

Unbounded Pontryagin numbers on nonnegatively curved spin manifolds

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Abstract

By studying a family of projective bundles over complex projective space, we show that any rational Pontryagin number that does not factor through Ochanine's elliptic genus is unbounded on closed connected spin manifolds of nonnegative sectional curvature. This result provides a positive answer to a question posed by M. Herrmann and N. Weisskopf, and implies that the only possible obstructions to nonnegative sectional curvature on spin manifolds in terms of bordism theory come from the elliptic genus.

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1 Introduction

A part of this thesis is presented in a forthcoming paper [HK21] jointly written by the author and her advisor, which is available on arXiv at the time of writing.

The study of obstructions to nonnegative curvature on Riemannian manifolds has long been of natural interest to mathematicians. For though curvature is a local phenomenon, the requirement that a manifold possess upper or lower bounds on curvature often yields global insights onto its topology. One of the most notable results of this nature is Gromov's Betti Number Theorem, concerning the existence of a universal bound to the Betti numbers of closed connected manifolds with nonnegative sectional curvature:

Theorem 1.1 (Gromov, [Gro81]). *There exists a constant $\mathcal{C} = \mathcal{C}(n)$, such that every compact connected n -dimensional Riemannian manifold M of nonnegative sectional curvature satisfies*

$$\sum_{i=0}^n b_i(M) \leq \mathcal{C}.$$

The signature $\sigma(M)$ of a closed oriented $4m$ -manifold M , defined via the intersection form, is bounded by the middle dimensional Betti number $b_{2m}(M)$. If we regard the signature $\sigma : \Omega_*^{SO} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ as the L -genus on the rational oriented cobordism ring, so that each image $\sigma(M)$ is a Pontryagin number of M , the above result immediately implies a universal bound for σ on all nonnegatively curved closed oriented manifolds in each dimension.

It was shown that in fact, the signature and its multiples are the only Pontryagin numbers that are bounded on nonnegatively curved closed connected oriented manifolds.

Theorem 1.2 (Kotschick, [Kot10]). *A rational linear combination of Pontryagin numbers is bounded on connected closed oriented manifolds of nonnegative sectional curvature if and only if it is a multiple of the signature.*

As a corollary, the signature is the only Pontryagin number that can be bounded in terms of Betti numbers.

Apart from implications on nonnegative sectional curvature, it is worthy to note that many results on positive Ricci and scalar curvatures such as that of Bonnet-Myers [Car92] and Gromov-Lawson, Stolz [Sto92] have also shed light on the global structure on various Riemannian manifolds. Among these, we have the index-theoretic obstruction to positive scalar curvature on spin manifolds due to Lichnerowicz:

Theorem 1.3 (Lichnerowicz, [Lic63]). *Let M be a compact spin manifold admitting positive scalar curvature, then the \hat{A} -genus of M vanishes, i.e. $\hat{A}(M) = 0$.*

Since scalar curvature is the average of sectional curvature over all possible two-planes at a point, nonnegative sectional curvature implies positive scalar curvature unless it is identically zero everywhere, in which case Chern-Weil Theory suggests that all characteristic numbers are zero. This means that contrary to the oriented case, on spin manifolds with nonnegative sectional curvature, the signature is not the only Pontryagin number that can be bounded by a universal constant, one must also include the \hat{A} -genus. Moreover, since the \hat{A} -genus is proportional to the signature σ in dimension 4, the statement also implies that it is impossible to find a nonnegatively curved 4-dimensional generator of the rational oriented cobordism ring.

Following Dessai [Des07], who conjectured that the elliptic genus, a ring homomorphism from the spin cobordism ring to the weight zero modular forms of level 2

$$\varphi : \Omega_*^{spin} \otimes \mathbb{Q} \rightarrow M_*(\Gamma_0(2)),$$

is constant (i.e. all non-constant coefficients vanish) on any spin manifold admitting positive sectional curvature, Herrmann and Weisskopf [HW18] raised the question as to whether the obstruction to nonnegative sectional curvature is given by the elliptic genus. This is motivated by the fact that the elliptic genus is uniquely determined by the q -expansion of its weight zero form Φ_0 at the two cusps of $\Gamma_0(2)$, and degenerates into a series of twisted L -genus or twisted \hat{A} -genus respectively. To our knowledge, this question remains open, and we formulate it here as a conjecture:

Conjecture 1.4. *A rational linear combination of Pontryagin numbers is bounded on connected closed spin manifolds of nonnegative sectional curvature if and only if it lies in the span of the elliptic genus.*

This statement, if true, would yield an analogous result to that of Kotschick's above. Herrmann and Weisskopf [HW18] attempted the conjecture in one direction, asking whether rational Pontryagin numbers not contained in the span of the elliptic genus are unbounded on connected nonnegatively curved spin manifolds. Using case by case calculations, they were able to provide a positive answer up to dimension 20.

In this work we provide a positive answer to their question above in all generality, solving one direction of the conjecture:

Theorem 1.5 (Hsiao, Kotschick, [HK21]). *Any rational linear combination of Pontryagin numbers that does not factor through the elliptic genus is unbounded on connected closed spin manifolds of nonnegative sectional curvature.*

The proof is a variation of the approach in the oriented case [Kot10] and relies on constructing a suitable family of basis sequences $(\alpha_m(c))_{m \geq 1}$ of $\Omega_*^{spin} \otimes \mathbb{Q}$ dependent on a parameter c , such

that each generator $\alpha_{m \neq 1}$ admits nonnegative sectional curvature and for dimensions ≥ 12 , it lies in the kernel of the elliptic genus. In addition, by varying the parameter arbitrarily, $\alpha_m(c)$ have Pontryagin numbers unbounded in terms of c .

The logic of our approach then unfolds as follows: Fix a dimension $4m$ and pick a Pontryagin number p . By assumption the chosen Pontryagin number p does not factor through the elliptic genus φ , so it must be nonzero on a monomial in ring generators involving at least one $\alpha_{m \geq 3}(c)$ in the kernel of φ . If this monomial is nonnegatively curved, the Pontryagin number is unbounded as one varies the parameter c and we would be done. However, due to Lichnerowicz, it is not possible to choose a spin generator with nonnegative curvature in dimension 4. We therefore can not guarantee that the monomial in question has nonnegative curvature, as it is a product of manifolds of which some may be negatively curved. This difficulty is overcome by uncovering certain polynomial relations between different generators of the cobordism ring via the introduction of a characteristic number we call the q -number, which allows us to eliminate the presence of the ring generator in dimension 4 altogether.

As a corollary, Theorem 1.5 shows that the only potential obstruction to nonnegative sectional curvature on spin manifolds comes from the elliptic genus. It is not yet clear if the other direction to Conjecture 1.6 holds, that is, if every Pontryagin number in the span of the elliptic genus has a universal bound. We formulate this open question here as a conjecture, a positive answer of which together with Theorem 1.5 would provide a positive solution to the Conjecture 1.4 above:

Conjecture 1.6. *Any rational linear combination of Pontryagin numbers factoring through the elliptic genus has a universal bound on nonnegatively curved spin manifolds.*

The following is an outline of this study: In section 2 we briefly review the fact that rational spin cobordism is classified by Pontryagin numbers and introduce the elliptic genus. In section 3 we construct a family of projective bundles and discuss properties of its Pontryagin numbers. In section 4 we give a proof of the main theorem by considering the family of basis sequences built from the projective bundles in the previous section, in the course of which we will encounter the q -number. In section 5 we provide some outlooks to the converse, and discuss why the basis sequence fails to provide a similar proof in the case of complex spin manifolds.

2 Preliminaries

2.1 Pontryagin numbers and rational spin cobordism

The study of Pontryagin numbers on spin manifolds can be done by considering classes in the rational spin cobordism ring $\Omega_*^{spin} \otimes \mathbb{Q}$, as these are completely classified by Pontryagin numbers. In other words, two manifolds in $\Omega_*^{spin} \otimes \mathbb{Q}$ are bordant if and only if their Pontryagin numbers agree. An introduction to this classification is best done via the rational oriented cobordism ring $\Omega_*^{SO} \otimes \mathbb{Q}$, which turns out to be isomorphic to $\Omega_*^{spin} \otimes \mathbb{Q}$.

Definition 2.1. *Two oriented manifolds M, N of dimension n are called (oriented) **bordant** if there exists an $(n + 1)$ -dimensional oriented manifold W whose boundary ∂W is diffeomorphic to the disjoint union $M \sqcup \bar{N}$ with respect to an orientation-preserving map.*

An oriented manifold M which is the boundary of another oriented manifold is bordant to the empty manifold, and in this case we say that M is **nullbordant** or that M **bounds**.

Now oriented bordism is an equivalence relation, and the quotient of all n -dimensional oriented manifolds under this relation forms the n -th **oriented bordism group** Ω_n^{SO} , where the identity element is the nullbordant class and the group structure $+$ is given by taking disjoint unions.

The direct sum of all the oriented bordism groups has the structure of a graded ring:

Definition 2.2. *The **oriented cobordism ring***

$$\Omega_*^{SO} = \bigoplus_{n=0} \Omega_n^{SO}$$

is a graded ring with unital element $1 \in \Omega_0^{SO}$ and multiplication induced by Cartesian products. The ring $\Omega_^{SO} \otimes \mathbb{Q}$ is called the **rational oriented cobordism ring**.*

Here the unital element in the 0-th bordism group is the bordism class of a point, which generate $\Omega_0^{SO} \cong \mathbb{Z}$. Note that multiplication is indeed graded commutative

$$M^m \times N^n = (-1)^{mn} N^n \times M^m.$$

Meanwhile, to study Pontryagin numbers, we will first introduce Pontryagin classes, or more generally, characteristic classes of vector bundles. Given a ring R and a vector bundle $E \rightarrow B$, a characteristic class is an R -cohomology class $c(E) \in H^k(B; R)$ that respects pullbacks, i.e. for some continuous map $f : X \rightarrow B$ of the base, $c(f^*E) = f^*(c(E))$. In certain cases it also

makes sense to speak about characteristic classes of fiber bundles when the bundles are projectivizations of vector bundles or when the fibers are spheres, as we will encounter in later sections.

Pontryagin classes are a subset of characteristic classes of a real vector bundle $E \rightarrow B$, defined most conveniently in terms of the Chern classes of its complexification $E \otimes \mathbb{C}$. There are several ways of defining Chern classes. Here we take the axiomatic approach:

Definition 2.3. *Given a rank n complex vector bundle $E \rightarrow B$, its i -th **Chern class** is the unique integral cohomology class $c_i \in H^{2i}(B; \mathbb{Z})$ satisfying the following axioms:*

- i) Naturality: $c(f^*(E)) = f^*(c(E))$ for a pullback $f^*(E)$.*
- ii) If $F \rightarrow B$ is another complex vector bundle, the **total Chern class** defined to be the sum of Chern classes $c = c_0 + c_1 + c_2 + \cdots + c_n$, satisfies*

$$c(E \oplus F) = c(E) \cdot c(F).$$

- iii) $c_0(E) = 1$, and $c_i(E) = 0$ for $i > n$.*

- iv) c_1 of the hyperplane bundle over $\mathbb{C}P^\infty$ is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.*

Example 2.4. *The total Chern class of $\mathbb{C}P^n$ is computed to be $(1+x)^n$, where $x \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is the negative generator.*

Definition 2.5. *Given a rank n real vector bundle $E \rightarrow B$, its i -th **Pontryagin class** is the integral cohomology class $p_i \in H^{4i}(B; \mathbb{Z})$ defined via the Chern classes of $E \otimes \mathbb{C}$,*

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}).$$

*The **total Pontryagin class** is defined to be the sum $p(E) = \sum_i p_i(E)$.*

The characteristic class of a manifold is generally defined to be that of its tangent bundle. For a complex manifold X , $c_i(X)$ is defined to be the Chern classes of its tangent bundle $TX \rightarrow X$. Similarly, for a smooth manifold M , $p_i(M)$ is defined to be the Pontryagin classes of its tangent bundle $TM \rightarrow M$.

The characteristic number of a manifold is the evaluation of its corresponding characteristic class on the fundamental class.

Definition 2.6. *Let M be a smooth, compact, oriented manifold of dimension $4m$. Let $I = (i_1, \dots, i_r)$ be a partition of m . Then the I -th **Pontryagin number** $p_I[M]$ is defined to be the integer*

$$p_I[M] = \langle p_{i_1}(M) \cdots p_{i_r}(M), [M] \rangle.$$

Note that only classes of suitable degree have possible nonzero characteristic numbers.

In this work, a Pontryagin number is meant to be any linear combination of **monomial Pontryagin numbers**, where those refer to Pontryagin numbers associated to a partition I as per Definition 2.6.

Example 2.7. *We compute the Pontryagin numbers of $\mathbb{H}P^2$, which will be used in Proposition 3.14. The total Pontryagin class of $\mathbb{H}P^k$ is obtained (Section 1.3, [HBJ92]) through the fibration $\mathbb{C}P^{2k+1} \rightarrow \mathbb{H}P^k$ with fiber $\mathbb{C}P^1$ to be*

$$p(\mathbb{H}P^k) = (1 + u)^{2k+2}(1 - 4u)^{-1},$$

where $u \in H^4(\mathbb{H}P^k; \mathbb{Z})$ is a generator and $(1 - 4u)^{-1}$ is expanded into a Taylor series and truncated after degree k .

