phi^2 = l^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.
Now, in order to perform the Hamiltonian analysis, one rewrites the components of the metric $l_{\mu\nu}$ by using the $3+1$ decomposition due to Arnowitt, Deser and Misner (ADM) [37]

$$l_{00} = -N^2 + N_i h^{ij} N_j, \; l_{0i} = N_i, \; l_{ij} = h_{ij} \tag{6.6}$$

with inverse metric components given by

$$l^{00} = -\frac{1}{N^2}, \; l^{0i} = \frac{N^i}{N^2}, \; l^{ij} = h^{ij} - \frac{N^i N^j}{N^2}, \tag{6.7}$$

where N is the lapse function, N_i are the components of the shift vector and h_{ij} is the induced metric on the Cauchy surface Σ_t at each time t, with h^{ij} is its inverse.
The 4-dimensional scalar curvature in 3+1 formalism has the form

$$R(l_{\mu\nu}) = K_{ij}\mathcal{G}^{ijkl}K_{kl} + {}^3R + \frac{2}{\sqrt{-l}}\partial_\mu(\sqrt{-l}n^\mu K) - \frac{2}{\sqrt{h}N}\partial_i(\sqrt{h}h^{ij}\partial_j N) \tag{6.8}$$

where ${}^3R$ is Ricci curvature associated with the hypersurface Σ_t and its extrinsic curvature is defined as

$$K_{ij} = \frac{1}{2N}\Big(\frac{\partial h_{ij}}{\partial t} - D_i N_j - D_j N_i\Big) \tag{6.9}$$

with D_i being the covariant derivative determined by the metric h_{ij}, and where

$$\mathcal{G}^{ijkl} = \frac{1}{2}(h^{ik}h^{jl} + h^{il}h^{jk}) - h^{ij}h^{kl} \tag{6.10}$$

is the de Witt metric, the metric on the space of the metrics on compact spaces, with inverse

$$\mathcal{G}_{ijkl} = \frac{1}{2}(h_{ik}h_{jl} + h_{il}h_{jk}) - \frac{1}{2}h_{ij}h_{kl} \tag{6.11}$$

obeying the relation

$$\mathcal{G}_{ijkl}\mathcal{G}^{klmn} = \frac{1}{2}(\delta_i^m\delta_j^n + \delta_i^n\delta_j^m). \tag{6.12}$$

Furthermore, n^μ is the future-pointing unit normal vector to the hypersurface Σ_t, which is written in terms of the ADM variables introduced above as

$$n^0 = \sqrt{-l^{00}} = \frac{1}{N},\, n^i = -\frac{l^{0i}}{\sqrt{-l^{00}}} = -\frac{N^i}{N} \tag{6.13}$$

By substituting these expressions in the 3+1 formalism into the action (6.5), the following form for the action is found

$$\begin{aligned} S[N, N^i, h_{ij}\Phi, \lambda, \phi] = \frac{1}{2}\int dt d^3x \sqrt{h} N [K_{ij}\mathcal{G}^{ijkl}K_{kl}\Phi^2 + {}^3R\Phi^2 - 4K\Phi\nabla_n\Phi \\ - \frac{2}{\sqrt{h}N}\partial_i(\sqrt{h}h^{ij}\partial_j\Phi^2) - 6(\nabla_n\Phi)^2 + 6h^{ij}\partial_i\Phi\partial_j\Phi \\ - \lambda\Phi^2 + \lambda(\nabla_n\phi)^2 - \lambda h^{ij}\partial_i\phi\partial_j\phi], \end{aligned} \tag{6.14}$$

where the derivative operator ∇_n is defined as

$$\nabla_n = \frac{1}{N}(\partial_t - N^i\partial_i). \tag{6.15}$$

The conjugate momenta to h_{ij}, Φ, λ and ϕ, derived from the action (6.14), are given by

$$\pi^{ij} = \frac{1}{2}\sqrt{g}\mathcal{G}^{ijkl}K_{kl}\Phi^2 - \sqrt{h}h^{ij}\partial_n\Phi\Phi, \tag{6.16}$$

$$p_\Phi = -2K\Phi\sqrt{h} - 6\sqrt{h}\nabla_n\Phi, \tag{6.17}$$

$$p_\lambda \approx 0, \tag{6.18}$$

and

$$p_\phi = \sqrt{h}\lambda\nabla_n\phi. \tag{6.19}$$

By using these relations, the following primary constraint is obtained

$$\mathcal{D} = p_\Phi\Phi - 2\pi^{ij}h_{ij} \approx 0. \tag{6.20}$$

By performing a Legendre transformation of the Lagrangian in the action (6.14), the Hamitonian of the model is

$$H = \int d^3x(N\mathcal{H}_T + N^i\mathcal{H}_i + v_\mathcal{D}\mathcal{D} + v_N\pi_N + v^i\pi_i + v_\lambda p_\lambda), \tag{6.21}$$

where boundary terms are ignored because they contribute to the global gravitational energy. For a detailed discussion of this point, see [34].

6.1.1 Constraints and degrees of freedom

The Hamiltonian (6.21) results to be a sum of constraints that vanish for any physical configuration on the constraint surface in the phase space of the theory, where v_λ and $v_\mathcal{D}$ are Lagrange multipliers enforcing the primary constraints (6.18) and (6.20) respectively, v_N and v^i are Lagrange multipliers enforcing those primary constraints associated with the conjugate momenta of the lapse function N and the shift vector N^i

$$\pi_N \approx 0, \quad \pi_i \approx 0, \tag{6.22}$$

and the expression for $\mathcal{H}_T$ and $\mathcal{H}_i$ are given by

$$\begin{aligned}\mathcal{H}_T = &\frac{2}{\sqrt{h}\Phi^2}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1}{2}\sqrt{h}\,{}^3R\Phi^2 + \frac{1}{2\sqrt{h}\lambda}p_\phi^2 + \partial_i(\sqrt{h}h^{ij}\partial_j\Phi^2)\\ &- 3\sqrt{h}h^{ij}\partial_i\Phi\partial_j\Phi + \frac{1}{2}\sqrt{h}\lambda(\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi)\end{aligned} \tag{6.23}$$

and

$$\mathcal{H}_i = p_\Phi\partial_i\Phi + p_\phi\partial_i\phi - 2h_{ij}D_k\pi^{jk}, \tag{6.24}$$

respectively.
The next step is the analysis of the preservation of the primary constraints (6.18), (6.20) and (6.22).
As in GR, the requirement of preserving the constraints (6.22) implies the secondary constraints

$$\mathcal{H}_T \approx 0,\ \mathcal{H}_i \approx 0. \tag{6.25}$$

To show that these constraints are first-class, their smeared form is introduced

$$\mathbf{T}_T(N) = \int d^3x N\mathcal{H}_T, \tag{6.26}$$

$$\mathbf{T}_S(N^i) = \int d^3x(N^i\mathcal{H}_i + p_\lambda\partial_i\lambda). \tag{6.27}$$

The usual Dirac algebra of GR remains valid also for Mimetic Gravity as can be seen from the following Poisson brackets between $\mathcal{H}_i$ and $\mathcal{H}_T$ by using their smeared form (6.26) and (6.27)

$$\begin{aligned}\{\mathbf{T}_T(N), \mathbf{T}_T(M)\} = &\mathbf{T}_S((N\partial_iM - M\partial_iN)h^{ij})\\ &- \int d^3x(\partial_iMN - N\partial_iM)h^{ij}\frac{\partial_j\Phi}{\Phi}\mathcal{D}\end{aligned} \tag{6.28}$$

where the last term vanishes on the constraint surface because of (6.20);

$$\{\mathbf{T}_S(N^i), \mathbf{T}_S(M^i)\} = \mathbf{T}_S((N^i\partial_iM^j - M^i\partial_iN^j)); \tag{6.29}$$

and

$$\{\mathbf{T}_S(N^i), \mathbf{T}_T(M)\} = \mathbf{T}_T(N^i\partial_iM) \tag{6.30}$$

Hence, the Hamiltonian and momentum constraints (6.25) are preserved under time evolution and from the closure of the algebra it is inferred that (6.25) are first-class constraints.
Now the new primary constraints, present in Mimetic Gravity but not in GR, are considered.
The preservation requirement of the primary constraint (6.18) implies

$$\frac{1}{N}\partial_t p_\lambda = \frac{1}{N}\{p_\lambda, H\} = \frac{1}{2\sqrt{h}\lambda^2}p_\phi^2 - \frac{1}{2}\sqrt{h}(\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi) \equiv \mathcal{C}_\lambda \simeq 0. \quad (6.31)$$

Hence, $p_\lambda \simeq 0$ and $\mathcal{C}_\lambda \simeq 0$ are second-class constraint and further constraints are necessary for them to vanish.
In order to study the preservation in time of the primary constraint (6.20), the following linear combination is used

$$\tilde{\mathcal{D}} = \mathcal{D} + 2p_\lambda\lambda = p_\Phi\Phi - 2\pi^{ij}h_{ij} + 2p_\lambda\lambda. \quad (6.32)$$

The last constraint has the following non-zero Poisson brackets:

$$\begin{aligned}
\{\tilde{\mathcal{D}}(x), h_{ij}(y)\} &= 2h_{ij}(x)\delta(x-y) \\
\{\tilde{\mathcal{D}}(x), \pi^{ij}(y)\} &= -2\pi^{ij}(x)\delta(x-y) \\
\{\tilde{\mathcal{D}}(x), \Phi(y)\} &= -\Phi(x)\delta(x-y) \\
\{\tilde{\mathcal{D}}(x), p_\Phi(y)\} &= p_\Phi(x)\delta(x-y) \\
\{\tilde{\mathcal{D}}(x), \lambda(y)\} &= -2\lambda(x)\delta(x-y) \\
\{\tilde{\mathcal{D}}(x), p_\lambda(y)\} &= 2p_\lambda(x)\delta(x-y),
\end{aligned} \quad (6.33)$$

which give the following Poisson brackets with (6.26) and (6.27)

$$\{\tilde{\mathcal{D}}, \mathbf{T}_T(N)\} = -N\mathcal{H}_T. \quad (6.34)$$

and

$$\{\tilde{\mathcal{D}}, \mathbf{T}_S(N^i)\} = \partial_i(N^i\tilde{\mathcal{D}}). \quad (6.35)$$

By collecting all these results

$$\partial_t\tilde{\mathcal{D}} = \{\tilde{\mathcal{D}}, H\} = -N\mathcal{H}_T + \partial_i(N^i\tilde{\mathcal{D}}) + 2v_\lambda p_\lambda \approx 0. \quad (6.36)$$

Hence, $\tilde{\mathcal{D}}$ is preserved without imposing any additional constraint, representing another first class constraint in the theory.
From the analysis of the constraints above, it follows that, besides to 20 degrees of freedom due to the 3-metric h^{ij}, its conjugate momenta π_{ij}, the lapse function N, the shift vector N^i and their conjugate momenta π_N and π_i and the 8 first-class constraints (6.22) and (6.25) of GR, six extra canonical variables $(\phi, p_\phi, \Phi, p_\Phi, \lambda, p_\lambda)$, one extra first class constraint $\tilde{\mathcal{D}} \approx 0$ and

two extra second class constraints $p_\lambda \approx 0$, $\mathcal{C}_\lambda \approx 0$ are present.
So, by using the Dirac formula [34,35]

$$\#\text{canonical variables}/2 - \#\text{first class constraints} - \#\text{second class constraints}/2 \tag{6.37}$$

it is seen that there exist one extra physical degree of freedom, with respect to the two degrees of freedom of GR.

