Formulation of Batalin-Vilkovisky Field Theories as Homotopy Lie Algebras

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Abstract

This thesis provides a very detailed practitioner's guide for obtaining a homotopy Lie algebra from a given polynomial Batalin-Vilkovisky field theory action. We thoroughly introduce the algebraic framework of field theories paying special care to the proper treatment of fermionic fields. In the setting of differential graded vector spaces, a homotopy Lie algebra is equivalent to multilinear L_∞ products obeying a set of L_∞ relations. The core of this thesis is our presentation of very explicit formulas for deriving L_∞ products from a given action. Subsequently, we apply the developed procedure to two exemplary field theories.

Contents

1 Introduction

The difficult task of mathematical physics is to tame nature into the beautiful corset of mathematics. To the outside observer, this endeavour appears at times as though the mathematical physicist has lost their proclaimed touch with the real world and is digging deeper and deeper into the rabbit hole of abstract formalisms and concepts. However, it is the core belief of mathematical physics that the many detours of yet one further step of abstraction will eventually lead to the discovery of true insight. The bright light pointing the path is the beauty of mathematics itself. Though in order not to be blinded, it is good advice to test one's abstract ideas now and then on some toy model - the mathematical physicist's lifeline back to reality.

The particular detour we are presenting in this thesis is the formulation of field theories as homotopy Lie algebras. And the toy model which we cling to is classical Abelian Yang-Mills field theory including minimally coupled Dirac fermions (i.e. quantum electrodynamics at the classical level). Although our ansatz is much more general, we present its application in this clarifying example.

Put simply, a homotopy Lie algebra is a set of multilinear products. The inputs are multiple fields of the respective field theory, the output is a single field that depends on the inputs. In this respect, the homotopy Lie algebra may be compared to a scattering experiment, in which the incoming particles form a single outgoing particle. Indeed, the multilinear products describe interactions of the fields. Even more, the homotopy Lie algebra provides a complete description of the field theory.

Usually, field theories are described by an action. The equations of motion are derived from this action by a variation principle. The field theory can be quantised via canonical quantisation or through path integral quantisation. Both approaches finally lead to scattering matrix elements which describe amplitudes for certain interactions in scattering processes. And both rely on the field theory action.

In the algebraic formulation of field theories, the fundamental object is not the action but the homotopy Lie algebra. It encodes the classical equations of motion, the gauge transformations and the Noether identities of the field theory [1]. Scattering matrix elements can be computed through a transfer from the full field space to the sub-space of physical fields which satisfy the equations of motion [2]. To be precise, the homotopy Lie algebra encodes only tree-level scattering, i.e. it does not capture quantum effects. However, a generalisation which takes quantum effects into account exists and is called quantum homotopy Lie algebra or loop homotopy Lie algebra (cf. [3] and [4]).

The formulation of field theories as homotopy Lie algebras is the last step on a long path that started with the problem of quantising gauge theories. The first successful ansatz was the Fadeev-Popov quantisation procedure in the 1960s, followed by the discovery of the Becchi-Rouet-Stora-Tyutin symmetry (abbreviated BRST) in the 1970s, and further abstracted in the Batalin-Vilkovisky formalism (abbreviated BV) in the 1980s [5]. Finally, homotopy Lie algebras were introduced to physics in the 1990s, originally in the setting of string theory by Zwiebach [6]. A few years later, Alexandrov, Kontsevich, Schwarz and Zaboronsky described how BV field theories can be formulated as homotopy Lie algebras [7].

Since then, a lot of progress has been made in this area and nowadays, much can be done in the algebraic framework of field theories. A particularly useful mathematical theorem is the so-called homotopy transfer theorem. It is used in various applications, including the computation of effective actions [8], gauge-invariant perturbation theory [9] or regularisation of the field theory action [10].

Any perturbative field theory can be written as a homotopy Lie algebra [7]. Some explicit examples have been demonstrated, most importantly an algebraic formulation of Yang-Mills theory [11]. However, we found that much of the discussion is presented in an abstract notation which hides computational subtleties. The greatest difficulties arise from properly keeping track of signs, which originate from plentiful sources. In this thesis, we aim to be as explicit as possible and discuss the various subtleties that often make the actual use of the algebraic framework for field theories quite involved.

After a quick recapitulation of BV theory, we present in great detail the algebraic framework for field theories. We pay much attention to explaining the right treatment of fermionic (i.e. anti-commuting) fields. Subsequently, we present our general ansatz for how a homotopy Lie algebra structure can be derived from a given field theory action. The detailed presentation of this approach in section 4.1 is the main goal of this thesis. We go on to show two explicit examples: scalar field theory and abelian Yang-Mills theory with minimally coupled fermions.

2 Short Recapitulation of the Batalin-Vilkovisky Formalism

In this section, we quickly recapitulate the BV formalism. The reader who is already familiar with the BV formalism may skip this section. We make the discussion self-contained, yet we only cover what is relevant to this thesis. For an extensive introduction, consult section 4 of the excellent review by Gomis, Paris and Samuel [12] or chapter 15 of the standard textbook by Weinberg [5]. The following presentation of the BV formalism is built on these two references.

The BV formalism is a procedure for quantising gauge theories, i.e. field theories with a local symmetry. A symmetry of a field theory is called local if the symmetry transformations vary at each space-time point. While a global symmetry transformation is described by a fixed parameter (for example an angle or a phase shift), a local symmetry is described by a smooth function (for example an angle that differs at each space-time point). Both the Standard Model of particle physics and the theory of general relativity have local symmetries. Hence, gauge theories play a very important role in high-energy physics.

The local symmetry becomes an issue once we quantise a gauge theory. Consider the path integral for a gauge theory. In a naive approach, one would integrate over all fields including gauge transformations. Since two fields that differ only by a gauge transformation are physically equivalent, such a path integral could be split into two parts. First, we perform the integration over fields up to gauge transformations, i.e. a gauge fixed integration where we choose a specific transformation parameter at each space-time point. Secondly, the result of this integration must be integrated over all infinite possible choices of local gauge parameters. Assuming the first step produces a finite result, then the second integration will diverge. Yet, we are only interested in the result of the first step anyway since a change of gauge has no physical relevance. Thus, it suffices to compute the gauge fixed path integral.

The Fadeev-Popov quantisation procedure gives a recipe for computing gauge fixed path integrals. The specific gauge is fixed by a generalised Dirac delta function. Through a computational trick, one rewrites this delta function as another path integral over newly introduced fields and combines the original path integral over ordinary fields with the new path integral. These new fields must be anti-commuting scalar fields. Since scalar fields have zero spin, the spin-statistics theorem implies that they should be bosonic, i.e. commuting. Thus, the newly introduced fields appear to defy basic principles of quantum field theory. However, this is not an issue since they only manifest a computational trick and never appear in any physical quantities. Therefore, they are called unphysical ghost fields.

After gauge fixing, the path integral is well-defined (up to the usual singularities appearing in quantum field theory). However, one has destroyed the local symmetry. At first sight, it is not obvious, whether a different choice of gauge fixing would have produced the same result. In answering this question, one finds a new symmetry of the gauge fixed path integral, which is referred to as BRST symmetry. The BRST symmetry encapsulates the original gauge symmetry by including the ghost fields in the symmetry transformation. It transforms bosonic fields into fermionic ones and vice-versa. The existence of the BRST symmetry justifies the gauge fixing of the path integral.

The BV formalism takes a step back and considers the classical starting point. Although not necessary, one can introduce the ghost fields already in the classical theory before quantisation. Then, also the BRST symmetry is present at the classical level. Unlike the gauge transformation, the BRST symmetry is (multi-)linear in the fields, which poses a computational advantage. The BV formalism introduces not only the ghost fields at the classical level but subsequently, doubles the number of fields. For each field (ordinary or ghost), a so-called anti-field is added. The resulting field space is even-dimensional and therefore allows for a symplectic structure. Hence, one can set up a generalised Poisson bracket which generates the BRST transformation as well as the equations of motion. The BV formalism is capable of coping with much more general local symmetries than the original Fadeev-Popov procedure. For example, it plays a very important role in string field theory [5].

After this abstract prologue, we go into more detail. We denote a field of the BV theory by a bold letter $\mathbf{\Phi}$ and the corresponding field configuration (i.e. the smooth function on the space-time manifold describing the field) by the same non-bold letter Φ. Making this difference between a field and its field configuration is unusual but it will become clear later in this thesis why we do so. The fields in a BV theory are the ghosts (usually denoted by the letter $\boldsymbol{c}$), the ordinary fields (e.g. scalar fields $\boldsymbol{\phi}$, spinor fields $\boldsymbol{\psi}_\alpha$ and vector fields $\boldsymbol{A}_\mu$), the corresponding anti-fields ($\boldsymbol{\phi}^*$, $\boldsymbol{\psi}^{*\beta}$, $\boldsymbol{A}^{*\nu}$), and the ghost anti-fields ($\boldsymbol{c}^*$). A generic field is denoted $\mathbf{\Phi}_a$, where the index a stands collectively for any index carried by a specific field. The corresponding anti-field is denoted $\mathbf{\Phi}^{*a}$.

Each field is assigned a number called ghost degree gh. The ghosts have ghost degree $\mathrm{gh}(\boldsymbol{c}) = 1$, the ordinary fields have $\mathrm{gh}(\boldsymbol{\phi}) = \mathrm{gh}(\boldsymbol{\psi}_\alpha) = \mathrm{gh}(\boldsymbol{A}_\mu) = 0$. The corresponding anti-fields have $\mathrm{gh}(\mathbf{\Phi}^{*a}) = -\,\mathrm{gh}(\mathbf{\Phi}_a) - 1$. Furthermore, the anti-field corresponding to a field of fermionic or bosonic statistics is of opposite statistics. In general, one may have also ghosts for ghosts of degree 2, and ghosts for ghosts for ghosts of degree 3 and so on. These become necessary if the gauge symmetry is reducible [12].

Very crucial to the BV theory is the so-called BV bracket, a generalised Poisson-bracket on functionals F, G of the fields. It is defined as

$$\{\mathrm{F}, \mathrm{G}\} = \sum_{\substack{\text{fields } \mathbf{\Phi}_a \\ \text{with } \mathrm{gh}(\mathbf{\Phi}_a)\geq 0}} \int \left[\frac{\delta_R \, \mathrm{F}}{\delta \Phi_a} \frac{\delta_L \, \mathrm{G}}{\delta \Phi^{*a}} - \frac{\delta_R \, \mathrm{F}}{\delta \Phi^{*a}} \frac{\delta_L \, \mathrm{G}}{\delta \Phi_a} \right] , \tag{2.1}$$

where δ_R is the right functional derivative, and δ_L is the left functional derivative. Right and left functional derivatives are defined as follows: the field configuration of a bosonic field is commuting, while the configuration of a fermionic field is anti-commuting. Obeying the appropriate commutation behaviour, the specific field configuration is moved to the very right (or left) within a term. Only then it is differentiated.

The BV bracket is graded symmetric: if both F and G are bosonic, the bracket is symmetric, i.e. $\{\mathrm{F}, \mathrm{G}\} = \{\mathrm{G}, \mathrm{F}\}$. Otherwise, it is anti-symmetric, i.e $\{\mathrm{F}, \mathrm{G}\} = -\{\mathrm{G}, \mathrm{F}\}$. Furthermore, it satisfies a graded Jacobi-identity.

Let S be the field theory action, i.e. a functional of ghosts, ordinary fields, anti-fields and anti-ghosts. Then the BRST transformation is generated by the BRST charge

$$Q = \{S, \cdot\} = \sum_{\substack{\text{fields } \mathbf{\Phi}_a \\ \text{with } \mathrm{gh}(\mathbf{\Phi}_a)\geq 0}} \int \left[\frac{\delta_R \, \mathrm{S}}{\delta \Phi_a} \frac{\delta_L}{\delta \Phi^{*a}} - \frac{\delta_R \, \mathrm{S}}{\delta \Phi^{*a}} \frac{\delta_L}{\delta \Phi_a} \right] . \tag{2.2}$$

The BRST charge is nilpotent, i.e. the square of the BRST charge vanishes. This is an important property, which will be useful in the algebraic treatment of BV field theories. Since the action is invariant under the BRST transformation, it satisfies

$$0 = Q\,S = \{S, S\} = \sum_{\substack{\text{fields } \mathbf{\Phi}_a \\ \text{with } \text{gh}(\mathbf{\Phi}_a)\geq 0}} 2 \cdot \int \frac{\delta_R\,\mathrm{S}}{\delta\Phi_a}\,\frac{\delta_L\,\mathrm{S}}{\delta\Phi^{*a}} \ . \tag{2.3}$$

This is known as the BV master equation. It defines the notion of a BV action: a bosonic functional of ghost degree 0 is a BV action if it satisfies the master equation.

So far, we have only discussed the classical level. One can quantise the BV theory via standard path integral quantisation. The quantum theory is anomaly-free if the BV action satisfies the so-called quantum master equation

$$0 = \{S, S\} + 2i\hbar\Delta S \ . \tag{2.4}$$

Here, Δ is the so-called BV Laplacian

$$\Delta = \sum_{\substack{\text{fields } \mathbf{\Phi}_a \\ \text{with } \text{gh}(\mathbf{\Phi}_a)\geq 0}} \int \frac{\delta_R}{\delta\Phi_a}\,\frac{\delta_L}{\delta\Phi^{*a}} \ . \tag{2.5}$$

This concludes our recapitulation. We will not touch upon the issue of gauge fixing in BV formalism. However, we make one final remark about the nature of the unphysical ghosts and anti-fields. As mentioned at the beginning of this section, the presence of ghost fields, anti-fields and ghost anti-fields in the BV theory is an ingenious computational trick. However, unlike the ordinary fields, they must not appear in any physical quantities. In computations of physical quantities, these unphysical fields either cancel or are replaced by expressions in the ordinary fields through the process of gauge fixing.

3 Algebraic Framework

3.1 Differential Graded Field Space

In this section, we discuss in great detail the field space of a BV theory. A lot of the discussion is standard and is meant as an introduction to differential graded vector spaces. However, we also present very thoroughly how fermionic fields fit into the picture. For this, we establish less common notation which will be very useful later on in this thesis.

3.1.1 Field Space

In field theories in physics, a field is mathematically modelled by a function on a smooth manifold $\mathcal{M}$ i.e. it takes values in a suitable linear space $\mathcal{F}$ of real or complex functions on the manifold. For example, $\mathcal{F} = \mathcal{C}^\infty_{\mathbb{R}}(\mathcal{M})$, but other regularity assumptions may be imposed.

A field may have multiple components each being described a priori independently by a function in $\mathcal{F}$. The different components of a field may be related by symmetries – otherwise, we could just treat them as independent fields. To describe this situation mathematically, we define the *field space* $\mathcal{V}$ as the direct sum $\mathcal{F} \oplus \cdots \oplus \mathcal{F}$ with one copy of $\mathcal{F}$ for each field component. The symmetries are implemented on this field space through a suitable representation of the full symmetry group. In fact, the dimension of the specific representation defines the number of field components. Finally, we often deal with multiple independent fields $\mathbf{\Phi}, \mathbf{\Psi}, \ldots$, so that the full field space is defined to be the direct sum $\mathcal{V} = \mathcal{V}_\Phi \oplus \mathcal{V}_\Psi \oplus \ldots$ of the individual field spaces.

The field space is a linear function space and as such an infinite-dimensional space. Nevertheless, through the construction detailed above, it also displays a finite-dimensional structure induced by the finite number of fields and field components. Consider replacing the underlying infinite-dimensional function space $\mathcal{F}$ by the one-dimensional linear space of real numbers $\mathbb{R}$. This amounts to restricting the manifold $\mathcal{M}$ to a single point. In this field theory at a single point, the field space is obviously finite-dimensional and its dimension is just given by the number of all components of all fields. The dimensionality is induced by the construction of the field space through direct sums. Since this construction is independent of what we choose for $\mathcal{F}$, the field space carries an induced finite-dimensional structure even if $\mathcal{F}$ – and thus $\mathcal{V}$ – are infinite-dimensional.

To highlight this finite-dimensional structure and because we will later make heavy use of it, we introduce a finite-dimensional *pseudo-basis*. Assume our field theory describes fields $\mathbf{\Phi}_\alpha, \mathbf{\Psi}_\beta, \ldots$, where the indices $\alpha, \beta, \ldots$ label the various components of the fields. The field space is

$$\mathcal{V} = \mathcal{V}_\Phi \oplus \mathcal{V}_\Psi \oplus \cdots = (\mathcal{V}_{\Phi_1} \oplus \mathcal{V}_{\Phi_2} \oplus \ldots) \oplus (\mathcal{V}_{\Psi_1} \oplus \mathcal{V}_{\Psi_2} \oplus \ldots) \oplus \ldots \; . \tag{3.1}$$

By definition, the pseudo-basis is the set $\{\boldsymbol{e}_{\Phi,1}, \boldsymbol{e}_{\Phi,2}, \ldots, \boldsymbol{e}_{\Psi,1}, \boldsymbol{e}_{\Psi,2}, \ldots\}$ of artificial objects $\boldsymbol{e}$ indexed by field labels and component labels. We will treat them like basis vectors for the finite-dimensional structure of the field space in the following way. We define

$$\mathcal{V}_{\Phi_\alpha} := \mathcal{F} \cdot \boldsymbol{e}_{\Phi,\alpha} := \{ y_{\Phi,\alpha}\, \boldsymbol{e}_{\Phi,\alpha} \mid y_{\Phi,\alpha} \in \mathcal{F} \} \;, \tag{3.2}$$

and in the same way for the other field component spaces. In this notation, a field $\mathbf{\Phi}$ is described by the vector $y_\Phi\, \boldsymbol{e}_\Phi$. The function $y_\Phi \in \mathcal{F}$ describes the configuration of the field and the pseudo-basis vector $\boldsymbol{e}_\Phi$ the type of field. In general, an arbitrary field vector in $\mathcal{V}$ can be written in the form

$$\boldsymbol{y} = y_\Phi{}^\alpha \, \boldsymbol{e}_{\Phi,\alpha} + y_\Psi{}^\beta \, \boldsymbol{e}_{\Psi,\beta} + \dots \tag{3.3}$$

with functions $y_{\Phi,\alpha}, y_{\Psi,\beta}, \dots \in \mathcal{F}$. Here and in the following, we use the Einstein summation convention according to which repeated indices are summed over if one is denoted as subscript and the other as superscript. In principle, we could denote the functions $y_{\Phi,\alpha}, y_{\Psi,\beta}, \dots$ directly by the letters $\Phi_\alpha, \Psi_\beta, \dots$ that we use for labelling the field components. However, to avoid confusion, it is often instructive not to do so.

Two remarks are in order. Firstly, the reader may wonder, why we go to such lengths to detail the field space. We found that often in the literature, heavy use is made of the finite-dimensional structure of the field space without much explanation or justification. Since this will also be the practice in this thesis, we found it necessary to explain how an inherently infinite-dimensional field space displays a finite-dimensional structure. Secondly, the reader may wonder, why we work with a real (or complex) infinite-dimensional function space instead of defining the field space as a finite-dimensional module over the ring of functions $\mathcal{F}$. The reason is that we need differentiation to be a linear map on the field space and integration to be a linear function. Since both are $\mathbb{R}$-linear (or $\mathbb{C}$-linear) but not $\mathcal{F}$-linear, the field space cannot be an $\mathcal{F}$-linear module.

3.1.2 Grading

In a BV field theory, the fields are assigned a ghost degree as explained in section 2. Hence, we introduce a grading $\mathrm{gh} \in \mathbb{Z}$ on the field space which reflects the ghost degree. The field space can be decomposed into the direct sum $\mathcal{V} = \bigoplus_{\mathrm{gh}\in\mathbb{Z}} \mathcal{V}_{(\mathrm{gh})}$ of sub-spaces of fixed ghost degree. The sub-space $\mathcal{V}_{(\mathrm{gh})}$ is the direct sum of sub-spaces of fields with ghost degree gh. For $\mathrm{gh} \geq 2$ and $\mathrm{gh} \leq -3$, we set $\mathcal{V}_{(\mathrm{gh})} = \{0\}$.