The total Pontryagin class of $\mathbb{H}P^2$ is given by,

$$\begin{aligned} p(\mathbb{H}P^2) &= (1 + u)^6(1 - 4u)^{-1} \\ &= (1 + u)^6(1 - 4u + 16u^2) \\ &= 1 + 2u - 7u^2 \\ &= 1 + p_1(\mathbb{H}P^2) + p_2(\mathbb{H}P^2). \end{aligned}$$

Consequently, the monomial Pontryagin numbers of $\mathbb{H}P^2$ are

$$p_1^2[\mathbb{H}P^2] = 4, \quad p_2[\mathbb{H}P^2] = 7.$$

A powerful tool for computing Chern and Pontryagin classes is through Hirzebruch's splitting principle:

Theorem 2.8 (Hirzebruch, [BT82]). *For every complex vector bundle $E \rightarrow B$, there exists a splitting manifold $f : N \rightarrow M$ with the following properties:*

- i) $f^*E \cong L_1 \oplus \cdots \oplus L_k$, where the L_i are line bundles.*
- ii) f^* is injective on $H^*(M; \mathbb{Z})$.*

By the naturality of characteristic classes, the splitting principle implies that the total Chern class of any complex vector bundle splits into a product

$$c(E) = c(L_1) \cdots c(L_k) = (1 + c_1(L_1)) \cdots (1 + c_1(L_k)),$$

and is completely determined by the first Chern classes of the line bundles L_j . The $c_1(L_j)$ are called the **Chern roots** of E , and the i -th Chern class of E is the i -th elementary symmetric

polynomial in the Chern roots.

The same holds for the total Pontryagin class

$$p(E) = p(L_1) \cdots p(L_k).$$

The first Pontryagin class can be worked out in terms of Chern classes $p_1 = c_1^2 - 2c_2$. In the case of line bundles $c_2(L_j) = 0$, and we have $p_1(L_j) = c_1^2(L_j)$. Therefore, the total Pontryagin class

$$p(E) = (1 + p_1(L_1)) \cdots (1 + p_1(L_k))$$

is completely determined by the squares of the Chern roots $c_i^2(L_j)$, which we call the **Pontryagin roots**. The Pontryagin classes are elementary symmetric polynomials of the Pontryagin roots.

The following statement concerning the classification of oriented cobordism classes yields a valuable corollary on the classification of rational oriented cobordism classes:

Theorem 2.9 (C.T.C Wall, [Wal60]). *Any two oriented manifolds are bordant in Ω_*^{SO} if and only if their Pontryagin numbers and Stiefel-Whitney numbers agree.*

The Stiefel-Whitney classes are characteristic classes taking values in \mathbb{Z}_2 and are defined for real vector bundles with axioms analogous to Chern classes. Therefore, they vanish in $\Omega_*^{SO} \otimes \mathbb{Q}$.

Corollary 2.10 (C.T.C Wall, [Wal60]). *Any two oriented manifolds are bordant in $\Omega_*^{SO} \otimes \mathbb{Q}$ if and only if their Pontryagin numbers agree.*

In short, Pontryagin numbers $\Omega_*^{SO} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$, regarded as elements in the dual vector space of $\Omega_*^{SO} \otimes \mathbb{Q}$ in each dimension, classify the rational oriented cobordism classes. Using this classification, it can be shown (Chapter 18, [MS74]) that the rational cobordism ring is a polynomial algebra over \mathbb{Q} with independent generators $\alpha_m \in \Omega_{4m}^{SO} \otimes \mathbb{Q}$, one for each dimension $4m$. $4m$ -Dimensional monomials of generators form a basis of the vector space $\Omega_{4m}^{SO} \otimes \mathbb{Q}$. Such a sequence of manifolds $(\alpha_m)_{m \geq 1}$ is called a **basis sequence**.

Theorem 2.11. *The rational oriented cobordism ring is isomorphic to a \mathbb{Q} -polynomial ring generated by the cobordism classes $(\alpha_m)_{m \geq 1}$, $\alpha_m \in \Omega_{4m}^{SO} \otimes \mathbb{Q}$,*

$$\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\alpha_1, \alpha_2, \dots].$$

The following theorem of Thom gives a criterion for a sequence of manifolds to be a basis sequence by examining the nonvanishing of the Thom number, defined by the evaluation

$$s_m(\alpha_m) = \left\langle \sum_{i=1}^p y_i^{2m}, [\alpha_m] \right\rangle, \quad (1)$$

where y_1^2, \dots, y_p^2 are the Pontryagin roots of α_m .

Theorem 2.12 (Thom, [HBJ92]). *A sequence $(\alpha_m)_{m \geq 1}$ of manifolds is a basis sequence if and only if the Thom number $s_m(\alpha_m) \neq 0$ for all m .*

It is clear for dimension reasons that the Thom number vanishes on product manifolds $M \cdot N$ of dimension $4m$, since each Pontryagin root is raised to the m -th power but subsequently evaluated on fundamental classes $[M]$ and $[N]$ of dimensions strictly smaller than $4m$. Conversely, due to the ring structure of $\Omega_*^{SO} \otimes \mathbb{Q}$, if the Thom number vanishes on some class $M \in \Omega_{4m}^{SO} \otimes \mathbb{Q}$ in the cobordism ring, M itself cannot be a generator, but instead can be written as a homogeneous polynomial in generators in dimensions $< 4m$.

Example 2.13. *The even complex projective spaces $(\mathbb{C}P^{2k})_{k \geq 1}$ form a basis sequence of $\Omega_*^{SO} \otimes \mathbb{Q}$. Using the Pontryagin roots obtained from squaring the Chern roots of $c(\mathbb{C}P^{2k}) = (1+x)^{2k}$, the Thom number is computed to be $s_k(\mathbb{C}P^{2k}) = 2k + 1$, which is nonzero.*

The spin cobordism ring is defined in a way similar to the oriented cobordism ring but with spin manifolds and with respect to the spin bordism relation.

Definition 2.14. *A **spin structure** on an oriented manifold M is a principal $\text{Spin}(n)$ -bundle $P \rightarrow M$ that equivariantly double covers the oriented frame bundle $\text{Fr}(M)$ on each fiber in a nontrivial way. Two spin structures $P, P' \rightarrow M$ are isomorphic if there is an isomorphism $P \rightarrow P'$ of principal $\text{Spin}(n)$ -bundles that covers the identity over $\text{Fr}(M)$, that is, the following diagram commutes:*

$$\begin{array}{ccc}
 P & \xrightarrow{\cong} & P' \\
 & \searrow & \swarrow \\
 & \text{Fr}(M) & \\
 & \downarrow & \\
 & M &
 \end{array}$$

*Oriented manifolds admitting spin structures are called **spin manifolds**.*

Now an orientation-preserving diffeomorphism $f : M \rightarrow M$ induces an isomorphism of $\text{Fr}(M)$ which in turn induces a permutation of spin structures on M . If the spin structure remains fixed under the induced map, f is called a **spin structure-preserving diffeomorphism**. Furthermore, restricting the double covering of $P \rightarrow \text{Fr}(M)$ to a double covering of its boundary $\text{Fr}(\partial M)$ induces a spin structure on ∂M .

Definition 2.15. *Two n -dimensional spin manifolds M, N are called **spin bordant** if they bound an $(n + 1)$ -dimensional spin manifold W , such that ∂W is diffeomorphic to the disjoint union $M \sqcup N$ with respect to a spin structure-preserving map, where the orientation and spin structure of ∂W are induced from W .*

The n -th **spin cobordism group** Ω_n^{spin} is the quotient of all n -dimensional spin manifolds by the spin bordism relation, with addition given by disjoint unions and identity element the vacuous manifold.

Definition 2.16. *The direct sum of all the spin cobordism groups forms the **spin cobordism ring***

$$\Omega_*^{spin} = \bigoplus_{n=0} \Omega_n^{spin}$$

with unital element $1 \in \Omega_0^{spin}$ and Cartesian products inducing graded multiplication. The ring $\Omega_*^{spin} \otimes \mathbb{Q}$ is called the **rational spin cobordism ring**.

Now spin manifolds are by definition oriented, and there is a canonical map $\Omega_*^{spin} \rightarrow \Omega_*^{SO}$ whose kernel is \mathbb{Z}_2 in dimensions $8k+1$ and $8k+2$ (p.47, [Sto68]). Therefore, there is an isomorphism of rational cobordism rings

$$\Omega_*^{spin} \otimes \mathbb{Q} \cong \Omega_*^{SO} \otimes \mathbb{Q}.$$

Furthermore, the rational spin cobordism classes are also classified by Pontryagin numbers (Corollary 2.3, [ABP67]). Therefore, we may still apply Thom's criteria (Theorem 2.12) to obtain basis sequences. This way we reduce the problem of boundedness of Pontryagin numbers on spin manifolds to a problem on spin cobordism classes.

2.2 Ochanine's elliptic genus

We introduce another main ingredient, the elliptic genus of Ochanine. A genus is a ring homomorphism $\phi : \Omega_* \otimes \mathbb{Q} \rightarrow R$ from some rational cobordism ring of choice to an integral domain R over \mathbb{Q} , normalized so that $\phi(1) = 1$.

By Theorem 2.11 and results in the previous section any genus ϕ of $\Omega_*^{SO} \otimes \mathbb{Q}$ is uniquely determined by its values on a basis sequence. The information containing these values is contained in terms of an associated **logarithmic power series** $g(y)$, defined by

$$g(y) = \sum_{n=0}^{\infty} \phi(\mathbb{C}P^{2n}) \cdot \frac{y^{2n+1}}{2n+1},$$

whose derivative satisfies

$$g'(y) = \sum_{n=0}^{\infty} \phi(\mathbb{C}P^{2n}) \cdot y^{2n+1}.$$

On the other hand, the classification of oriented cobordism classes by Pontryagin numbers provides for any such genus $\phi : \Omega_*^{SO} \otimes \mathbb{Q} \rightarrow R$ a sequence of polynomials $\{K_n(p_1, \dots, p_n)\}_{n \geq 0}$ in

the Pontryagin classes p_i , normalized with $K_0 = 1$ and with coefficients in R , so that for each $M \in \Omega_*^{SO} \otimes \mathbb{Q}$ of dimension $4n$, the value of the genus on M is given by

$$\phi(M) := \langle K_n(p_1(M), \dots, p_n(M)), [M] \rangle.$$

These polynomials K_n are called the **multiplicative sequence** of ϕ , as their formal sum $K = \sum_n K_n$ satisfy the multiplicative property:

$$K(M \times N) = K(M) \cdot K(N).$$

Since the Pontryagin classes are elementary symmetric polynomials in the Pontryagin roots x_i^2 , each K_n can be written as a symmetric product of single variable polynomials dependent on x_i^2 . If we consider the formal sum K , these polynomials become formal power series. That is, there exists an even power series

$$Q(x) = 1 + a_2x^2 + a_4x^4 + \dots$$

normalized with constant coefficient 1, and with coefficients in R such that for any $n \in \mathbb{N}$

$$Q(x_1) \cdots Q(x_n) = 1 + K_1(p_1) + \cdots + K_n(p_1, \dots, p_n) + K_{n+1}(p_1, \dots, p_n, 0) + \cdots.$$

Such a power series $Q(x)$, called the **characteristic series**, is unique and completely determines the genus ϕ (Section 1.6, [HBJ92]).

Thus, we have two complete characterizations of the genus ϕ , one via the associated logarithmic series $g(y)$ and the other through the characteristic series $Q(x)$. These two series are related in the following way (Section 1.6, [HBJ92]): Let $f(x) = x/Q(x)$ be the odd power series beginning with x . Then $g(y)$ is the inverse of $f(x)$.

Example 2.17. (i) *The signature L-genus has characteristic series*

$$Q(x) = x / \tanh x,$$

yielding $f(x) = \tanh x$ and $g(y) = \tanh^{-1}(x)$, we therefore obtain

$$g'(y) = \frac{1}{1-y^2} = \sum_{n=0}^{\infty} y^{2n}.$$

Consequently, we have that $\sigma(\mathbb{C}P^{2n}) = 1$, as expected from computations via the intersection form.

(ii) *The \hat{A} -genus has characteristic series*

$$Q(x) = \frac{x/2}{\sinh(x/2)}.$$

Hence $f(x) = 2 \sinh(x/2)$ and $g(y) = 2 \sinh^{-1}(y/2)$, we therefore obtain the series

$$g'(y) = \frac{2}{\sqrt{4+y^2}} = 1 - \frac{1}{8}y^2 + \frac{3}{128}y^4 + \dots$$

With this machinery in mind, we give a definition of the elliptic genus, which was first studied in depth by S. Ochanine, and often bears his name:

Definition 2.18. Let δ and ϵ be two formal variables of degrees 4 and 8. The **elliptic genus** is a ring homomorphism $\varphi : \Omega_*^{spin} \otimes \mathbb{Q} \rightarrow \mathbb{Q}[\delta, \epsilon]$, whose logarithmic power series $g_\varphi(y)$ is defined by the elliptic integral

$$g_\varphi(y) = \int_0^y \frac{dt}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}}.$$

Note that the elliptic genus turns out to be the familiar L -genus and \hat{A} -genus when the formal variables δ and ϵ take on specific values:

When $\epsilon = \delta = 1$, φ is the signature L -genus

$$g'_\varphi(y) = \frac{1}{1-y^2}.$$

When $\epsilon = 0$ and $\delta = -1/8$, φ is the \hat{A} -genus

$$g'_\varphi(y) = \frac{2}{\sqrt{4+y^2}}.$$

More generally, we can treat the formal variables δ, ϵ as modular forms of weight 2 and 4 with respect to the modular group $\Gamma_0(2)$, which has two cusps 0 and ∞ . Then the elliptic genus φ maps into the modular forms $M_*(\Gamma_0(2))$. At the two cusps, φ degenerates to a series of twisted \hat{A} -genus and the equivariant signature of the loop space. More on this can be found in Chapter 6 of [HBJ92].

The results of Ochanine show that the elliptic genus φ surjects onto the \mathbb{Q} -algebra $\mathbb{Q}[\delta, \epsilon]$. Let \mathcal{I} denote the kernel of φ . Then \mathcal{I} must be generated by classes in dimensions ≥ 12 . To be more precise, the following theorem shows that \mathcal{I} is the ideal in $\Omega_*^{spin} \otimes \mathbb{Q}$ generated by projectivizations $\mathbb{P}(E)$ of even-ranked complex vector bundles, or alternatively, $\mathbb{C}P^{\text{odd}}$ -vector bundles. And conversely, any genus whose kernel contains all $\mathbb{C}P^{\text{odd}}$ -vector bundles must be an elliptic genus.