6.1.2 The Ostrogradski instability problem

A further step is noticing that the pair of canonical variables λ, p_λ can be eliminated from the formalism. In fact, $p_\lambda \approx 0$ and $\mathcal{C}_\lambda \approx 0$ are second-class constraints that can be set to vanish strongly.
By solving the constraint $\mathcal{C}_\lambda = 0$ with respect to λ

$$\lambda = \pm \frac{p_\phi}{\sqrt{h}\sqrt{\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi}} \tag{6.38}$$

and inserting these solutions in (6.23), the Hamiltonian constraint becomes

$$\begin{aligned}\mathcal{H}_T = &\frac{2}{\sqrt{h}\Phi^2}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1}{2}\sqrt{h}\,{}^3R\Phi^2 + \partial_i(\sqrt{h}h^{ij}\partial_j\Phi^2) \\ &- 3\sqrt{h}h^{ij}\partial_i\Phi\partial_j\Phi \pm p_\phi\sqrt{\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi}\end{aligned} \tag{6.39}$$

The Hamiltonian dependence on p_ϕ is linear and this can lead to negative energies: this is the so-called Ostrogradski instability [38].
As shown in [34], the two alternative Hamiltonians given in the last expression actually describe the same physical system. Therefore it suffices to consider the dynamics for one of the cases, and here the Hamiltonian with positive sign in front of p_ϕ is chosen.
This choice leads to interpret physically the momentum p_ϕ as proportional to the energy density of the mimetic dust on the spatial hypersurface Σ_t, i.e. p_ϕ is interpreted as the rest mass density of the mimetic dust per coordinate volume element d^3x as measured by the Eulerian observers with four-velocity n^μ.
Since p_ϕ has the physical meaning of density of rest mass, its initial configuration must satisfy $p_\phi \geq 0$ everywhere on the initial Cauchy surface Σ_0 at time $t = 0$. Moreover, the physical meaning of ϕ is that its gradient $\partial_\mu\phi$ is the direction of the rest mass current of the mimetic dust in spacetime.
The equation of motion for ϕ is given by

$$\partial_t\phi = \{\phi, H\} = N\sqrt{\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi} + N^i\partial_i\phi, \tag{6.40}$$

where the time evolution of ϕ is not driven by its canonical conjugate momentum p_ϕ.

By choosing a gauge where $N = \text{const.}$ and $\Phi = \text{const.}$ both greater than zero, and $N^i = 0$, ϕ grows monotonically, increasing under time evolution. The rate of growth has the minimal value $\partial_t \phi = N\Phi$ and then increases when spatial gradient of ϕ contribute.
The equation of motion for the momentum p_ϕ has the form

$$\partial_t p_\phi = \{p_\phi, H\} = \partial_i \left(\frac{N p_\phi h^{ij} \partial_j \phi}{\sqrt{\Phi^2 + h^{ij} \partial_i \phi \partial_j \phi}} + N^i p_\phi \right) \tag{6.41}$$

This continuity equation for the rest mass current of the mimetic dust ensures that the total rest mass on the spatial hypersurface Σ_t is conserved under time evolution.
Firstly, the configuration of this system that can be interpreted as the ground state such that the time derivative of p_ϕ is equal to zero, has to be found. The equation (6.41) can be rewritten as

$$\partial_t p_\phi = p_\phi \partial_i \left(\frac{N h^{ij} \partial_j \phi}{\sqrt{\Phi^2 + h^{ij} \partial_i \phi \partial_j \phi}} + N^i \right) + \partial_i p_\phi \left(\frac{N h^{ij} \partial_j \phi}{\sqrt{\Phi^2 + h^{ij} \partial_i \phi \partial_j \phi}} + N^i \right), \tag{6.42}$$

by using simply the Leibniz rule of derivatives.
From the last equation, there exists a ground state where $p_\phi = 0$. If there exists a region of space where $p_\phi = 0$, then inside that region p_ϕ remains zero, since the vanishing momentum p_ϕ and their spatial derivatives imply $\partial_t p_\phi = 0$.
If an initial configuration where in some region $p_\phi > 0$ holds, which corresponds to the presence of mimetic dust, is considered, now, the fundamental issue regards the fact p_ϕ could evolve to $p_\phi < 0$, making the system unstable. Negative p_ϕ is not desirable because dust acquires negative rest mass and this leads to the problem that an infinite amount of radiation, matter or dust could be created without violating the conservation of energy in a decay process.
The problem can be divided in two steps: whether p_ϕ can evolve to zero or not, and if it possible, if it can become negative.
By assuming the above mentioned gauge with $N = \text{const.}$ and $\Phi = \text{const.}$ both greater than zero, and $N^i = 0$ the equation (6.42) reads

$$\partial_t p_\phi = N p_\phi \partial_i \left(\frac{h^{ij} \partial_j \phi}{\sqrt{\Phi^2 + h^{ij} \partial_i \phi \partial_j \phi}} \right) + \frac{N h^{ij} \partial_i p_\phi \partial_j \phi}{\sqrt{\Phi^2 + h^{ij} \partial_i \phi \partial_j \phi}}. \tag{6.43}$$

The equation of motion (6.43) is dominated by its second term, while the first term is negligible in a region of space where the metric h_{ij} and the gradient $\partial_i \phi$ are nearly constant.
If the case in which the gradient $\partial_i p_\phi$ is such that $h^{ij} \partial_i p_\phi \partial_j \phi < 0$ holds, is taken into account, by considering that the given point is a local minimum of p_ϕ, $\partial_i p_\phi$ is pointing away from the given point. Hence, since $\partial_t p_\phi$ can be

negative, p_ϕ can evolve towards zero, regardless of how close is to zero and since the time evolution of p_ϕ does not necessarily change the direction of the gradient $\partial_i p_\phi$, nothing can avoid the evolution p_ϕ to zero.
Now what happens assuming that p_ϕ has evolved to zero is studied.
The equation of motion (6.43) simplifies to

$$\partial_t p_\phi = \frac{N h^{ij} \partial_i p_\phi \partial_j \phi}{\sqrt{\Phi^2 + h^{ij} \partial_i \phi \partial_j \phi}} \tag{6.44}$$

When the directions of the gradients of p_ϕ and ϕ are such that $h^{ij} \partial_i p_\phi \partial_j \phi < 0$, one has $\partial_t p_\phi < 0$ and consequently p_ϕ becomes negative. Thus, the argument shows that under certain circumstances, the energy density of the mimetic dust can become negative, and consequently the system can become unstable.
If, instead of the plus sign in (6.39), the negative sign in front of p_ϕ is chosen, $-p_\phi$ is identified as the rest mass density of the mimetic dust: it is required that initially p_ϕ must be negative and the argument goes as above.
The equations of motion are

$$\partial_t \phi = \{\phi, H\} = -N\sqrt{\Phi^2 + h^{ij} \partial_i \phi \partial_j \phi} + N^i \partial_i \phi, \tag{6.45}$$

and

$$\partial_t p_\phi = \{p_\phi, H\} = \partial_i \left(-\frac{N h^{ij} p_\phi \partial_j \phi}{\sqrt{\Phi^2 + h^{ij} \partial_i \phi \partial_j \phi}} + N^i p_\phi \right) \tag{6.46}$$

and these equations are the mirror image, obtained via the transformation $(\phi, p_\phi) \to (-\phi, -p_\phi)$, of the equations (6.40) and (6.41), respectively.
If a potential term for the scalar field ϕ is added, the term $\sqrt{h}\Phi^4 V(\phi)$ should be included into the Hamiltonian constraint $\mathcal{H}_T$, and the term $-N\sqrt{h}\Phi^4 \frac{dV(\phi)}{d\phi}$ appears into the equation of motion (6.41). Even in the case this term is positive, the system can still become unstable depending on its initial configurations. In conclusion, the original theory of MDM remains stable and can describe physical dust as long as one takes into account only those initial configurations for which $p_\phi > 0$ at all times.
In the same paper [30], the same Hamiltonian analysis is done for a vector field model of MDM, where the gradient of the mimetic scalar field $\partial_\mu \phi$ is replaced by a vector field u^μ, satisfying the same kinematical constraint and with a Maxwell kinetic term, and for this second model it is shown such instabilities are not present because the corresponding Hamiltonian $\mathcal{H}_T$ results to be in the conjugate momentum p^i.