Field vectors of distinct degree are called homogeneous. Any field vector can be written as a linear combination of homogeneous vectors. Thus, in many definitions and computations, we consider only homogeneous vectors and assume that the definition extends by linearity to the full field space.

The degree enters many computations through a sign $(-1)^{\mathrm{gh}}$, often abbreviated as $(-)^{\mathrm{gh}}$. Signs usually appear due to the Koszul sign rule discussed in section 3.2.1, or when we commute fields in algebraic expressions. In formulating field theories as homotopy Lie algebras, signs are a constant source of mistakes and confusion. Throughout this thesis, we aim to be as transparent with signs as possible. We will showcase computations in full detail to explain our sign conventions well and to point out any subtleties otherwise often hidden in the details.

We consider the grading to be associated with the pseudo-basis, i.e. $\mathrm{gh}(\boldsymbol{y}) = \mathrm{gh}(y_\Phi \, \boldsymbol{e}_\Phi) = \mathrm{gh}(\boldsymbol{e}_\Phi)$. The function $y_\Phi \in \mathcal{F}$ carries no degree and products of functions in $\mathcal{F}$ are commutative, just as usual, i.e. for $y_\Phi, y_\Psi \in \mathcal{F}$, we have $y_\Phi \cdot y_\Psi = y_\Psi \cdot y_\Phi$.

For quite some time, the right treatment of fermionic fields was a big puzzle for the author of this thesis. A solution to this problem is given by A. Zeitlin in [13] and [11]. In the following, we will explain Zeitlin's solution. First, we describe how Dirac fermions can be included in the field space. Later, we explain how anti-commuting functions can be introduced to describe fermionic fields.

Dirac fermions are ordinary fields and hence, have ghost degree $\mathrm{gh} = 0$. However, as fermions, they should anti-commute among each other and with the ghost fields. This is not reflected in their degree since so far, they contribute a sign $(-)^0 = +1$. Contrary to the expected anti-commuting behaviour, the sign is positive. To cure this defect, we introduce another grading $\varepsilon \in \{-1, 0, 1\}$ called *fermion degree*. Dirac fermion fields are assigned fermion degree $\varepsilon = 1$, Dirac conjugate fermion fields get $\varepsilon = -1$, and all other fields have $\varepsilon = 0$. The anti-field Φ^* of some field Φ has fermion degree $-\,\varepsilon(\Phi)$. The fermion degree is again associated with the pseudo-basis. Fermion degree and ghost degree enter signs jointly as $(-)^{|\cdot|}$, where we have defined $|\cdot| := \mathrm{gh}(\cdot) + \varepsilon(\cdot)$.

To give an example, the field space of QED, including a ghost field $\boldsymbol{c}$, gauge field $\boldsymbol{A}$, Dirac fermion $\boldsymbol{\psi}$ and Dirac conjugate fermion $\bar{\boldsymbol{\psi}}$, along with their anti-fields, can be decomposed as:

$$
\begin{array}{ccccccccc c}
 & & \mathcal{V}_{\bar{\psi}} & \oplus & \mathcal{V}_{\psi^*} & & & & -1 & \downarrow\varepsilon \\
 & & \oplus & & \oplus & & & & & \\
\mathcal{V}_c & \oplus & \mathcal{V}_A & \oplus & \mathcal{V}_{A^*} & \oplus & \mathcal{V}_{c^*} & & 0 & \\
 & & \oplus & & \oplus & & & & & \\
 & & \mathcal{V}_{\psi} & \oplus & \mathcal{V}_{\bar{\psi}^*} & & & & 1 & \\
 & & & & & & & & & \\
1 & & 0 & & -1 & & -2 & & & \leftarrow \mathrm{gh}
\end{array} .
$$

By now, the field space correctly incorporates all types of fields. However, so far, all fields are described by commuting functions in $\mathcal{F}$. Yet, fermionic fields should be described by anti-commuting functions. An anti-commuting function is a map from the manifold $\mathcal{M}$ to a Grassmann algebra – colloquially referred to as "anti-commuting numbers". A Grassmann algebra is an exterior tensor algebra over a real (or complex) linear space.

We want to treat both commuting and anti-commuting functions simultaneously. Thus, following Zeitlin [13], we introduce a generalised Grassmann algebra $\mathcal{A}$ as the symmetric tensor algebra over a $\mathbb{Z}$-graded linear space. For a discussion of symmetric and exterior tensor algebras, see section 3.2.1. The specifics of the underlying $\mathbb{Z}$-graded linear space are not relevant. We note only that elements $\xi^{(2n)}$ of even degree are commuting and elements $\xi^{(2n+1)}$ of odd degree are anti-commuting. This follows from the graded symmetry of the symmetric tensor product. Often we use the following shorthand notation:

$$\xi^{(n+m)} := \xi^{(n)}\xi^{(m)} = (-)^{n\cdot m}\,\xi^{(m)}\xi^{(n)} := (-)^{n\cdot m}\,\xi^{(m+n)}\ . \tag{3.4}$$

By restricting the underlying linear space to the sub-space of degree 1, one would regain the definition of an ordinary Grassmann algebra.

A linear space of commuting and anti-commuting functions is finally given by the tensor product $\mathcal{A}\otimes\mathcal{F}$. We define the product of functions $f^{(n)} := \xi^{(n)}\, f$ in $\mathcal{A}\otimes\mathcal{F}$ in the following way:

$$f_1^{(n)} \cdot f_2^{(m)} = (\xi_1^{(n)}\, f_1) \cdot (\xi_2^{(m)}\, f_2) := \xi_1^{(n+m)}\,(f_1 \cdot f_2) = (f_1 \cdot f_2)^{(n+m)}\ . \tag{3.5}$$

It is simple to check that

$$f_1^{(n)} \cdot f_2^{(m)} = (-)^{n\cdot m} f_2^{(m)} \cdot f_1^{(n)}\ , \tag{3.6}$$

as desired. We call the grading on $\mathcal{A}\otimes\mathcal{F}$ *Grassmann parity* λ and define $|\xi^{(n)}| := \lambda(\xi^{(n)}) = n$.

We are now in the position to define a field space of bosonic and fermionic fields which allows us to describe the former with commuting and the latter with anti-commuting functions. We will denote this space by $\mathcal{H}$ to differentiate it from the commuting-functions-only field space $\mathcal{V}$. $\mathcal{H}$ is built through direct sums of multiple copies of $\mathcal{A} \otimes \mathcal{F}$ in the same way that $\mathcal{V}$ was built from direct sums of multiple copies of $\mathcal{F}$. Equivalently, one may view $\mathcal{H}$ as the space $\mathcal{A} \otimes \mathcal{V}$. Thus, $\mathcal{V}$ and $\mathcal{H}$ share the finite-dimensional structure detailed in section 3.1.1 above. Despite the obvious short-coming of lacking anti-commuting functions, we will sometimes work with $\mathcal{V}$ if we are mainly interested in this finite-dimensional structure. We will do so because computations in $\mathcal{V}$ are often clearer since fewer degree-related signs appear.

A generic vector in $\mathcal{H}$ can be written in the form

$$\boldsymbol{y} = \sum_{\text{fields } \boldsymbol{\Phi}} \sum_{\text{parities } n_\Phi} y_\Phi^{(n_\Phi)} \boldsymbol{e}_\Phi \ , \tag{3.7}$$

with the outer sum running over all fields in the BV theory including anti-fields. We define the total degree

$$|y_\Phi^{(n_\Phi)} \boldsymbol{e}_\Phi| := |y_\Phi^{(n_\Phi)}| + |\boldsymbol{e}_\Phi| = \lambda\left(y_\Phi^{(n_\Phi)}\right) + \mathrm{gh}(\boldsymbol{e}_\Phi) + \varepsilon(\boldsymbol{e}_\Phi) \ . \tag{3.8}$$

If we want to compute physical results, we have to input physically meaningful vectors, i.e. the Grassmann parity of the function describing a field must match the expected commutation behaviour. We ensure that by working with vectors of total degree 0, such that each term $y_\Phi^{(n_\Phi)} \boldsymbol{e}_\Phi$ in a linear combination satisfies $\lambda\left(y_\Phi^{(n_\Phi)}\right) = -\big(\mathrm{gh}(\boldsymbol{e}_\Phi) + \varepsilon(\boldsymbol{e}_\Phi)\big)$.

To simplify notation, we will sometimes denote the function describing a field $\boldsymbol{\Phi}$ directly by the same letter Φ instead of the more cumbersome $y_\Phi^{(n_\Phi)}$. However, we will only do so if we work with physically meaningful vectors. We want to give an example: consider a ghost field $\boldsymbol{c}$, which is of ghost degree $\mathrm{gh}(\boldsymbol{c}) = 1$ and of fermion degree $\varepsilon(\boldsymbol{c}) = 0$. If we describe the field configuration of this ghost field by a function $c \in \mathcal{A} \otimes \mathcal{F}$, this function is understood to be of Grassmann parity $\lambda(c) = -1$, in particular, it is anti-commuting. In contrast, we may also work with a field configuration $y_c^{(n)} \in \mathcal{A} \otimes \mathcal{F}$, which is of arbitrary Grassmann parity n – although it is a priori not obvious how this should be interpreted physically. Hence, a physically meaningful vector is always of the form

$$\boldsymbol{y}^{\text{phys}} = \sum_{\text{fields } \boldsymbol{\Phi}} y_\Phi^{(-|\boldsymbol{e}_\Phi|)} \boldsymbol{e}_\Phi =: \sum_{\text{fields } \boldsymbol{\Phi}} \Phi \, \boldsymbol{e}_\Phi \ . \tag{3.9}$$

One may wonder why we allow arbitrary Grassmann parities at all, but we will see soon that this is necessary for the algebraic description of a BV field theory.

In the following, some definitions will depend on whether we consider a field space $\mathcal{V}$ with commuting functions only and with grading $\mathrm{gh} + \varepsilon$ or a field space $\mathcal{H} = \mathcal{A} \otimes \mathcal{V}$ with commuting and anti-commuting functions and with grading $\mathrm{gh} + \varepsilon + \lambda$. We will point out whenever the Grassmann parity λ has to be treated with extra care.

3.1.3 Differential

Consider two field spaces $\mathcal{V}$ and $\mathcal{V}'$, each with a decomposition into homogeneous sub-spaces. We call a collection $\{\mathrm{f}_{(i)}\}_{i \in \mathbb{Z}}$ of linear maps $\mathrm{f}_{(i)} : \mathcal{V}_{(i)} \to \mathcal{V}'_{(i+k)}$ a *homomorphism of degree k*.

The generalisation to $\mathcal{H}$ is straightforward but not trivial. We consider only homomorphisms of zero Grassmann parity. Let $\mathrm{f}_\mathcal{V} : \mathcal{V} \to \mathcal{V}'$ be a homomorphism of degree $|\mathrm{f}_\mathcal{V}|$. The corresponding homomorphism on $\mathcal{H}$ is defined by $\mathrm{f}_\mathcal{H} = \mathbb{1}_\mathcal{A} \otimes \mathrm{f}_\mathcal{V}$. When we apply this map on a vector in $\mathcal{H}$, we have to move the Grassmann element through the map $\mathrm{f}_\mathcal{V}$ and thereby, pick up a sign. Thus,

$$\mathrm{f}_\mathcal{H}\left(y_\Phi^{(n)} \boldsymbol{e}_\Phi\right) := (-)^{n\,|\mathrm{f}_\mathcal{V}|} \cdot \xi^{(n)} \ \mathrm{f}_\mathcal{V}\left(y_\Phi \boldsymbol{e}_\Phi\right) \ . \tag{3.10}$$

Note that a homomorphism of non-zero degree maps a physically meaningful vector with total degree zero to a vector with non-zero total degree, i.e. a vector which is not physically meaningful. This is the reason, why we allow for arbitrary Grassmann parities on $\mathcal{H}$.

Next, we define a differential on the field space. This is a particularly important homomorphism and makes the first step in setting up the algebraic structure for a BV field theory. A differential is a degree -1 homomorphism $\mathrm{d} : \mathcal{H} \to \mathcal{H}$ such that $\mathrm{d}^2 = 0$, i.e. $\mathrm{d}_{(i-1)} \circ \mathrm{d}_{(i)} = 0$ holds for all $i \in \mathbb{Z}$. In field theory applications, the differential will always only affect the ghost degree and leave fermion degree and Grassmann parity invariant. Given a differential $d_{\mathcal{V}}$ on $\mathcal{V}$, the corresponding map $d_{\mathcal{H}}$ on $\mathcal{H}$, defined through (3.10), is a differential on $\mathcal{H}$. A graded vector space with a differential is called *differential graded vector space*, abbreviated dgv.

A dgv constitutes a *chain complex*, i.e. an indexed family of vector spaces with a linear map that maps from the space indexed i to the space indexed $i-1$ and that squares to zero. Two different conventions for the degree of the differential exist. We can replace the ghost degree gh on the field space by $-\,\mathrm{gh}$ to obtain a differential of degree $+1$. A lot of the literature, especially on algebraic structures in string theory, follows this "inverted" degree-convention (for example, cf. [14]). A complex with a degree $+1$ differential is referred to as *cochain complex*, and most of the related terminology is prefixed with "co". It is conventional to denote the degree of constituents of a chain complex with subscripts, as we are doing for the ghost degree and the fermion degree. For a cochain complex, degrees are denoted as superscripts, as we are doing for the Grassmann parity, which in a sense is inverse to ghost degree and fermion degree.

The differential d on the field space will encode the gauge symmetry, the free equations of motion excluding interactions and the Noether identities in the following way [1]:

$$\mathrm{d}: \quad \boxed{\begin{matrix}\text{ghost}\\ \text{fields}\end{matrix}} \xrightarrow[\text{trafos}]{\text{gauge}} \boxed{\begin{matrix}\text{ordinary}\\ \text{fields}\end{matrix}} \xrightarrow[\text{of motion}]{\text{equations}} \boxed{\begin{matrix}\text{anti-}\\ \text{fields}\end{matrix}} \xrightarrow[\text{identities}]{\text{Noether}} \boxed{\begin{matrix}\text{ghost}\\ \text{anti-fields}\end{matrix}} \;. \tag{3.11}$$

In a BV field theory, this information is contained in the linearisation of the BRST charge $Q_1 = \{S_2, \cdot\}$. Here, $\{\cdot,\cdot\}$ denotes the BV bracket, and the subscripts on the charge Q and on the action S denote the polynomial order in the fields of the respective terms considered. We present in section 4.1 how the differential can be derived from the BRST charge.

As mentioned above, cf. equation (3.11), the differential is set up to encode gauge transformations and the equations of motion. Consider a physically meaningful vector $c\,\boldsymbol{e}_c$ describing a ghost field in $\mathcal{H}_{(\mathrm{gh}=1)}$. Then, a pure gauge is given by $\mathrm{d}(c\,\boldsymbol{e}_c) \in \mathcal{H}_{(\mathrm{gh}=0)}$. In the terminology of chain complexes, a vector that lies in the image im(d) of the differential is called *exact*. Thus, gauge transformations are exact field vectors of ghost degree $\mathrm{gh} = 0$.

Gauge transformations express a redundancy in the description of a field theory. Two field configurations are physically equivalent if they are related by a gauge transformation. Hence, two vectors $A_1^\mu \boldsymbol{e}_{A,\mu}$ and $A_2^\mu \boldsymbol{e}_{A,\mu}$ in $\mathcal{H}_{(\mathrm{gh}=0)}$ are equivalent from a physical point of view if they only differ by the addition of an exact part $\mathrm{d}(c\,\boldsymbol{e}_c)$.

A gauge field, represented by a field vector $A^\mu \boldsymbol{e}_{A,\mu} \in \mathcal{H}_{(\mathrm{gh}=0)}$, satisfies the free equations of motion if $\mathrm{d}(A^\mu \boldsymbol{e}_{A,\mu})$ vanishes in $\mathcal{H}_{(\mathrm{gh}=-1)}$. A vector that lies in the kernel ker(d) of the differential is referred to as *closed*. Closed vectors of ghost degree $\mathrm{gh} = 0$ describe fields that satisfy the free equations of motion, also called "on-shell" fields. Due to the nilpotency of d, exact vectors are automatically closed, while the opposite is usually not true. We can consider the sub-space of closed vectors in $\mathcal{H}$ and treat two closed vectors as equivalent if they only differ by an exact part.

Mathematically, this sub-space is the quotient space

$$\mathcal{H}om(\mathrm{d}) = \frac{\ker(\mathrm{d})}{\mathrm{im}(\mathrm{d})} \ , \tag{3.12}$$

called the *homology* of d in $\mathcal{H}$. It inherits the grading of $\mathcal{H}$, namely the ghost degree, the fermion degree and the Grassmann parity. The homology $\mathcal{H}om_{(\mathrm{gh}=0)}(\mathrm{d})$ in ghost degree zero is the sub-space of physical fields. It contains all ordinary fields up to gauge transformations that satisfy the free equations of motion.

In physics, we are almost exclusively interested in the sub-space of physical fields. The remaining parts of the field space, including "off-shell" fields (as opposed to on-shell fields), ghosts and anti-fields are deemed unphysical auxiliary tools for computations. However, any measurable quantity must not depend on these unphysical fields. Hence, the homology plays an important role in the algebraic formulation of field theories.

3.1.4 (Quasi-)Isomorphisms

At the beginning of section 3.1.3, we introduced homomorphisms of graded vector spaces as linear maps that are compatible with the grading. After having defined a differential on the field space, we want to discuss what kind of maps are compatible with the differential. Consider two dgv $(\mathcal{H}, \mathrm{d})$ and $(\mathcal{H}', \mathrm{d}')$. Then, a *chain map* is a degree 0 homomorphism $\mathrm{f} : \mathcal{H} \to \mathcal{H}'$ that satisfies

$$\mathrm{f} \circ \mathrm{d} = \mathrm{d}' \circ \mathrm{f} \ . \tag{3.13}$$

We call f an isomorphism if it is bijective and its inverse f^{-1} is again a homomorphism of graded vector spaces. A quick computation shows that f^{-1} is also a chain map. Two dgv are *isomorphic* if an isomorphism exists between them.

As mentioned at the end of section 3.1.3, we are only interested in the sub-space of physical fields, i.e. in the homology. We consider two physical theories equivalent if their homologies are isomorphic but not necessarily the full field spaces. In this case, we say that the field spaces are *quasi-isomorphic*. Two dgv $(\mathcal{H}, \mathrm{d})$ and $(\mathcal{H}', \mathrm{d}')$ are quasi-isomorphic if a chain map $\mathrm{f} : \mathcal{H} \to \mathcal{H}'$ exists whose restriction to homologies $\mathrm{f}\,|_{\mathcal{H}om(\mathrm{d})} : \mathcal{H}om(\mathrm{d}) \to \mathcal{H}om(\mathrm{d}')$ is bijective and whose inverse on homologies $\mathrm{f}^{-1}\,|_{\mathcal{H}om(\mathrm{d}')}$ extends to a chain map on the full space. Unsurprisingly, such a chain map is called *quasi-isomorphism.*

Two chain maps $\mathrm{f}, \mathrm{g} : \mathcal{H} \to \mathcal{H}'$ are called *chain homotopic* if a degree +1 homomorphism $\mathrm{h} : \mathcal{H} \to \mathcal{H}'$ exists, such that

$$\mathrm{h} \circ \mathrm{d} + \mathrm{d}' \circ \mathrm{h} = \mathrm{f} - \mathrm{g} \ . \tag{3.14}$$

The homomorphism h is called *chain homotopy.* A good explanation of how the notion of chain homotopy is related to the more common homotopies in topology is given in [15]. Chain homotopies play an important role in the algebraic treatment of field theories, as will become clear later in this thesis.