Theorem 2.19 (Ochanine, [Och87]). *A genus ϕ annihilates the projectivizations $\mathbb{P}(E)$ of even-ranked complex vector bundles if and only if its associated logarithmic series is given by an elliptic integral.*

This theorem turns out to be quite important for us, because to prove Theorem 1.5 we must show that any Pontryagin number p_I not factoring through Ochanine's elliptic genus φ is unbounded on nonnegatively curved spin manifolds. That is, we wish to find a sequence of nonnegatively curved spin manifolds on which the chosen Pontryagin number p_I becomes arbitrarily large. Since p_I does not factor through φ , there must exist an element in \mathcal{I} on which p_I is nonzero. Understanding of the kernel \mathcal{I} then, is crucial.

Since φ surjects onto the generators in dimensions 4 and 8, there exists a basis sequence $(\alpha_m)_{m \geq 1}$ of $\Omega_*^{spin} \otimes \mathbb{Q}$ such that \mathcal{I} as an ideal is generated by the α_m in dimensions ≥ 12 . The spin cobordism ring structure then allows us to quickly come up with elements in \mathcal{I} . It must be noted however, that curvature is not a cobordism invariant. To prove unboundedness of Pontryagin numbers on nonnegatively curved spin manifolds, it is still essential to find concrete examples of spin manifolds with nonnegative sectional curvature.

By Theorem 2.19, these $\alpha_{m \geq 3}$ must all be $\mathbb{C}P^{\text{odd}}$ -bundles. Therefore, it suffices to investigate nonnegatively curved $\mathbb{C}P^{\text{odd}}$ -bundles in dimensions ≥ 12 . In particular, we discuss these over a complex projective base. That is, we will construct $X_n^k(c) \in \Omega_{4m}^{spin} \otimes \mathbb{Q}$, a nonnegatively curved family of $\mathbb{C}P^k$ -bundles over $\mathbb{C}P^n$, from which we will build a family of basis sequences of $\Omega_*^{spin} \otimes \mathbb{Q}$. The Pontryagin numbers of $X_n^k(c)$ will be shown to be odd polynomials in c , and since the $X_n^k(c)$ belong to the kernel \mathcal{I} , any Pontryagin number not factoring through the elliptic genus is unbounded on some monomial in ring generators involving at least one factor of $X_n^k(c)$.

3 A family of spin projective bundles over projective space

3.1 Cohomology and spin criteria of projective bundles over projective space

Let $\mathbb{P}(E)$ be the projectivization of a complex rank $(k + 1)$ -bundle E over $\mathbb{C}P^n$. Its integral cohomology ring $H^*(\mathbb{P}(E))$ can be determined by the Leray-Hirsch Theorem:

Theorem 3.1 (Leray-Hirsch, [Hat02]). *Let $F \rightarrow E \rightarrow B$ be a fiber bundle such that, for some commutative ring R , $H^n(F; R) = H^n(F)$ is finite dimensional for each n and the inclusion $\iota : F \rightarrow E$ induces a surjection $H^*(E) \rightarrow H^*(F)$, then there is an isomorphism of $H^*(B)$ -modules,*

$$H^*(B) \otimes H^*(F) \cong H^*(E).$$

We are in a position to apply this result since the cohomology of the fiber $H^*(\mathbb{C}P^k)$ is finite dimensional for each degree. Moreover, the hyperplane bundle L over $\mathbb{P}(E)$ pulls back under inclusion to the fiberwise hyperplane bundle $i^*L = \mathcal{O}(1)$ over $\mathbb{C}P^k$, mapping the first Chern class $c_1(L) = y \in H^2(\mathbb{P}(E))$ via naturality to the positive generator of $H^*(\mathbb{C}P^k)$ in each fiber.

Therefore, $H^*(\mathbb{P}(E))$ is generated by two classes $x, y \in H^2(\mathbb{P}(E))$, where the former is the pullback of the positive generator of $H^*(\mathbb{C}P^n)$ and the latter is the first Chern class of the hyperplane bundle $c_1(L) \in H^2(\mathbb{P}(E))$.

By Leray-Hirsch, the pullback of $\pi : \mathbb{P}(E) \rightarrow \mathbb{C}P^n$ is injective, and we identify the cohomology of the base with its pullback image so that $x^{n+1} = 0$. The relation between the generators is given by Grothendieck's definition of Chern classes.

Definition 3.2 (Grothendieck, [Gro58]). *Let $E \rightarrow B$ be a complex vector bundle, and $\mathbb{P}(E)$ its projectivization, with $y = c_1(L) \in H^2(\mathbb{P}(E))$. Then the **Chern classes** of E are the unique classes $c_i(E) \in H^{2i}(B)$ satisfying the equations*

$$\sum_{i=0}^{k+1} c_i(E)y^{k+1-i} = 0, \quad c_0(E) = 1, \quad c_i(E) = 0 \text{ for } i > r.$$

This construction is natural and fulfills the axioms of Chern classes, which agrees with the usual notion of Chern classes defined axiomatically in the previous section.

Summarizing the above, we have the following lemma:

Lemma 3.3. *The cohomology ring $H^*(\mathbb{P}(E))$ is generated by two classes $x, y \in H^2(\mathbb{P}(E))$ subject to the relations:*

$$y^{k+1} + c_1(E)y^k + \cdots + c_{k+1}(E) = 0, \quad x^{n+1} = 0.$$

Note that the top cohomology class of $H^*(\mathbb{P}(E))$ is generated by x^ny^k , which evaluates to 1 on the fundamental class,

$$\langle x^ny^k, [\mathbb{P}(E)] \rangle = 1.$$

Next we compute the characteristic classes of $\mathbb{P}(E)$.

Lemma 3.4. *The total Chern class of $\mathbb{P}(E)$ is given by*

$$\begin{aligned} c(\mathbb{P}(E)) &= c(T\pi) \cdot c(\pi^*TCP^n) \\ &= \left[\sum_{i=0}^{k+1} (1+y)^{k+1-i} c_i(E) \right] \cdot (1+x)^{n+1} \end{aligned} \quad (2)$$

where $T\pi$ is the tangent bundle along the fibers.

Proof. We have the decomposition $T\mathbb{P}(E) = T\pi \oplus \pi^*TCP^n$. By Leray-Hirsch the pullback π^* is injective and computes the latter factor $\pi^*c(\mathbb{C}P^n) = (1+x)^{n+1}$.

It remains to calculate $c(T\pi)$. Suppose E have Chern roots $x_i, i = 1, \dots, k+1$ with corresponding line bundles L_i from the splitting principle. The relative Euler sequence gives

$$0 \rightarrow L^{-1} \rightarrow \pi^*E \rightarrow L^{-1} \otimes T\pi \rightarrow 0$$

where L is the hyperplane bundle over $\mathbb{P}(E)$ with $c(L) = 1+y$. Tensoring by L on the left yields an isomorphism of bundles $L \otimes \pi^*E \cong T\pi \oplus \mathbb{C}$.

By injectivity of π^* , this shows

$$c(T\pi) = c(L \otimes \pi^*E) = \prod_{i=1}^{k+1} c(L \otimes L_i) = \prod_{i=1}^{k+1} (1+y+x_i) = \sum_{i=0}^{k+1} (1+y)^{k+1-i} c_i(E)$$

where for the last equality we used the fact that $c_i(E)$ is the i -th elementary symmetric polynomial of the Chern roots x_i . This completes the proof. \square

From the total Chern class, we derive a criterion for when $\mathbb{P}(E)$ is spin. In order for $\mathbb{P}(E)$ to have nontrivial Pontryagin numbers, we pay special attention the case when the dimension is divisible by 4.

Lemma 3.5. *$\mathbb{P}(E)$ is spin if and only if k is odd and $c_1(E) + (n+1)x$ as an element of $H^*(\mathbb{C}P^n) \cong \mathbb{Z}$ is even. In particular, $\mathbb{P}(E)$ of real dimension $4m$ is spin if and only if both k and n are odd and $c_1(E)$ is even.*

Proof. Recall that a complex manifold is spin if and only if its first Chern class $c_1(\mathbb{P}(E))$ is even in integral cohomology. For $\mathbb{P}(E)$ formula (2) gives us

$$c_1(\mathbb{P}(E)) = c_1(E) + (n+1)x + (k+1)y.$$

This is even if and only if both $k+1$ and $c_1(E) + (n+1)x$ are even.

If $\mathbb{P}(E)$ has real dimension $2(k+n) = 4m$, k being odd forces n to be odd, which in turn implies that $c_1(E)$ must be even. \square

We have thus shown that any projective bundle over $\mathbb{C}P^n$ which is also spin must be a $\mathbb{C}P^{\text{odd}}$ -fiber bundle, hence belongs to the kernel \mathcal{I} of the elliptic genus by Theorem 2.19.

3.2 The family of projective bundles $X_n^k(c)$

Consider the complex rank $(k+1)$ vector bundle $E_c \rightarrow \mathbb{C}P^n$ of the form $E_c = H^c \oplus \underline{\mathbb{C}}^k$, where H^c is the c -fold tensor product of the hyperplane bundle, and let $X_n^k(c) = \mathbb{P}(E_c)$ denote its projectivization. The vector bundle E_c has total Chern class $c(E_c) = 1 + cx$. Then $X_n^k(c)$ is a family of complex manifolds with real dimension $2(n+k)$ parametrized by the first Chern class c of E_c and the complex dimensions k, n of its fiber and base.

Lemma 3.3 yields the structure of the cohomology ring $H^*(X_n^k(c))$.

Lemma 3.6. *The cohomology ring $H^*(X_n^k(c))$ is generated by degree two classes $x, y \in H^2(X_n^k(c))$. The two generators satisfies the cohomology relations*

$$x^{n+1} = 0, \quad y^{k+1} + cxy^k = 0.$$

As we remarked in the previous section, the top class of $X_n^k(c)$ is generated by $x^n y^k$ by Leray-Hirsch. Meanwhile, formula (2) gives us the total Chern class of $X_n^k(c)$, which in turn provides a formula for the total Pontryagin class.

Lemma 3.7. *The total Chern class of $X_n^k(c)$ is given by*

$$c(X_n^k(c)) = (1+x)^{n+1}(1+y)^k(1+y+cx). \quad (3)$$

Therefore, its total Pontryagin class is given by

$$p(X_n^k(c)) = (1+x^2)^{n+1}(1+y^2)^k(1+(y+cx)^2). \quad (4)$$

Note that we have Chern roots

$$\underbrace{x, \dots, x}_{n+1}, \underbrace{y, \dots, y}_k, (y+cx).$$

The Pontryagin roots are obtained by squaring the Chern roots,

$$\underbrace{x^2, \dots, x^2}_{n+1}, \underbrace{y^2, \dots, y^2}_k, (y + cx)^2.$$

Knowledge of these Pontryagin roots will be used to compute the Thom number and later the q -number.

Lemma 3.5 tells us under which conditions $X_n^k(c)$ of dimension $4m$ is spin.

Lemma 3.8. *$X_n^k(c)$ of dimension $4m$ is spin if and only if both k and n are odd and c is even.*

In the remaining parts of sections 3 and 4, we will always take $X_n^k(c)$ to have dimension $4m$ and be spin, thereby fixing the parity of n, k and c .

3.3 Nonnegative sectional curvature on $X_n^k(c)$

In this section we show that the $X_n^k(c)$ admits nonnegative sectional curvature by regarding it as a Riemannian submersion of a nonnegatively curved manifold.

The vector bundle $E_c = H^c \oplus \underline{\mathbb{C}}^k$ has structure group S^1 , and is therefore associated to an S^1 -bundle $P_c \rightarrow \mathbb{C}P^n$.

Proposition 3.9. *S^1 -bundles over $\mathbb{C}P^n$ are uniquely classified by their Euler class $c \cdot x \in H^2(\mathbb{C}P^n)$.*

Proof. Recall that complex line bundles L over $\mathbb{C}P^n$ are completely classified by their first Chern class $c_1(L)$. Now each S^1 -bundle P is uniquely associated to a line bundle L via the tautological representation $\rho : S^1 \rightarrow GL_1(\mathbb{C})$ given by inclusion, and vice versa each line bundle L corresponds to a unique S^1 -bundle. The Euler class of P is mapped to the first Chern class of L under the pushforward ρ_* . Therefore, S^1 -bundles over $\mathbb{C}P^n$ are completely classified by their Euler classes. \square

Proposition 3.10. *The total space $P_c \rightarrow \mathbb{C}P^n$ of the S^1 -bundle associated to E_c is the lens space $L_c = S^{2n+1}/\mathbb{Z}_{|c|}$.*

Proof. By the classification in Proposition 3.9, it suffices to show that the Euler class of the lens space L_c is $c \cdot x$. Now the Euler class of S^1 -bundles over $\mathbb{C}P^n$ is defined to be the image of $1 \in H^0(\mathbb{C}P^n)$ in the Gysin-sequence

$$0 \rightarrow H^0(\mathbb{C}P^n) \rightarrow H^2(\mathbb{C}P^n) \rightarrow H^2(L_c) \rightarrow H^1(\mathbb{C}P^n) \rightarrow \dots$$

Since $\mathbb{C}P^n$ has no cohomology in degree one, the first terms of the Gysin sequence forms a short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_{|c|} \rightarrow 0,$$

where we have used that $H^0(\mathbb{C}P^n) = H^2(\mathbb{C}P^2) = \mathbb{Z}$ and the second cohomology of the lens space is well-known to be $H^2(L_c) = \mathbb{Z}_{|c|}$.