6.2 Alternative Hamiltonian analysis

A different Hamiltonian formulation of the original proposal of mimetic DM is studied in [39]. The difference with the formulation reviewed in the previous section regards the fact that here the analysis is done in Einstein frame,

starting directly with the constraint on the scalar field added to the Einstein-Hilbert action. Despite of these differences, the same conclusions already made in the previous section regarding the number of degrees of freedom in the theory and the closure of the GR Dirac algebra.
First, the full action for MDM model is recalled

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2\kappa}R + \frac{1}{2}\lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + \mathcal{L}_M\right). \tag{6.47}$$

To construct the canonical formalism of this theory, one can rewrite the action in a $3+1$ formalism, introduced in the previous section.
The part of the action for the scalar field ϕ in this formalism becomes

$$\begin{aligned} S_\phi = -\int d^4x\frac{1}{2}N\sqrt{h}\lambda\Big(1 - g^{00}\partial_0\phi\partial_0\phi - 2g^{0i}\partial_0\phi\partial_i\phi + h^{ij}\partial_i\phi\partial_j\phi \\ - \frac{N^iN^j}{N^2}\partial_i\phi\partial_j\phi\Big) + N\sqrt{h}V(\phi) \end{aligned} \tag{6.48}$$

To find the Hamiltonian, the conjugate momenta have to be derived. The momentum conjugate to ϕ is given by

$$p = \frac{\partial L}{\partial\dot{\phi}} = N\sqrt{h}\lambda(g^{00}\partial_0\phi + g^{0i}\partial_i\phi), \tag{6.49}$$

while the momentum conjugate to λ is obviously zero,

$$p_\lambda = \frac{\partial L}{\partial\dot{\lambda}} = 0. \tag{6.50}$$

Hence, $p_\lambda = 0$ is a primary constraint, and this implies a secondary constraint by requiring that it will be constant in time

$$\dot{p}_\lambda = \{p_\lambda, H\} = 0. \tag{6.51}$$

By inverting equation (6.49), $\dot{\phi}$ can be expressed in terms of its conjugate momentum p, and then the Hamiltonian reads

$$H = \frac{Np^2}{2\sqrt{h}\lambda} + \frac{1}{2}N\sqrt{h}\lambda[1 + h^{ij}\partial_i\phi\partial_j\phi] + pN^i\partial_i\phi + N\sqrt{h}V(\phi). \tag{6.52}$$

By solving the secondary constraint (6.51) one gets

$$\lambda = \frac{p}{\sqrt{h}\sqrt{1 + h^{ij}\partial_i\phi\partial_j\phi}} \tag{6.53}$$

and substituting in (6.52), the dependence of the Hamiltonian on the Lagrange multiplier λ can be eliminated.

Hence, by Legendre transforming back the Hamiltonian (6.52), the total action takes the form

$$S = \int d^4x \Big(L_{ADM} + p\dot{\phi} - Np\sqrt{1 + h^{ij}\partial_i\phi\partial_j\phi} - pN^i\partial_i\phi - N\sqrt{h}V(\phi) \Big) \quad (6.54)$$

where the Arnowitt-Deser-Misner Lagrangian of GR is given by [40]

$$L_{ADM} = \dot{h}^{ij}\pi_{ij} - NR^0 - N^iR_i. \quad (6.55)$$

R^0 and R^i are the intrinsic curvatures given by

$$R^0 \equiv -\sqrt{h}\Big[\,{}^3R + h^{-1}\Big(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\Big)\Big] \quad (6.56)$$

$$R_i \equiv -2h_{ik}\pi^{kj}_{|j} \quad (6.57)$$

From the total action (6.55), the equations of motion of the theory can be found by varying with respect to π_{ij} and h^{ij} respectively

$$\dot{h}^{ij} = \{h^{ij}, H\} = 2Nh^{-1/2}\Big(\pi^{ij} - \frac{1}{2}h^{ij}\pi\Big) + N^{i|j} + N^{j|i}, \quad (6.58)$$

$$\begin{aligned}\dot{\pi}_{ij} = \{\pi_{ij}, H\} = &-N\sqrt{h}\Big({}^3R_{ij} - \frac{1}{2}h_{ij}\,{}^3R\Big) + \frac{1}{2}Nh^{-1/2}h_{ij}\Big(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2\Big)\\ &- 2Nh^{-1/2}\Big(\pi_{im}\pi^m_j - \frac{1}{2}\pi\pi_{ij}\Big) + \sqrt{h}(N_{|ij} - h_{ij}N^{|m}_{|m}) + (\pi_{ij}N^m)_{|m}\\ &- N^{|m}_i\pi_{mj} - N^{|m}_j\pi_{mi} + \frac{Np\partial_i\phi\partial_j\phi}{2\sqrt{1 + h^{kl}\partial_k\phi\partial_l\phi}} - \frac{1}{2}N\sqrt{h}V(\phi)h_{ij}, \end{aligned} \quad (6.59)$$

where the notation $_{|i}$ stands for covariant derivative defined with respect to the 3-metric h^{ij}.
The first equation is independent of the mimetic scalar field ϕ since the action S_ϕ is independent of π_{ij}, while the second equation contains two terms as a function of ϕ which are not present in GR.
By varyng with respect to N and N^i one gets the four modified constraints

$$R^0 + p\sqrt{1 + h^{ij}\partial_i\phi\partial_j\phi} + \sqrt{h}V(\phi) = H_{grav} + H_\phi = 0 \quad (6.60)$$

and

$$R_i + p\partial_i\phi = H_{igrav} + H_{i\phi} = 0 \quad (6.61)$$

Finally, the equations of motion for the phase variables (ϕ, p) are given by

$$\dot{\phi} - N\sqrt{1 + h^{ij}\partial_i\phi\partial_j\phi} - N^i\partial_i\phi = 0 \quad (6.62)$$

and

$$\dot{p} - \partial_k\Big(\frac{Nph^{kl}\partial_l\phi}{\sqrt{1 + h^{ij}\partial_i\phi\partial_j\phi}} + N^kp\Big) + N\sqrt{h}\frac{dV(\phi)}{d\phi} = 0 \quad (6.63)$$

Therefore, the number of equations of motion are those of GR plus two more equations. The former equation is the mimetic kinematical constraint (3.8) rewritten in terms of ADM variables, while the second equation is the Bianchi identity $\nabla_\mu T^{\mu i} = 0$ for the energy-momentum tensor associated with the mimetic scalar field, as it is shown in [39]. The closedness of GR Dirac algebra (6.28)-(6.30) in this equivalent Hamiltonian formulation is also proven in the same paper, showing that the modified constraints (6.60) and (6.61) are first class and the number of first class constraints of GR is not changed in the Hamiltonian formulation of this section. However, there are four extra degrees of freedom $(\lambda, p_\lambda, \phi, p)$ and two extra second class constraint $p_\lambda \approx 0$ and $\dot{p}_\lambda \approx 0$ and this implies that, by using equation (6.37), it is confirmed that there is one more degree of freedom in the theory.

Chapter 7

Quantum Cosmology

As seen in Chapter 4, already at the classical level, Mimetic Gravity can lead to resolution of cosmological singularities.
It is believed that it is possible to avoid the initial singularity, by taking into account the quantum gravity effects at the Planck scale. However, since no full theory of quantum gravity is still available, the cosmological singularity can be resolved by using the quantum cosmological approach.
In this chapter the resolution of cosmological singularities is studied at the quantum level in the Mimetic Gravity framework, identified with the Dust Field formalism [41,42], and compared with the results obtained in the Gauge-fixed picture [43], within the Quantum Cosmology framework.

7.1 Quantum Geometrodynamics

The Hamiltonian formulation of GR reviewed in the Hamiltonian analysis, in the previous chapter, can be the starting pointing for the quantization of the gravitational field dynamics performed following the usual Dirac procedure in Quantum Field Theory [35].
The space of states is that of functionals of configuration variables N, N^i, h_{ij}

$$\Psi = \Psi[N, N^i, h_{ij}]. \tag{7.1}$$

The configuration variables and the conjugate momenta are promoted to operators that in configuration space representation, acting on the functional Ψ, reads

$$h_{ij}(x) \to \hat{h}_{ij}(x) \equiv h_{ij}(x), \quad \Pi^{ij}(x) \to \hat{\Pi}^{ij}(x) \equiv -i\hbar\frac{\delta}{\delta h_{ij}(x)}, \tag{7.2}$$

$$N(x) \to \hat{N}(x) \equiv N(x), \quad \Pi(x) \to \hat{\Pi}(x) \equiv -i\hbar\frac{\delta}{\delta N(x)}, \tag{7.3}$$

$$N^i(x) \to \hat{N}^i(x) \equiv N^i(x), \quad \Pi_i(x) \to \hat{\Pi}_i(x) \equiv -i\hbar\frac{\delta}{\delta N^i(x)}. \tag{7.4}$$

Then canonical commutation relations are established

$$[\hat{h}_{ij}(x), \hat{\Pi}^{kl}] = i\hbar\frac{1}{2}(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l)\delta^{(3)}(x-y), \tag{7.5}$$

$$[\hat{N}(x), \hat{\Pi}(y)] = i\hbar\delta^{(3)}(x-y), \tag{7.6}$$

$$[\hat{N}^i(x), \hat{\Pi}_j(y)] = i\hbar\delta_j^i\delta^{(3)}(x-y) \tag{7.7}$$

while the other commutators vanish.
The next steps of the Dirac prescription for the quantization of constrained systems are quantizing the constraints present in the theory and defining the physical states as those annihilated by the operators associated with these constraints.
For the primary constraints at the quantum level one has

$$-i\hbar\frac{\delta}{\delta N}\Psi[N, N^i, h_{ij}] = 0, \quad -i\hbar\frac{\delta}{\delta N^i}\Psi[N, N^i, h_{ij}] = 0. \tag{7.8}$$

These conditions implies that physical states are functionals of 3-metric only, i.e.

$$\Psi = \Psi[h_{ij}]. \tag{7.9}$$

Then, the momentum and hamiltonian contraints are quantized. The momentum constraint (6.24) restricted only to the GR part, is imposed as follows

$$\hat{\mathcal{H}}_i\Psi \equiv D_j\left[\frac{\delta\Psi}{\delta h_{ij}(x)}\right] = 0 \tag{7.10}$$