A chain map $\mathrm{p} : \mathcal{H} \to \mathcal{H}'$ is a quasi-isomorphism of dgv if and only if another chain map $\mathrm{i} : \mathcal{H}' \to \mathcal{H}$ exists such that $\mathrm{i} \circ \mathrm{p}$ is chain homotopic to $\mathbb{1}_{\mathcal{H}}$ [16]. Then, i is automatically a quasi-isomorphism, too, and p and i are inverse on homology. Let $\mathrm{h} : \mathcal{H} \to \mathcal{H}$ be the defining homotopy, then we can summarise this situation diagrammatically as

$$\mathrm{h} \circlearrowright (\mathcal{H}, \mathrm{d}) \underset{\mathrm{i}}{\overset{\mathrm{p}}{\rightleftarrows}} (\mathcal{H}', \mathrm{d}') \ , \tag{3.15}$$

with

$$\mathrm{h} \circ \mathrm{d} + \mathrm{d} \circ \mathrm{h} = \mathbb{1}_{\mathcal{H}} - \mathrm{i} \circ \mathrm{p} \ . \tag{3.16}$$

This is called a *homotopy retract*.

3.1.5 Lie Algebras

We mentioned before that the field space is endowed with a representation of the full symmetry group of the field theory. If the field theory has a gauge symmetry associated with some Lie algebra, the Lie bracket of this algebra is carried over by the symmetry representation to the field space. We demand that the differential d and the Lie bracket $[\cdot,\cdot]$ are compatible, i.e. they satisfy

$$\mathrm{d}\,[\boldsymbol{y}_1, \boldsymbol{y}_2] + [\mathrm{d}\,\boldsymbol{y}_1, \boldsymbol{y}_2] + (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|}\,[\mathrm{d}\,\boldsymbol{y}_2, \boldsymbol{y}_1] = 0 \tag{3.17}$$

for all (homogeneous) vectors $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{H}$. This requirement ensures that the Lie bracket also descends to the homology, i.e. the Lie bracket of two closed vectors is again closed and does not depend on exact parts. A dgv with a compatible Lie bracket is called *differential graded Lie algebra*.

Here, a remark on grading, symmetry and signs is necessary. The compatibility requirement as stated above is given for a graded-symmetric degree -1 Lie bracket, i.e.

$$[\boldsymbol{y}_1, \boldsymbol{y}_2] = (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|}\,[\boldsymbol{y}_2, \boldsymbol{y}_1] \ . \tag{3.18}$$

In particular, the Lie bracket is symmetric for two ordinary degree 0 fields. The Lie bracket satisfies the graded Jacobi identity

$$\Big[[\boldsymbol{y}_1, \boldsymbol{y}_2], \boldsymbol{y}_3\Big] + (-)^{|\boldsymbol{y}_2||\boldsymbol{y}_3|}\Big[[\boldsymbol{y}_1, \boldsymbol{y}_3], \boldsymbol{y}_2\Big] + (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|+|\boldsymbol{y}_1||\boldsymbol{y}_3|}\Big[[\boldsymbol{y}_2, \boldsymbol{y}_3], \boldsymbol{y}_1\Big] = 0 \ . \tag{3.19}$$

For physicists, these sign conventions are unfamiliar. We have chosen them here in this way since the discussion of the Lie algebra structure prepares the definition of the homotopy Lie algebra structure, which is at the core of this thesis. We will discuss in the following section 3.2.3 that by a so-called degree-shift on the field space, one can transform the Lie bracket into a graded-anti-symmetric degree 0 mapping with more conventional signs in the Jacobi identity and the compatibility condition. Hohm and Zwiebach extensively discuss these sign conventions in the context of homotopy Lie algebras in [17].

We do not go into detail about how the Lie bracket $[\cdot,\cdot]$ of a field theory is implemented on the corresponding field space. Instead, we want to discuss whether a given Lie bracket on some field space can be transferred to another field space through a homotopy retract. Since two field spaces related by a homotopy retract are quasi-isomorphic and thus physically equivalent, one would expect such a transfer to be possible. A similar discussion on the transfer of an associative algebra was presented in [18].

Consider two field spaces $(\mathcal{H}, \mathrm{d})$ and $(\mathcal{H}', \mathrm{d}')$ related by a homotopy retract with quasi-isomorphisms $\mathrm{p} : \mathcal{H} \to \mathcal{H}'$ and $\mathrm{i} : \mathcal{H}' \to \mathcal{H}$ and homotopy $\mathrm{h} : \mathcal{H} \to \mathcal{H}$. Let $[\cdot,\cdot]$ be a compatible degree -1 Lie bracket on $(\mathcal{H}, \mathrm{d})$. Since the restrictions of p and i to homology are isomorphisms, we can easily define a new Lie bracket $[\cdot,\cdot]'$ on the homology $\mathcal{H}om(\mathrm{d}')$ through

$$[\cdot,\cdot]' := \mathrm{p} \circ [\cdot,\cdot] \circ (\mathrm{i} \otimes \mathrm{i}) \ . \tag{3.20}$$

We map the inputs to the original homology $\mathcal{H}om(\mathrm{d})$, where we compute the original Lie bracket $[\cdot,\cdot]$, and then map the output back to $\mathcal{H}om(\mathrm{d}')$. The new Lie bracket satisfies the Jacobi identity (3.19) as required.

Next, we try to lift this new Lie bracket from the homology $\mathcal{H}om(\mathrm{d}')$ to the full space $\mathcal{H}'$. If we do not assume p and i to be isomorphisms on the full spaces, we must carefully check whether the new bracket still satisfies the Jacobi identity. However, a quick computation shows that this is not the case in general. Let

$$\mathrm{J}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3) := \Big[[\boldsymbol{y}_1,\boldsymbol{y}_2]',\boldsymbol{y}_3\Big]' + (-)^{|\boldsymbol{y}_2||\boldsymbol{y}_3|}\Big[[\boldsymbol{y}_1,\boldsymbol{y}_3]',\boldsymbol{y}_2\Big]' \tag{3.21}$$

$$+ (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|+|\boldsymbol{y}_1||\boldsymbol{y}_3|}\Big[[\boldsymbol{y}_2,\boldsymbol{y}_3]',\boldsymbol{y}_1\Big]' \tag{3.22}$$

be the so-called "Jacobiator". Then, defining on $\mathcal{H}'$ a new graded-symmetric degree -1 bracket $[\cdot,\cdot,\cdot]'$ with three inputs, we can express the Jacobiator as

$$\begin{aligned} -\mathrm{J}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3) =& \,\mathrm{d}'\,[\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3]' + [\mathrm{d}'\,\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3]' + (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|}[\mathrm{d}'\,\boldsymbol{y}_2,\boldsymbol{y}_1,\boldsymbol{y}_3]' \\ &+ (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_3|+|\boldsymbol{y}_2||\boldsymbol{y}_3|}[\mathrm{d}'\,\boldsymbol{y}_3,\boldsymbol{y}_1,\boldsymbol{y}_2]' \;. \end{aligned} \tag{3.23}$$

Schematically, this has the form

$$[\cdot,\cdot,\cdot]'\circ\mathrm{d}' + \mathrm{d}'\circ[\cdot,\cdot,\cdot]' = 0 - \mathrm{J}\;, \tag{3.24}$$

i.e. the three-inputs bracket takes the role of a higher homotopy and the Jacobiator is homotopic to 0. The Jacobiator is said to vanish only "up to homotopy".

The three-inputs bracket is schematically given by

$$[\cdot,\cdot,\cdot]' = \mathrm{p}\circ[\cdot,\cdot]\circ\Big\{\big(\mathrm{h}\circ[\cdot,\cdot]\circ(\mathrm{i}\otimes\mathrm{i})\big)\otimes\mathrm{i}\Big\}\;. \tag{3.25}$$

Especially in the mathematical literature on the topic, it is common to stay at the schematic level. Since in this thesis, we are mainly interested in computations in the algebraic framework, we always try to give very explicit results for computations. We hope that this level of detail can be useful to the reader. The explicit form for the three-inputs bracket is

$$\begin{aligned} [\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3]' =& \,\mathrm{p}\left(\Big[\mathrm{h}\Big(\big[\,\mathrm{i}(\boldsymbol{y}_1),\mathrm{i}(\boldsymbol{y}_2)\big]\Big),\ \mathrm{i}(\boldsymbol{y}_3)\Big]\right) \\ &+ (-)^{|\boldsymbol{y}_2||\boldsymbol{y}_3|}\ \mathrm{p}\left(\Big[\mathrm{h}\Big(\big[\,\mathrm{i}(\boldsymbol{y}_1),\mathrm{i}(\boldsymbol{y}_3)\big]\Big),\ \mathrm{i}(\boldsymbol{y}_2)\Big]\right) \\ &+ (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|+|\boldsymbol{y}_1||\boldsymbol{y}_3|}\ \mathrm{p}\left(\Big[\mathrm{h}\Big(\big[\,\mathrm{i}(\boldsymbol{y}_2),\mathrm{i}(\boldsymbol{y}_3)\big]\Big),\ \mathrm{i}(\boldsymbol{y}_1)\Big]\right)\;. \end{aligned} \tag{3.26}$$

In summary, we found that one cannot simply transfer the Lie bracket from some field space to another physically equivalent field space. The transferred structure is not a Lie bracket anymore since it does not fully satisfy the Jacobi identity. However, we will discuss in section 3.3 that the transferred structure manifests a homotopy Lie algebra. It consists of a two-inputs bracket defined in (3.20), a three-inputs bracket defined in (3.25) and an infinite number of higher brackets for arbitrary number of inputs. The brackets all satisfy Jacobi-like identities that hold up to higher homotopies, which are described by higher brackets.

A Lie algebra is just a special form of a homotopy Lie algebra with the Lie bracket as the two-inputs bracket and all higher brackets equal to zero. The fact that one can transfer a homotopy Lie algebra from one field space to a physically equivalent other one is called the homotopy transfer theorem (cf. [16]). The powerful homotopy transfer theorem is one of the main motivations to formulate field theories as homotopy Lie algebras.

3.2 Tensor (Co)algebra

We now introduce all the mathematical tools necessary for rigorously defining homotopy Lie algebras. At least four equivalent definitions of a homotopy Lie algebra exist in mathematics (see [19] for all four of them). In this thesis, we will follow the so-called bar construction, which offers the most useful access for physicists. From this point of view, a homotopy Lie algebra is a co-derivative on the symmetric tensor co-algebra on the field space. The definition will be given in more detail in the subsequent section 3.3 together with a discussion of how it relates to the infinite brackets mentioned above.

3.2.1 Tensor Algebra

The *tensor product space* of two graded linear spaces $\mathcal{H}$ and $\mathcal{H}'$ is the graded linear space $\mathcal{H}\otimes\mathcal{H}'$ with homogeneous sub-spaces

$$(\mathcal{H}\otimes\mathcal{H}')_{(m)} = \bigoplus_{k+l=m} \mathcal{H}_{(k)} \otimes \mathcal{H}'_{(l)} \ . \tag{3.27}$$

It is defined through choosing a tensor product, which is a bilinear map

$$\begin{aligned} \otimes : \mathcal{H} \times \mathcal{H}' &\to \mathcal{H}\otimes\mathcal{H}' \\ (\boldsymbol{y}, \boldsymbol{y}') &\mapsto \boldsymbol{y}\otimes\boldsymbol{y}' \end{aligned} \tag{3.28}$$

from the Cartesian product into the tensor product space. The tensor product spaces corresponding to different choices of tensor products are isomorphic [20]. Hence, we do not have to pick a specific choice.

In general, a tensor product is not commutative, since the vectors $\boldsymbol{y}\otimes\boldsymbol{y}' \in \mathcal{H}\otimes\mathcal{H}'$ and $\boldsymbol{y}'\otimes\boldsymbol{y} \in \mathcal{H}'\otimes\mathcal{H}$ live in different spaces. Nevertheless, a natural isomorphism exists between these two spaces:

$$\tau_{\mathcal{H},\mathcal{H}'} : \boldsymbol{y}\otimes\boldsymbol{y}' \mapsto (-)^{|\boldsymbol{y}||\boldsymbol{y}'|}\, \boldsymbol{y}'\otimes\boldsymbol{y} \ . \tag{3.29}$$

It is referred to as *switching map.*

Let f be a degree k homomorphism on $\mathcal{H}$ and g be a degree l homomorphism on $\mathcal{H}'$. Then, the degree $k+l$ homomorphism $\mathrm{f}\otimes\mathrm{g}$ on $\mathcal{H}\otimes\mathcal{H}'$ is defined as

$$\mathrm{f}\otimes\mathrm{g} : \boldsymbol{y}\otimes\boldsymbol{y}' \mapsto (-)^{|\mathrm{g}||\boldsymbol{y}|}\, \mathrm{f}(\boldsymbol{y})\otimes\mathrm{g}(\boldsymbol{y}') \ . \tag{3.30}$$

One may think of the sign in this definition as originating from "pulling" the vector $\boldsymbol{y}$ through the map g. This sign convention is usually referred to as *Koszul sign rule* and is used both in mathematics and physics when one works with graded linear spaces [21]. We have already used this rule in section 3.1.3, when we introduced homomorphisms on $\mathcal{H} = \mathcal{A}\otimes\mathcal{V}$.

For $s\in\mathbb{N}$, we define the *tensor power* $\mathcal{H}^{\otimes s}$ of a graded linear space $\mathcal{H}$ as the tensor product space of s copies of $\mathcal{H}$. Furthermore, if $\mathbb{K}$ is the ground field of the linear space (in our case $\mathbb{K}$

is either $\mathbb{R}$ or $\mathbb{C}$), let $\mathcal{H}^{\otimes 0} := \mathbb{K}$. Then, the free tensor algebra over the graded linear space $\mathcal{H}$ is the space $T(\mathcal{H}) := \bigoplus_{s\geq 0} \mathcal{H}^{\otimes s}$ with the tensor product as algebraic structure.

$T(\mathcal{H})$ consists of linear combinations of finite tensor products $\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s$ of vectors in $\mathcal{H}$. We define an action of the permutation group $\mathcal{S}_s = \{\sigma \mid \sigma \text{ permutation of } 1, \dots, s\}$ on the s'th tensor power of $\mathcal{H}$:

$$\begin{aligned} \sigma :\ \mathcal{H}^{\otimes s} &\to \mathcal{H}^{\otimes s} \\ \boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s &\mapsto \boldsymbol{y}_{\sigma(1)} \otimes \cdots \otimes \boldsymbol{y}_{\sigma(s)} \ , \end{aligned} \tag{3.31}$$

where vectors in the tensor product are permuted using the switching map (3.29). For a given permutation and a specific tensor product, the *Koszul sign* $\epsilon(\sigma; \boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s)$ is the sign obtained from the permutation in equation (3.31). Often, we will simply write $\epsilon(\sigma)$. However, one should keep in mind that the Koszul sign depends not only on the permutation but also on the specific vectors that are permuted. The Koszul sign $\epsilon(\sigma)$ should not be confused with the fermion degree ε, which is also denoted by the Greek letter epsilon.

One can symmetrise or anti-symmetrise the tensor product to get to the *symmetric tensor algebra* $S(\mathcal{H})$ or the *exterior tensor algebra* $\Lambda(\mathcal{H})$, respectively. The symmetric tensor algebra is the direct sum $S(\mathcal{H}) = \bigoplus_{s\geq 0} \mathcal{H}^{\odot s}$ of finite symmetric tensor powers $\mathcal{H}^{\odot s}$ and has as algebraic structure the symmetric tensor product

$$\begin{aligned} \odot : \mathcal{H}^{\odot s} \times \mathcal{H}^{\odot t} &\to \mathcal{H}^{\odot s+t} \\ (\,\boldsymbol{y}_1 \odot \cdots \odot \boldsymbol{y}_s,\, \boldsymbol{y}_{s+1} \odot \cdots \odot \boldsymbol{y}_{s+t}\,) &\mapsto \sum_{\sigma\in\mathcal{S}_{s+t}} \epsilon(\sigma)\ \boldsymbol{y}_{\sigma(1)} \otimes \cdots \otimes \boldsymbol{y}_{\sigma(s+t)} \ . \end{aligned} \tag{3.32}$$

The symmetric tensor product satisfies

$$\boldsymbol{y}_{\sigma(1)} \odot \cdots \odot \boldsymbol{y}_{\sigma(s)} = \epsilon(\sigma)\ \boldsymbol{y}_1 \odot \cdots \odot \boldsymbol{y}_s \ . \tag{3.33}$$

The definition of the exterior tensor algebra is analogous, only the exterior tensor product $\wedge$ is graded-anti-symmetric:

$$\boldsymbol{y}_{\sigma(1)} \wedge \cdots \wedge \boldsymbol{y}_{\sigma(s)} = \mathrm{sgn}(\sigma)\ \epsilon(\sigma)\ \boldsymbol{y}_1 \wedge \cdots \wedge \boldsymbol{y}_s \ , \tag{3.34}$$

with $\mathrm{sgn}(\sigma)$ positive for an even permutation and negative for an odd one.

3.2.2 Multilinear Maps

By the so-called universal property of tensor products, any s-multilinear map on $\mathcal{H}$ can be represented by a linear map on the tensor power $\mathcal{H}^{\otimes s}$ [20]. Symmetric maps can be represented by linear maps on $\mathcal{H}^{\odot s}$, and anti-symmetric maps by linear maps on $\mathcal{H}^{\wedge s}$.

As with linear homomorphisms on the field space, we have to take extra care of the Grassmann parity. Let $\mathrm{f}_{\mathcal{V}} : \mathcal{V}^{\otimes s} \to \mathcal{V}$ be an s-multilinear map on a field space of commuting functions. The corresponding multilinear map on the field space $\mathcal{H}$ of commuting and anti-commuting functions is given by $\mathrm{f}_{\mathcal{H}} = \bigotimes_{\mathcal{A}} \otimes \mathrm{f}_{\mathcal{V}}$. Here $\bigotimes_{\mathcal{A}}$ describes the product of Grassmann elements $\xi_1^{(n_1)}, \dots, \xi_s^{(n_s)}$. Explicitly, we have

$$\mathrm{f}_{\mathcal{H}}\left(y_1^{(n_1)}\boldsymbol{e}_1, \dots, y_s^{(n_s)}\boldsymbol{e}_s\right) = (-)^{\sum_i n_i|\mathrm{f}_{\mathcal{V}}| + \sum_{i>j} n_i|\boldsymbol{e}_j|}\ \xi^{(\sum_i n_i)}\, \mathrm{f}_{\mathcal{V}}\left(y_1\boldsymbol{e}_1, \dots, y_s\boldsymbol{e}_s\right) \ . \tag{3.35}$$

The sign factor in front is due to the Koszul sign rule (3.30). We have to move all $\xi_1, \dots, \xi_s$ through the pseudo-basis vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_s$ in front of them and through the map $\mathrm{f}_{\mathcal{V}}$. Keeping this in mind is important in computations.

Assume $\mathrm{f}_{\mathcal{V}}$ is graded-(anti-)symmetric with respect to the grading $\mathrm{gh}+\varepsilon$ on $\mathcal{V}$. Then $\mathrm{f}_{\mathcal{H}}$ is graded-(anti-)symmetric with respect to the total grading $\mathrm{gh}+\varepsilon+\lambda$ on $\mathcal{H}$. The proof is given in appendix A.