Therefore the image of $1 \in H^0(\mathbb{C}P^n)$ is $\pm c \cdot x \in H^2(\mathbb{C}P^n)$ a multiple of the generator. Hence L_c has Euler class $c \cdot x$, while L_{-c} with the opposite orientation has Euler class $-c \cdot x$. \square

Since E_c is associated to P_c , its projectivization is

$$X_n^k(c) = \mathbb{P}(E) = (L_c \times \mathbb{C}P^n)/S^1.$$

The lens space L_c has universal cover S^{2n+1} , on which S^1 acts by isometries. Therefore, L_c carries an S^1 -invariant metric g of constant positive curvature descended from the one on S^{2n+1} . Meanwhile, $\mathbb{C}P^n$ is equipped with the positively curved Fubini-Study metric, which has isometry group $U(k+1)$. Regarding $S^1 = U(1)$ included as a subgroup in $U(k+1)$, it acts isometrically, freely and properly on the nonnegatively curved product $P_c \times \mathbb{C}P^n$.

Therefore, the quotient $P_c \times \mathbb{C}P^n \rightarrow X_n^k(c)$ is a Riemannian submersion, and the non-decreasing property of curvature under submersions implies that $X_n^k(c)$ is nonnegatively curved. The above argument shows that:

Proposition 3.11. *The projective bundle $X_n^k(c)$ admits a metric of nonnegative sectional curvature.*

3.4 The Thom number of $X_n^k(c)$

We show that each of the $X_n^k(c)$ can be taken as a generator of the spin cobordism ring in dimension $4m \geq 12$, so that with a suitable choice of generators in dimensions 4 and 8, we can construct a basis sequence for $\Omega_*^{spin} \otimes \mathbb{Q}$. Recall by Theorem 2.12 this amounts to checking that the Thom number of $X_n^k(c)$ is nonzero.

The following proposition computes the Thom number of $X_n^k(c)$.

Lemma 3.12. *Let $X_n^k(c)$ be spin of dimension $4m$. The Thom number of $X_n^k(c)$ is given by*

$$s_m(X_n^k(c)) = -c^n \left[k - \binom{k+n-1}{n} \right]. \quad (5)$$

In particular, $s_m(X_n^k(c)) \neq 0$ whenever $n, k \geq 3$ and $c \neq 0$.

Proof. Using the Pontryagin roots observed in formula (4), the definition of the Thom number gives

$$s_m(X_n^k(c)) = \langle ky^{n+k} + (n+1)x^{n+k} + (y+cx)^{n+k}, [X_n^k(c)] \rangle$$

Recall the relation between x and y given in Lemma 3.6. The second term vanishes since $x^{n+1} = 0$. For the third term we expand $(y+cx)^{n+k}$ using the binomial formula, and drop all terms of x with exponent greater than n . Finally, by applying the relation $y^k = -cxy^{k-1}$ repeatedly, we obtain

$$\begin{aligned} ky^{n+k} + (n+1)x^{n+k} + (y+cx)^{n+k} &= (-1)^n k c^n x^n y^k + \sum_{i=0}^n (-1)^{n-i} \binom{n+k}{i} c^n x^n y^k \\ &= (-1)^n c^n \left[k + \sum_{i=0}^n (-1)^i \binom{n+k}{i} \right] x^n y^k. \end{aligned}$$

Using explicitly the fact that n is odd, $(-1)^n = -1$. The top cohomology class of $X_n^k(c)$ evaluates to 1 on the fundamental class $\langle x^n y^k, [X_n^k(c)] \rangle = 1$, therefore

$$s_m(X_n^k(c)) = -c^n \left[k + \sum_{i=0}^n (-1)^i \binom{n+k}{i} \right].$$

Replace each binomial coefficient with $i > 0$ in the sum using the recursion

$$\binom{m}{i} = \binom{m-1}{i-1} + \binom{m-1}{i}. \quad (6)$$

Since every binomial coefficient appears twice with opposite signs, all the other summands cancel in pairs, except for the last term $\binom{n+k-1}{n}$. This gives us the formula as claimed

$$s_m(X_n^k(c)) = -c^n \left[k - \binom{n+k-1}{n} \right].$$

This number is nonzero since we have the strict inequality

$$k < \binom{n+k-1}{n} = \frac{(n+k-1) \cdots (k+1) \cdot k}{n!}.$$

Dividing both sides through by k , this follows from the fact that

$$n! < (k+n-1) \cdots (k+1)$$

where if we maintain the ordering, each factor on the left is strictly smaller than the factor on the right. \square

The cases when either $k = 1$ or $n = 1$ we must exclude since formula (5) gives zero. When $k = 1$, the fiber $\mathbb{C}P^1 \cong S^2$ is a 2-sphere. Therefore, X_n^1 is the boundary of a 3-ball bundle. The case $n = 1$ manifests the phenomenon that Pontryagin numbers of $X_n^k(c)$ are polynomials in c with vanishing linear term, see Proposition 3.17.

Applying Theorem 2.12, the nonvanishing of $s_m(X_n^k(c))$ implies the following:

Proposition 3.13. *Any $X_n^k(c)$ spin of dimension $4m \geq 12$ with both n and $k \geq 3$ and $c \neq 0$ can be taken as a generator of the spin cobordism ring.*

The next proposition constructs from the $X_n^k(c)$ a family of basis sequences of the rational spin cobordism ring. Note that there are in fact multiple choices of $X_n^k(c)$ in each dimension $4m$, yielding different families of basis sequences parametrized by c .

Proposition 3.14. *Let $\alpha_1 = K3$, $\alpha_2 = \mathbb{H}P^2$. For $m \geq 3$, fix a $X_n^k(c)$ spin of dimension $4m$ as in the previous proposition, and let $\alpha_m(c) = X_n^k(c)$. Then for each nonzero even c , $(\alpha_m(c))_{m \geq 1}$ form a basis sequence of $\Omega_*^{spin} \otimes \mathbb{Q}$.*

Proof. Owing to Proposition 3.13, it suffices to show that $K3$ and $\mathbb{H}P^2$ are generators of the spin cobordism ring. The $K3$ surface is a compact simply connected complex surface with vanishing first Chern class, so it is spin by definition. Moreover, the Thom number must be proportional to the only Pontryagin number in dimension 4, which is the signature σ , and $\sigma(K3) = -16$ (Proposition 3.5, [Huy16]), so the Thom number of $K3$ is nonzero.

On the other hand, $\mathbb{H}P^2$ does not have degree 2 cohomology, hence is spin. Its Thom number can be computed to be nonzero by the monomial Pontryagin numbers $p_1^2[\mathbb{H}P^2] = 4$, $p_2[\mathbb{H}P^2] = 7$, discussed in Example 2.7,

$$s_2(\mathbb{H}P^2) = (p_1^2 - p_2)[\mathbb{H}P^2] = -10.$$

This shows that $K3$ and $\mathbb{H}P^2$ can be taken as generators of the spin cobordism ring in dimensions 4 and 8, hence for each suitable fixed c , $(\alpha_m(c))_{m \geq 1}$ is a basis sequence of $\Omega_*^{spin} \otimes \mathbb{Q}$. \square

Notice that $\mathbb{H}P^2$ admits a metric nonnegative sectional curvature descended from the submersion $\mathbb{C}P^{2k+1} \rightarrow \mathbb{H}P^2$, so that apart from $\alpha_1 = K3$, each generator $\alpha_m(c)$ admits nonnegative sectional curvature. This agrees with the fact that in dimension 4, it is impossible to find a nonnegatively curved spin manifold of nonzero signature. Therefore, our choice of generators is optimal.

Moreover, the ring generators $(\alpha_m(c))_{m \geq 3}$, being $\mathbb{C}P^{\text{odd}}$ -fiber bundles, generate the kernel \mathcal{I} of the elliptic genus φ as an ideal in $\Omega_*^{spin} \otimes \mathbb{Q}$. This is also the best we can hope for, because the two ring generators $K3$ and $\mathbb{H}P^2$ of dimensions 4 and 8 must generate the \mathbb{Q} -algebra $\mathbb{Q}[\delta, \epsilon]$ on which φ surjects, therefore cannot belong to \mathcal{I} . From a different perspective, any Pontryagin number of spin manifolds in dimensions 4 and 8 must factor through the elliptic genus.

3.5 Pontryagin numbers of $X_n^k(c)$

In this section, we characterize the Pontryagin numbers of $X_n^k(c)$ as odd polynomials in c .

Proposition 3.15. *As a function of c , every Pontryagin number of $X_n^k(c)$ is an odd polynomial of degree at most n .*

In fact, by Proposition 3.12 the maximal degree n does occur for the Thom number.

Proof. Every Pontryagin number is given by the evaluations of a homogeneous polynomial of degree $m = \frac{1}{2}(n + k)$ in the Pontryagin roots $x^2, y^2, (y + cx)^2$. These are evaluations of linear combinations of monomials of the form

$$x^{2a} \cdot y^{2b} \cdot (y + cx)^{2(m-a-b)},$$

with nonnegative exponents. The third factor we expand via the binomial theorem

$$(y + cx)^{2(m-a-b)} = \sum_{i=0}^{2(m-a-b)} y^i \cdot (cx)^{2(m-a-b)-i}$$

to a linear combination of monomials

$$x^{2m-2b-i} \cdot y^{2b+i} \cdot c^{2(m-a-b)-i},$$

for $0 \leq i \leq 2(m - a - b)$. Setting $j = 2m - 2b - i$, we can write the monomials as

$$x^j \cdot y^{2m-j} \cdot c^{j-2b}$$

for $0 \leq j \leq n$. Now employing the cohomology relation $y^k = -cxy^{k-1}$ repeatedly, we trade for each power of y larger than k for a factor of $-cx$,

$$x^j \cdot y^{2m-j} \cdot c^{j-2b} = (-1)^{n-j} x^n \cdot y^k \cdot c^{n-2b}.$$

Since $x^n y^k$ is the top class, each monomial evaluates on the fundamental class to $(-1)^{n-j} c^{n-2b}$. The spin criterion implies n is odd, so $n - 2b$ is odd and bounded above by n . This shows that any Pontryagin number of $X_n^k(c)$ is an odd polynomial in c of degree $\leq n$. \square

Proposition 3.16. *Any Pontryagin number p_I that does not factor through the elliptic genus φ is unbounded as a polynomial in c on some cobordism class $Z(c)$ in \mathcal{I} .*

Proof. If p_I does not factor through the elliptic genus, there exists an $Z(c) = X_n^k(c) \cdot M \in \mathcal{I}$ on which p_I does not vanish where M is a spin cobordism class that has no c dependence. Then the Pontryagin number $p_I[X_n^k(c) \cdot M]$ becomes a linear combination of Pontryagin numbers $p_J[M] \cdot p_K[X_n^k(c)]$ of $X_n^k(c)$, where $JK = I$. By Proposition 3.15 the Pontryagin numbers of $X_n^k(c)$ are odd polynomials in c , therefore $p_I[Z(c)]$ is unbounded as one varies c arbitrarily. \square

For each nonzero even c , consider a basis sequence $(\alpha_m(c))_{m \geq 1}$ in Proposition 3.14. If $Z(c)$ is a product of generators $\alpha_m(c)$ not involving $K3$, it admits nonnegative sectional curvature.

We thus have a sequence of nonnegatively curved spin manifolds on which p_I is unbounded, as required for Theorem 1.5. It remains to discuss the case when $Z(c)$ involves factors of $K3$, when we could not ascertain the curvature of $Z(c)$, which we deal with in section 4.

The rest of this section is devoted to proving the following proposition, which illuminates further properties of the Pontryagin numbers of $X_n^k(c)$, but is not directly used in the proof of Theorem 1.5.

Proposition 3.17. *As polynomials in c , the Pontryagin numbers of $X_n^k(c)$ have vanishing linear coefficients.*

Recall we have the total Pontryagin class

$$p(X_n^k(c)) = (1 + y^2)^k (1 + x^2)^{n+1} (1 + (y + cx)^2).$$

Based on whether a factor of $(y + cx)^2$ is involved, the i -th Pontryagin class are homogeneous polynomials of degree $2i$ in the Pontryagin roots, and can be written as

$$p_i(X_n^k(c)) = q_i(x, y) + (y + cx)^2 q_{i-1}(x, y) \tag{7}$$

where

$$q_i(x, y) = \sum_{a+b=i} \binom{n+1}{a} \binom{k}{b} x^{2a} y^{2b}$$

corresponds to choosing a powers of x^2 and b powers of y^2 . Clearly, $q_0 = 1$.

Now let $I = (i_1, \dots, i_r)$ be a partition of m and consider the product $p_I = p_{i_1} \cdots p_{i_r}$. If we simply multiply the formula from (7), we would obtain $q_I = q_{i_1} \cdots q_{i_r}$ and terms involving powers of $(y + cx)^2$ coupled with $q_{i_1} \cdots q_{i_{j-1}} \cdots q_{i_r}$ where some subscripts of q_i are reduced by 1. In fact, as can be seen from (7), if there are s powers of $(y + cx)^2$, then precisely s nonzero subscripts are reduced by 1.

Roughly speaking, we would like to write these terms as

$$(y + cx)^{2s} q_{I-s},$$

where $I - s$ is some partition where s nonzero entries are reduced by 1, and then sum over these partitions. However, $I - s$ may fail to be a partition, since it could have entries that are zero. This motivates the following definition:

Definition 3.18. *A **pseudo-partition** of integer m of fixed length r is a partition of m that allows 0 in its entries.*

Furthermore, all following pseudo-partitions or partitions are ordered, so that where the length agrees, it makes sense add and subtract (where suitable) entry-wise. The set of **ordered pseudo-partitions** of m of length r is a monoid under addition and is denoted $\mathcal{S}(m, r)$.

Definition 3.19. For an ordered partition I and $s \leq |I|$, denote $I - s$ the set of ordered pseudo-partitions where exactly s nonzero entries are reduced by 1.

With this set up, the Pontryagin numbers can be written as

$$p_I(X_n^k(c)) = \sum_{s=0}^{|I|} \left(\sum_{J \in I-s} (y+cx)^{2s} q_J(x, y) \right). \quad (8)$$

Proof of Proposition 3.17. Fix a partition $I = (i_1, \dots, i_r)$ of m . When evaluating $p_I(X_n^k(c))$ upon the fundamental class, the only terms contributing to the linear coefficients are

$$cx^n y^k, \quad x^{n-1} y^{k+1},$$

which evaluate to c and $-c$ respectively.