This condition implies that the wave functionals depend on the three-geometry $\{h_{ij}\}$ only, and not on one of its representation.
The Hamiltonian constraint (6.23) without the terms due to the addition of the mimetic scalar field in the theory, gives the following equation

$$\hat{\mathcal{H}}_T\Psi \equiv \left[-\frac{1}{\sqrt{h}}G_{ijkl}[h_{mn}]\frac{\delta^2\Psi}{\delta\hat{h}_{ij}(x)\delta\hat{h}_{kl}(x)} - \sqrt{h}^{\,3}R(x)\right]\Psi = 0. \tag{7.11}$$

This is the Wheeler-DeWitt equation, which is the fundamental equation giving the quantum dynamics for the gravitational fields. Moreover, for the first term a proper ordering must be chosen such that the Dirac algebra (6.28)-(6.30) is preserved.
There are some problems in adopting this quantization scheme for GR.
First of all, a Hilbert space structure in the space of the solutions of the constraints (7.10) and (7.11) has not been found. The difficulties are related with finding a basis in the physical Hilbert space due to non linearities of the Wheeler-DeWitt equation, and with defining a proper scalar product on it. Secondly, a problem in dealing with these equations arises from the functional nature of the theory, which complicates further its mathematical

treatment.
Finally, the most relevant problem with this quantization procedure concerns time.
The Hamiltonian is the generator of time displacements in phase-space but the Hamiltonian of the gravitation field is a linear combination of constraints (6.21) which, according to the Dirac prescription, annihilates physical states

$$\hat{H}\Psi = 0. \tag{7.12}$$

The last equation can be seen as the Schrödinger equation for a quantum state not depending on time. This means that quantum states do not evolve, suggesting that there is no quantum dynamics. The problem arises from the implicit attempt to identify two notions of time which are distinct: time in Quantum Mechanics is a fixed external parameter, time in GR is a coordinate.
A physical time can be chosen by fixing a particular foliation of the spacetime as fundamental and by labeling events on a manifold according to some physical clock. This can be done either before or after the quantization, or by some phenomenological considerations, in a model where time plays no precise role. For a general in depth discussion of the several approaches to solve the problem of time, see [44].

7.2 Mini-superspace

Given the shortcomings discussed above, instead of working with Quantum Geometrodynamics dealing with infinite dimensions of the full space of the 3-geometries, also called superspace, it is preferable to work with a finite number of degrees of freedom and construct a quantum model on a finite-dimensional Mini-superspace [45-47].
The reduction of the number of degrees of freedom corresponds to a symmetry reduction implemented at the classical level. This reduction can be done for homogeneous geometries such as FLRW spacetimes and its quantization is generally agreed to be only a toy model of a quantum theory of geometry. To see how this works, in the next section the Mini-superspace of a minimally coupled scalar field in the gravitational field described by the usual Einstein-Hilbert action will be discussed, following the reasoning in [43].

7.2.1 Classical Mini-superspace model

The metric considered is a flat FLRW spacetime

$$ds^2 = N^2(t)dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{7.13}$$

written in Cartesian coordinates, where N is an arbitrary function of time, coinciding with the lapse function introduced in Chapter 6.

The first part of the action is the usual Einstein-Hilbert action (2.1) without the matter Lagrangian $\mathcal{L}_M$ The Ricci scalar R can be written in terms of the scale factor a and the lapse function N

$$R = -6\Big[\frac{\dot{N}\dot{a}}{N^3 a} - \frac{\dot{a}^2}{a^2 N^2} - \frac{\ddot{a}}{aN^2}\Big]. \tag{7.14}$$

So the Einstein-Hilbert action reads

$$S[N,a] = -\frac{3}{\kappa}\int d^4x \Big[\frac{\dot{N}\dot{a}a^2}{N^2} - \frac{\dot{a}^2 a}{N} - \frac{\ddot{a}a^2}{N}\Big] \tag{7.15}$$

where it is used $\sqrt{-g} = Na^3$.
By performing an integration by parts in the last term and assuming that the total derivative term vanishes at the boundaries, the action simplifies to

$$S[N,a] = -\frac{3}{\kappa}\int d^4x \frac{\dot{a}^2 a}{N}. \tag{7.16}$$

This is the classical Mini-superspace model action for the gravitational part. GR is a diffeomorphism Diff($\mathcal{M}$)-invariant theory implying that the equations of the theory transform covariantly under spacetime coordinate transformations. However, the only symmetry left in classical Mini-superspace is given by invariance of the action (7.16) under time re-parametrizations $t = f(\tau)$.
The action for a scalar field minimally coupled to gravity, in a potential V is given by

$$S_M = \frac{1}{2}\int d^4x [g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi)] \tag{7.17}$$

which is invariant under time re-parametrizations because the field Φ is a scalar under time transformations, i.e. $\tilde{\Phi}(\tau, x) = \Phi(\tau, x)$.
By using the definition of the energy-momentum tensor (2.4), it is easy to find that it is

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}(g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - V(\Phi))g_{\mu\nu}, \tag{7.18}$$

and by comparing with the energy-momentum tensor of a perfect fluid (2.9), the pressure, the normalized 4-velocity and the energy density read

$$p = \frac{1}{2}\partial_\alpha\Phi\partial^\alpha\Phi - V(\Phi), \tag{7.19}$$

$$u_\mu = \frac{\partial_\mu\Phi}{|\partial_\mu\Phi|}, \tag{7.20}$$

and

$$\epsilon = \frac{1}{2}\partial_\alpha\Phi\partial^\alpha\Phi + V(\Phi), \tag{7.21}$$

respectively. In the comoving coordinate system, the 4-velocity of the fluid is $u^\mu = \delta_0^\mu / N$.
Then, the pressure p and the energy density ϵ simplify to

$$p = \frac{1}{2}\dot{\Phi}^2 N^{-2} - V(\Phi), \tag{7.22}$$

and

$$\epsilon = \frac{1}{2}\dot{\Phi}^2 N^{-2} + V(\Phi), \tag{7.23}$$

analogously to what happens for Inflation and Quintessence in Chapter 2.
In summary, the total action of this classical mini-superspace model is given by

$$S = V_0 \int dt \Big[-\frac{3}{N\kappa}\dot{a}^2 a + \frac{1}{2N}\dot{\Phi}^2 a^3 - V(\Phi) N a^3 \Big], \tag{7.24}$$

where $V_0 = \int d^3x$ is the comoving volume and it is a multiplicative constant that can be ignored. So the Lagrangian of the system is

$$\mathcal{L} = -\frac{3}{N\kappa}\dot{a}^2 a + \frac{1}{2N}\dot{\Phi}^2 a^3 - V(\Phi) N a^3, \tag{7.25}$$

As usual, the Euler-Lagrange equations for a system with a finite number of degrees of freedom, read

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) = 0. \tag{7.26}$$

where q_i are the dynamical variables of our system, i.e., a, N and Φ.
The Lagrangian (7.25) is independent of the time derivative of the lapse function N. This means that the lapse function plays the role of a gauge, and its choice has no physical consequences for the system. Two choices are usually done: $N = 1$, called the physical time gauge, used in Chapter 2, and $N = a$, called the conformal time gauge.
The equation of motion for the lapse function N is

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa}{3} N^2 \epsilon. \tag{7.27}$$

This equation gives the correct first Friedmann equation (2.6), whatever gauge is chosen.
The equation of motion for the scale factor a reads

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \kappa N^2 p = 2\frac{\dot{N}}{N}\frac{\dot{a}}{a} \tag{7.28}$$

which, by using equation (7.27) is the second Friedmann equation (2.7) in both gauges.
The last equation of motion for the scalar field Φ is given by

$$\ddot{\Phi} + \Big(3\frac{\dot{a}}{a} - \frac{\dot{N}}{N}\Big)\dot{\Phi} + V'(\Phi) N^2 = 0 \tag{7.29}$$

where $V'(\Phi)$ is the derivative of the potential with respect to the field Φ. By combining this equation with the Friedmann equations (7.27) and (7.28) gives the continuity equation (2.10) independent of the gauge chosen. A Legendre transformation gives the Hamiltonian

$$\mathcal{H} = \dot{a}p_a + \dot{\Phi}p_\Phi - \mathcal{L} = N\Big(-\frac{\kappa p_a^2}{12a} + \frac{p_\Phi^2}{2a^3} + V(\Phi)a^3\Big) \equiv NH. \tag{7.30}$$

where p_a and p_Φ are the conjugate momenta of a and Φ, respectively.
As pointed out previously, the Lagrangian (7.25) does not depend on the time derivative of N, so $p_N = 0$ is the primary constraint of the theory. So the total Hamiltonian is given by

$$\mathcal{H}_T = \mathcal{H} + cp_N \tag{7.31}$$

where c is a Lagrange multiplier.
By ensuring that the primary constraint is a constraint for all time, i.e. $\dot{p}_N = 0$, the Poisson brackets is computed

$$\dot{p}_N \approx \{p_N, \mathcal{H}_T\} = -H = 0 \tag{7.32}$$

and the secondary constraint

$$H = -\frac{\kappa p_a^2}{12a} + \frac{p_\Phi^2}{2a^3} + V(\Phi)a^3 = 0, \tag{7.33}$$

is found. The equation $\dot{H} = 0$ is identically satisfied, so there are no more secondary constraints.
Moreover, the dynamical variable N is a Lagrange multiplier so the whole Hamiltonian, $\mathcal{H}_T = cp_N + NH \approx 0$, is a constraint. This fact causes difficulties in quantizing the theory, as already seen in the discussion regarding the full superspace.
The two constraints H and p_N are first class because their Poisson bracket vanishes and the Hamiltonian, besides generating time translations, is also the generator of time re-parametrizations.
The equations of motion in the Hamiltonian formalism are the Poisson brackets of the dynamical variables and their conjugate momenta with the Hamiltonian

$$\dot{g} = \{g, H\}, \tag{7.34}$$

for $g = a$, p_a, Φ and p_Φ.
Of course, the Hamiltonian should give the same equations of motion given by the Lagrangian to be consistent with it and this can be easily verified, by suitable linear combinations of equations (7.34), supplemented by the Hamiltonian constraint (7.33).
In this case, for the Hamiltonian being generator of time re-parametrizations,

a dynamical variable should have vanishing Poisson brackets with this generator in order to be a physical observable.
However, the Poisson brackets of a, p_a, Φ and p_Φ do not vanish, so it is necessary to fix the gauge before quantizing the system to be able to calculate the quantum values of these quantities.
From this need, this model is called Gauge-fixed picture.