We give an example to clarify the above statement: Let $\mathrm{f}_{\mathcal{V}}$ be a degree $+1$ graded-symmetric bilinear map on a field space $\mathcal{V}$ of commuting functions. Assume the field space contains only a gauge field $\boldsymbol{A}_\mu$ of ghost degree 0 and its anti-field $\boldsymbol{A}^{*\nu}$ of ghost degree -1. Both fields are of zero fermion degree. The map on $\mathcal{V}$ be explicitly given by

$$\mathrm{f}_{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2) = \big(y_{1A,\mu}\, y_{2A^*}{}^{\mu} \;+\; y_{2A,\mu}\, y_{1A^*}{}^{\mu}\big)\,\boldsymbol{e}_A \ . \tag{3.36}$$

Then, on $\mathcal{H}$, the corresponding map is given by

$$\begin{aligned}\mathrm{f}_{\mathcal{H}}(\boldsymbol{y}_1,\boldsymbol{y}_2) &= \Big(\quad (-)^{n_{1A}|\mathrm{f}_{\mathcal{V}}|+n_{2A^*}(|\mathrm{f}_{\mathcal{V}}|+|\boldsymbol{e}_A|)}\,\xi^{(n_{1A}+n_{2A^*})}\,y_{1A,\mu}\,y_{2A^*}{}^{\mu}\\ &\qquad + (-)^{n_{1A^*}|\mathrm{f}_{\mathcal{V}}|+n_{2A}(|\mathrm{f}_{\mathcal{V}}|+|\boldsymbol{e}_{A^*}|)}\,\xi^{(n_{1A^*}+n_{2A})}\,y_{2A,\mu}\,y_{1A^*}{}^{\mu}\Big)\,\boldsymbol{e}_A\\ &= \Big(\quad (-)^{n_{1A}+n_{2A^*}}\,\xi^{(n_{1A}+n_{2A^*})}\,y_{1A,\mu}\,y_{2A^*}{}^{\mu}\\ &\qquad + (-)^{n_{1A^*}+n_{1A^*}n_{2A}}\,\xi^{(n_{2A}+n_{1A^*})}\,y_{2A,\mu}\,y_{1A^*}{}^{\mu}\Big)\,\boldsymbol{e}_A \ .\end{aligned}$$

In the second step, we have plugged in the degrees $|\mathrm{f}_{\mathcal{V}}|$, $|\boldsymbol{e}_A|$ and $|\boldsymbol{e}_{A^*}|$ and commuted the product of Grassmann elements in the second term. If we restrict $\boldsymbol{y}_1,\boldsymbol{y}_2$ to be physically meaningful field vectors, we find

$$\begin{aligned}\mathrm{f}_{\mathcal{H}}\left(\boldsymbol{y}_1^{\mathrm{phys}},\,\boldsymbol{y}_2^{\mathrm{phys}}\right) &= \xi^{(-|\boldsymbol{e}_A|-|\boldsymbol{e}_{A^*}|)}\Big(\quad (-)^{-|\boldsymbol{e}_A|-|\boldsymbol{e}_{A^*}|}\,y_{1A,\mu}\,y_{2A^*}{}^{\mu}\\ &\qquad\qquad + (-)^{-|\boldsymbol{e}_{A^*}|+|\boldsymbol{e}_A||\boldsymbol{e}_{A^*}|}\,y_{2A,\mu}\,y_{1A^*}{}^{\mu}\Big)\,\boldsymbol{e}_A\\ &= -\,\xi^{(-|\boldsymbol{e}_A|-|\boldsymbol{e}_{A^*}|)}\Big(y_{1A,\mu}\,y_{2A^*}{}^{\mu} + y_{2A,\mu}\,y_{1A^*}{}^{\mu}\Big)\,\boldsymbol{e}_A\\ &= -\Big(A_{1\mu}\,A_2^{*\mu} + A_{2\mu}\,A_1^{*\mu}\Big)\,\boldsymbol{e}_A\end{aligned}$$

In the last step, we have pulled the Grassmann element $\xi^{(-|\boldsymbol{e}_A|-|\boldsymbol{e}_{A^*}|)}$ back into the parenthesis and split it into the product $\xi^{(-|\boldsymbol{e}_A|)}\,\xi^{(-|\boldsymbol{e}_{A^*}|)}$. Finally, we denoted the commuting function $\xi^{(-|\boldsymbol{e}_A|)}\,y_{A,\mu}$ by A_μ and the anti-commuting function $\xi^{(-|\boldsymbol{e}_{A^*}|)}\,y_{A^*}{}^{\mu}$ by $A^*{}_{\mu}$.

Somewhat surprisingly, the map acquires an overall sign when we evaluate it on physically meaningful field vectors as opposed to field vectors with commuting functions only. This is an example of the tedious subtleties hidden in the Grassmann parity. We will later refer to this as the "surprising sign".

We can generalise this observation. Given a multilinear map $\mathrm{f}_{\mathcal{V}}$ on $\mathcal{V}$ of degree $|\mathrm{f}_{\mathcal{V}}|$ and of zero Grassmann parity, the corresponding multilinear map $\mathrm{f}_{\mathcal{H}}$ on $\mathcal{H}$ takes the following form when evaluated on physically meaningful field vectors:

$$\mathrm{f}_{\mathcal{H}}\left(y_1^{(-|\boldsymbol{e}_1|)}\boldsymbol{e}_1,\ldots,y_s^{(-|\boldsymbol{e}_s|)}\boldsymbol{e}_s\right) = (-)^{\mathrm{phys}}\;\xi^{(-\sum_i|\boldsymbol{e}_i|)}\,\mathrm{f}_{\mathcal{V}}\left(y_1\boldsymbol{e}_1,\ldots,y_s\boldsymbol{e}_s\right) \ . \tag{3.37}$$

The "surprising sign" is the factor

$$(-)^{\mathrm{phys}} := (-)^{\sum_i|\boldsymbol{e}_i||\mathrm{f}_{\mathcal{V}}| + \sum_{i>j}|\boldsymbol{e}_i||\boldsymbol{e}_j|} \ . \tag{3.38}$$

It is independent of permutations of the inputs. Of course, the "surprising sign" is not at all surprising but follows straightforwardly from equation (3.35). However, it comes as a surprise if one does not treat fermionic field configurations properly. Note that $\mathrm{f}_{\mathcal{H}}$ is symmetric on physically meaningful inputs since it is graded-symmetric in general and physically meaningful inputs have total degree zero.

3.2.3 Degree Shift

In this section, we discuss the degree-shift, which was mentioned before in section 3.1.5. The tensor space $T(\mathcal{H})$ of a field space $\mathcal{H}$ inherits the grading from the field space. The degree of a tensor product of field vectors is the sum of the degrees of the single vectors:

$$\left|\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s\right| = \sum_{i=1}^{s} |\boldsymbol{y}_i| \ . \tag{3.39}$$

Let $\mathcal{H}[s]$ be the field space $\mathcal{H}$ but with a shift in the degree, such that

$$\mathcal{H}[s]_{(t)} = \mathcal{H}_{(s+t)} \ . \tag{3.40}$$

The shifted field space $\mathcal{H}[-1]$ is naturally isomorphic to the original field space $\mathcal{H}$ via the *degree-shift* isomorphism

$$\begin{aligned} \downarrow\colon \mathcal{H} &\xrightarrow{\sim} \mathcal{H}[-1] \\ \boldsymbol{y} &\mapsto (-)^{|\boldsymbol{y}|}\, \boldsymbol{y}_{[-1]} \ . \end{aligned} \tag{3.41}$$

Since the total degree on the field space is made up of the ghost degree, fermion degree and Grassmann parity, we could in principle define three different types of degree shift, one for each grading. However, we will only be interested in shifts of the ghost degree and do not consider other degree shifts here. Thus, $\mathrm{gh}(\downarrow\boldsymbol{y}) = \mathrm{gh}(\boldsymbol{y}) - 1$. The inverse map $\uparrow\colon \mathcal{H}[-1] \to \mathcal{H}$ increases the ghost degree by one.

Consider a tensor product $\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s$ of s vectors and compare it to the same tensor product $\downarrow\boldsymbol{y}_1 \otimes \cdots \otimes \downarrow\boldsymbol{y}_s$ of shifted vectors. The degree of the tensor product is shifted by $-s$ since the degree of a tensor product is the sum of the individual degrees and the degree of each individual vector is shifted by -1. This is expressed by the *decalage isomorphism* [22]

$$\begin{aligned} \mathcal{H}[-1] \otimes \cdots \otimes \mathcal{H}[-1] &\xrightarrow{\sim} \big(\mathcal{H} \otimes \cdots \otimes \mathcal{H}\big)[-s] \\ \boldsymbol{y}_{1[-1]} \otimes \cdots \otimes \boldsymbol{y}_{s[-1]} &\mapsto (-)^{\sum_{i=1}^{s}(s-i)|\boldsymbol{y}_i|}\, \big(\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s\big)_{[-s]} \ . \end{aligned} \tag{3.42}$$

A degree-shift affects all degree related signs as $(-)^{|\boldsymbol{y}_{[-1]}|} = (-)^{|\boldsymbol{y}|-1} = -(-)^{|\boldsymbol{y}|}$. Hence, the degree-shift interchanges the commutation behaviour of previously even-degree and odd-degree vectors. Accordingly, the decalage isomorphism induces an isomorphism between the symmetric and exterior tensor power:

$$\big(\mathcal{H}[-1]\big)^{\odot s} \ \xrightarrow{\sim}\ \big(\mathcal{H}^{\wedge s}\big)[-s] \ . \tag{3.43}$$

We will not exploit this isomorphism in this thesis. However, it is useful to understand different grading and sign conventions present in physics.

In section 3.1.5, we remarked on the symmetry or anti-symmetry of the Lie bracket. This can be made precise now. By the universal property of tensor products, symmetric maps can be represented by linear maps on $\mathcal{H}^{\odot s}$, and anti-symmetric maps by linear maps on $\mathcal{H}^{\wedge s}$. Hence, due to the decalage isomorphism, a degree-shift turns symmetric maps into anti-symmetric maps and vice versa [22].

3.2.4 Tensor Coalgebra

The bar construction of homotopy Lie algebras presented in this thesis works on the symmetric tensor coalgebra of the field space. In general, a (graded) *coalgebra* is a (graded) $\mathbb{K}$-linear space $\mathcal{C}$ endowed with a (degree 0) linear map $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ which is coassociative, i.e. it satisfies $(\Delta \otimes \mathbb{1}_{\mathcal{C}}) \circ \Delta = (\mathbb{1}_{\mathcal{C}} \otimes \Delta) \circ \Delta$. This map is referred to as *coproduct.*

We can equip the tensor algebra $T(\mathcal{H})$ of a dgv $\mathcal{H}$ with a deconcatenation map

$$T(\mathcal{H}) \ni \boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s \;\mapsto\; \sum_{i=0}^{s} \big(\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_i\big) \otimes \big(\boldsymbol{y}_{i+1} \otimes \cdots \otimes \boldsymbol{y}_s\big) \;\in\; T(\mathcal{H}) \otimes T(\mathcal{H}) \;. \tag{3.44}$$

The first term in this sum is $1 \otimes \big(\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s\big)$ and the last term is $\big(\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_s\big) \otimes 1$. Hence, on scalars $k \in \mathbb{K}$, this map acts as $k \mapsto k \otimes k$. The deconcatenation map is a coassociative coproduct and turns the tensor algebra into a tensor coalgebra [16]. From now on, we denote the deconcatenation coproduct by Δ and the tensor coalgebra with this coproduct by $T^c(\mathcal{H})$.

Similarly, we can define a coproduct on the symmetric tensor algebra. This yields the *symmetric tensor coalgebra* $S^c(\mathcal{H})$. The symmetric coproduct is given by

$$\Delta\big(\boldsymbol{y}_1 \odot \cdots \odot \boldsymbol{y}_s\big) = \sum_{i=0}^{s} \sum_{\sigma \in \mathcal{S}_{i,s-i}} \epsilon(\sigma) \,\big(\boldsymbol{y}_{\sigma(1)} \odot \cdots \odot \boldsymbol{y}_{\sigma(i)}\big) \otimes \big(\boldsymbol{y}_{\sigma(i+1)} \odot \cdots \odot \boldsymbol{y}_{\sigma(s)}\big) \;, \tag{3.45}$$

where we additionally sum over all $(i, s-i)$-unshuffles [16]. An (i, j)-*unshuffle* is a permutation of the set $\{1, \ldots, i+j\}$ with the two additional conditions $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(i+j)$. To put it in words, an unshuffle permutes the set $\{1, \ldots, i+j\}$ in such a way that elements from the two subsets $\{1, \ldots, i\}$ and $\{i+1, \ldots, i+j\}$ stay in the original order relative to other elements from the same subset. We denote the set of all (i, j)-unshuffles by $\mathcal{S}_{i,j}$. We will make frequent use of unshuffles again at a later point in this thesis, although in a different context.

3.2.5 Coderivation

On a coalgebra $(\mathcal{C}, \Delta)$, a linear map $\mathrm{D} : \mathcal{C} \to \mathcal{C}$ is called *coderivation* if it satisfies the co-Leibniz rule

$$\Delta \circ \mathrm{D} = \big(\, \mathrm{D} \otimes \mathbb{1}_{\mathcal{C}} + \mathbb{1}_{\mathcal{C}} \otimes \mathrm{D} \,\big) \circ \Delta \;. \tag{3.46}$$

A graded coalgebra with a nilpotent degree -1 coderivation, i.e. $\mathrm{D}^2 = 0$, is called *differential graded coalgebra.*

Let $\mathcal{H}$ be a dgv with differential d, then the tensor coalgebra $T^c(\mathcal{H})$ is a differential graded coalgebra with nilpotent coderivation given by

$$\mathrm{D} : \boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_t \;\mapsto\; \sum_{i=0}^{t-1} (-)^{|\boldsymbol{y}_1| + \cdots + |\boldsymbol{y}_i|}\, \boldsymbol{y}_1 \otimes \cdots \otimes \mathrm{d}(\boldsymbol{y}_{i+1}) \otimes \cdots \otimes \boldsymbol{y}_t \;. \tag{3.47}$$

The sign follows from the Koszul sign rule since we have to move d past the first $i-1$ vectors. A quick computation shows that the coderivation defined in this way squares to zero. Similarly, the symmetric tensor coalgebra $S^c(\mathcal{H})$ becomes a differential graded coalgebra with a symmetrised version of (3.47).

3.3 Homotopy Lie Algebra

At least four different but equivalent definitions of a homotopy Lie algebra exist [19]. Particularly useful to applications in physics is the so-called bar construction of homotopy Lie algebras. The homotopy Lie algebra induces a set of multilinear products on the field space which are related by a set of conditions. These products provide the algebraic formulation of a BV field theory. We have already had a glimpse at this formulation in section 3.1.5, where we computed the higher brackets produced by the homotopy transfer of an ordinary Lie algebra structure from one dgv to another. These higher brackets are exactly the multilinear products of the resulting homotopy Lie algebra.

3.3.1 Bar Construction

By definition, a *homotopy Lie algebra*, abbreviated L_∞ algebra, is a dgv $(\mathcal{H}, \mathrm{d})$ with a degree -1 coderivation M on the symmetric tensor coalgebra $S^c(\mathcal{H})$ of $\mathcal{H}$, such that

i) $\mathrm{M}\,|_{\mathbb{K}\odot\mathcal{H}} = 0$

ii) $(\mathrm{D}+\mathrm{M})^2 = 0$.

To express the definition in words, the coderivation M extends the existing degree -1 coderivation D on $S^c(\mathcal{H})$ defined through the vector space differential d in such a way that the extended coderivation $\mathrm{D}+\mathrm{M}$ squares to zero [19].

One could simplify the definition in the following way: Forget about the differential d. Instead, define the homotopy Lie algebra structure directly as a degree -1 coderivation $\tilde{\mathrm{M}}$ that turns $S^c(\mathcal{H})$ into a differential graded coalgebra and whose restriction $\tilde{\mathrm{M}}|_{\mathbb{K}}$ to scalars vanishes [17]. However, we want to make the underlying dgv explicit, since the field space is the starting point of formulating a field theory algebraically. From this point of view, we prefer the definition we have given above.

In field theory applications, the differential d encodes the free equations of motion and field independent gauge transformations. The coderivation M encodes all interactions of fields and the field-dependent gauge transformations. Like the differential d, the coderivation M is always of zero fermion degree and zero Grassmann parity.

Just as a Lie algebra is a specific kind of algebra, a homotopy Lie algebra is a specific kind of homotopy algebra. While L_∞ algebras seem to play the most important role in field theory applications, some field theories can also be described in terms of a *homotopy associative algebra*, abbreviated A_∞ algebra [17]. The definitions of L_∞ and A_∞ algebras are closely related: We can replace the symmetric tensor algebra $S^c(\mathcal{H})$ in the definition above by the ordinary tensor algebra $T^c(\mathcal{H})$ to get the definition of an A_∞ algebra [19]. Indeed, the symmetrisation of a homotopy associative algebra is a homotopy Lie algebra [23].

3.3.2 L_∞ Products

In section 3.2.5, we have seen how the linear differential on a dgv induces a coderivation on the tensor algebra. This result generalises in the following proposition. Any s-multilinear map $\mathrm{f} : \mathcal{H}^{\otimes s} \to \mathcal{H}$ defines a coderivation $\mathrm{F} : T^c(\mathcal{H}) \to T^c(\mathcal{H})$ through

$$\mathrm{F} : \boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_t \;\mapsto\; \sum_{i=0}^{t-s} (-)^{|\mathrm{f}|(|\boldsymbol{y}_1|+\cdots+|\boldsymbol{y}_i|)}\, \boldsymbol{y}_1 \otimes \cdots \otimes \mathrm{f}(\boldsymbol{y}_{i+1}, \ldots, \boldsymbol{y}_{i+s}) \otimes \cdots \otimes \boldsymbol{y}_t \,. \tag{3.48}$$

Furthermore, any coderivation F is uniquely determined by a linear map $T^c(\mathcal{H}) \to \mathcal{H}$ from the tensor space into the underlying vector space [16]. It is given by projecting the output of the coderivation to the sub-space $\mathcal{H} \subset T^c(\mathcal{H})$. Since the tensor space is the direct sum $\bigoplus_{s\geq 0} \mathcal{H}^{\otimes s}$ of finite tensor powers of $\mathcal{H}$, a coderivation F is given by a sum $\sum_{s\geq 0} \mathrm{f}_s$ of multilinear maps $\mathrm{f}_s : \mathcal{H}^{\otimes s} \to \mathcal{H}$.

This proposition carries over to coderivations on the symmetric tensor algebra $S^c(\mathcal{H})$, which can be represented by graded-symmetric multilinear maps. A multilinear map on $\mathcal{H}$ defines a symmetric coderivation via a symmetrised form of equation (3.48).

We use this proposition to represent an L_∞ algebra structure M by a set of graded-symmetric multilinear maps $\mathrm{m}_s : \mathcal{H}^{\otimes s} \to \mathcal{H}$. These maps are called L_∞ *products*. They are of degree -1, since M is of degree -1. By the assumption $\mathrm{M}\,|_{\mathbb{K}\odot\mathcal{H}} = 0$, the zero-inputs product m_0 and the one-input product m_1 vanish. Since we are interested in the coderivation $\mathrm{D}+\mathrm{M}$ and D corresponds to the linear differential d, i.e. a linear map on $\mathcal{H}$, we will often denote d as the one-input product m_1.

In the literature, also other conventions are present. It is possible to work with graded-anti-symmetric products of degree $s-2$. Hohm and collaborators refer to this convention as the "ℓ-picture" of L_∞ algebras and to the convention used in this thesis as the "b-picture" (cf. [17], [8]). The two conventions – or pictures – are related by a degree shift on the underlying dgv. They are best differentiated by the symmetry or anti-symmetry of the products and by their degree. Furthermore, one could "invert" the grading to work with cochain complexes instead of chain complexes. All possible conventions are summarised in table 1.