How often these two terms show up can be calculated from combinatorics, which depends on how many factors of the Pontryagin root $(y+cx)^2$ are involved. We organize the contributions from $(y+cx)^{2s}$ in the chart below:

	$cx^n y^k$	$x^{n-1} y^{k+1}$
$s = 0$	0	1
$s \geq 1$	$s \cdot 2cxy \cdot y^{2s-2}$	y^{2s}

The way to read this chart is as follows:

(i) For $s = 0$, there are no factors of $(y+cx)^2$. Any such term does not contain factors of c , therefore there can be no terms of $cx^n y^k$. The term $x^{n-1} y^{k+1}$ does occur with the frequency documented in an overall combinatorics C_s below.

(ii) For $s \geq 1$, there a factor of $(y+cx)^{2s}$. Now the term $cx^n y^k$ does occur, with the factor of c coming from $(y+cx)^{2s} = (y^2 + 2cxy + c^2x^2)^s$. There are s ways to pick $2cxy$ from one of the factors, and for the rest we must choose y^2 since we do not want any more powers of c . This explains the factor

$$s \cdot 2cxy \cdot y^{2s-2}$$

in the chart. The frequency of $cx^n y^k$ occurring then depends on how often the combination $x^{n-1} y^{k+1-2s}$ shows up in $q_J(x, y)$, so that overall we have

$$s \cdot 2cxy \cdot y^{2s-2} \times x^{n-1} y^{k+1-2s} = 2s \cdot cx^n y^k.$$

This frequency is documented in an overall combinatorics C_s below and is summed over the corresponding frequencies for each ordered pseudo-partition $J \in I - s$ and each s .

The term $x^{n-1} y^{k+1}$ also occurs, with the factor of y^{2s} coming from $(y + cx)^2$, because we could not choose factors involving c . The frequency of $x^{n-1} y^{k+1}$ occurring also depends on the ways to choose $x^{n-1} y^{k+1-2s}$ from $q_J(x, y)$, so that overall we have

$$y^{2s} \times x^{n-1} y^{k+1-2s} = x^{n-1} y^{k+1}.$$

This frequency is documented in an overall combinatorics C_s below and is summed over the corresponding frequencies for each ordered pseudo-partition $J \in I - s$ and each s .

Treating $x^{n-1} y^{k+1} = -cx^n y^k$, we combine the positive and negative contributions to the linear term from the chart above and take the sum with respect to s , the number of factors of $(y + cx)^2$. The linear coefficient is then given by,

$$\sum_{s=0}^r (2s - 1) cxy^{2s-1} \times C_s \cdot x^{n-1} y^{k+1-2s} = \sum_{s=0}^r (2s - 1) C_s \cdot cx^n y^k. \quad (9)$$

Recall that r is the length of the partition I , which is the maximum value s can take according to formula (8). Here C_s for each fixed s is the collection of combinatorial factors that arise from choosing $(n-1)/2$ powers of x^2 and $(k+1-2s)/2$ powers of y^2 in $q_J(x, y)$, summed over $J \in I - s$.

Next we compute the overall combinatorics C_s . For a fixed s , we turn to the form of $q_J(x, y)$ for $J = (j_1, \dots, j_r)$, which is the product

$$q_J(x, y) = \prod_{j_i} q_{j_i}(x, y) = \prod_{j_i} \sum_{a_i + b_i = j_i} \binom{n+1}{a_i} \binom{k}{b_i} x^{2a_i} y^{2b_i}.$$

We want the combinatorial factor for the term $x^{n-1} y^{k+1-2s}$, which occurs whenever

$$\sum_i a_i = (n-1)/2, \quad \sum_i b_i = (k+1-2s)/2.$$

To simplify this calculation, we write $(A, B)_s$ for pairs of ordered pseudo-partitions $A+B \in I-s$ such that

$$A = (a_1, \dots, a_r) \in \mathcal{S}\left(\frac{n-1}{2}, r\right),$$

$$B = (b_1, \dots, b_r) \in \mathcal{S}\left(\frac{k+1-2s}{2}, r\right).$$

Each pair $(A, B)_s$ gives rise to the combinatorial term

$$\binom{n+1}{a_1} \binom{k}{b_1} x^{2a_1} y^{2b_1} \dots \binom{n+1}{a_r} \binom{k}{b_r} x^{2a_r} y^{2b_r} = \prod_{i=1}^r \binom{n+1}{a_i} \binom{k}{b_i} x^{n-1} y^{k+1-2s}.$$

We introduce some notation and write

$$\prod_{i=1}^r \binom{n+1}{a_i} \binom{k}{b_i} := \binom{n+1}{A} \binom{k}{B}.$$

The coefficient C_s is then these combinatorial terms summed over all pairs $(A, B)_s$,

$$C_s = \sum_{(A,B)_s} \binom{n+1}{A} \binom{k}{B}.$$

Evaluating (9) on the fundamental class, the linear coefficient of $p_I[X_n^k(c)]$, denoted with a subscript c , is given by

$$p_I[X_n^k(c)]_c = \sum_{s=0}^r (2s-1) C_s = \sum_{s=0}^r \sum_{(A,B)_s} (2s-1) \binom{n+1}{A} \binom{k}{B}.$$

Notice that the ordered pseudo-partition independent A is independent of s . Now we want to show that if we fix A and sum over all possible B the resulting double sum is zero. Let $A \in \mathcal{S}((n-1)/2, r)$ be a sub-partition of I , i.e. $a_j \leq i_j$ for all $1 \leq j \leq r$. Then there is a $L \in \mathcal{S}((k+1)/2, r)$ such that $A+L=I$. The possible ordered pseudo-partition B are those that belong to $L-s$ for each s .

Without loss of generality, we may omit the zero entries in L and let L be an ordered partition of length $|L| \leq r$. This is possible since a zero entry in L contributes to a factor of 1 in the computation. Applying Lemma 3.22 for L an ordered partition of $(k+1)/2$, we have the desired result:

$$\sum_{s=0}^{|L|} \left[\sum_{B \in L-s} (2s-1) \binom{k}{B} \right] = 0.$$

Finally, summing over the different possibilities of A , the linear coefficient of $p_I[X_n^k(c)]$ vanishes. \square

Before proving Lemma 3.22, let's first consider some concrete calculations.

Example 3.20. Consider the simplest case $k = 3$. Here L is an ordered partition of $(k+1)/2 = 2$, after omitting possible zero entries, and falls into two cases.

Case (i): $L = (2)$, the sum runs over $L - 0 = \{(2)\}$ and $L - 1 = \{(1)\}$, we have

$$-\binom{3}{2} + \binom{3}{1} = 0.$$

Case (ii): $L = (1, 1)$, the sum runs over

$$\begin{aligned} L - 0 &= \{(1, 1)\}, \\ L - 1 &= \{(0, 1), (1, 0)\}, \\ L - 2 &= \{(0, 0)\}, \end{aligned}$$

with their respective weights,

$$-\binom{3}{1}\binom{3}{1} + \binom{3}{1}\binom{3}{0} + \binom{3}{0}\binom{3}{1} + 3\binom{3}{0}\binom{3}{0} = 0.$$

Example 3.21. As a second, more complicated example, consider $k = 5$. Here L is an ordered partition of $(k+1)/2 = 3$, after omitting possible zero entries, and falls into three cases.

Case (i): $L = (3)$, the sum runs over $L - 0 = \{(3)\}$ and $L - 1 = \{(2)\}$,

$$-\binom{5}{3} + \binom{5}{2} = 0.$$

Case (ii): $L = (1, 2)$, the sum runs over

$$\begin{aligned} L - 0 &= \{(1, 2)\}, \\ L - 1 &= \{(0, 2), (1, 1)\}, \\ L - 2 &= \{(0, 1)\}, \end{aligned}$$

with their respective weights,

$$-\binom{5}{1}\binom{5}{2} + \binom{5}{0}\binom{5}{2} + \binom{5}{1}\binom{5}{1} + 3\binom{5}{0}\binom{5}{1} = 0.$$

Note here that reordering L does not alter the resulting sum. The ordering of L is only imposed to make sense of the condition $A + L = I$ above.

Case (iii): $L = (1, 1, 1)$, the sum runs over

$$\begin{aligned} L - 0 &= \{(1, 1, 1)\}, \\ L - 1 &= \{(0, 1, 1), (1, 0, 1), (0, 1, 1)\}, \\ L - 2 &= \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}, \\ L - 3 &= \{(0, 0, 0)\}, \end{aligned}$$

with respective weights,

$$-\binom{5}{1}\binom{5}{1}\binom{5}{1} + 3\binom{5}{1}\binom{5}{1}\binom{5}{0} + 3 \cdot 3\binom{5}{1}\binom{5}{0}\binom{5}{0} + 5\binom{5}{0}\binom{5}{0}\binom{5}{0} = 0.$$

Lemma 3.22. *Let $L = (l_1, \dots, l_r)$ an ordered partition of the positive integer K . Then*

$$\sum_{s=0}^r \left[\sum_{B \in L-s} (2s-1) \binom{2K-1}{B} \right] = 0. \quad (10)$$

Proof. (i) First note that each entry in $B \in L-s$ is either l_i or l_i-1 , and that (10) is symmetric with respect to the entries of L .

Claim: The desired equality in (10) is equivalent to

$$\begin{aligned} 0 &= (-1)(2K-l_1) \cdots (2K-l_r) \\ &\quad + l_1(2K-l_2) \cdots (2K-l_r) + \text{perm. of the } l_i \\ &\quad + 3l_1l_2(2K-l_3) \cdots (2K-l_r) + \text{perm. of the } l_i \\ &\quad + \cdots \\ &\quad + (2r-1)l_1 \cdots l_r. \end{aligned} \quad (11)$$

Proof. Consider the following equation with arbitrary coefficients c_1, c_2 ,

$$c_1 \binom{2K-1}{l_i} + c_2 \binom{2K-1}{l_i-1} = 0. \quad (12)$$

Written out in full, this is

$$c_1 \cdot \frac{(2K-1) \cdots (2K-l_i)}{l_i!} + c_2 \cdot \frac{(2K-1) \cdots (2K-l_i-1)}{(l_i-1)!} = 0.$$

Upon canceling common factors, (12) is equivalent to

$$c_1(2K-l_i) + c_2 l_i = 0.$$

Since each term in (10) contains a factor of either $\binom{2K-1}{l_i}$ or $\binom{2K-1}{l_i-1}$, after cancelling common factors we may replace every $\binom{2K-1}{l_i}$ in (10) by $(2K-l_i)$ and every $\binom{2K-1}{l_i-1}$ by l_i . Finally, symmetrizing with respect to l_1, \dots, l_r , we obtain the claim.

(ii) It suffices to prove (11). We write each line of (11) in terms of K and the elementary

symmetric polynomials $\sigma_i(L) := \sigma_i(l_1, \dots, l_r)$,

$$\begin{aligned}
(-1)(2K - l_1) \cdots (2K - l_r) &= (-1) \sum_{i=0}^r (-1)^i \sigma_i(L) \cdot (2K)^{r-i}, \\
l_1(2K - l_2) \cdots (2K - l_r) + \text{perm.} &= \sum_{i=1}^r (-1)^{i-1} \binom{i}{1} \sigma_i(L) \cdot (2K)^{r-i}, \\
3l_1l_2(2K - l_3) \cdots (2K - l_r) + \text{perm.} &= 3 \sum_{i=2}^r (-1)^{i-2} \binom{i}{2} \sigma_i(L) \cdot (2K)^{r-i}, \\
&\vdots \\
(2r - 1) l_1 \cdots l_r &= (2r - 1) \sigma_r(L).
\end{aligned}$$

Summing these terms with their respective weights, the equality (11) becomes

$$\begin{aligned}
0 &= -(2K)^r \sigma_0(L) + (2K)^{r-1} \sigma_1(L) \cdot (-(-1) + 1) \\
&\quad + \sum_{i=2}^r (2K)^{r-i} \sigma_i(L) \cdot \left[\sum_{j=0}^i (-1)^{i-j} \cdot (2j - 1) \binom{i}{j} \right].
\end{aligned} \tag{13}$$

The first line of (13) vanishes, since by assumption L is a partition of K ,

$$\sigma_1(L) = l_1 + \cdots + l_r = K.$$

Meanwhile, the second line vanishes as the term in brackets does by the following lemma. \square

Lemma 3.23. *For $i \geq 2$:*

$$\sum_{j=0}^i (-1)^{i-j} \cdot (2j - 1) \binom{i}{j} = 0.$$

Proof. Consider the function $(y^2 - 1)^i$ for $i \geq 2$. It has extremum at $y = 1$,

$$\frac{d}{dy} \Big|_{y=1} (y^2 - 1)^i = 0.$$

Expanding in binomial coefficients, we have

$$\begin{aligned}
&\frac{d}{dy} \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} y^{2j} \Big|_{y=1} \\
&= \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} 2j \cdot y^{2j-1} \Big|_{y=1} \\
&= \sum_{j=0}^i (-1)^{i-j} \cdot 2j \binom{i}{j} = 0.
\end{aligned}$$

Therefore,

$$\sum_{j=0}^i (-1)^{i-j} \cdot (2j-1) \binom{i}{j} = \sum_{j=0}^i (-1)^{i-j} \cdot 2j \binom{i}{j} - \sum_{j=0}^i (-1)^{i-j} \binom{i}{j} = 0.$$

□

These two lemmas conclude the proof of Proposition 3.17.

3.6 Other projective bundles over projective space

In this section, we record examples of not-so-useful projective bundles that we came across along the way. This showcases constructions of nonnegatively curved spin manifolds that fail to provide a proof for Theorem 1.5, either because the Thom number vanishes or because the calculations become too complicated to pursue further.