7.2.2 Quantum Mini-superspace model

In this section, the quantization of the system presented in the previous section is described.
The Lagrangian found in the previous section is recalled

$$\mathcal{L} = -\frac{3}{N}\dot{a}^2 a + \frac{1}{2N}\dot{\Phi}^2 a^3 - V(\Phi) N a^3. \tag{7.35}$$

By combining the Friedmann equations (7.27) and (7.28) written in physical time gauge with the equation of state (2.11), the following equation of motion is found

$$\frac{\ddot{a}}{a} = -\frac{3\omega + 1}{2}\frac{\dot{a}^2}{a^2}. \tag{7.36}$$

The Lagrangian (7.35) can be rewritten as a Lagrangian describing a particle with variable mass $M(a)$ moving in a potential V, of the form

$$\mathcal{L} = \frac{1}{2}M(a)\dot{a}^2 - V(a) \tag{7.37}$$

that gives the following equation of motion

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\frac{\partial \ln M(a)}{\partial \ln a}\frac{\dot{a}^2}{a^2} - \frac{1}{M(a)a}V'(a). \tag{7.38}$$

By demanding the equality between the last equation and equation (7.36), the form for the mass M and the potential V are found

$$M = a^{3\omega+1}, \qquad V(a) = 0, \tag{7.39}$$

such that the dynamics produced by the two Lagrangians is the same.
Then the Lagrangian (7.37) becomes

$$\mathcal{L} = a^{3\omega+1}\dot{a}^2 \tag{7.40}$$

and substitutes the previous Lagrangian (7.35). Instead of two equations of motion (7.27) and (7.28) and the continuity equation (7.29), in the physical time gauge, a not time re-parametrization invariant Lagrangian (7.40) which has only an equation of motion (7.36) is used: the gauge freedom has been removed and the system is written in terms of its true degrees of freedom.

Hence, this Lagrangian can be used to quantize the system.
The conjugate momentum of the scale factor is

$$p_a = 2a^{3\omega+1}\dot{a} \tag{7.41}$$

and the Hamiltonian, after a Legendre transformation, is given by

$$\mathcal{H} = \frac{p_a^2}{4a^{3\omega+1}}. \tag{7.42}$$

Now, the dynamical variables are promoted to operators and the following operator ordering in the Hamiltonian, upon quantizing, is chosen

$$\frac{p_a^2}{4a^{3\omega+1}} = \frac{1}{8}[\hat{a}^{-3\omega-1}\hat{p}_a^2 + \hat{p}_a\hat{a}^{-3\omega-1}\hat{p}_a]. \tag{7.43}$$

After the gauge-fixing procedure, the Hamiltonian evolves in time, so the Schrödinger equation in this setting is given by

$$\mathcal{H}\Psi(a,t) = i\partial_t\Psi(a,t), \tag{7.44}$$

with $\hbar = 1$. By substituting in it the operator ordered Hamiltonian (7.43) and writing the operator $\hat{p}_a$ in the position representation, the Schrödinger equation takes the form

$$\frac{1}{4}\Big(\frac{3\omega+1}{2}a^{-3\omega-2}\partial_a - a^{-3\omega-1}\partial_a^2\Big)\Psi(a,t) = i\partial_t\Psi(a,t). \tag{7.45}$$

To solve the Schrödinger equation, the following ansatz for the wavefunction is made

$$\Psi(a,t) = f(a)e^{-iEt}, \tag{7.46}$$

leading to a second order differential equation for f

$$a^{-3\omega-1}f''(a) - \frac{3\omega+1}{2}a^{-3\omega-2}f'(a) + 4Ef(a) = 0. \tag{7.47}$$

The solution to this equation is

$$f(a) = c_1 \sin\Big[\frac{4\sqrt{E}}{3(\omega+1)}a^{\frac{3(\omega+1)}{2}}\Big] + c_2\cos\Big[\frac{4\sqrt{E}}{3(\omega+1)}a^{\frac{3(\omega+1)}{2}}\Big] \tag{7.48}$$

where c_1 and c_2 are constants of integration.
By imposing the boundary condition $f(a=0) = 0$, $c_2 = 0$ and the eigenfunctions of the Schrödinger equation in the energy level E are

$$\Psi_E(a,t) = \sin\Big[\frac{4\sqrt{E}}{3(\omega+1)}a^{\frac{3(\omega+1)}{2}}\Big]e^{-iEt}. \tag{7.49}$$

The eigenfunctions are not normalizable and so they are unphysical states. Hence a wave packet is needed to obtain a physical sensible wave function

of the Universe.
In order to construct the wave packet, one takes a superposition of all the eigenfunctions using a suitable weight function $A(E)$

$$\Psi(a,t) = \int_0^\infty A(E)\Psi_E(a,t)dE \tag{7.50}$$

The choice of the weight function $A(E)$ is nearly arbitrary and, to make the integrals duable here, the weight function is chosen to be a quasi-Gaussian function $A(E) = e^{-\gamma E}$, with $\gamma > 0$.
Therefore, the wave packet is given by

$$\Psi(a,t) = \mathcal{N}\frac{a^{\frac{3(\omega+1)}{2}}}{(\omega+1)(\gamma+it)^{3/2}}\exp\Big[-\frac{4a^{3(\omega+1)}}{9(\omega+1)^2(\gamma+it)}\Big] \tag{7.51}$$

with $\mathcal{N}$ a normalization constant.

7.2.3 Interlude: interpretations of Quantum Mechanics

The wave function (7.51) describing the Universe as a whole has been obtained and to try to grasp its physical significance, a short discussion about which interpretation of Quantum Mechanics (QM) has to be used in this context is necessary. A way to interpret the wave function of the Universe is the most widely accepted interpretation: the Copenhagen interpretation. However, in the context of Quantum Cosmology, the Copenhagen interpretation is not suitable because of the problems arising from interpretation of the measurement procedure.
In fact, in the Copenhagen interpretation the measurement of an observable of a system is made by an external classical observer, an observer in the classical domain, outside the system under investigation. If the whole Universe is considered, there is no place for such an observer to make a measurement on the system. This makes Copenhagen interpretation inapplicable to Quantum Cosmology.
An alternative interpretation is the Many Worlds Interpretation (MWI) due to Everett [48,49].
The Hilbert space structure and the self-adjoint operators are the same as those of Copenhagen interpretation and in particular, the expectation value of a dynamical variable in MWI is given by

$$<\hat{q}>=\frac{<\Psi|\hat{q}|\Psi>}{<\Psi|\Psi>} = \frac{\int_0^\infty \Psi^*(a,t)q(a,\partial_a)\Psi(a,t)da}{\int_0^\infty \Psi^*(a,t)\Psi(a,t)da}. \tag{7.52}$$

as in Copenhagen interpretation, but what changes is the interpretation of the physical reality. The collapse of the wave function is avoided because when an observer inside the system, makes a measurement of an observable then every eigenvalue and the corresponding eigenstate is actually realized by creating new Universes that do not interact with each other, describing independent worlds. Allowing the coexistence of the eigenvalues which do not play any role in our world, does not cause any trouble and therefore the problem is solved.

An alternative interpretation which differs from both Copenhagen interpretation and MWI is Bohmian Mechanics [50-52], also known as the causal or ontological interpretation of QM. In this theory one tries to solve the 1-dimensional Schrödinger equation

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\Psi(x,t). \tag{7.53}$$

by using the ansatz

$$\Psi(x,t) = R(x,t)e^{iS(x,t)/\hbar}, \tag{7.54}$$

where $R(x,t)$ and $S(x,t)$ are two real functions of time t and coordinates x. By considering separately the real and the imaginary parts of the Schrödinger equation one gets two independent equations

$$\frac{\partial R^2}{\partial t} + \nabla\left(R^2\frac{\nabla S}{m}\right) = 0, \tag{7.55}$$

and

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0. \tag{7.56}$$

The first equation is a continuity equation for R^2 interpreted as the probability density of an ensemble of particles, if the velocity of the particles is defined as

$$\dot{x} = \frac{\nabla S(x,t)}{m}. \tag{7.57}$$

By exploiting the same definition, the second equation is a Hamilton-Jacobi equation with an extra term

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{7.58}$$

which is called the quantum potential and vanishes in the classical limit $\hbar \to 0$, being responsible for all the quantum effects in this theory.

An important difference with Copenhagen interpretation is that the notion of particle trajectory survives. Having defined the velocity in (7.57), the guidance equation

$$p = m\dot{x} = \nabla S(x,t), \tag{7.59}$$

is found. The name is due to the fact that the wave function, satisfying the Schrödinger equation, through its phase $S(x,t)$, guides the particles in its path, which follow trajectories independent of observations: The integration

of the guidance equation (7.59) gives the trajectory of the particle depending only on the initial position of the particle. This unknown variable is what distinguishes the particles in the ensemble and the reason why it is considered the hidden variable of the theory.
As already pointed out, from equation (7.55) R^2 can be seen as the probability density of the position of the ensemble of particles: one only needs to postulate that the initial distribution of the particles is equal to R^2. However, even if this is not the case, from equations (7.55) and (7.59), R^2 will rapidly tend to the position probability density distribution P.
Hence, depending on the initial position distribution one can make statistical predictions of the position of the particles and calculate the expectation values for the position or the momenta of the ensemble of particles, whose mean values agree with the calculations of the Copenhagen interpretation.
However, differently from what happens in the Copenhagen interpretation, the ignorance of the position of one of the particles in the ensemble is not real, but it is only due to the ignorance of the initial position of the particle. Once the initial position is determined then one can obtain accurately the path of the particle.
Therefore, the probability that Bohmian Mechanics provides is the probability of the particle to be in a position, not just to measure it there: the position of the particle is independent of whether one does a measurement or not. For further discussions, see [52].