$\lvert\mathrm{m}_s\rvert$	$(-)^\sigma$	name
-1	$\epsilon(\sigma)$	homological b-picture (used in this thesis)
$+1$	$\epsilon(\sigma)$	cohomological b-picture
$s-2$	$\mathrm{sgn}(\sigma)\epsilon(\sigma)$	homological ℓ-picture
$2-s$	$\mathrm{sgn}(\sigma)\epsilon(\sigma)$	cohomological ℓ-picture

Table 1: Summary of different conventions for L_∞ products. The first column gives the degree of the L_∞ products in a certain convention. The second column states the symmetry under permutation, i.e. it lists the sign in $\mathrm{m}_s\left(\boldsymbol{y}_{\sigma(1)},\ldots,\boldsymbol{y}_{\sigma(s)}\right) = (-)^\sigma\,\mathrm{m}_s\left(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_s\right)$. The third column states the colloquial name of the respective convention.

The condition of nilpotency of the coderivation $\mathrm{D}+\mathrm{M}$ translates into so-called L_∞ *relations* of the products $\{\mathrm{m}_s\}_{s\in\mathbb{N}}$. Schematically, these are the following conditions:

$$\sum_{s+t=u+1} \mathrm{m}_s \circ \mathrm{m}_t = 0 \ , \tag{3.49}$$

which must hold for all $u \in \mathbb{N}$. In the following, we discuss the explicit formula in the b-picture. An explicit formula with the right signs for the ℓ-picture can be found in [17].

First, we recap the definition of an unshuffle. An (i,j)-unshuffle is a permutation σ of the set $\{1,\ldots,i+j\}$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(i+j)$. We denote the set of (i,j)-unshuffles by $\mathcal{S}_{i,j}$. Then, for arbitrary input vectors $\boldsymbol{y}_1,\ldots\boldsymbol{y}_u$, the L_∞ relations are

$$\sum_{s+t=u+1}\ \sum_{\sigma\in\mathcal{S}_{t,s-1}} \epsilon(\sigma)\ \mathrm{m}_s\Big(\mathrm{m}_t\left(\boldsymbol{y}_{\sigma(1)},\ldots,\boldsymbol{y}_{\sigma(t)}\right),\,\boldsymbol{y}_{\sigma(t+1)},\ldots,\boldsymbol{y}_{\sigma(u)}\Big) = 0 \ . \tag{3.50}$$

The relations stated here hold whether we consider a field space $\mathcal{V}$ with commuting functions only or the corresponding field space $\mathcal{H}$ with both commuting and anti-commuting functions. In the former case, all signs are induced by the grading $\mathrm{gh}+\varepsilon$, in the latter, by the total grading $\mathrm{gh}+\varepsilon+\lambda$. Furthermore, the L_∞ relations are satisfied for products m_s on $\mathcal{H}$ if and only if the corresponding products $\mathrm{m}_s^{\mathcal{V}}$ on $\mathcal{V}$ satisfy the relations. In section 3.1, we have emphasised that the Grassmann parity λ must be treated with care. Hence, the fact that the L_∞ relations translate so simple from $\mathcal{V}$ to $\mathcal{H}$ is not trivial. Of course, the whole setup presented in the previous sections was aimed at this. In appendix A, we prove this statement.

In the literature, it is more common to state the L_∞ relations as they were presented in the original paper by Zwiebach [6] with the product m_t entering in the last input of m_s instead of in the first input. However, one then has to deal with additional signs coming from the Koszul sign rule. A quick computation shows that indeed both conventions can be transformed into each other and are thus equivalent. We perform this computation in appendix A.

Indeed, we have discussed some of the L_∞ relations before in section 3.1.5. For $u=1$, we have

$$0=\mathrm{m}_1\left(\mathrm{m}_1(\boldsymbol{y})\right)\,. \tag{3.51}$$

Hence, the first relation states that $\mathrm{m}_1=\mathrm{d}$ is a differential on the field space. The second relation

$$0=\mathrm{m}_1\left(\mathrm{m}_2(\boldsymbol{y}_1,\boldsymbol{y}_2)\right)+\mathrm{m}_2\left(\mathrm{m}_1(\boldsymbol{y}_1),\boldsymbol{y}_2\right)+(-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|}\,\mathrm{m}_2\left(\mathrm{m}_1(\boldsymbol{y}_2),\boldsymbol{y}_1\right) \tag{3.52}$$

is the condition of compatibility of the two-inputs product with the differential. The relation for $u=3$ encodes the vanishing of the Jacobiator of the m_2 product up to the higher homotopy m_3:

$$\begin{aligned}
0=\;&\mathrm{m}_1\left(\mathrm{m}_3(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3)\right)\\
&+\mathrm{m}_3\left(\mathrm{m}_1(\boldsymbol{y}_1),\boldsymbol{y}_2,\boldsymbol{y}_3\right)+(-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|}\,\mathrm{m}_3\left(\mathrm{m}_1(\boldsymbol{y}_2),\boldsymbol{y}_1,\boldsymbol{y}_3\right)\\
&+(-)^{|\boldsymbol{y}_1||\boldsymbol{y}_3|+|\boldsymbol{y}_2||\boldsymbol{y}_3|}\,\mathrm{m}_3\left(\mathrm{m}_1(\boldsymbol{y}_3),\boldsymbol{y}_1,\boldsymbol{y}_2\right)\\
&+\mathrm{m}_2\left(\mathrm{m}_2(\boldsymbol{y}_1,\boldsymbol{y}_2),\boldsymbol{y}_3\right)+(-)^{|\boldsymbol{y}_2||\boldsymbol{y}_3|}\,\mathrm{m}_2\left(\mathrm{m}_2(\boldsymbol{y}_1,\boldsymbol{y}_3),\boldsymbol{y}_2\right)\\
&+(-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|+|\boldsymbol{y}_1||\boldsymbol{y}_3|}\,\mathrm{m}_2\left(\mathrm{m}_2(\boldsymbol{y}_2,\boldsymbol{y}_3),\boldsymbol{y}_1\right)\,.
\end{aligned} \tag{3.53}$$

The higher relations are generalised Jacobiators for higher products, which hold up to higher homotopies.

In section 3.1.3, we have discussed that the differential on the field space is induced by the linearisation of the BRST charge Q. In the same way, we infer the L_∞ products from the BRST charge. Hence, the full coderivation $\mathrm{D}+\mathrm{M}$ is defined by the BRST charge. This suggests preferring the modified definition of a homotopy Lie algebra structure in terms of a single nilpotent coderivation $\tilde{\mathrm{M}}$. However, splitting off the linear part, as we have done in our discussion, suits the usual approach to quantum field theory of considering interactions as perturbations of the free equations of motion. In this spirit, we have chosen to present the differential graded field space, which encodes the free equations of motion, as the given basis for setting up the homotopy Lie algebra structure, which encodes the full equations of motion including interactions. The procedure of defining a differential and L_∞ products from the BRST charge is detailed in section 4.1.

3.3.3 Cyclic Homotopy Lie Algebra

Usually, field theories are described by an action. In our approach, the algebraic formulation of a field theory is inferred from the BRST charge, which is defined in terms of the action. In that sense, the algebraic formulation follows from the field theory action. However, this is not necessarily the case. One may as well set up the L_∞ products from fundamental considerations with the requirement that they satisfy the L_∞ relations (for example, cf. [17]). This would make the homotopy Lie algebra the fundamental description of a field theory.

Both formulations – action-based and algebra-based – should produce the same computational results. And we can only take advantage of having two different, yet equivalent formulations if it is possible to go back and forth between them. We show in section 4.1 how the homotopy Lie algebra is constructed from the action. The reverse way, from algebra to action, follows via a suitable inner product on the field space. Having such an inner product is crucial for physically interpreting the homotopy Lie algebra [11].

Let $(\mathcal{V}, \mathrm{d})$ be a dgv. An *inner product* on $\mathcal{V}$ is a bilinear degree $+1$ map $\langle\cdot,\cdot\rangle : \mathcal{V}^{\otimes 2} \to \mathbb{K}$ which is non-degenerate. That is, for every $\boldsymbol{y}_1 \in \mathcal{V}$ exists $\boldsymbol{y}_2 \in \mathcal{V}$ such that $\langle \boldsymbol{y}_1, \boldsymbol{y}_2\rangle \neq 0$. Under permutation of the inputs, one requires [17]

$$\langle \boldsymbol{y}_1, \boldsymbol{y}_2\rangle = (-)^{(|\boldsymbol{y}_1|+1)(|\boldsymbol{y}_2|+1)} \langle \boldsymbol{y}_2, \boldsymbol{y}_1\rangle \ , \tag{3.54}$$

so that the inner product is graded-symmetric but with respect to shifted degrees. Furthermore, we demand that the inner product be compatible with the differential d, i.e.

$$\langle \mathrm{d}\, \boldsymbol{y}_1, \boldsymbol{y}_2\rangle = (-)^{|\boldsymbol{y}_1|} \langle \boldsymbol{y}_1, \mathrm{d}\, \boldsymbol{y}_2\rangle \ . \tag{3.55}$$

In field theory applications, we consider only inner products of zero fermion degree and zero Grassmann parity. To determine the degree of the inner product, we define the field of scalars $\mathbb{K}$ to be of zero ghost degree, zero fermion degree and zero Grassmann parity. Thus, $\langle \boldsymbol{y}_1, \boldsymbol{y}_2\rangle$ is only non-zero if $|\boldsymbol{y}_1| = -|\boldsymbol{y}_2| - 1$. This suggests that the inner product pairs fields with their respective anti-fields.

Once more, the generalisation to a field space $\mathcal{H}$ with commuting and anti-commuting functions is not trivial. Let $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ be an inner product on the field space $\mathcal{V}$, then the corresponding inner product on $\mathcal{H}$ is the degree $+1$ map $\langle\cdot,\cdot\rangle_{\mathcal{H}} : \mathcal{H}^{\otimes 2} \to \mathcal{A} \otimes \mathbb{K}$ defined by

$$\left\langle y_1^{(n_1)} \boldsymbol{e}_1, y_2^{(n_2)} \boldsymbol{e}_2 \right\rangle_{\mathcal{H}} = (-)^{n_2 + n_2 |\boldsymbol{e}_1|}\, \xi^{(n+m)} \left\langle y_1 \boldsymbol{e}_1, y_2 \boldsymbol{e}_2 \right\rangle_{\mathcal{V}} \ . \tag{3.56}$$

Note that we have to deviate here from the usual Koszul sign rule and there is no factor $(-)^{n_1}$ on the left-hand side. This is necessary so that the inner product on $\mathcal{H}$ has the same degree-shifted symmetry as the inner product on $\mathcal{V}$. Since the transition from $\mathcal{V}$ to $\mathcal{H}$ is different for the inner product than for multilinear maps on the field space, also the "surprising sign" is slightly different. If we plug in physically meaningful vectors, we get a sign

$$(-)^{|\boldsymbol{e}_2| + |\boldsymbol{e}_2||\boldsymbol{e}_1|} = (-)^{|\boldsymbol{e}_2|} \ , \tag{3.57}$$

where we have used $|\boldsymbol{e}_1| = -|\boldsymbol{e}_2| - 1$ so that the second term in the exponent is always even.

A homotopy Lie algebra with L_∞ products $\{\mathrm{m}_s\}_{s\in\mathbb{N}}$ is called *cyclic* if an inner product $\langle\cdot,\cdot\rangle$ exists on the underlying field space, such that the maps $\left\langle \mathrm{m}_s(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_s), \boldsymbol{y}_{s+1} \right\rangle$ are graded-symmetric in all $s+1$ inputs with respect to the ordinary, unshifted degrees.

Finally, we use the inner product on a cyclic homotopy Lie algebra to define a field theory action. The action is given by

$$S_\infty := \sum_{s\in\mathbb{N}} \frac{1}{(s+1)!} \left\langle \mathrm{m}_s(\boldsymbol{y}^{\mathrm{phys}}, \dots, \boldsymbol{y}^{\mathrm{phys}}),\, \boldsymbol{y}^{\mathrm{phys}} \right\rangle , \tag{3.58}$$

where all input vectors are set to be identical and physically meaningful. It is also referred to as *homotopy Chern-Simons action*, due to its superficial similarity with the original Chern-Simons action [13].

We make one last remark on conventions. In the previously mentioned ℓ-picture, i.e. when one is working with graded-anti-symmetric L_∞ products of degree $s-2$, the pairing $\langle \cdot, \cdot \rangle$ must also be graded-anti-symmetric. In this case, the L_∞ algebra is cyclic if $\left\langle \mathrm{m}_s(\boldsymbol{y}_1, \dots, \boldsymbol{y}_s), \boldsymbol{y}_{s+1} \right\rangle$ are graded-anti-symmetric in all $s+1$ inputs. Usually, one then calls the pairing not an inner product but a symplectic form. Just as the L_∞ products in b- and ℓ-picture are related by a degree shift, so are the inner product and the symplectic form.

4 Application to Batalin-Vilkovisky Field Theories

4.1 General Ansatz

In the following, we explain how the L_∞ algebra structure can be derived from the BRST charge $Q = \{S, \cdot\}$, where $\{\cdot, \cdot\}$ denotes the BV bracket and S is the field theory action. This procedure relies on a polynomial action. If a given action is not polynomial in the fields, one has to expand it around a background. While some authors set up the algebraic description of a field theory from fundamental considerations paired with the requirement for mathematical consistency (for example, cf. [17]), we follow the approach to infer all algebraic structure from the BRST charge, originally described by Alexandrov, Kontsevich, Schwarz and Zaboronsky in [7].

From equation (2.2), the BRST charge is given by

$$Q = \{S, \cdot\} = \sum_{\text{fields } \mathbf{\Psi}} (\pm) \frac{\delta_R S}{\delta \Psi} \frac{\delta_L}{\delta \Psi^*} , \tag{4.1}$$

where $(\pm)$ is a positive sign if the field $\mathbf{\Psi}$ has positive or zero ghost degree, and else it is negative. The BRST charge is a functional differential operator on a space of functionals of the fields. One can work directly with such an operator in the language of so-called QP-manifolds, as was originally done in [7] and very pedagogically explained in [24]. Instead, we want to stay in the framework of dgv introduced in the sections above. These two points of view are often referred to as dual pictures, and their connection is worked out well by Arvanitakis and collaborators in [8]. Our procedure of translating the BRST charge into a differential and a set of L_∞ products on the field space $\mathcal{H}$ follows their discussion. However, we add a lot more details and aim to be very explicit about signs.

We consider a field theory with action S. The fields $\mathbf{\Phi}, \mathbf{\Psi}, \ldots$ in this theory are of ghost degree $\mathrm{gh}(\mathbf{\Phi}), \mathrm{gh}(\mathbf{\Psi}), \ldots$. If the field theory contains Dirac fermion fields, they have an additional fermion degree $\varepsilon = \pm 1$, while all other fields have zero fermion degree. For each field $\mathbf{\Phi}$, there is a corresponding anti-field $\mathbf{\Phi}^*$ of degree $-|\mathbf{\Phi}| - 1$. The anti-field $(\mathbf{\Phi}^*)^*$ of the anti-field $\mathbf{\Phi}^*$ of a field $\mathbf{\Phi}$ is again the field $\mathbf{\Phi}$. Usually, one calls fields of positive or zero ghost degree simply "fields", and fields of negative ghost degree "anti-fields". As long as we stay in this general setting and do not discuss a particular example of field theory, we will restrain from that habit to avoid confusion.

The fields $\mathbf{\Phi}, \mathbf{\Psi}, \ldots$ appearing in the action are always physically meaningful. That means a field $\mathbf{\Phi}$ is represented by a function $\Phi \in \mathcal{A} \otimes \mathcal{F}$ of Grassmann parity $\lambda(\Phi) = -|\mathbf{\Phi}|$, and the action S is a functional of commuting or anti-commuting functions $\Phi, \Psi, \ldots$. The action is always of zero Grassmann parity.

4.1.1 Differential

Let S_s denote the terms of polynomial order s in the action. The functional derivative $\frac{\delta_R S_s}{\delta \Psi}$ is a polynomial of functions $\Phi, \Psi, \ldots$ of polynomial order $s - 1$. We can consider it – or rather its result upon plugging in functions – again as a function in $\mathcal{A} \otimes \mathcal{F}$. We start with the quadratic terms S_2 in the action. Replacing the functional differential operator $\delta_L / \delta \Psi^*$ in the BRST charge by the pseudo-basis vector $\boldsymbol{e}_{\Psi^*}$, we achieve a translation from functional differential operator to linear map on the field space $\mathcal{H}$:

$$\frac{\delta_R S_2}{\delta\Psi}\,\frac{\delta_L}{\delta\Psi^*} \quad \xrightarrow[\text{to dual picture}]{\text{translation}} \quad \left(y_\Phi^{(n)} \boldsymbol{e}_\Phi \;\mapsto\; (-)^n\, \xi^{(n)}\, \frac{\delta_R S_2}{\delta\Psi}\bigg|_{\Phi=y_\Phi} \boldsymbol{e}_{\Psi^*} \right) . \tag{4.2}$$

The sign $(-)^n$ comes as usual from pulling out the Grassmann element, which itself is not affected by the mapping, and $\frac{\delta_R S_2}{\delta\Psi}|_{\Phi=y_\Phi}$ is a function in $\mathcal{F}$.

Since the action is of zero Grassmann parity, $\frac{\delta_R S_2}{\delta\Psi}$ is of parity $-\lambda(\Psi)$. Because it is linear in the fields, it only depends on functions Φ of parity $\lambda(\Phi) = -\lambda(\Psi)$. This implies that the mapping is non-zero only for input vectors $y_\Phi^{(n)}\boldsymbol{e}_\Phi$ with $|\boldsymbol{e}_\Phi| = -\lambda(\Phi) = \lambda(\Psi)$. Finally, the output is of degree $|\boldsymbol{e}_{\Psi^*}| = -|\boldsymbol{e}_\Psi| - 1 = \lambda(\Psi) - 1$. Hence, the mapping is of degree -1 as required.

Before we proceed to the definition of the differential, we quickly recap the issue of the "surprising sign" explained in section 3.2.2. If we evaluate a (multi-)linear map on physically meaningful inputs, we find that the result equals a product of Grassmann elements times the result of the corresponding map on the simpler commuting-functions-only field space times the "surprising sign" (equation (3.38))

$$(-)^{\text{phys}} = (-)^{\sum_i |\boldsymbol{e}_i||\mathrm{f}_\mathcal{V}| + \sum_{i>j} |\boldsymbol{e}_i||\boldsymbol{e}_j|} .$$

The "surprising sign" depends on the number of inputs with odd-degree pseudo-vector. Assume that $|\mathrm{f}_\mathcal{V}|$ is odd, as is the case for the differential and the L_∞ products. Then the "surprising sign" is negative if the number of inputs with odd-degree pseudo-vector is 1, 2, 5, 6, 9, 10, ... and so on. Due to reasons that will become clear later on, we need to define the differential and the L_∞ products such that the "surprising sign" cancels when we evaluate the differential on physically meaningful inputs.

After this brief digression, we define the differential on the graded field space $\mathcal{H}$ as the linear map

$$\mathrm{d}: \quad y_\Phi^{(n)}\boldsymbol{e}_\Phi \;\mapsto\; (-)^{\text{phys}}\,(-)^n\,\xi^{(n)} \sum_{\text{fields } \boldsymbol{\Psi}} (\pm)\, \frac{\delta_R S_2}{\delta\Psi}\bigg|_{\Phi=y_\Phi} \boldsymbol{e}_{\Psi^*} . \tag{4.3}$$

The sign $(\pm)$ is positive if the field $\boldsymbol{\Psi}$ has positive or zero ghost degree, and else it is negative. The "surprising sign" pops up in this definition for the reason explained above. When we evaluate the differential on physically meaningful inputs, the "surprising sign" vanishes.