The first example is a family of projectivizations parametrized by the second Chern class of a complex vector bundle. Consider the projectivization $W_c = \mathbb{P}(V_c)$ of the rank $(k+1)$ vector bundle $V_c \rightarrow \mathbb{C}P^n$,

$$V_c = H^c \oplus H^{-c} \oplus \underline{\mathbb{C}}^{k-1}, \quad k \geq 1.$$

The total Chern class of V_c is given by $c(V_c) = 1 - c^2 x^2$. Therefore, by formula (2) the total Chern class of W_c is computed to be

$$\begin{aligned} c(W_c) &= \left[(1+y)^{k+1} - (1+y)^{k-1} c^2 x^2 \right] \cdot (1+x)^{n+1} \\ &= (1+y)^{k-1} (1+x)^{n+1} (1+2y+y^2 - c^2 x^2). \end{aligned}$$

We can write the last factor per the splitting principle as

$$1+2y+y^2 - c^2 x^2 = (1+y_1)(1+y_2)$$

with $y_1 + y_2 = 2y$, $y_1 y_2 = y^2 - c^2 x^2$. Then the total Pontryagin class can be obtained by squaring the formal Chern roots

$$p(W_c) = (1+y^2)^{k-1} (1+x^2)^{n+1} (1+y_1^2)(1+y_2^2)$$

The cohomology classes x and y satisfy the relations

$$x^{n+1} = 0, \quad y^{k+1} = c^2 x^2 y^{k-1}. \quad (14)$$

As V_c is a sum of line bundles, W_c admits nonnegative sectional curvature [HW18]. Furthermore, if n and k are both odd, the first Chern class

$$\begin{aligned} c_1(W_c) &= (k-1)y + (n+1)x + y_1 + y_2 \\ &= (k+1)y + (n+1)x \end{aligned}$$

is even, and W_c is spin. However, the Thom number of W_c vanishes, therefore it is not a suitable choice as a generator of $\Omega_{4m}^{spin} \otimes \mathbb{Q}$.

Lemma 3.24. *Let W_c spin be of dimension $4m$, then $s_m(W_c) = 0$.*

Proof. We show that no term in

$$s_m(W_c) = (k-1)y^{2m} + (n+1)x^{2m} + y_1^{2m} + y_2^{2m}$$

is proportional to the top class $x^n y^k$ so that evaluation on the fundamental class yields no contribution.

Since $y_1 y_2 = y^2 - c^2 x^2$, y_1, y_2 must be linear in x and y . Raised to the even power $2m$, y_1^{2m} and y_2^{2m} have only terms that are products of even powers of x and y . But relation (14) only allows us to exchange two powers of y for two powers of x . So there is no term in $y_1^{n+k} + y_2^{n+k}$ proportional to the top class $x^n y^k$, as both n and k are odd. Similarly, neither can x^{2m} nor y^{2m} be proportional to the top class. \square

The proof generalizes to any projectivization of vector bundles without odd degree Chern classes, as the relation between generators of the cohomology ring is given by Lemma 3.3.

Remark 3.25. *Any projectivization $\mathbb{P}(E)$ of vector bundles E having only even degree Chern classes has vanishing Thom number.*

The second example is a family of projectivizations parametrized by the third Chern class of a complex vector bundle. Consider S^6 and its tangent bundle TS^6 , which has an almost complex structure. Now take a degree c map $f: \mathbb{C}P^3 \rightarrow S^6$ mapping the top cell of $\mathbb{C}P^3$ to the top cell of S^6 and sending all other cells of $\mathbb{C}P^3$ to a point. Let $V_c \rightarrow \mathbb{C}P^3$ be the rank $(k+1)$ pullback bundle

$$V_c = f^* TS^6 \oplus \underline{\mathbb{C}}^{k-3}, \quad k \geq 3.$$

Then $c(V_c) = 1 + cx^3$ has nontrivial third Chern class, where $x \in H^2(\mathbb{C}P^3; \mathbb{Z})$ is a generator. Let $W_c = \mathbb{P}(V_c)$ be the projectivization. By formula (2) we can compute the total Chern class

$$c(W_c) = \left[(1+y)^{k+1} + (1+y)^{k-2} \cdot cx^3 \right] \cdot (1+x)^4.$$

The cohomology of W_c satisfies the relations

$$x^4 = 0, \quad y^{k+1} + cx^3 y^{k-2} = 0.$$

However, the Pontryagin numbers of W_c are much harder to compute as the factorization of $c(W_c)$ into its Chern roots are more involved. Therefore, we did not use W_c in our proof of Theorem 1.5.

4 Proof of the main theorem

In this section we give a proof of Theorem 1.5. By Proposition 3.16 in the previous section, it suffices to discuss the case when $Z(c)$ contains factors of $K3$. Therefore the main task here is to work around the lack of a nonnegatively curved generator in dimension 4.

The proof relies on uncovering certain polynomial relations between different families of generators $X_n^k(c)$. These are detected by the characteristic number q , which vanishes on product manifolds when the dimensions of the factors are insufficient. The crucial ingredient is the existence of a certain linear combination of the $X_n^k(c)$, which has vanishing Thom number but nonzero q -number.

Recall from Proposition 3.14 we have multiple choices of generator $X_n^k(c)$ in each dimension $4m$. Instead of fixing one $X_n^k(c)$ in each dimension $4m \geq 12$, by throwing in redundant generators into the cobordism ring, we can rewrite terms containing factors of $K3$ as terms without factors of $K3$, thereby eliminating the presence of the negatively curved $K3$ altogether.

4.1 Computations of polynomial relations in low dimensions

We compute concretely the Thom number and monomial Pontryagin numbers of $X_n^k(c)$ in dimensions 16 and 20 using the cohomology relations in Lemma 3.6. Using these computations, we uncover polynomial relations between the $X_n^k(c)$ in low dimensions, leading to the discovery of the q -number.

Notice that the resulting Pontryagin numbers are all odd polynomials in c of degree $\leq n$ with vanishing linear term, which agree with the polynomial properties in Proposition 3.15 and Proposition 3.17.

Example 4.1. *The only $X_n^k(c)$ in dimension 12 is $X_3^3(c)$, which has Thom number and monomial Pontryagin numbers:*

$$\begin{aligned} s_3(X_3^3(c)) &= 7c^3, \\ p_1^3[X_3^3(c)] &= -8c^3, \\ p_1p_2[X_3^3(c)] &= -6c^3, \\ p_3[X_3^3(c)] &= -c^3. \end{aligned}$$

Example 4.2. *The following is a list of monomial Pontryagin numbers and Thom numbers of $X_n^k(c)$ in dimension 16.*

$$\begin{array}{ll}
s_4(X_3^5(c)) = 30c^3, & s_4(X_5^3(c)) = 18c^5, \\
p_1^4[X_3^5(c)] = -240c^3, & p_1^4[X_5^3(c)] = -48c^5 - 192c^3, \\
p_1^2 p_2[X_3^5(c)] = -140c^3, & p_1^2 p_2[X_5^3(c)] = -24c^5 - 120c^3, \\
p_2^2[X_3^5(c)] = -75c^3, & p_2^2[X_5^3(c)] = -9c^5 - 72c^3, \\
p_1 p_3[X_3^5(c)] = -40c^3, & p_1 p_3[X_5^3(c)] = -3c^5 - 42c^3, \\
p_4[X_3^5(c)] = -5c^3, & p_4[X_5^3(c)] = -6c^3.
\end{array}$$

From the information about the Thom numbers, observe that the linear combination

$$Y_4(c) = 5X_5^3(c) - 3c^2 X_3^5(c)$$

has vanishing Thom number. Therefore, it can not be taken as a generator of the spin cobordism ring in dimension 16, but rather can be expressed as a polynomial in the generators in dimensions ≤ 12 .

Consider a basis sequence $\alpha_m(c)$ from Proposition 3.14 with $\alpha_3(c) = X_3^3(c)$. As a linear combination of the $X_n^k(c)$, $Y_4(c)$ belongs to the kernel \mathcal{I} of the elliptic genus. Hence $Y_4(c)$ must be proportional to $K3 \cdot X_3^3(c)$, the only monomial in generators in dimensions ≤ 12 that lies in the kernel \mathcal{I} .

Solving for the proportionality factor from the Pontryagin numbers in Examples 4.1 and 4.2, we find

$$Y_4(c) = \frac{5}{16}(c^2 - 2)K3 \cdot X_3^3(c). \quad (15)$$

Example 4.3. *The following is a list of monomial Pontryagin numbers and Thom numbers of $X_n^k(c)$ in dimension 20.*

	$X_3^7(c)$	$X_5^5(c)$	$X_7^3(c)$
s_5	$77c^3$	$121c^5$	$33c^7$
p_1^5	$-7168c^3$	$-16(139c^5 + 450c^3)$	$-64(3c^7 + 30c^5 + 80c^3)$
$p_1^3 p_2$	$-3808c^3$	$-1060c^5 - 3960c^3$	$-2(39c^7 + 480c^5 + 1456c^3)$
$p_1 p_2^2$	$-1960c^3$	$-475c^5 - 2130c^3$	$-27c^7 - 456c^5 - 1616c^3$
$p_1^2 p_3$	$-1386c^3$	$-240c^5 - 1320c^3$	$-3(3c^7 + 80c^5 + 352c^3)$
$p_1 p_4$	$-210c^3$	$-25c^5 - 270c^3$	$-8(3c^5 + 29c^3)$
$p_2 p_3$	$-588c^3$	$-100c^5 - 690c^3$	$-3c^7 - 96c^5 - 580c^3$
p_5	$-21c^3$	$-c^5 - 30c^3$	$-28c^3$

Observe that the linear combination

$$Y_5(c) = 7X_5^5(c) - 11c^2 X_3^7(c)$$

has vanishing Thom number, therefore it can be expressed as a polynomial in generators in dimensions ≤ 16 . Consider a basis sequence $\alpha_m(c)$ from Proposition 3.14 with $\alpha_3(c) = X_3^3(c)$ and $\alpha_4(c) = X_3^5(c)$. Since the elliptic genus vanishes on $Y_5(c)$, the only monomials that can enter its expression are those in the kernel \mathcal{I} , namely $K3 \cdot X_3^5(c)$ and $\mathbb{H}P^2 \cdot X_3^3(c)$.

Solving for the relation via the Pontryagin numbers in Examples 4.3 and 4.2, we find,

$$Y_5(c) = \frac{7}{80} (13c^2 - 10) K3 \cdot X_3^5(c) + 7c^2 \mathbb{H}P^2 \cdot X_3^3(c). \quad (16)$$

Note that by rearranging equations (15) and (16) we can effectively rewrite $K3 \cdot X_3^3(c)$ and $K3 \cdot X_3^5(c)$ as polynomials of ring generators not involving $K3$. However, this comes at the expense of taking two generators in the same dimension.

4.2 The q -number

The ability to express $K3 \cdot X_n^k(c)$ as a polynomial of ring generators not involving $K3$ can be detected by the nonvanishing of the q -number, which we now define.

Definition 4.4. *Let M be a closed connected oriented manifold of dimension $4m$. The q -number of M is the Pontryagin number defined by*

$$q_m(M) = \left\langle \sum_{i=1}^p \sum_{j \neq i} y_i^2 y_j^{2m-2}, [M] \right\rangle$$

where y_1, \dots, y_p denote the Pontryagin roots of M .

For dimension reasons, the q -number vanishes on $4m$ -dimensional product manifolds, in which no factor has dimension greater than or equal to $4m-4$. Note that by construction the q -number of a product manifold $M^4 \times N^{4m-4}$ is simply the product of the first Pontryagin number of M and the Thom number of N ,

$$q_m(M \times N) = p_1[M] s_{m-1}(N),$$

as no other combinations contribute.

The following proposition computes the q -number of $X_n^k(c)$.

Lemma 4.5. *Let $X_n^k(c)$ be spin of dimension $4m$. Then the q -number of $X_n^k(c)$ is given by*

$$q_m(X_n^k(c)) = k \left[\binom{n+k-3}{n} - (k-1) \right] c^n + (n+1) \left[\binom{n+k-3}{n-2} - k \right] c^{n-2}. \quad (17)$$

Proof. We use the Pontryagin roots observed in formula (4) to compute the q -number:

$$\begin{aligned} q_m(X_n^k(c)) &= \langle (n+1)x^2 \left[ky^{n+k-2} + nx^{n+k-2} + (y+cx)^{n+k-2} \right] \\ &\quad + ky^2 \left[(k-1)y^{n+k-2} + (n+1)x^{n+k-2} + (y+cx)^{n+k-2} \right] \\ &\quad + (y+cx)^2 \left[ky^{n+k-2} + nx^{n+k-2} \right], [X_n^k(c)] \rangle. \end{aligned}$$

The cohomology relations from Lemma 3.6 give $x^{n+k-2} = 0$ and $(y+cx) \cdot y^k = 0$, so several terms have no contribution, giving

$$q_m(X_n^k(c)) = \langle (n+1)x^2 \left[ky^{n+k-2} + (y+cx)^{n+k-2} \right] + ky^2 \left[(k-1)y^{n+k-2} + (y+cx)^{n+k-2} \right], [X_n^k(c)] \rangle$$

We expand $(y+cx)^{n+k}$ using the binomial formula, and drop all terms of x with exponent greater than n . Applying the relation $y^k = -cxy^{k-1}$ repeatedly, and evaluating on the fundamental class using $\langle x^n y^k, [X_n^k(c)] \rangle = 1$, we obtain

$$q_m(X_n^k(c)) = (n+1) \left[-k + \sum_{i=0}^{n-2} (-1)^{i+1} \binom{n+k-2}{i} \right] c^{n-2} + k \left[-(k-1) + \sum_{i=0}^n (-1)^{i+1} \binom{n+k-2}{i} \right] c^n.$$

The first sum can be simplified by the recursive identity (6),

$$\sum_{i=0}^{n-2} (-1)^{i+1} \binom{n+k-2}{i} = \binom{n+k-3}{n-2},$$

and similarly for the second sum. This completes the proof. \square

Notice that for $n = 3$, the linear term of the q -number vanishes, which agrees with the statement in Proposition 3.17.