7.2.4 Many Worlds Interpretation (MWI)

The dynamical variables, which are physically interesting for the theory, are the scale factor a, the Hubble constant H and the energy density ϵ. These three variables are promoted to operators and the last two physical variables, in the position representation, read

$$\hat{H} = -\frac{i}{4}[\partial_a a^{-3\omega-2} + a^{-3\omega-2}\partial_a] \tag{7.60}$$

and

$$\hat{\epsilon} = \epsilon_0 a^{-3(\omega+1)} \tag{7.61}$$

respectively, where the last expression is the solution of the continuity equation (2.10) and ϵ_0 is a constant.
Their expectation values, by using equation (7.52), are found to be

$$< \hat{a} > (t) = B(\omega)(\gamma^2 + t^2)^{\frac{1}{3(1+\omega)}} \tag{7.62}$$

with

$$B(\omega) = \Big(\frac{9}{8}\frac{(\omega+1)^2}{\gamma}\Big)^{\frac{1}{3(1+\omega)}} \frac{\Gamma\Big(\frac{5+3\omega}{3(1+\omega)}\Big)}{\Gamma\Big(\frac{4+3\omega}{3(1+\omega)}\Big)}, \tag{7.63}$$

$$< \hat{H} >= \frac{2t}{3(\omega+1)(\gamma^2+t^2)}, \tag{7.64}$$

and

$$< \hat{\epsilon} >= \frac{8\epsilon_0\gamma}{3(\omega+1)(\gamma^2+t^2)}, \tag{7.65}$$

respectively.

7.2.5 Bohmian Mechanics

The same dynamical variables in the context of the causal interpretation of QM can be computed. By using the ansatz (7.54) for the wavefunction and substituting this expression in the Schrödinger equation one gets the continuity equation

$$\frac{\partial R}{\partial t} - \frac{1}{4}\Big[\frac{3\omega+1}{2}a^{-3\omega-2}R\frac{\partial S}{\partial a} - 2a^{-3\omega-1}\frac{\partial S}{\partial a}\frac{\partial R}{\partial a} - a^{-3\omega-1}\frac{\partial^2 S}{\partial a^2}\Big] = 0 \tag{7.66}$$

and the modified Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{4}a^{-3\omega-1}\Big(\frac{\partial S}{\partial a}\Big)^2 + Q = 0 \tag{7.67}$$

with the quantum potential Q given by

$$Q = \frac{1}{4R}\Big[\frac{3\omega+1}{2}a^{-3\omega-2}\frac{\partial R}{\partial a} - a^{-3\omega-1}\frac{\partial^2 R}{\partial a^2}\Big]. \tag{7.68}$$

The R and S functions can easily be found by using the wave packet (7.51) and $p_a = \partial_a S$ and get

$$H_b(t) = \frac{2}{3(\omega+1)}\frac{t}{\gamma^2+t^2} \tag{7.69}$$

By integrating the last equation an expression for the scale factor is obtained

$$a_b(t) = a_0(\gamma^2+t^2)^{\frac{1}{3(\omega+1)}}, \tag{7.70}$$

where a_0 is an integration constant.
By using the last equation, the energy density, given by (7.61), reads

$$\epsilon_b(t) = \frac{\epsilon_0}{a_0^{3(\omega+1)}(\gamma^2+t^2)} \tag{7.71}$$

Finally, the Quantum potential (7.68), by using the expression for a_b (7.70) and the wave packet (7.51), takes the form

$$Q(t) = \frac{27\gamma(1+\omega)^2 - 8\gamma^2}{18(1+\omega)^2(\gamma^2+t^2)}. \tag{7.72}$$

7.3 Dust Field formalism

The action of MDM was already introduced in a different context: the so called Dust Field formalism proposed in [41,42] By following the notation used in [42], the action reads

$$S = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}R + \frac{1}{2}\int d^4x\sqrt{-g}M(g^{\alpha\beta}\partial_\alpha T\partial_\beta T - 1). \tag{7.73}$$

which looks identical to that of MDM (3.36), where M is a Lagrange multiplier forcing the gradient of the field T to be timelike. By following the same steps to obtain the Mini-superspace version of the previous model, the Lagrangian becomes

$$\mathcal{L} = \frac{3a\dot{a}^2}{N} + \frac{a^3M}{2N}\dot{T}^2 - \frac{Na^3M}{2} \tag{7.74}$$

where $\kappa = 1$ for convenience.
From deriving the equations of motion from this Mini-superspace model Lagrangian, it is easy to see that the scalar field T has precisely the same features of the mimetic field ϕ of the original proposal, i.e.

1. pressureless,
2. identical to cosmological time in synchronous reference frame,
3. with energy density equal to the Lagrange multiplier M,
4. satisfying a continuity equation.

1. The equation of motion for a is given by

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + \frac{M}{2}(\dot{T}^2 - N^2) = 2\frac{\dot{N}}{N}\frac{\dot{a}}{a} \tag{7.75}$$

   and by comparing it with the second Friedmann equation (7.28), the pressure should be equal to $p = \frac{M}{2}(\dot{T}^2/N^2 - 1)$.
   The equation of motion for the Lagrange multiplier M reads

$$\frac{3a^2}{2N}\dot{T}^2 = \frac{3a^2N}{2} \implies \dot{T}^2 = N^2 \tag{7.76}$$

   thus making the pressure vanish.

2. From the last equation, it is easy to see that time can be identified with the dust field variable in the physical time gauge, $N = 1$, which corresponds precisely to a synchronous reference frame.

3. The equation of motion for the other Lagrange multiplier gives
$$\frac{\dot{a}^2}{a^2} = \frac{M}{3}\frac{\dot{T}^2 + N^2}{2} = \frac{M}{3} \tag{7.77}$$
that is the first Friedmann equation (7.27) with the Lagrange multiplier playing the role of the energy density

4. The equation of motion for the field itself is
$$\ddot{T} - \Big(3\frac{\dot{a}}{a} + \frac{\dot{M}}{M} - \frac{\dot{N}}{N}\Big)\dot{T} = 0 \tag{7.78}$$
which, for $\dot{T} = N$ and $M = \epsilon$ is the right continuity equation for the pressure-less dust field.

The equations of motion (7.77)-(7.75)-(7.76)-(7.78) are precisely the tt component and the rr component of the modified Einstein field equations, the mimetic constraint and the continuity equation, respectively, found in Chapter 3 for the mimetic scalar field.
From the Lagrangian, the corresponding Hamiltonian can be obtained in the usual way.
The conjugate momenta read
$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -6\dot{a}a \tag{7.79}$$
$$p_T = \frac{\partial \mathcal{L}}{\partial \dot{T}} = a^3 M\dot{T} \tag{7.80}$$
$$p_M = 0, \quad p_N = 0 \tag{7.81}$$
and the Hamiltonian
$$\mathcal{H} = N\Big(- \frac{p_a^2}{12a} + \frac{p_T^2}{2a^3 M} + \frac{a^3 M}{2}\Big). \tag{7.82}$$
The Hamiltonian equation of motion for the conjugate momenta of T will give $\dot{p}_T = \{p_T, \mathcal{H}\} = 0$, which means that it is constant.
In the physical time gauge and using the fact that $\dot{T} = N$, the Lagrange multiplier M can be written as
$$M = \frac{p_T}{a^3} \tag{7.83}$$
and the Hamiltonian
$$\mathcal{H} = -\frac{p_a^2}{12a} + p_T = 0 \tag{7.84}$$
that vanishes because it is still a constraint.
The dependence of the Hamiltonian (7.84) on the momentum conjugated to T, p_T is linear and T, already identified with the physical time t, plays this role in the quantized theory.
The Hamiltonian equation of motion for p_T gives $\dot{p}_T = \{p_T, H\} = 0$, which implies that $p_T = \epsilon_0$, where ϵ_0 is a constant, connected with the energy density as can be easily seen from (7.83).

7.3.1 Quantization

The solution to the Wheeler-De Witt equation for a Mini-superspace model

$$\hat{\mathcal{H}}\Psi(a,t) = 0 \tag{7.85}$$

with the Hamiltonian (7.84), can be found without operator ordering needed. The starting point for the discussion is the canonical transformation

$$\tilde{p}_{\tilde{a}} = \frac{p_a}{\sqrt{12a}}, \quad \tilde{a} = \frac{4a^{3/2}}{\sqrt{3}}, \tag{7.86}$$

under which the Poisson brackets of the theory are preserved.
The Hamiltonian, rewritten in the new variables, takes the form

$$\tilde{\mathcal{H}} = -\tilde{p}_{\tilde{a}}^2 + p_T = 0 \tag{7.87}$$

Then, the new variables are promoted to operators and the equation (7.84) in position representation is given by

$$\hbar^2 \partial_{\tilde{a}}^2 \tilde{\Psi}(\tilde{a},t) - i\hbar\partial_T \tilde{\Psi}(\tilde{a},t) = 0. \tag{7.88}$$

Here, for later convenience, $\hbar$ is not set to one. By making a Fourier transformation

$$\tilde{\Psi}(\tilde{a},t) = \int_{-\infty}^{\infty} \exp(ik\tilde{a})\tilde{\Psi}_k(t)dk, \tag{7.89}$$

and by substituting the Fourier transform in (7.88), the following differential equation for the mode $\tilde{\Psi}_k$ is found

$$\dot{\tilde{\Psi}}_k(t) - i\hbar k^2 \tilde{\Psi}_k(t) = 0, \tag{7.90}$$

which has to be satisfied for every k.
The solution to this equation is given by

$$\tilde{\Psi}_k(t) = \tilde{f}(k)e^{ik^2\hbar t} \tag{7.91}$$

where $\tilde{f}(k)$ is a completely arbitrary function of k. The function $\tilde{f}(k)$ can be assumed of the form

$$\tilde{f}(k) = ke^{-\gamma\hbar^2 k^2} \tag{7.92}$$

with $\gamma > 0$.
Then

$$\tilde{\Psi}(\tilde{a},t) = \mathcal{N}\frac{\tilde{a}}{4(\gamma\hbar^2 - i\hbar t)^{3/2}} \exp\Big[-\frac{\tilde{a}^2}{4(\gamma\hbar^2 - i\hbar t)}\Big]. \tag{7.93}$$