All we need to show is that this is nilpotent, i.e. it squares to zero. If we apply the map twice on a vector $y_\Phi^{(n)}\boldsymbol{e}_\Phi$, we get

$$\mathrm{d}^2\left(y_\Phi^{(n)}\boldsymbol{e}_\Phi\right) = -\,(-)^{2n}\,\xi^{(n)} \sum_{\text{fields } \boldsymbol{\Psi},\boldsymbol{\Theta}} (\pm)\, \frac{\delta_R S_2}{\delta\Theta}\bigg|_{\Psi^*=\tilde{y}_{\Psi^*}} \boldsymbol{e}_{\Theta^*} , \tag{4.4}$$

with

$$\tilde{y}_{\Psi^*} = \frac{\delta_R S_2}{\delta\Psi}\bigg|_{\Phi=y_\Phi} . \tag{4.5}$$

The overall minus sign in (4.4) comes from the factor $(-)^{\text{phys}}$ in the definition of the differential. The sign $(\pm)$ is negative only if $\mathrm{gh}(\boldsymbol{\Theta}) = 0$ independent of $\mathrm{gh}(\boldsymbol{\Psi})$.

We have to show that all coefficients in the linear combination in (4.4) vanish. We pick one arbitrary coefficient and compute

$$\sum_{\mathbf{\Psi}} \frac{\delta_R S_2}{\delta \Theta}\bigg|_{\Psi^*=\tilde{y}_{\Psi^*}} = \sum_{\mathbf{\Psi}} \tilde{y}_{\Psi^*} \cdot \frac{\delta_L}{\delta \Psi^*} \frac{\delta_R S_2}{\delta \Theta} \tag{4.6a}$$

$$= \sum_{\mathbf{\Psi}} \frac{\delta_R S_2}{\delta \Psi}\bigg|_{\Phi=y_\Phi} \cdot \frac{\delta_L}{\delta \Psi^*} \frac{\delta_R S_2}{\delta \Theta} \tag{4.6b}$$

$$= \sum_{\mathbf{\Psi}} \frac{\delta_R S_2}{\delta \Psi} \cdot \frac{\delta_L}{\delta \Psi^*} \frac{\delta_R S_2}{\delta \Theta}\bigg|_{\Phi=y_\Phi} \tag{4.6c}$$

The first step holds since all terms of $\frac{\delta_R S_2}{\delta \Theta}$ are linear in a specific field Ψ^*. In the second step, we have plugged in equation (4.5) for $\tilde{y}_{\Psi^*}$. In the third step, we have used that the second functional derivative of the quadratic terms in the action is field independent. Finally, we can compare this to the vanishing square of the BRST charge:

$$\begin{aligned} 0 &= \left(Q_1\right)^2 = \{S_2, \{S_2, \cdot\}\} \\ &= \sum_{\mathbf{\Theta}} \sum_{\mathbf{\Psi}} \left[(\pm) \frac{\delta_R S_2}{\delta \Psi} \cdot \frac{\delta_L}{\delta \Psi^*} \frac{\delta_R S_2}{\delta \Theta} \cdot \frac{\delta_L}{\delta \Theta^*} + (\pm) \frac{\delta_R S_2}{\delta \Psi} \cdot \frac{\delta_R S_2}{\delta \Theta} \cdot \frac{\delta_L^2}{\delta \Psi^* \delta \Theta^*} \right] . \end{aligned} \tag{4.7}$$

The first and the second term must vanish independently since first- and second-order functional differential operators are linearly independent. In the first term, $(\pm)$ is a negative sign if $\mathrm{gh}(\mathbf{\Theta}) = 0$. In particular, it does not depend on $\mathbf{\Psi}$. Furthermore, the first term must vanish independently for each $\mathbf{\Theta}$, since the functional differential operators for different fields are linearly independent. Thus, we find for arbitrary $\mathbf{\Theta}$,

$$\sum_{\mathbf{\Psi}} \frac{\delta_R S_2}{\delta \Psi} \cdot \frac{\delta_L}{\delta \Psi^*} \frac{\delta_R S_2}{\delta \Theta} = 0 . \tag{4.8}$$

We use this in equation (4.6c) to complete the proof that the differential d as defined in equation (4.3) squares to zero.

4.1.2 L_∞ Products

Next, we derive the L_∞ products from the BRST charge. We do this in essentially the same way as we have done for the differential. However, the definition of the L_∞ products is more involved since for $s \geq 2$, $\frac{\delta_R S_{s+1}}{\delta \Psi}$ is not linear in the fields. $\frac{\delta_R S_{s+1}}{\delta \Psi}$ is of polynomial order s. Hence, s fields $\mathbf{\Theta}_1, \ldots, \mathbf{\Theta}_s$ enter in any term, some of them possibly identical. For example, for the monomial $\Phi^2\Psi$, we denote $\Theta_1 = \Phi, \Theta_2 = \Phi, \Theta_3 = \Psi$.

We start with a definition of the m_s product on the commuting-functions-only field space $\mathcal{V}$. The s input vectors are all of the form $\boldsymbol{y}_i = \sum_{\mathbf{\Phi}} y_{i\Phi}\, \boldsymbol{e}_\Phi$. Then, the m_s product is defined on $\mathcal{V}$ as:

$$\boxed{\mathrm{m}_s^{\mathcal{V}}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_s) := \sum_{\mathbf{\Psi}} (\pm)\, (-)^{\mathrm{phys}} \sum_{\sigma \in \mathcal{S}_s} \epsilon(\sigma) \left[\frac{\delta_R S_{s+1}}{\delta \Psi}\bigg|_{\substack{\Theta_1 = y_{\sigma(1)\Theta_1}, \\ \Theta_2 = y_{\sigma(2)\Theta_2}, \\ \ldots}} \right] \boldsymbol{e}_{\Psi^*}} . \tag{4.9}$$

As before, the sign $(\pm)$ is positive if the field $\mathbf{\Psi}$ has positive or zero ghost degree, and else it is negative. The sum over all permutations $\sigma \in \mathcal{S}_s$ and the Koszul sign $\epsilon(\sigma)$ ensure that the L_∞ product is indeed graded-symmetric on $\mathcal{V}$. For the first function Θ_1 in $\frac{\delta_R S_{s+1}}{\delta \Psi}$, we insert the function $y_{\sigma(1)\Theta_1}$ from the $\sigma(1)$'th input, and so on. The products so defined are of degree -1 by the same argument as above for the differential. Once more, the factor $(-)^{\mathrm{phys}}$ is meant to cancel the "surprising sign".

Next, we transfer this definition to the full field space $\mathcal{H}$ of commuting and anti-commuting functions. The s input vectors are now of the form $\boldsymbol{y}_i = \sum_{\boldsymbol{\Phi}} \xi^{(n_{i\Phi})}\, y_{i\Phi}\, \boldsymbol{e}_{\Phi}$. We have to take special care of the Grassmann parity. Otherwise, the definition remains unchanged. Using formula (3.35), we find

$$\mathrm{m}_s(\boldsymbol{y}_1, \dots, \boldsymbol{y}_s) =$$

$$\sum_{\boldsymbol{\Psi}} (\pm)\, (-)^{\mathrm{phys}} \sum_{\sigma \in \mathcal{S}_s} \epsilon_{\mathrm{gh}\,+\,\varepsilon}(\sigma)\ (-)^{\cdots}\ \xi^{\left(\sum_i n_{i\Theta_{\sigma^{-1}(i)}}\right)} \left[\frac{\delta_R S_{s+1}}{\delta \Psi}\bigg|_{\substack{\Theta_1 = y_{\sigma(1)\Theta_1},\\ \Theta_2 = y_{\sigma(2)\Theta_2},\\ \cdots}} \right] \boldsymbol{e}_{\Psi^*}\ , \tag{4.10}$$

with Grassmann parity induced sign factor

$$(-)^{\cdots} = (-)^{\sum_i n_{i\Theta_{\sigma^{-1}(i)}} + \sum_{i>j} n_{i\Theta_{\sigma^{-1}(i)}} \left|\boldsymbol{e}_{j\Theta_{\sigma^{-1}(j)}}\right|}\ . \tag{4.11}$$

We express in words what this formula describes: For the t'th function Θ_t, we insert into $\frac{\delta_R S_{s+1}}{\delta \Psi}$ the commuting function $y_{\sigma(t)\Theta_t}$ from the $\sigma(t)$'th input. Thus, from the input vector $\boldsymbol{y}_i$, we only use the function $y_{i\Theta_\sigma^{-1}(i)}$ and the corresponding Grassmann element. The overall Grassmann element is the product of the input Grassmann elements. The sign factor involving Grassmann parities follows from moving the i'th Grassmann element through all the previous pseudo-basis vectors and the map $\mathrm{m}_s^{\mathcal{V}}$ of degree -1. The other sign factors we took from the previous definition of $\mathrm{m}_s^{\mathcal{V}}$. Note the important subtlety that the Koszul sign $\epsilon_{\mathrm{gh}\,+\,\varepsilon}(\sigma)$ is only with regards to ghost degree and fermion degree.

We can permute two input vectors and compare the result thus obtained to the original result without permuting the inputs. We will get one extra sign from the change in the factor $(-)^{\cdots}$. We will get a second sign from the permutation of Grassmann elements, and we will get a third sign from a change in the Koszul sign $\epsilon_{\mathrm{gh}\,+\,\varepsilon}(\sigma)$. These three contributions combine to produce the desired overall Koszul sign $\epsilon(\sigma)$ which takes into account the total degree $\mathrm{gh} + \varepsilon + \lambda$. Hence, the products m_s are graded-symmetric on $\mathcal{H}$ with respect to the total degree, as desired.

For $s = 1$, the above definition coincides with the definition (4.3) of the differential d. This justifies once more to denote d as the m_1 product.

Finally, we give a formula for the L_∞ products (and for the differential) for physically meaningful input vectors. The formula for physically meaningful inputs is much clearer since almost no degree-related signs occur:

$$\boxed{\mathrm{m}_s(\boldsymbol{y}_1^{\mathrm{phys}}, \dots, \boldsymbol{y}_s^{\mathrm{phys}}) = \sum_{\boldsymbol{\Psi}} (\pm)\, \xi^{(|\boldsymbol{e}_\Psi|)} \sum_{\sigma \in \mathcal{S}_s} \left[\frac{\delta_R S_{s+1}}{\delta \Psi}\bigg|_{\substack{\Theta_1 = y_{\sigma(1)\Theta_1},\\ \Theta_2 = y_{\sigma(2)\Theta_2},\\ \cdots}} \right] \boldsymbol{e}_{\Psi^*}}\ , \tag{4.12}$$

with $(\pm)$ positive if the field $\boldsymbol{\Psi}$ has positive or zero ghost degree, and else negative. The Grassmann element $\xi^{(|\boldsymbol{e}_\Psi|)} = \xi^{(\sum_i |\boldsymbol{\Theta}_i|)}$ is determined by the order of the fields $\boldsymbol{\Theta}_i$ in the action and is independent of permutations of the inputs. By construction, the "surprising sign" vanishes.

So far, we are missing a general proof that the products defined above constitute a homotopy Lie algebra for any given BV action. That this is the case seems plausible from the nilpotency of the BRST charge. We used this argument in the proof that the differential $\mathrm{d} = \mathrm{m}_1$ squares to zero. Hence, we already proved that the first L_∞ relation holds for any given BV action. In principle, it should be possible to extend this to show that all higher L_∞ relations are satisfied, too. However, this extension is not straightforward since our proof made heavy use of the linearity of $\frac{\delta_R S_{s+1}}{\delta \Phi}$. Thus, the above discussion must be seen as a claim that should always be checked in applications.

We shortly remark on the practical usefulness of the definitions (4.9), (4.10) and (4.12). Although the definition (4.10) of the m_s products on the full field space $\mathcal{H}$ is the most general, we do not need it in applications. We did show in section 3.3.2 that L_∞ products on $\mathcal{H}$ satisfy the L_∞ relations if the corresponding products on $\mathcal{V}$ do so. Therefore, it suffices to check the L_∞ relations on the simpler $\mathrm{m}_s^{\mathcal{V}}$ products, defined in equation (4.9). Furthermore, almost all computations can be done using these simpler products. Only if we want to compute the field theory action, do we need the products on $\mathcal{H}$. However, we only need the products for physically meaningful inputs. Thus, (4.12) is sufficient in this case.

In section 4.2, we present a very simple example to clarify our notation and sign conventions. Subsequently, in section 4.3, we demonstrate very explicitly how quantum electrodynamics can be formulated in terms of L_∞ products.

4.1.3 Inner Product

The final ingredient in the algebraic formulation of field theories is an inner product on the field space which makes the previously defined homotopy Lie algebra cyclic. We use the inner product to compute a homotopy Chern-Simons action from a given homotopy Lie algebra. If this L_∞ algebra was originally derived from a field theory action, we want the homotopy Chern-Simons action to equal the original action. Thus, the inner product is supposed to "invert" the procedure by which we derived the L_∞ products from the action.

Let $\langle \cdot, \cdot \rangle$ be an inner product on the field space. Then, the homotopy Chern-Simons action is given by (equation (3.58))

$$S_\infty := \sum_{s \in \mathbb{N}} \frac{1}{(s+1)!} \left\langle \mathrm{m}_s(\boldsymbol{y}^{\mathrm{phys}}, \ldots, \boldsymbol{y}^{\mathrm{phys}}),\, \boldsymbol{y}^{\mathrm{phys}} \right\rangle .$$

Conversely, the L_∞ products are schematically given by

$$m_s \approx \sum_{\boldsymbol{\Phi}} (\pm) \frac{\delta_R S_{s+1}}{\delta \Phi} \boldsymbol{e}_{\Phi^*} ,$$

i.e. we store the result of the right functional derivative of the action S by a function Φ in the $\boldsymbol{\Phi}^*$-component of the output vector. Therefore, heuristically, we must multiply the $\boldsymbol{\Phi}^*$-component of m_s by Φ from the right to regain the original action.

In section 3.3.3, we have remarked that a degree $+1$ inner product pairs fields $\boldsymbol{\Phi}, \boldsymbol{\Psi}$ of degrees $|\boldsymbol{\Phi}| = -|\boldsymbol{\Psi}| - 1$. This is in good accordance with the heuristic idea presented above. Hence, we define an inner product on the field space $\mathcal{H}$ by

$$\boxed{\langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle = \sum_{\boldsymbol{\Phi}} (\pm)\, (-)^{|\boldsymbol{e}_\Phi|}\, (-)^{n_{2\Phi} + n_{2\Phi}|\boldsymbol{e}_{\Phi^*}|}\ \xi^{(n_{1\Phi^*} + n_{2\Phi})} \int y_{1\Phi^*} y_{2\Phi}} \ . \tag{4.13}$$

Again, the sign $(\pm)$ is positive if the field $\mathbf{\Phi}$ has positive or zero ghost degree, and else it is negative. The sign $(-)^{|e_\Phi|}$ cancels the "surprising sign", which differs a bit from the case of multilinear maps on $\mathcal{H}$, see equation (3.57). It also plays another important role: we need the factor $(-)^{|e_\Phi|}$ to balance the anti-symmetric $(\pm)$ sign and give the inner product the expected degree-shifted symmetry.

Plugging m_s into the inner product and setting all input vectors to be identical and physically meaningful, we find the homotopy Chern-Simons action to be equal to

$$\begin{aligned} S_\infty &= \sum_{s\in\mathbb{N}} \frac{1}{(s+1)!} \sum_{\mathbf{\Phi}} (\pm) \, (-)^{|e_\Phi|-|e_\Phi|-|e_\Phi||e_{\Phi^*}|} \int (\pm) \, s! \, \frac{\delta_R S_{s+1}}{\delta \Phi} \Phi \\ &= S \; . \end{aligned}$$

We have used that the exponent of the sign factor is always even and the two occurrences of $(\pm)$ cancel each other out. Furthermore, S_{s+1} is a polynomial of order $s+1$, so that $\sum_{\mathbf{\Phi}} \int \frac{\delta_R S_{s+1}}{\delta \Phi} \Phi = (s+1) \, S_{s+1}$. In the end, we find that the homotopy Chern-Simons action S_∞ equals the original action S if the L_∞ algebra is set up according to the procedure we have prescribed in the previous section. It was crucial that we took care of the "surprising sign" in the various steps. Otherwise, one would not get back to the original action.

The above computation shows that our approach is fully consistent. It can be used to easily go hence and forth between homotopy Lie algebra and action. The necessary formulas are given in equations (4.9), (4.12) and (4.13). They are the most important results of this thesis.

4.2 Scalar Field Theory

In this section, we discuss the example of scalar ϕ^3 theory. This is the simplest interacting field theory. It does not have any gauge symmetry so it is not necessary to describe the theory as a BV field theory. Nevertheless, there is also no objection to doing so.

Since the algebraic structure of the theory is very simple, we will use this example to clarify the notation and conventions introduced in section 3. It marks the first application in which we present the approach for deriving the L_∞ structure from a given BV action that was presented in section 4.1. A much more interesting example both from the physical and mathematical point of view will be discussed in the subsequent section.

The action we start from is given by

$$S^{\phi^3} = \int \left[\frac{1}{2} \phi \, \Box \, \phi + \frac{1}{3} \phi^3 \right] \; , \tag{4.14}$$

where $\Box = \partial_\mu \partial^\mu$ denotes the d'Alembert operator on four-dimensional Minkowski space-time. The action satisfies the BV master equation

$$\left\{ S^{\phi^3}, S^{\phi^3} \right\} = 2 \int \frac{\delta_R S^{\phi^3}}{\delta \phi} \frac{\delta_L S^{\phi^3}}{\delta \phi^*} = 0 \; , \tag{4.15}$$

because S^{ϕ^3} is independent of ϕ^*.

From the action, we take that the theory contains a real scalar field $\boldsymbol{\phi}$. Since we treat the theory as a BV field theory, there is also the corresponding scalar anti-field $\boldsymbol{\phi}^*$. The scalar field is an ordinary field of bosonic statistics. Hence, it has ghost degree $\mathrm{gh}(\boldsymbol{\phi}) = 0$ and fermion degree $\varepsilon(\boldsymbol{\phi}) = 0$. Accordingly, the scalar anti-field has $\mathrm{gh}(\boldsymbol{\phi}^*) = -\,\mathrm{gh}(\boldsymbol{\phi}) - 1 = -1$ and $\varepsilon(\boldsymbol{\phi}^*) = -\,\varepsilon(\boldsymbol{\phi}) = 0$. The commuting-functions-only field space is

$$\mathcal{V} = \mathcal{V}_\phi \oplus \mathcal{V}_{\phi^*} \ . \tag{4.16}$$

The full field space $\mathcal{H}$, including anti-commuting functions, is the tensor product $\mathcal{A} \otimes \mathcal{V}$ of a Grassmann algebra $\mathcal{A}$ with the commuting-functions-only field space $\mathcal{V}$.

A generic field vector in $\mathcal{H}$ is given by

$$\boldsymbol{y} = y_\phi^{(n_\phi)}\, \boldsymbol{e}_\phi + y_{\phi^*}^{(n_{\phi^*})}\, \boldsymbol{e}_{\phi^*} \ , \tag{4.17}$$

where $y_\phi^{(n_\phi)}$ is a function on Minkowski space of Grassmann parity $\lambda\left(y_\phi^{(n_\phi)}\right) = n_\phi$. If the parity is even, the function is commuting. If the parity is odd, it is anti-commuting. The function describes the field configuration of the scalar field $\boldsymbol{\phi}$, but it carries neither ghost degree nor fermion degree. Instead, ghost degree and fermion degree are carried by the pseudo-basis vector $\boldsymbol{e}_\phi$, which on the other hand has no Grassmann parity. The total degree of a field vector $\boldsymbol{y} = y_\phi^{(n_\phi)} \boldsymbol{e}_\phi$ is

$$|\boldsymbol{y}| = \left|y_\phi^{(n_\phi)}\right| + |\boldsymbol{e}_\phi| = \lambda\left(y_\phi^{(n_\phi)}\right) + \mathrm{gh}(\boldsymbol{e}_\phi) + \varepsilon(\boldsymbol{e}_\phi) \ . \tag{4.18}$$

For a generic vector, the Grassmann parity is not fixed, i.e. the function $y_\phi^{(n_\phi)}$ may be of arbitrary degree and in particular, the function may be anti-commuting. However, for physically meaningful vectors, we demand $\lambda = -(\mathrm{gh} + \varepsilon)$. We denote physically meaningful vectors as

$$\boldsymbol{y}^{\mathrm{phys}} = \phi\, \boldsymbol{e}_\phi + \phi^*\, \boldsymbol{e}_{\phi^*} \ , \tag{4.19}$$

where ϕ is a commuting function of parity $\lambda(\phi) = -\big(\mathrm{gh}(\boldsymbol{e}_\phi) + \varepsilon(\boldsymbol{e}_\phi)\big) = 0$ and ϕ^* is an anti-commuting function of parity $\lambda(\phi^*) = -\big(\mathrm{gh}(\boldsymbol{e}_{\phi^*}) + \varepsilon(\boldsymbol{e}_{\phi^*})\big) = 1$.