Example 4.6. We can immediately apply the q -number to recover the proportionality between $Y_4(c)$ and $K3 \cdot X_3^3(c)$:

$$Y_4(c) = \mu(c) \cdot K3 \cdot X_3^3(c).$$

Evaluating the q -number on both sides of the equation, we find

$$\begin{aligned} q_4(Y_4(c)) &= 5q_4(X_5^3(c)) - 3c^2 q_4(X_3^5(c)) \\ &= 5 \cdot (-3c^2 + 42c^3) - 3c^2 \cdot 30c^3 \\ &= -105c^3(c^2 - 2). \\ \mu(c) \cdot q_4(K3 \cdot X_3^3(c)) &= \mu(c) \cdot p_1[K3] \cdot s_3(X_3^3(c)) \\ &= \mu(c) \cdot (-48) \cdot 7c^3. \end{aligned}$$

Therefore, we obtain $\mu(c) = \frac{5}{16}(c^2 - 2)$, which agrees with the result we computed above explicitly with the monomial Pontryagin numbers.

Example 4.7. For $Y_5(c)$, there is no direct proportionality, but since the q -number vanishes on $\mathbb{H}P^2 \cdot X_3^3(c)$, we can recover the coefficient of $K3 \cdot X_3^5(c)$ in the equation:

$$Y_5(c) = \mu(c) \cdot K3 \cdot X_3^5(c) + \rho(c) \cdot \mathbb{H}P^2 \cdot X_3^3(c).$$

Evaluating the q -number on the left hand side of the equation,

$$\begin{aligned} q_5(Y_5(c)) &= 7q_5(X_3^5(c)) - 11c^2 q_5(X_3^7(c)) \\ &= 7 \cdot (85c^5 + 180c^3) - 11c^2 \cdot (203c^3) \\ &= -1638c^5 + 1260c^3. \end{aligned}$$

Meanwhile, q -number on the right hand side is

$$\begin{aligned} q_5(\mu(c) \cdot K3 \cdot X_3^5(c) + \rho(c) \cdot \mathbb{H}P^2 \cdot X_3^3(c)) &= \mu(c) \cdot p_1[K3] \cdot s_3(X_3^5(c)) \\ &= \mu(c) \cdot (-48) \cdot 30c^3. \end{aligned}$$

Therefore, we obtain $\mu(c) = \frac{7}{80}(13c^2 - 10)$, as we did previously using the monomial Pontryagin numbers.

Observe that in dimensions 16 and 20, the nonvanishing of the coefficient $\mu(c)$ is a direct result of the q -number of $Y_m(c)$ not being zero. This allows us to rearrange the equation and rewrite $K3 \cdot X_p^q(c)$ in terms of $Y_m(c)$ and generators in dimensions $\leq 4m - 8$, where $X_p^q(c)$ is the chosen generator in dimension $4m - 4$.

This idea is easily generalized to arbitrary dimension $4m$. Consider the linear combination

$$Y_n^k(c) := X_n^k(c) - \lambda_n^k(c) \cdot X_3^{2m-3}(c),$$

where we define the coefficient

$$\lambda_n^k(c) = \lambda_n^k \cdot c^{n-3} := \frac{s_m(X_n^k(c))}{s_m(X_3^{2m-3}(c))}$$

to be the quotient of their Thom numbers. Here we require $X_n^k(c) \neq X_3^{2m-3}(c)$ so that $Y_n^k(c)$ is nontrivial.

By construction, the Thom number of $Y_n^k(c)$ vanishes

$$s_m(Y_n^k(c)) = s_m(X_n^k(c)) - \lambda_n^k(c) \cdot s_m(X_3^{2m-3}(c)) = 0.$$

Meanwhile, by the q -number formula in Lemma 4.5, the q -number of $Y_n^k(c)$ is given by

$$\begin{aligned} q_m(Y_n^k(c)) &= q_m(X_n^k(c)) - \lambda_n^k(c) \cdot q_m(X_3^{2m-3}(c)) \\ &= k \left[\binom{n+k-3}{n} - (k-1) \right] c^n + (n+1) \left[\binom{n+k-3}{n-2} - k \right] c^{n-2} \\ &\quad - \lambda_n^k \cdot (2m-3) \cdot \left[\binom{2m-3}{3} - (2m-4) \right] c^n. \end{aligned}$$

Here recall that $q_m(X_3^{2m-3}(c))$ has no linear term, therefore the c^{n-2} term of $q_m(Y_n^k(c))$ is solely that of $q_m(X_n^k(c))$. By a similar reasoning as before with the Thom number, this term

$$(n+1) \left[\binom{n+k-3}{n-2} - k \right] c^{n-2}$$

is nonzero because of the strict inequality

$$k < \binom{n+k-3}{n-2} = \frac{(n+k-3) \cdots (k+1) \cdot k}{(n-2)!}.$$

To summarize, we have shown the following proposition:

Proposition 4.8. *For $X_{n \neq 3}^k(c)$ spin of dimension $4m$, the linear combination*

$$Y_n^k(c) = X_n^k(c) - \lambda_n^k(c) \cdot X_3^{2m-3}(c)$$

has vanishing Thom number but nonzero q -number.

With this proposition we arrive at the general statement that for all $m \geq 4$, if $X_p^q(c)$ is the chosen generator in dimension $4m-4$, we can rewrite $K3 \cdot X_p^q(c)$ in terms of $Y_n^k(c)$ and generators in dimensions $\leq 4m-8$. This will be made clear in the next section.

4.3 Proof of Theorem 1.5

We are now in the position to prove our main theorem. Using the machinery developed until now, we first show that the presence of $K3$ in the cobordism ring can be compensated by throwing in redundant ring generators $X_n^k(c)$ in dimensions ≥ 12 .

Proposition 4.9. *Let $X_p^q(c)$ be spin of dimension $4m \geq 12$. Then in the spin cobordism ring $\Omega_*^{spin} \otimes \mathbb{Q}$, the product $X_p^q(c) \cdot K3$ equals a polynomial in $\mathbb{H}P^2$ and the various $X_n^k(c)$, but not involving $K3$. Therefore, by inducting on the number of factors of $K3$, any presence of the negatively curved $K3$ can be eliminated in favor of the nonnegatively curved generators $X_n^k(c)$ of $\Omega_*^{spin} \otimes \mathbb{Q}$.*

Proof. The proof is by induction on the dimension $4m = 2(p+q)$. The base case is in dimension 12, where $n = k = 3$, and we look at $X_3^3(c) \cdot K3$ in dimension 16. This is taken care of in Example 4.6,

$$Y_4(c) = 5X_5^3(c) - 3c^2X_3^5(c) = \mu(c) \cdot K3 \cdot X_3^3(c).$$

where we established the relation with $\mu(c) \neq 0$.

Let $X_p^q(c)$ be of dimension $4m \geq 16$, and assume the inductive hypothesis in dimensions $\leq 4m - 4$. For this inductive step, we take a family of basis sequences $(\alpha_i(c))_{i \geq 1}$ in Proposition 3.14, with the choice $\alpha_m(c) = X_p^q(c)$. Let $Y_n^k(c)$ as defined in Proposition 4.8 be of dimension $4m + 4$. Since it has vanishing Thom number, Theorem 2.12 tells us that it can be written as a polynomial in generators in dimensions $\leq 4m$. But the nonvanishing of $q_m(Y_n^k(c))$ implies that the monomial $K3 \cdot X_p^q(c)$ must appear with a nonzero coefficient $\mu(c)$.

This is the only term involving the generator in dimension $4m$, the remaining terms are polynomials in generators in dimensions $\leq 4m - 4$. Therefore, by the inductive hypothesis, any other term in this polynomial with factors of $K3$ can be replaced iteratively by terms not containing $K3$. We can then solve the resulting equation and express $K3 \cdot X_p^q(c)$ as linear combinations of $Y_n^k(c)$ and monomials of the $X_a^b(c)$ in dimensions $< 4m$. This completes the inductive step. \square

We now give a proof of Theorem 1.5.

Theorem 1.5. *Any rational linear combination of Pontryagin numbers that does not factor through the elliptic genus is unbounded on connected closed spin manifolds of nonnegative sectional curvature.*

Proof of Theorem 1.5. Let p_I be a Pontryagin number that does not factor through the elliptic genus φ . We take a family of basis sequences $(\alpha_m(c))_{m \geq 1}$ given in Proposition 3.14. By Proposition 3.16, there is a class $Z(c)$ in \mathcal{I} on which p_I is unbounded. As we argued in Section 3, if $Z(c)$ does not involve factors of $K3$, we are done, for we have found a family of nonnegatively curved manifolds on which p_I is unbounded. Otherwise, by Proposition 4.9, we write $Z(c)$ as a linear combination of nonnegatively curved monomials that does not involve $K3$. Amongst the summands of $Z(c)$, there exists one nonnegatively curved term parametrized by c , on which p_I is unbounded. This completes the proof of our main theorem. \square

5 The complex case

Since the $X_n^k(c)$ are complex spin manifolds, one may ask if an analogous result to Theorem 1.5 holds for Chern classes on nonnegatively curved complex spin manifolds with respect to some complex elliptic genus defined on the rational complex spin cobordism ring $\Omega_*^{c-s} \otimes \mathbb{Q}$.

There are different ways to define a complex elliptic genus that specializes to Ochanine's elliptic genus under the projection $\Omega_*^U \rightarrow \Omega_*^{SO}$. We discuss here the complex elliptic genus due to Schreieder [Sch14], which is originally defined on the rational complex cobordism ring $\Omega_*^U \otimes \mathbb{Q}$.

Definition 5.1. *The **complex cobordism ring** Ω_*^U is the cobordism ring of closed stably almost complex manifolds modulo boundaries of compact stably almost complex manifolds.*

The **complex spin cobordism ring** Ω_*^{c-s} is defined almost verbatim with the additional requirement that the manifolds be spin.

Due to the following theorem, there is an isomorphism of rational cobordism rings

$$\Omega_*^{c-s} \otimes \mathbb{Q} \cong \Omega_*^U \otimes \mathbb{Q}.$$

Theorem 5.2 (Stong, [Sto67]). *The homomorphism $\Omega_*^{c-s} \rightarrow \Omega_*^U$ corresponding to the map on classifying spaces is an isomorphism of cobordism rings modulo 2 primary groups.*

Therefore, a genus on $\Omega_*^U \otimes \mathbb{Q}$ can be treated as a genus on $\Omega_*^{c-s} \otimes \mathbb{Q}$.

Definition 5.3 (Schreieder, [Sch14]). *For formal variables q_1, q_2, q_3, q_4 of degree $|q_i| = 2i$, the **complex elliptic genus** $\psi : \Omega_*^U \otimes \mathbb{Q} \rightarrow \mathbb{Q}[q_1, q_2, q_3, q_4]$ is defined by its logarithmic series $g_\psi(y)$ via the integral,*

$$g_\psi(y) = \int_0^y \frac{dt}{\sqrt{1 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4}}.$$

Let $E \rightarrow B$ be a complex vector bundle over a stably almost complex base B , and let E^* denote its dual. Then ψ is dualization invariant in the sense that its kernel \mathcal{J} as an ideal in $\Omega_*^U \otimes \mathbb{Q}$ is generated by all differences $\mathbb{P}(E) - \mathbb{P}(E^*)$. In other words,

$$\psi(\mathbb{P}(E)) = \psi(\mathbb{P}(E^*)).$$

Furthermore, ψ is surjective onto $\mathbb{Q}[q_1, q_2, q_3, q_4]$ and under certain restrictions of the formal variables q_i , ψ specializes to the Ochanine's elliptic genus φ and the Hirzebruch genus $\chi_y : \Omega_*^U \otimes \mathbb{Q} \rightarrow \mathbb{Q}[y]$,

$$\chi_y(M) = \sum_{p=0}^n \chi_p(M) \cdot y^p = \sum_{p,q=0}^n (-1)^q h^{p,q}(M) \cdot y^p,$$

so that both the φ -genus and χ_y -genus factor through ψ . In addition, it can be shown that both genera are dualization invariant (p.21, [Sch14]).

In the same manner of Theorem 1.2, an analogous statement exists for boundedness of Chern numbers on smooth complex projective varieties:

Theorem 5.4 (Kotschick, [Kot12]). *A rational linear combination of Chern numbers of smooth complex projective varieties can be bounded in terms of Betti numbers if and only if it is a linear combination of the χ_p .*

This leads us to ask the natural question in the complex spin case:

Question 5.5. *Is every Chern number not in Schreieder's complex elliptic genus ψ unbounded on complex spin manifolds of nonnegative curvature?*

We wish to see if a similar approach using the $X_n^k(c)$ construction provides a proof to the question above. The rational complex cobordism ring $\Omega_*^U \otimes \mathbb{Q}$, and therefore $\Omega_*^{c-s} \otimes \mathbb{Q}$, is a \mathbb{Q} -polynomial ring generated by a basis sequence $(\gamma_m)_{m \geq 1}$,

$$\Omega_*^U \otimes \mathbb{Q} = \mathbb{Q}[\gamma_1, \gamma_2, \dots],$$

with one generator γ_m in each even dimension $\Omega_{2m}^U \otimes \mathbb{Q}$ (Chapter 7, [Sto68]).

The Thom number (1) defined using the Pontryagin roots y_i^2 is equally well-defined using the Chern roots y_i . Therefore, if we let $X_n^k(c)$ be generators of $\Omega_*^{c-s} \otimes \mathbb{Q}$ in the even complex dimensions, the natural candidate for generators in the odd complex dimensions is to take $X_n^k(c)$ with n even and c, k odd, as this choice of parity satisfies the spin criterion in Lemma 3.5.