To obtain the expectation values of the physical relevant quantities operators above, one has to rewrite them in terms of the new variables in position representation as

$$\hat{a} = \frac{\sqrt{3}}{4}\tilde{a}^{2/3}, \tag{7.94}$$

$$\hat{\epsilon} = \left(\frac{4}{\sqrt{3}}\right)^3 \epsilon_0 \tilde{a}^2 \tag{7.95}$$

$$\hat{H} = \frac{8\sqrt{2}i\hbar}{3^{3/4}}(\tilde{a}^{-1}\partial_{\tilde{a}} + \partial_{\tilde{a}}\tilde{a}^{-1}). \tag{7.96}$$

In the framework of the MWI, the expectation values are given by

$$< \hat{a} >= \sqrt{\frac{3}{\pi}} \frac{\Gamma\left(\frac{11}{6}\right)}{(4\gamma)^{1/3}} (\gamma^2\hbar^2 + t^2)^{1/3}, \tag{7.97}$$

$$< \hat{\epsilon} >= \frac{64\epsilon_0\gamma}{3\sqrt{3}(\gamma^2\hbar^2 + t^2)}, \tag{7.98}$$

and

$$< \hat{H} >= \frac{8\sqrt{2}}{3^{3/4}} \frac{t}{(\gamma^2\hbar^2 + t^2)}. \tag{7.99}$$

The same expectation values, by using Bohmian Mechanics, read

$$a_b(t) = a_0(\gamma^2\hbar^2 + t^2)^{1/3}, \tag{7.100}$$

$$\epsilon_b(t) = \frac{\epsilon_0}{a_0^3(\gamma^2\hbar^2 + t^2)}, \tag{7.101}$$

and

$$H_b(t) = \frac{2}{3} \frac{t}{(\gamma^2\hbar^2 + t^2)}. \tag{7.102}$$

Furthermore, the quantum potential Q in terms of the physical time t reads

$$Q(t) = \frac{\gamma\hbar^2\left(\frac{3}{2} - \frac{4}{3}\gamma a_0^3\right)}{\gamma^2\hbar^2 + t^2}. \tag{7.103}$$

7.4 Comparison of the results

The tables 7.1 and 7.2 summarize the results found in the Gauge-fixed picture and in the Dust Field formalism in the previous sections for the scale factor, the Hubble constant and the energy density of the Universe.

What can be seen is that the quantized version of the Dust Field formalism, which at the classical level coincides with MDM, gives almost the same results (modulo some numerical factors) as the Gauge-fixed picture in the Mini-superspace formulation, in both interpretations studied.

More precisely, the scale factor in Gauge-fixed picture depends on ω, i.e. the equation of state of the fluid that is the dominant matter component of the Universe in a particular epoch and it coincides for dust $\omega = 0$ with the Dust Field formalism, where there is no potential for the mimetic field and so the pressure is vanishing. Moreover, in the case of MWI there are some numerical factors that distinguishes the results obtained in the two

Table 7.1: Many Worlds Interpretation

Gauge-fixed picture	Dust Formalism
$< a > (t) = B(\omega)(\gamma^2\hbar^2 + t^2)^{\frac{1}{3(1+\omega)}}$	$< a >= \sqrt{\frac{3}{\pi}}\frac{\Gamma\left(\frac{11}{6}\right)}{(4\gamma)^{1/3}}(\gamma^2\hbar^2 + t^2)^{1/3}$
$< H >= \frac{2t}{3(\omega+1)(\gamma^2\hbar^2+t^2)}$	$< H >= \frac{8\sqrt{2}}{3^{3/4}}\frac{t}{(\gamma^2\hbar^2+t^2)}$
$< \epsilon >= \frac{8\epsilon_0\gamma}{3(\omega+1)(\gamma^2\hbar^2+t^2)}$	$< \epsilon >= \frac{64\epsilon_0\gamma}{3\sqrt{3}(\gamma^2\hbar^2+t^2)}$

Table 7.2: Bohmian Mechanics

Gauge-fixed picture	Dust Formalism
$a_b(t) = a_0(\gamma^2\hbar^2 + t^2)^{\frac{1}{3(\omega+1)}}$	$a_b(t) = a_0(\gamma^2\hbar^2 + t^2)^{1/3}$
$H_b(t) = \frac{2}{3(\omega+1)}\frac{t}{\gamma^2\hbar^2+t^2}$	$H_b(t) = \frac{2}{3}\frac{t}{(\gamma^2\hbar^2+t^2)}$
$\epsilon_b(t) = \frac{\epsilon_0}{a_0^{3(\omega+1)}(\gamma^2\hbar^2+t^2)}$	$\epsilon_b(t) = \frac{\epsilon_0}{a_0^3(\gamma^2\hbar^2+t^2)}$

approaches.
Therefore, the analysis can be restricted only to the Gauge-fixed picture.
As can be seen in the Figure 7.1, the scale factor has a minimum for $t = 0$, remaining finite and connecting a contracting phase for $t < 0$ with an expanding phase for $t > 0$. So, there is quantum bounce for the Universe history, while the behavior of the scale factor for $|t| >> 1$ is the same as the classical dust-dominated Universe $a \propto t^{\frac{2}{3}}$. The minimum of the scale factor and so the comoving size of the Universe at time $t = 0$ will depend on the parameter γ present in the wave function of the Universe.

In Figure 7.2, instead, three different scale factor evolutions are depicted:

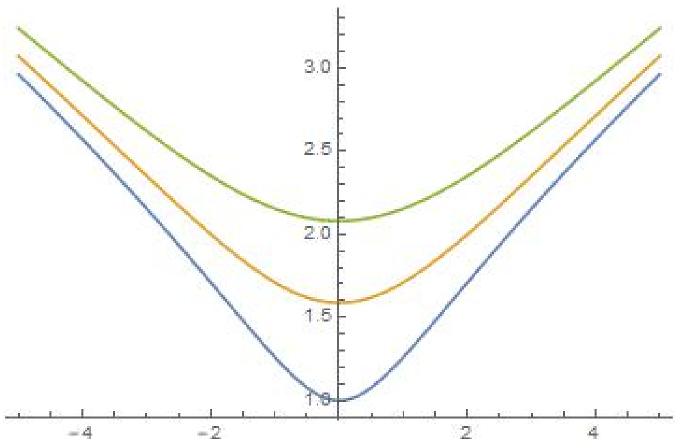

Figure 7.1: The expectation value of the scale factor for a pressure-less dust field $\omega = 0$, with different values of the parameter γ ($\gamma = 1, 2, 3$).

the steepest plot is the one corresponding to a dust-dominated Universe $\omega = 0$ for all time, the intermediate one represents a radiation-dominated Universe with $\omega = 1/3$ and the last one is the scale factor for a Universe with ultra-stiff matter with $\omega = 1$, appearing in some theories such as in [53].
According to the known facts about the Universe, the actual scale factor evolution should be a combination of the first two cases: the second plot remains valid until the radiation-dust matter equality is reached and then the first plot becomes more accurate in describing the rest of the history of the Universe.
The Quantum Bounce scenario is confirmed by the change of sign of the

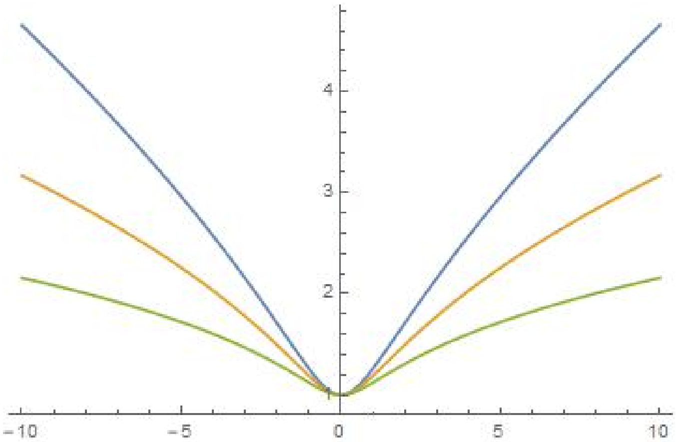

Figure 7.2: The expectation value of the scale factor for $\gamma = 1$ and for different values of ω ($\omega = 0, 1/3, 1$).

Hubble constant at $t = 0$ as can be seen in Figure 7.3 and in Figure 7.4. In particular, in Figure 7.3, it is shown how the choice of the parameter γ modulates the time dependence of the Hubble constant, displaying a growing flatness for increasing γ. Instead, the form of the Hubble constant curves remains quite the same for different values of ω and so the shape of the curve is almost independent of the dominant matter component of the Universe, despite the inequality of the maxima and minima and consequently the steepnesses for small and intermediate values of time t.
In Figure 7.5 and Figure 7.6, it can be seen what is expected in a Quantum Bounce scenario: the finiteness of the expectation value of the energy density when $t = 0$ resolving the problem of the initial singularity, at the quantum level. The same observations made for the plots of the Hubble constant apply here.
Furthermore, the Quantum potential in Bohmian Mechanics in the Gauge-

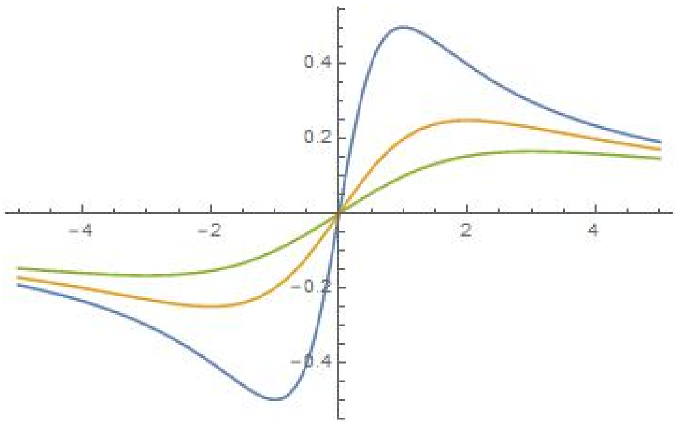

Figure 7.3: The expectation value of the Hubble constant for a pressure-less dust field $\omega = 0$, with different values of the parameter γ ($\gamma = 1, 2, 3$).