Next, we derive the differential on the field space $\mathcal{V}$. From the ansatz in equation (4.3), we compute

$$\mathrm{d}^{\mathcal{V}}(\boldsymbol{y}) = (-)^{|\boldsymbol{e}_\phi|}\, \Box\, y_\phi\, \boldsymbol{e}_{\phi^*} = \Box\, y_\phi\, \boldsymbol{e}_{\phi^*} \ . \tag{4.20}$$

The $\boldsymbol{\phi}$-component of d vanishes because the action is independent of ϕ^*. Furthermore, the differential is independent of the $\boldsymbol{\phi}^*$-component of the input vector. Hence, if we apply the differential twice, it vanishes. So d is nilpotent as required. To check the degree of d, we note that the $\boldsymbol{\phi}$-component of the input is mapped to the $\boldsymbol{\phi}^*$-component of the output. Thus, the differential is of degree -1 as expected.

We do not give the full definition of the differential on the full field space $\mathcal{H}$, since we do not need it in computations. The definition on $\mathcal{V}$ suffices in most cases. If we want to compute the homotopy Chern-Simons action from the L_∞ algebra, we need the differential on the full field space $\mathcal{H}$ but only on physically meaningful inputs. In this case, the differential takes the form

$$\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}}) = \Box\, \phi\, \boldsymbol{e}_{\phi^*} \ . \tag{4.21}$$

Note that the result is not physically meaningful because $\Box\,\phi$ is of Grassmann parity 0 but $\boldsymbol{e}_{\phi^*}$ is of degree -1.

Following the procedure detailed in section 4.1, the next step is to compute the L_∞ products. Only the m_2 product is non-trivial. All higher products vanish because the action contains only terms of polynomial degree up to 3. Following equation (4.9), the m_2 product on $\mathcal{V}$ is

$$\mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1, \boldsymbol{y}_2) = \big(y_{1\phi}\, y_{2\phi} + y_{2\phi}\, y_{1\phi}\big)\, \boldsymbol{e}_{\phi^*} \tag{4.22}$$

As for the differential, the $\boldsymbol{\phi}$-component of the m_2 product vanishes, and the product does not depend on the $\boldsymbol{\phi^*}$-components of the inputs. The product is of degree -1 and it is graded-symmetric, i.e. $\mathrm{m}_2^{\mathcal{Y}}(\boldsymbol{y}_1, \boldsymbol{y}_2) = (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|}\, \mathrm{m}_2^{\mathcal{Y}}(\boldsymbol{y}_2, \boldsymbol{y}_1)$.

Again, we do not need the definition of the product on the full space $\mathcal{H}$ other than for physically meaningful inputs. Already identifying the two inputs, we have

$$\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}}) := \mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}}, \boldsymbol{y}^{\mathrm{phys}}) = 2\,\phi^2\, \boldsymbol{e}_{\phi^*}\ . \tag{4.23}$$

Since the homotopy Lie algebra consists only of the two L_∞ products $\mathrm{m}_1 = \mathrm{d}$ and m_2, only the first three L_∞ relations are non-trivial. The non-trivial relations are

1. $0 = \mathrm{m}_1\big(\mathrm{m}_1(\boldsymbol{y})\big)$

2. $0 = \mathrm{m}_1\big(\mathrm{m}_2(\boldsymbol{y}_1, \boldsymbol{y}_2)\big) + \mathrm{m}_2\big(\mathrm{m}_1(\boldsymbol{y}_1), \boldsymbol{y}_2\big) + (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2|}\, \mathrm{m}_2\big(\mathrm{m}_1(\boldsymbol{y}_2), \boldsymbol{y}_1\big)$

3. $$\begin{aligned} 0 = {} & \mathrm{m}_2\big(\mathrm{m}_2(\boldsymbol{y}_1, \boldsymbol{y}_2), \boldsymbol{y}_3\big) + (-)^{|\boldsymbol{y}_2||\boldsymbol{y}_3|}\, \mathrm{m}_2\big(\mathrm{m}_2(\boldsymbol{y}_1, \boldsymbol{y}_3), \boldsymbol{y}_2\big) \\ & + (-)^{|\boldsymbol{y}_1||\boldsymbol{y}_2| + |\boldsymbol{y}_1||\boldsymbol{y}_3|}\, \mathrm{m}_2\big(\mathrm{m}_2(\boldsymbol{y}_2, \boldsymbol{y}_3), \boldsymbol{y}_1\big)\ . \end{aligned}$$

They are satisfied because both m_1 and m_2 have vanishing $\boldsymbol{\phi}$-components and are independent of the $\boldsymbol{\phi}^*$-components of the inputs.

Finally, we need an inner product on the field space which makes the homotopy Lie algebra cyclic. We always need the inner product on the full field space $\mathcal{H}$. From the inner product, we get the homotopy Chern-Simons action upon plugging in the L_∞ products in the first input. We did discuss above that for physically meaningful inputs, the output of an L_∞ product is not physically meaningful. Hence, we cannot restrict our definition of the inner product to physically meaningful vectors.

Using equation (4.13), the inner product on $\mathcal{H}$ is

$$\langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle = \xi^{(n_{1\phi^*} + n_{2\phi})} \int y_{1\phi^*}\, y_{2\phi} \;+\; (-)^{n_{2\phi^*}}\, \xi^{(n_{1\phi} + n_{2\phi^*})} \int y_{1\phi}\, y_{2\phi^*}\ . \tag{4.24}$$

The inner product has the expected symmetry. It is of degree $+1$ and it is non-degenerate. Furthermore, $\langle \mathrm{m}_1(\boldsymbol{y}_1), \boldsymbol{y}_2 \rangle$ is graded-symmetric with respect to both inputs, and $\langle \mathrm{m}_2(\boldsymbol{y}_1, \boldsymbol{y}_2), \boldsymbol{y}_3 \rangle$ is graded-symmetric with respect to all three inputs. So the homotopy Lie algebra with this inner product is cyclic.

As a final check, we compute the homotopy Chern-Simons action to show that it equals the original action (4.14) we started with:

$$\begin{aligned} S_\infty &= \sum_{s\in\mathbb{N}} \frac{1}{(s+1)!} \Big\langle \mathrm{m}_s(\boldsymbol{y}^{\mathrm{phys}}, \ldots, \boldsymbol{y}^{\mathrm{phys}}),\, \boldsymbol{y}^{\mathrm{phys}} \Big\rangle \\ &= \frac{1}{2!} \int (\Box\,\phi)\phi + \frac{1}{3!} \int (2\,\phi^2)\phi \\ &= \int \Big[\frac{1}{2} \phi\,\Box\,\phi + \frac{1}{3} \phi^3 \Big] \\ &= S^{\phi^3}\ . \end{aligned}$$

This completes our discussion of the algebraic framework for scalar ϕ^3 theory.

4.3 Quantum Electrodynamics

The second exemplary application that we discuss is quantum electrodynamics (abbreviated QED), i.e. abelian gauge theory with minimally coupled Dirac fermions. However, we will stay at the classical level. To incorporate quantum effects, one has to work with so-called loop homotopy Lie algebras (cf. [4]). Instead, we focus on the application of ordinary homotopy Lie algebras.

The QED BV action is

$$S^{QED} = \int \left[i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}A_\mu(\Box A^\mu - \partial^\mu\partial_\nu A^\nu) + \bar{\psi}A\!\!\!/\psi - A^{*\nu}\partial_\nu c + i\psi^* c\psi + i\bar{\psi}c\bar{\psi}^* \right] . \qquad (4.25)$$

As before, $\Box = \partial_\mu\partial^\mu$ denotes the d'Alembert operator on four-dimensional Minkowski space. Furthermore, $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ is the contraction of the partial differential operator with the gamma matrix, and similarly $A\!\!\!/ = \gamma^\mu A_\mu$. By $\bar{\psi} = \psi^\dagger\gamma^0$ we denote the Dirac conjugate spinor.

A quick computation shows that the action satisfies the BV master equation:

$$\begin{aligned}
\left\{S^{QED}, S^{QED}\right\} = 2 \int \Bigg[&\frac{\delta_R S^{QED}}{\delta c}\frac{\delta_L S^{QED}}{\delta c^*} + \frac{\delta_R S^{QED}}{\delta A_\mu}\frac{\delta_L S^{QED}}{\delta A^{*\mu}} \\
&+ \frac{\delta_R S^{QED}}{\delta \psi}\frac{\delta_L S^{QED}}{\delta \psi^*} + \frac{\delta_R S^{QED}}{\delta \bar{\psi}}\frac{\delta_L S^{QED}}{\delta \bar{\psi}^*} \Bigg] \\
= 2 \int \Big[&\big(\underset{1}{-i\psi^*\psi} \underset{2}{+i\bar{\psi}\bar{\psi}^*}\big)\cdot 0 + \big(\underset{3}{\Box A^\mu} \underset{4}{- \partial^\mu\partial_\nu A^\nu} \underset{5}{+ \bar{\psi}\gamma^\mu\psi}\big)\cdot\big(-\partial_\mu c\big) \\
&+ \big(\underset{6}{-i\bar{\psi}\overleftarrow{\partial\!\!\!/}} \underset{7}{+\bar{\psi}A\!\!\!/} \underset{8}{+ i\psi^* c}\big)\cdot\big(ic\psi\big) + \big(\underset{9}{-i\partial\!\!\!/\psi} \underset{10}{-A\!\!\!/\psi} \underset{11}{- ic\bar{\psi}^*}\big)\cdot\big(i\bar{\psi}c\big)\Big] \\
= 0\ .&
\end{aligned}$$

The first two terms vanish because the action does not depend on c^*. Terms 3 and 4 cancel after integration by parts of the differential operator ∂_μ. This leaves ∂_μ(term 5) which is cancelled by the sum of terms 6 and 10. Terms 7 and 9 cancel each other. Finally, terms 8 and 11 vanish because they are quadratic in the anti-commuting c. In term 6, we have denoted a left-arrow above the partial differential to indicate that it acts to the left, i.e. on $\bar{\psi}$.

We start with setting up the field space. The theory contains a ghost field $\boldsymbol{c}$, gauge field $\boldsymbol{A}$, Dirac fermion field $\boldsymbol{\psi}$, Dirac conjugate fermion field $\bar{\boldsymbol{\psi}}$, and the respective anti-fields. In section 3.1.2, we have discussed how the field space is created as a direct sum of the sub-spaces corresponding to single fields. We do not repeat this discussion here. Instead, we summarise the ghost degree and fermion degree of all the fields in table 2.

As usual, for a generic vector, the Grassmann parity is not fixed. However, for physically meaningful vectors, we demand $\lambda = -(\mathrm{gh} + \varepsilon)$. To give an example, the vector $c\,\boldsymbol{e}_c$ describes a ghost field with anti-commuting field configuration c. In contrast, $y_c^{(n)}\boldsymbol{e}_c$ describes a ghost field with field configuration of arbitrary Grassmann parity n, in particular not necessarily anti-commuting.

field	gh	ε	$\lvert\cdot\rvert = \mathrm{gh}(\cdot) + \varepsilon(\cdot)$	physical behaviour
$\boldsymbol{c}$	1	0	1	odd
$\boldsymbol{A}$	0	0	0	even
$\boldsymbol{\psi}$	0	1	1	odd
$\bar{\boldsymbol{\psi}}$	0	-1	-1	odd
$\bar{\boldsymbol{\psi}}^*$	-1	-1	-2	even
$\boldsymbol{\psi}^*$	-1	1	0	even
$\boldsymbol{A}^*$	-1	0	-1	odd
$\boldsymbol{c}^*$	-2	0	-2	even

Table 2: Summary of the field content of QED. Along with the fields, we list the respective ghost degree and fermion degree. The fourth column shows the sum of ghost degree and fermion degree. The last column states whether the physically meaningful field configurations are commuting (even) or anti-commuting (odd).

Next, we derive the differential and the L_∞ product m_2 on the field space $\mathcal{V}$. All higher products vanish since the QED action has no terms of polynomial order larger than 3. Again, it is not necessary to determine the differential and the product on the full space $\mathcal{H}$. All we need are the definitions for inputs with zero Grassmann parity, i.e. vectors in $\mathcal{V}$, and for inputs with zero total degree, i.e. for physically meaningful vectors in $\mathcal{H}$. We compute the differential from equation (4.3). Let

$$\mathrm{d}(\boldsymbol{y}) = \sum_{\text{fields } \boldsymbol{\Phi}} \Big[\mathrm{d}(\boldsymbol{y})\Big]_{\Phi} \boldsymbol{e}_\Phi \ , \tag{4.26}$$

then the differential is given by

$$\begin{array}{ll|ll}
\Big[\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\Big]_{c} &= 0 & \Big[\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}})\Big]_{c} &= 0 \\
\Big[\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\Big]_{A_\mu} &= \partial_\mu y_c & \Big[\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}})\Big]_{A_\mu} &= -\partial_\mu c \\
\Big[\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\Big]_{\psi} &= 0 & \Big[\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}})\Big]_{\psi} &= 0 \\
\Big[\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\Big]_{\bar\psi} &= 0 & \Big[\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}})\Big]_{\bar\psi} &= 0 \\
\Big[\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\Big]_{\bar\psi^*} &= i\partial\!\!\!/ y_\psi & \Big[\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}})\Big]_{\bar\psi^*} &= -i\partial\!\!\!/ \psi \\
\Big[\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\Big]_{\psi^*} &= i y_{\bar\psi} \overleftarrow{\partial\!\!\!/} & \Big[\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}})\Big]_{\psi^*} &= -i\bar\psi \overleftarrow{\partial\!\!\!/} \\
\Big[\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\Big]_{A^{*\nu}} &= \Box\, y_A{}^\nu - \partial^\nu \partial_\mu y_A{}^\mu & \Big[\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}})\Big]_{A^{*\nu}} &= \Box\, A^\nu - \partial^\nu \partial_\mu A^\mu \\
\Big[\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\Big]_{c^*} &= -\partial_\nu y_{A^*}{}^\nu & \Big[\mathrm{d}(\boldsymbol{y}^{\mathrm{phys}})\Big]_{c^*} &= \partial_\nu A^{*\nu} \\
\text{inputs with zero Grassmann parity} & & \text{physically meaningful inputs with zero total degree} &
\end{array} \ . \tag{4.27}$$

The differential is of degree -1 and is nilpotent. The computation showing that $\mathrm{d}^2 = 0$ is done in appendix B.

We compute the L_∞ product m_2 on $\mathcal{V}$ as prescribed by equation (4.9). Besides, we give the product m_2 on $\mathcal{H}$ for identical physically meaningful inputs according to equation (4.12). As before, let

$$\mathrm{m}_2(\boldsymbol{y}_1,\boldsymbol{y}_2) = \sum_{\text{fields } \boldsymbol{\Phi}} \Big[\mathrm{m}_2(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{\Phi} \boldsymbol{e}_\Phi \ . \tag{4.28}$$

We find for the m_2 product

$$\begin{aligned}
\Big[\mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{c} &= 0\\
\Big[\mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{A_\mu} &= 0\\
\Big[\mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{\psi} &= i\big(y_{1c}\, y_{2\psi} - y_{2c}\, y_{1\psi}\big)\\
\Big[\mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{\bar\psi} &= i\big(y_{1\bar\psi}\, y_{2c} - y_{2\bar\psi}\, y_{1c}\big)\\
\Big[\mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{\bar\psi^*} &= \big(y_{1A,\mu}\,\gamma^\mu\, y_{2\psi} + y_{2A,\mu}\,\gamma^\mu\, y_{1\psi}\big)\\
&\quad + i\big(y_{1c}\, y_{2\bar\psi^*} + y_{2c}\, y_{1\bar\psi^*}\big)\\
\Big[\mathrm{m}_2^{V}(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{\psi^*} &= -\big(y_{1\bar\psi}\, y_{2A,\mu}\,\gamma^\mu + y_{2\bar\psi}\, y_{1A,\mu}\,\gamma^\mu\big)\\
&\quad - i\big(y_{1\psi^*}\, y_{2c} + y_{2\psi^*}\, y_{1c}\big)\\
\Big[\mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{A^{*\nu}} &= -\big(y_{1\bar\psi}\,\gamma^\nu\, y_{2\psi} - y_{2\bar\psi}\,\gamma^\nu\, y_{1\psi}\big)\\
\Big[\mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2)\Big]_{c^*} &= i\big(y_{1\psi^*}\, y_{2\psi} + y_{2\psi^*}\, y_{1\psi}\big)\\
&\quad - i\big(y_{1\bar\psi}\, y_{2\bar\psi^*} + y_{2\bar\psi}\, y_{1\bar\psi^*}\big)
\end{aligned}$$

inputs with zero Grassmann parity

$$\begin{aligned}
\Big[\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}})\Big]_{c} &= 0\\
\Big[\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}})\Big]_{A_\mu} &= 0\\
\Big[\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}})\Big]_{\psi} &= -2ic\psi\\
\Big[\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}})\Big]_{\bar\psi} &= -2i\bar\psi c\\
\Big[\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}})\Big]_{\bar\psi^*} &= -2\big(A_\mu\gamma^\mu\psi + ic\bar\psi^*\big)\\
\Big[\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}})\Big]_{\psi^*} &= 2\big(\bar\psi A_\mu\gamma^\mu + i\psi^* c\big)\\
\Big[\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}})\Big]_{A^{*\nu}} &= 2\bar\psi\gamma^\nu\psi\\
\Big[\mathrm{m}_2(\boldsymbol{y}^{\mathrm{phys}})\Big]_{c^*} &= -2i\big(\psi^*\psi - \bar\psi\bar\psi^*\big)
\end{aligned} \ . \tag{4.29}$$

identical physically meaningful inputs with zero total degree

The product is graded-symmetric and of degree -1. In appendix B, we show that the L_∞ relations are satisfied. Therefore, what we have found here indeed constitutes a homotopy Lie algebra.

Finally, we give the inner product and compute the homotopy Chern-Simons action. According to equation (4.13), the inner product is

$$\begin{aligned}
\langle \boldsymbol{y}_1,\boldsymbol{y}_2\rangle = &- (-)^{n_{2c}}\, \xi^{(n_{1c^*}+n_{2c})} \int y_{1c^*}\, y_{2c} \ - \ \xi^{(n_{1c}+n_{2c^*})} \int y_{1c}\, y_{2c^*}\\
&+ \xi^{(n_{1A^*}+n_{2A})} \int y_{1A^*}{}^{\mu}\, y_{2A,\mu} \ + \ (-)^{n_{2A^*}}\, \xi^{(n_{1A}+n_{2A^*})} \int y_{1A,\mu}\, y_{2A^*}{}^{\mu}\\
&- (-)^{n_{2\psi}}\, \xi^{(n_{1\psi^*}+n_{2\psi})} \int y_{1\psi^*}\, y_{2\psi} \ - \ \xi^{(n_{1\psi}+n_{2\psi^*})} \int y_{1\psi}\, y_{2\psi^*}\\
&- (-)^{n_{2\bar\psi}}\, \xi^{(n_{1\bar\psi^*}+n_{2\bar\psi})} \int y_{1\bar\psi^*}\, y_{2\bar\psi} \ - \ \xi^{(n_{1\bar\psi}+n_{2\bar\psi^*})} \int y_{1\bar\psi}\, y_{2\bar\psi^*} \ .
\end{aligned} \tag{4.30}$$

It has the expected symmetry, is of degree -1 and is non-degenerate.