The first four ring generators form a \mathbb{Q} -algebra basis of the image, we take them to be $q_1 = \mathbb{C}P^1$, $q_2 = K3$, $q_3 = \mathbb{C}P^3$ and $q_4 = Q$, where Q is the quadric hypersurface embedded in $\mathbb{C}P^5$ cut out by a generic quadratic polynomial.

Now q_1 and q_3 are equipped with the positive Fubini-Study metric, while $q_4 = Q$ is isometric to $SO(6)/(SO(2) \times SO(4))$, a compact Hermitian symmetric space [Smy67], which admits non-negative sectional curvature by Cartan's classification of symmetric spaces. Hence all of the q_i except $q_2 = K3$ admit metrics of nonnegative curvature.

From previous calculations, we know q_1, q_2, q_3 have nonzero Thom number and therefore are generators of $\Omega_*^{c-s} \otimes \mathbb{Q}$. The following proposition shows that $q_4 = Q$ is indeed a generator of the $\Omega_*^{c-s} \otimes \mathbb{Q}$ in dimension 8:

Proposition 5.6. *The quadric hypersurface $Q \subset \mathbb{C}P^5$ is spin and has nonvanishing Thom number.*

Proof. The Chern classes of Q can be computed via the adjunction formula

$$T\mathbb{C}P^5|_Q = TQ \oplus \mathcal{O}(2).$$

Let a be a generator of $H^*(\mathbb{C}P^5)$. From the adjunction formula one computes

$$\begin{aligned} c_1(Q) &= 4a, \\ c_2(Q) &= 7a^2, \\ c_3(Q) &= 6a^3, \\ c_4(Q) &= 3a^4. \end{aligned}$$

Thus, the first Chern class is even and Q is spin.

To compute the Thom number we formally factor the total Chern class into its Chern roots

$$\begin{aligned} 1 + 4a + 7a^2 + 6a^3 + 3a^4 &= (1 + a + a^2)(1 + 3a + 3a^2) \\ &= (1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4), \end{aligned}$$

such that

$$\begin{aligned} x_1x_2 &= a^2, \quad x_1 + x_2 = a, \\ x_3x_4 &= 3a^2, \quad x_3 + x_4 = 3a. \end{aligned}$$

The Thom number is the integral

$$s_4(Q) = \int_Q x_1^4 + x_2^4 + x_3^4 + x_4^4.$$

Then we compute recursively,

$$\begin{aligned} x_1^2 + x_2^2 &= (x_1 + x_2)^2 - 2x_1x_2 = -a^2, \\ x_1^3 + x_2^3 &= (x_1^2 + x_2^2 - x_1x_2)(x_1 + x_2) = -2a^3, \\ x_1^4 + x_2^4 &= (x_1^3 + x_2^3)(x_1 + x_2) - x_1x_2(x_1^2 + x_2^2) = -a^4. \end{aligned}$$

Similarly,

$$\begin{aligned} x_3^2 + x_4^2 &= (x_3 + x_4)^2 - 2x_3x_4 = 3a^2, \\ x_3^3 + x_4^3 &= (x_3^2 + x_4^2 - x_3x_4)(x_3 + x_4) = 0, \\ x_3^4 + x_4^4 &= (x_3^3 + x_4^3)(x_3 + x_4) - x_3x_4(x_3^2 + x_4^2) = -9a^4. \end{aligned}$$

Hence the Thom number of Q is

$$s_4(Q) = -a^4 - 9a^4 = -10a^4 \neq 0.$$

□

To summarize, we have found ring generators of $\Omega_*^{c-s} \otimes \mathbb{Q}$ in dimensions 2,4,6 and 8, of which only the one in dimension 4 does not admit nonnegative sectional curvature.

As before, we would like to construct a family of basis sequences $(\gamma_m(c))_{m \geq 1}$ of $\Omega_*^{c-s} \otimes \mathbb{Q}$ such that the first four generators are $\gamma_i = q_i$ and $\gamma_{m \geq 5}(c)$ generate the kernel \mathcal{J} . It would be convenient if $\gamma_{m \geq 5}(c)$ could be written in terms of the $X_n^k(c)$ as they admit nonnegative sectional curvature and it is simple to show that their Chern numbers are unbounded with respect to c .

However, as a first check $X_n^k(c)$ in odd complex dimensions does not belong to \mathcal{J} because its Euler number does not vanish. It is a well known fact from algebraic topology (p.481, Theorem 1, [Spa66]) that the Euler number of a Serre fibration $F \rightarrow E \rightarrow B$ is multiplicative,

$$e(E) = e(B) \cdot e(F).$$

Therefore, the Euler number of $X_n^k(c)$

$$e(X_n^k(c)) = (k+1) \cdot (n+1),$$

is nonzero. However, the Euler class $c_n = \sum_p (-1)^p \chi_p = \chi_{-1}$ lies in the span of the χ_y -genus, so the Euler number must be dualization invariant, and should vanish on $X_n^k(c)$ were it in the kernel \mathcal{J} .

We could try to remedy this problem by replacing $X_n^k(c)$ with the difference

$$Y_n^k(c) := X_n^k(c) - \overline{X_n^k(c)},$$

where $\overline{X_n^k(c)} := \mathbb{P}(E_c^*)$, which manifestly belongs to \mathcal{J} . However, the Thom number of $Y_n^k(c)$, which is proportional to c^n , vanishes in odd complex dimensions, i.e. when n is even. Therefore, $Y_n^k(c)$ can not be taken as a generator of $\Omega_*^{c-s} \otimes \mathbb{Q}$.

This leads us to consider other differences of projectivizations $\mathbb{P}(E) - \mathbb{P}(E^*)$ of even-ranked complex vector bundles, hoping to find a candidate in odd complex dimensions that is spin and has nonvanishing Thom number. We arrived at a negative result. That is,

Proposition 5.7. *There are no even-ranked complex vector bundles $E \rightarrow B$ such that $\mathbb{P}(E) - \mathbb{P}(E^*)$ has odd complex dimensions and nonvanishing Thom number.*

Proof. Let $E \rightarrow B$ be a complex vector bundle of even rank $(k+1)$ and B has complex dimension n . Then its projectivization $\mathbb{P}(E)$ has total Chern class

$$\begin{aligned} c(\mathbb{P}(E)) &= c(T\pi) \cdot c(\pi^*B) \\ &= \left[\sum_{i=0}^{k+1} (1+y)^{k+1-i} c_i(E) \right] \cdot \pi^* c(B) \\ &= \prod_{i=1}^l (1+y+x_i) \cdot \prod_{i=1}^m (1+w_i), \end{aligned}$$

where y is a generator of $H^*(\mathbb{C}P^k)$, and x_i and w_i are the Chern roots of E and B respectively. The calculation for $c(\mathbb{P}(E))$ is analogous, and we let $\bar{x}_1, \dots, \bar{x}_l$ denote the Chern roots of E^* .

The Thom class of $\mathbb{P}(E)$ is a polynomial p in $c_i(E) \in H^*(\mathbb{P}(E))$ while the Thom class of $\mathbb{P}(E^*)$ is the same polynomial in $c_i(E^*) \in H^*(\mathbb{P}(E^*))$. Recall that the Chern classes of the dual vector bundle satisfy $c_i(E^*) = (-1)^i c_i(E)$. Hence the Thom class of $\mathbb{P}(E) - \mathbb{P}(E^*)$ is the difference of the polynomials

$$p(c_1(E), c_2(E), c_3(E), \dots, c_{k+1}(E)) - p(-c_1(E), c_2(E), -c_3(E), \dots, c_{k+1}(E)),$$

which equals the expression with each Chern root raised to the $(n+k)$ -th power,

$$\sum_{i=1}^l (y+x_i)^{n+k} - (y+\bar{x}_i)^{n+k},$$

where the common Chern roots w_i from $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ cancel each other and do not appear in the end.

Evaluating the Thom class on the fundamental class $[\mathbb{C}P^k] \cdot [B]$, we use the cohomology relation from Lemma 3.3

$$y^{k+1} = -c_1(E)y^k - \dots - c_{k+1}(E),$$

to get rid of possibly excess factors of y . By assumption $\mathbb{P}(E)$ has odd complex dimensions and the fibers are odd complex, so B is of even complex dimension n . The terms with nonzero contribution to $s(\mathbb{P}(E))$ are those of y^k multiplied with monomials $c_{i_1}(E) \cdots c_{i_r}(E)$ that corresponds to partitions of n . Since n is even, the odd Chern classes in the monomial always appear in pairs. Hence $s(\mathbb{P}(E)) = s(\mathbb{P}(E^*))$, and the difference $\mathbb{P}(E) - \mathbb{P}(E^*)$ has vanishing Thom number. \square

Proposition 5.7 shows that, in contrast to the case of complex cobordism, when \mathcal{J} is treated as an ideal in $\Omega_*^{c-s} \otimes \mathbb{Q}$, it cannot be entirely generated by differences $\mathbb{P}(E) - \mathbb{P}(E^*)$, as some of these fail to be both spin and a generator. But since $\Omega_*^U \otimes \mathbb{Q}$ is isomorphic to $\Omega_*^{c-s} \otimes \mathbb{Q}$, there

must in fact, be a generator of $\Omega_*^{c-s} \otimes \mathbb{Q}$ in each odd complex dimension whose image under the isomorphism maps to $X_n^k(c) - \overline{X_n^k}(c)$ for k even and n odd, which is a generator of \mathcal{J} in $\Omega_*^U \otimes \mathbb{Q}$ but is not spin. The problem is we do not know if these preimages are nonnegatively curved, so our question with respect to the complex elliptic genus is still open.

More generally, since it was shown that the complex elliptic genus of Schreieder is genuinely different from that of Krichever-Höhn (Corollary 8.2, [Sch14]), if some kind of statement does hold for the complex elliptic genus of one, it would be interesting to consider whether it holds for the other.

References

- [ABP67] D. W. Anderson, E. H. Brown Jr., and F. P. Peterson. “The structure of the Spin cobordism ring”. In: *Ann. of Math. (2)* 86 (1967), pp. 271–298. ISSN: 0003-486X.
- [BT82] R. Bott and L. W. Tu. *Differential forms in algebraic topology*. Vol. 82. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982, pp. xiv+331. ISBN: 0-387-90613-4.
- [Car92] M. P. do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Translated from the second Portuguese edition by Francis Flaherty. Birkhäuser Boston, Inc., Boston, MA, 1992, pp. xiv+300. ISBN: 0-8176-3490-8.
- [Des07] A. Dessai. “Obstructions to positive curvature and symmetry”. In: *Adv. Math.* 210.2 (2007), pp. 560–577. ISSN: 0001-8708.
- [Gro81] M. Gromov. “Curvature, diameter and Betti numbers”. In: *Comment. Math. Helv.* 56.2 (1981), pp. 179–195. ISSN: 0010-2571.
- [Gro58] A. Grothendieck. “La théorie des classes de Chern”. In: *Bull. Soc. Math. France* 86 (1958), pp. 137–154. ISSN: 0037-9484.
- [Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0.
- [HW18] M. Herrmann and N. Weisskopf. “Non-negative curvature, elliptic genus and unbounded Pontryagin numbers”. In: *Proc. Edinb. Math. Soc. (2)* 61.2 (2018), pp. 449–456. ISSN: 0013-0915.
- [HBJ92] F. Hirzebruch, T. Berger, and R. Jung. *Manifolds and modular forms*. Aspects of Mathematics, E20. With appendices by Nils-Peter Skoruppa and by Paul Baum. Friedr. Vieweg & Sohn, Braunschweig, 1992, pp. xii+211. ISBN: 3-528-06414-5.
- [HK21] E. Hsiao and D Kotschick. “Unbounded Pontryagin numbers on nonnegatively curved spin manifolds”. In: (2021). arXiv: 2111.07862 [math.DG].
- [Huy16] D. Huybrechts. *Lectures on K3 surfaces*. Vol. 158. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016, pp. xi+485. ISBN: 978-1-107-15304-2.
- [Kot10] D. Kotschick. “Pontryagin numbers and nonnegative curvature”. In: *J. Reine Angew. Math.* 646 (2010), pp. 135–140. ISSN: 0075-4102.
- [Kot12] D. Kotschick. “Topologically invariant Chern numbers of projective varieties”. In: *Adv. Math.* 229.2 (2012), pp. 1300–1312. ISSN: 0001-8708.
- [Lic63] A. Lichnerowicz. “Spineurs harmoniques”. In: *C. R. Acad. Sci. Paris* 257 (1963), pp. 7–9. ISSN: 0001-4036.

- [MS74] J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, pp. vii+331.
- [Och87] S. Ochanine. “Sur les genres multiplicatifs définis par des intégrales elliptiques”. In: *Topology* 26.2 (1987), pp. 143–151. ISSN: 0040-9383.
- [Sch14] S. Schreieder. “Dualization invariance and a new complex elliptic genus”. In: *J. Reine Angew. Math.* 692 (2014), pp. 77–108. ISSN: 0075-4102.
- [Smy67] B. Smyth. “Differential geometry of complex hypersurfaces”. In: *Ann. of Math. (2)* 85 (1967), pp. 246–266. ISSN: 0003-486X.
- [Spa66] E. H. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966, pp. xiv+528.
- [Sto92] S. Stolz. “Simply connected manifolds of positive scalar curvature”. In: *Ann. of Math. (2)* 136.3 (1992), pp. 511–540. ISSN: 0003-486X.
- [Sto67] R. E. Stong. “On complex-spin manifolds”. In: *Ann. of Math. (2)* 85 (1967), pp. 526–536. ISSN: 0003-486X.
- [Sto68] R. E. Stong. *Notes on cobordism theory*. Mathematical notes. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968, pp. v+354+lvi.
- [Wal60] C. T. C. Wall. “Determination of the cobordism ring”. In: *Ann. of Math. (2)* 72 (1960), pp. 292–311. ISSN: 0003-486X.

Statement of Authorship

I, Enya Hsiao, declare that I completed this thesis on my own and that information which has been directly or indirectly taken from other sources has been noted as such. Neither this nor a similar work has been presented to an examination committee.

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