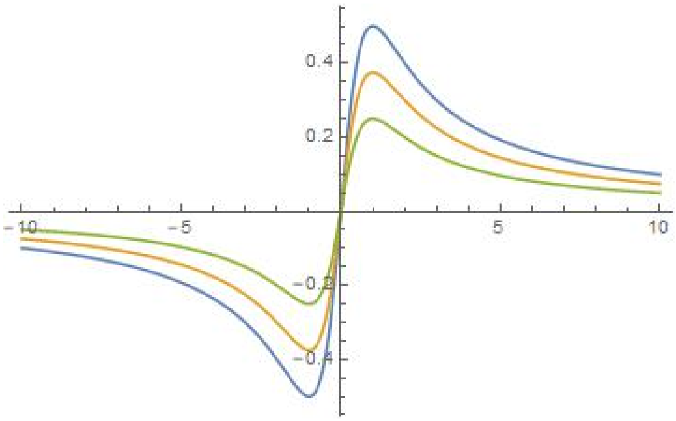

Figure 7.4: The expectation value of the Hubble constant for $\gamma = 1$ and for different values of ω ($\omega = 0, 1/3, 1$).

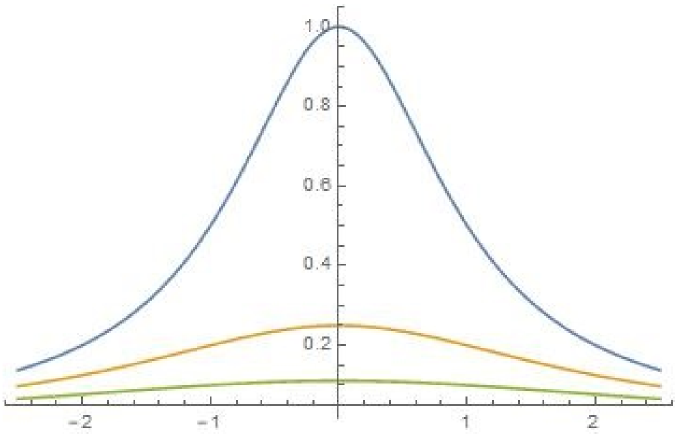

Figure 7.5: The expectation value of the energy density with $\epsilon_0 = 1$ for a pressure-less dust field $\omega = 0$, with different values of the parameter γ ($\gamma = 1, 2, 3$).

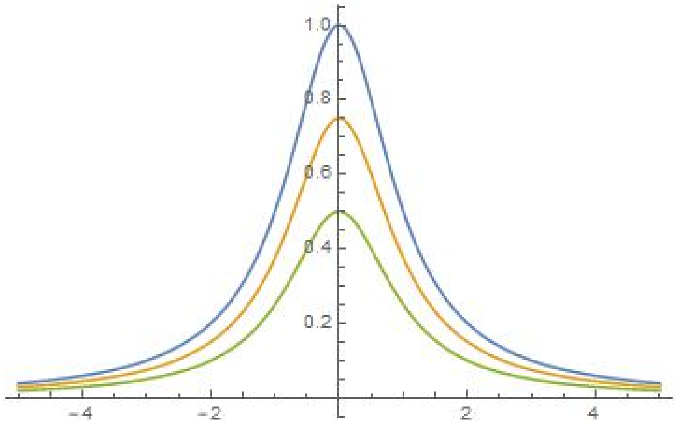

Figure 7.6: The expectation value of the energy density with $\epsilon_0 = 1$ for $\gamma = 1$ and for different values of ω ($\omega = 0, 1/3, 1$).

fixed picture (7.72) is recalled

$$Q(t) = \frac{27\gamma(1+\omega)^2 - 8\gamma^2}{18(1+\omega)^2}(\gamma^2+t^2)^{-1}. \tag{7.104}$$

In Figure 7.7, different plots of the Quantum potential for different values of the wave function parameter γ, while in Figure 7.8 the three plots are given for different dominant matter component equations of state. As it is easy to see in the graphs, the Quantum potential is relevant only for time scales near the Quantum Bounce and becomes negligible after an interval of time modulated by γ. Moreover, the Quantum potential becomes more and more negligible for all time as the parameter γ grows. For different values of ω, for fixed γ, the form of the Quantum potential remains the same, but the values of the maxima increase and as a consequence, its importance for early time as ω increases, at fixed γ.

Finally, the Quantum potential in Bohmian Mechanics for the Dust Field

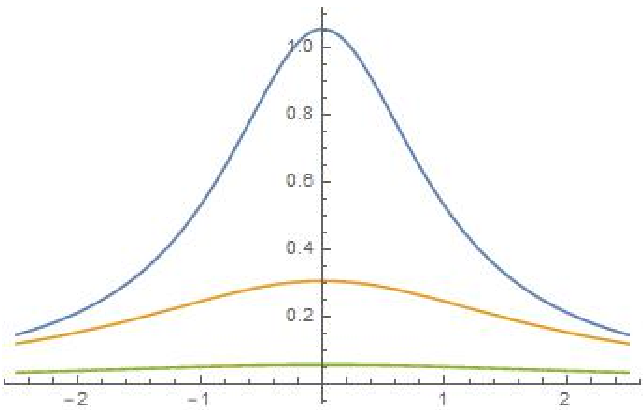

Figure 7.7: The Quantum potential for the Gauge-theory approach for a pressure-less dust field $\omega = 0$, with different values of the parameter γ ($\gamma = 1, 2, 3$).

formalism

$$Q(t) = \frac{\gamma\hbar^2\left(\frac{3}{2} - \frac{4}{3}\gamma a_0^3\right)}{\gamma^2\hbar^2 + t^2}. \tag{7.105}$$

is indendent of ω, as expected. However, the situation is different from what happens in the previous case. For $\gamma = \frac{9}{8}a_0^{-3}$ the Quantum potential vanishes, for $\gamma < \frac{9}{8}a_0^{-3}$ it is positive while it becomes negative for $\gamma > \frac{9}{8}a_0^{-3}$.

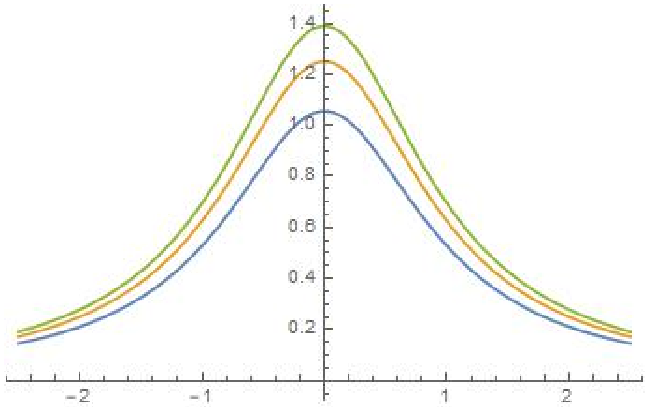

Figure 7.8: The expectation value of the scale factor for $\gamma = 1$ and for different values of ω ($\omega = 0, 1/3, 1$).

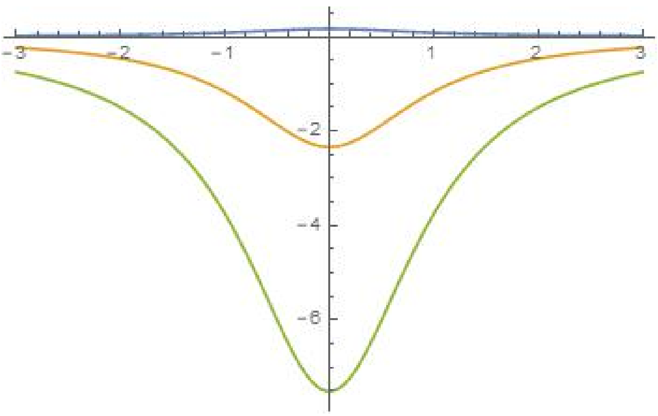

Figure 7.9: The quantum potential with $a_0 = 1$ for a pressure-less dust field $\omega = 0$ with different values of the parameter γ ($\gamma = 1, 2, 3$).

Chapter 8

Conclusions

Mimetic Gravity has emerged as an interesting alternative to General Relativity and has attracted since its birth an increasing interest.
In this thesis, it has been reviewed how the conformal degree of freedom of gravity has become dynamical through a singular disformal transformation, leading to a new degree of freedom as an effective dark matter component appearing on cosmological scales.
By addition of a potential it has been seen to give a well behaved early-time and late-time phenomenology, offering a unified description of dark components of the Universe, and present at the classical level a Big Bounce scenario, for which it has been described a very general way to avoid cosmological singularities, inspired from non-commutative geometry.
Furthermore, cosmological perturbations in this setting have been studied, evidencing the need for an extra term in order to have a nonvanishing speed of sound, reconciling the theory with the study of the quantum fluctuations in inflationary scenario and, from direct coupling of the mimetic scalar field with ordinary matter, a way to address physical processes in the early Universe such as Baryogenesis has been described.
As an interlude, it has been reviewed how Mimetic Gravity can reproduce the most important astrophysical evidence for Dark Matter: the galaxies rotation curves. This is done by looking for static spherically symmetric solutions associated with a mimetic scalar field potential via a reconstruction procedure, without invoking particle matter nature for Dark Matter.
By doing the Hamiltonian analysis of the theory it has been shown in a more formal way that one more degree of freedom is present in the theory, how the constrained Hamiltonian dynamics of GR is modified because of the mimetic degree of freedom and that Mimetic Gravity avoids Ostrogradski instability under certain conditions.
Then, the identification at the classical level with Dust Field formalism has been pursued and the associated quantized Mini-superspace model has been described and its results compared with those of the Gauge-fixed picture

case.
The results are the same, for the typical equation of state of dust, modulo some constants but Mimetic Gravity has the important feature to introduce a physical time in the Quantum Cosmology setting.
From the features presented in this thesis, the increasing appeal of this theory as an alternative to the particle Dark Matter appears justified and many developments appeared so far have to be analyzed further and still much has to be discovered.

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