The homotopy Chern-Simeons action is

$$
\begin{aligned}
S_\infty &= \sum_{s\in\mathbb{N}} \frac{1}{(s+1)!}\Big\langle \mathrm{m}_s(\boldsymbol{y}^{\mathrm{phys}},\dots,\boldsymbol{y}^{\mathrm{phys}}),\, \boldsymbol{y}^{\mathrm{phys}}\Big\rangle \\
&= \frac{1}{2!}\Bigg[\int (\partial_\nu A^{*\nu})c - \int 0\cdot c^* + \int \big(\Box A^\mu - \partial^\mu\partial_\nu A^\nu\big)A_\mu - \int \big(-\partial_\mu c\big)A^{*\mu} \\
&\qquad + \int \big(-i\bar\psi\overleftarrow{\not\partial}\big)\psi - \int 0\cdot\psi^* + \int\big(-i\not\partial\psi\big)\bar\psi - \int 0\cdot\bar\psi^*\Bigg] \\
&\quad + \frac{1}{3!}\Bigg[\int(-2i)\big(\psi^*\psi - \bar\psi\bar\psi^*\big)c - \int 0\cdot c^* + \int\big(2\bar\psi\gamma^\mu\psi\big)A_\mu - \int 0\cdot A^{*\mu} \\
&\qquad + \int 2\big(\bar\psi A_\mu\gamma^\mu + i\psi^* c\big)\psi - \int\big(-2ic\psi\big)\psi^* + \int(-2)\big(A_\mu\gamma^\mu\psi + ic\bar\psi^*\big)\bar\psi \\
&\qquad - \int\big(-2i\bar\psi c\big)\bar\psi^*\Bigg] \\
&= \int\Big[i\bar\psi\not\partial\psi + \frac{1}{2}A_\mu(\Box A^\mu - \partial^\mu\partial_\nu A^\nu) + \bar\psi\not A\psi - A^{*\nu}\partial_\nu c + i\psi^* c\psi + i\bar\psi c\bar\psi^*\Big] \\
&= S^{QED}\ .
\end{aligned}
$$

To get to the third step, we have integrated by parts in the first and fifth integral. Besides, in various terms, we have (anti-)commuted field configurations to bring everything into the right order. The respective commutation or anti-commutation behaviour for each field configuration was already listed in table 2.

The homotopy Chern-Simons action is identical to the QED action we started from – as it should be. This completes our discussion of the algebraic framework for QED at a classical level.

5 Summary and Outlook

In this thesis, we have presented in great detail a general procedure for how a homotopy Lie algebra can be derived from a given BV action. We assume that the action is polynomial in the fields and that it satisfies the BV master equation. Then, the differential on the field space and the L_∞ products follow directly from the BRST charge. This approach is not new. However, it had previously only been presented in the setting of QP-manifolds and with little focus on the proper treatment of fermionic fields. Hence, the important achievement of this thesis is to give a very explicit recipe that is useful for practical applications. We took great effort to explain subtleties otherwise often hidden in the notation, especially when it comes to getting all sign factors right.

The issue of signs is a tedious but important one. In setting up the L_∞ products it is crucial that they actually satisfy the L_∞ relations. From our own experience, we found that usually, an accidental sign mistake leads to a set of multilinear products which do not satisfy these relations and which, therefore, do not manifest a homotopy Lie algebra. Hence, this thesis is supposed to be a foolproof practitioner's guide. Particularly important is the derivation of the L_∞ products on the commuting-functions-only field space, given in equation (4.9), and on physically meaningful inputs, equation (4.12), and the general form of the inner product, equation (4.13), which makes the homotopy Lie algebra cyclic.

We have given the algebraic formulation of two exemplary field theories. Again, nothing new lies in these results other than the very explicit presentation of the procedure itself. The two examples illustrate our general practitioner's guide in action and help to clarify the less common notation of the field space.

With this thesis, we hope to add a practically useful reference to the current lively advancement of the topic, which often proceeds in more abstract discussions. We believe that the homotopy algebraic formulation of field theories is not only a beautiful detour but offers many opportunities for practical usage. This belief is based on the many applications of the homotopy transfer theorem that have been published in recent years.

We showed that our explicit formulas for the L_∞ products and the inner product on the field space are consistent and reproduce the original action as the homotopy Chern-Simons action. However, we are missing a general proof that our ansatz actually produces a homotopy Lie algebra. Such a general proof is very desirable and would relieve us from checking the L_∞ relations again in every new application.

Finally, even if one is relieved of checking the L_∞ relations in every new application, a lot of the necessary computations are still lengthy and error-prone. In this algebraic framework, most computations involve not more than sums and products of fields and inserting them into each other. The computations are often very algorithmic. So, it should be possible to write an algorithm on a computer, which implements the algebraic framework and takes the tedious task of computing L_∞ products and checking L_∞ relations. A practitioner would certainly benefit a lot from such a computer algorithm.

Appendix

In the following, we give the proofs and computations that were left out in the main text. In that, the appendix supports the statements and claims we have made in the main text. However, the computations performed here are also meant as a practical demonstration of our formalism and notations.

A Proofs of Statements in Section 3

Symmetry of Corresponding multilinear Maps on $\mathcal{V}$ and $\mathcal{H}$

Let $\mathrm{f}_{\mathcal{V}}$ be a multilinear map on the commuting-functions-only field space $\mathcal{V}$. The corresponding map on the full field space $\mathcal{H}$ is (equation (3.35))

$$\mathrm{f}_{\mathcal{H}}\left(y_1^{(n_1)}\boldsymbol{e}_1,\dots,y_s^{(n_s)}\boldsymbol{e}_s\right)=(-)^{\sum_i n_i|\mathrm{f}_{\mathcal{V}}|+\sum_{i>j}n_i|\boldsymbol{e}_j|}\ \xi^{(\sum_i n_i)}\,\mathrm{f}_{\mathcal{V}}\left(y_1\boldsymbol{e}_1,\dots,y_s\boldsymbol{e}_s\right)\ .$$

In section 3.2.2, we claimed that $\mathrm{f}_{\mathcal{H}}$ is graded-(anti-)symmetric with respect to the total grading $\mathrm{gh}+\varepsilon+\lambda$ on $\mathcal{H}$ if $\mathrm{f}_{\mathcal{V}}$ is graded-(anti-)symmetric with respect to the grading $\mathrm{gh}+\varepsilon$ on $\mathcal{V}$.

To show this, we consider a permutation $\sigma:[\dots,t,t+1,\dots]\mapsto[\dots,t+1,t,\dots]$ of two subsequent inputs. First, note that we can split the Koszul sign $\epsilon(\sigma)$ with respect to the total degree into three contributions:

$$\epsilon(\sigma)=(-)^{|\boldsymbol{y}_t||\boldsymbol{y}_{t+1}|}=(-)^{n_t n_{t+1}}\,(-)^{n_t|\boldsymbol{e}_{t+1}|+n_{t+1}|\boldsymbol{e}_t|}\,(-)^{|\boldsymbol{e}_t||\boldsymbol{e}_{t+1}|}\ . \tag{5.1}$$

The first sign factor comes from commuting the Grassmann elements of $\boldsymbol{y}_t$ and $\boldsymbol{y}_{t+1}$, the second from permuting the Grassmann elements with the pseudo-basis vectors, and the last one from permuting the pseudo-basis vectors. The last sign equals the Koszul sign $\epsilon(\sigma)_{\mathrm{gh}+\varepsilon}$ on $\mathcal{V}$ with respect to ghost degree and fermion degree only.

Next, we compute how $\mathrm{f}_{\mathcal{H}}$ changes after such a permutation of the inputs assuming $\mathrm{f}_{\mathcal{V}}$ to be symmetric:

$$\begin{aligned}\mathrm{f}_{\mathcal{H}}\left(y_{\sigma(1)}^{(n_{\sigma(1)})}\boldsymbol{e}_{\sigma(1)},\dots,y_{\sigma(s)}^{(n_{\sigma(s)})}\boldsymbol{e}_{\sigma(s)}\right)&=(-)^{\sum_i n_{\sigma(i)}|\mathrm{f}_{\mathcal{V}}|+\sum_{i>j}n_{\sigma(i)}|\boldsymbol{e}_{\sigma(j)}|}\ \xi^{(\sum_i n_{\sigma(i)})}\\&\quad\cdot\mathrm{f}_{\mathcal{V}}\left(y_{\sigma(1)}\boldsymbol{e}_{\sigma(1)},\dots,y_{\sigma(s)}\boldsymbol{e}_{\sigma(1)}\right) &(5.2\text{a})\\&=(-)^{\sum_i n_i|\mathrm{f}_{\mathcal{V}}|+\sum_{i>j}n_i|\boldsymbol{e}_j|+n_t|\boldsymbol{e}_{t+1}|+n_{t+1}|\boldsymbol{e}_t|}\\&\quad\cdot(-)^{n_t n_{t+1}}\ \xi^{(\sum_i n_i)}\ \epsilon(\sigma)_{\mathrm{gh}+\varepsilon}\ \mathrm{f}_{\mathcal{V}}\left(y_1\boldsymbol{e}_1,\dots,y_s\boldsymbol{e}_s\right) &(5.2\text{b})\\&=\epsilon(\sigma)\,\mathrm{f}_{\mathcal{H}}\left(y_1^{(n_1)}\boldsymbol{e}_1,\dots,y_s^{(n_s)}\boldsymbol{e}_s\right) &(5.2\text{c})\end{aligned}$$

In the first step, we have used the fact that σ permutes only the inputs t and $t+1$. In the sum $\sum_{i>j}$, we have to add the previously present term $n_t|\boldsymbol{e}_{t+1}|$ and subtract the previously missing term $n_{t+1}|\boldsymbol{e}_t|$. Since the exponent of $(-)$ is only relevant modulo 2, it does not matter whether we add or subtract a term. Furthermore, we get the sign $(-)^{n_t n_{t+1}}$ from commuting the product of Grassmann elements and the Koszul sign $\epsilon(\sigma)_{\mathrm{gh}+\varepsilon}$ from permuting the inputs in $\mathrm{f}_{\mathcal{V}}$. Putting all additional sign factors together, we find the desired Koszul sign $\epsilon(\sigma)$ with respect to the total grading $\mathrm{gh}+\varepsilon+\lambda$.

Finally, a general permutation can be represented as the concatenation of permutations of subsequent elements. This completes the proof. Of course, the proof holds for anti-symmetric maps in the same way, since only the Koszul sign is affected in going from $\mathcal{V}$ to $\mathcal{H}$. The usual sign $\operatorname{sgn}(\sigma)$ of a permutation is the same on both field spaces.

L_∞ Relations on $\mathcal{V}$ and $\mathcal{H}$

The L_∞ relations as usually stated in the literature (cf. [6] or [17]) are

$$\sum_{s+t=u+1} \sum_{\sigma\in\mathcal{S}_{s-1,t}} \epsilon(\sigma)\,(-)^{\sum_{i=1}^{s-1}\left|\boldsymbol{y}_{\sigma(i)}\right|}\;\mathrm{m}_s\left(\boldsymbol{y}_{\sigma(1)},\dots,\boldsymbol{y}_{\sigma(s-1)},\mathrm{m}_t(\boldsymbol{y}_{\sigma(s)},\dots,\boldsymbol{y}_{\sigma(u)})\right) \;=\; 0\;. \tag{5.3}$$

The m_t product enters the m_s product in the last input, and in front, we have an additional sign factor coming from the Koszul sign rule. The presence of this additional sign factor is annoying in computations. Thus, in the main text, we have presented the L_∞ relations with the m_t product entering in the first input to avoid the sign factor from the Koszul sign rule. We show now that the two conventions for the L_∞ products are indeed equivalent.

Since the L_∞ products are graded-symmetric, we can move the vector $\mathrm{m}_t(\boldsymbol{y}_{\sigma(s)},\dots,\boldsymbol{y}_{\sigma(u)})$ from the last input to the very first input. Thereby, we pick up a sign

$$(-)^{\left(\sum_i^{s-1}|\boldsymbol{y}_{\sigma(i)}|\right)\left(\sum_{j=s}^{u}|\boldsymbol{y}_{\sigma(j)}|-1\right)} = \epsilon(\tau)\,(-)^{-\sum_{i=1}^{s-1}|\boldsymbol{y}_{\sigma(i)}|}\;, \tag{5.4}$$

which we directly rewrote as the Koszul sign of a particular permutation τ times a sign factor which exactly cancels the undesirable sign in (5.3). The permutation τ acts as follows:

$$\tau : \boldsymbol{y}_{\sigma(1)},\dots,\boldsymbol{y}_{\sigma(s-1)},\boldsymbol{y}_{\sigma(s)},\dots,\boldsymbol{y}_{\sigma(u)} \;\mapsto\; \boldsymbol{y}_{\sigma(s)},\dots,\boldsymbol{y}_{\sigma(u)},\boldsymbol{y}_{\sigma(1)},\dots,\boldsymbol{y}_{\sigma(s-1)}\;, \tag{5.5}$$

We combine τ and σ into a new permutation $\sigma' := \tau\circ\sigma$ with Koszul sign $\epsilon(\sigma') = \epsilon(\tau)\,\epsilon(\sigma)$. Obviously, σ' is an $(t,s-1)$-unshuffle. Thus, we find

$$\sum_{s+t=u+1}\sum_{\sigma'\in\mathcal{S}_{t,s-1}} \epsilon(\sigma')\;\mathrm{m}_s\left(\mathrm{m}_t(\boldsymbol{y}_{\sigma'(1)},\dots,\boldsymbol{y}_{\sigma'(t)}),\,\boldsymbol{y}_{\sigma'(t+1)},\dots,\boldsymbol{y}_{\sigma'(u)}\right) = 0\;. \tag{5.6}$$

These are the L_∞ relations as stated in section 3.3.2.

In the main text, we claimed that the L_∞ relations are satisfied for products m_s on $\mathcal{H}$ if and only if the corresponding products $\mathrm{m}_s^{\mathcal{V}}$ on $\mathcal{V}$ satisfy the relations.

To prove this, we show that the L_∞ relations on $\mathcal{H}$ equal the relations on $\mathcal{V}$ up to an overall sign which does not matter. Without loss of generality, we assume all inputs to be homogeneous in ghost degree, fermion degree and Grassmann parity. In the sum over all unshuffles, consider the term for which the unshuffle is the identity. We pull all the Grassmann elements out of the L_∞ products:

$$\begin{aligned}
&\sum_{s+t=u+1} \mathrm{m}_s\left(\mathrm{m}_t(\boldsymbol{y}_1,\dots,\boldsymbol{y}_t),\boldsymbol{y}_{t+1},\dots,\boldsymbol{y}_u\right)\\
&= \sum_{s+t=u+1} \underset{1}{(-)^{\sum_{t\geq i>j} n_i|\boldsymbol{e}_j|}}\;\underset{2}{(-)^{\sum_{i=1}^{t} n_i}}\;\underset{3}{(-)^{\sum_{i=1}^{t} n_i}}\;\underset{4}{(-)^{\sum_{k>l>t} n_k|\boldsymbol{e}_l|}}\;\underset{5}{(-)^{\sum_{k=t+1}^{u} n_k\left(\sum_{i=1}^{t}|\boldsymbol{e}_i|-1\right)}}\\
&\qquad\cdot\underset{6}{(-)^{\sum_{k=t+1}^{u} n_k}}\;\xi^{\left(\sum_{i=1}^{u} n_i\right)}\;\mathrm{m}_s^{\mathcal{V}}\left(\mathrm{m}_t^{\mathcal{V}}(\boldsymbol{y}_1^{(0)},\dots,\boldsymbol{y}_t^{(0)}),\boldsymbol{y}_{t+1}^{(0)},\dots,\boldsymbol{y}_u^{(0)})\right)
\end{aligned} \tag{5.7a}$$

$$= \sum_{s+t=u+1} (-)^{\sum_{i>j}^{u} n_i|\boldsymbol{e}_j|}\;\xi^{\left(\sum_{i=1}^{u} n_i\right)}\;\mathrm{m}_s^{\mathcal{V}}\left(\mathrm{m}_t^{\mathcal{V}}(\boldsymbol{y}_1^{(0)},\dots,\boldsymbol{y}_t^{(0)}),\boldsymbol{y}_{t+1}^{(0)},\dots,\boldsymbol{y}_u^{(0)})\right)\;. \tag{5.7b}$$

In (5.7a), sign factors 1 and 2 come from pulling the first t Grassmann elements through the previous pseudo-basis vectors and out of the product m_t; factor 3 from pulling these first t Grassmann elements out of m_s; factors 4 and 5 from moving the latter $s-1$ Grassmann elements through the previous $s-1$ pseudo-basis vectors and through the pseudo-vector of $\mathrm{m}_t(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_t)$; and finally, factor 6 arises from pulling these $s-1$ Grassmann elements through m_s.

To make the step to (5.7b), we cancel factor 2 with 3, and factor 6 with the second part of 5. Then, we combine the remaining factors 1 and 4 and the first part of 5 into the sign factor that is left over in (5.7b). Now, the important observation is that this leftover sign and the Grassmann element are independent of s and t and we can move them in front of the sum over s and t. All we need to show is that we can find the same factor and the same Grassmann element for arbitrary unshuffles of the inputs. Then, we could also pull them out of the sum over unshuffles.

Consider an arbitrary unshuffle σ with Koszul sign $\epsilon(\sigma)$ with respect to the total grading $\mathrm{gh} + \varepsilon + \lambda$. Then, reversing the argument used in the previous proof above in equation (5.2b),

$$\epsilon(\sigma)\,(-)^{\sum_{i>j}^{u} n_{\sigma(i)}|\boldsymbol{e}_{\sigma(j)}|}\,\xi^{(\sum_{i=1}^{u} n_{\sigma(i)})} \;=\; \epsilon(\sigma)_{\mathrm{gh}\,+\,\varepsilon}\,(-)^{\sum_{i>j}^{u} n_i|\boldsymbol{e}_j|}\,\xi^{\left(\sum_{i=1}^{u} n_i\right)}\;. \tag{5.8}$$

Using this, we can rewrite the L_∞ relations on $\mathcal{H}$ as

$$\begin{aligned}
&\underbrace{\sum_{s+t=u+1}\;\sum_{\sigma\in\mathcal{S}_{t,s-1}} \epsilon(\sigma)\;\mathrm{m}_s\left(\mathrm{m}_t(\boldsymbol{y}_{\sigma(1)}, \ldots, \boldsymbol{y}_{\sigma(t)}), \boldsymbol{y}_{\sigma(t+1)}, \ldots, \boldsymbol{y}_{\sigma(u)}\right)}_{L_\infty \text{ relation on } \mathcal{H}} \\
&= (-)^{\sum_{i>j}^{u} n_i|\boldsymbol{e}_j|}\,\xi^{\left(\sum_{i=1}^{u} n_i\right)} \\
&\quad\cdot \underbrace{\sum_{s+t=u+1}\;\sum_{\sigma\in\mathcal{S}_{t,s-1}} \epsilon(\sigma)_{\mathrm{gh}\,+\,\varepsilon}\;\mathrm{m}_s^{\mathcal{V}}\left(\mathrm{m}_t^{\mathcal{V}}(\boldsymbol{y}_{\sigma(1)}^{(0)}, \ldots, \boldsymbol{y}_{\sigma(t)}^{(0)}), \boldsymbol{y}_{\sigma(t+1)}^{(0)}, \ldots, \boldsymbol{y}_{\sigma(u)}^{(0)})\right)}_{L_\infty \text{ relation on } \mathcal{V}}\;.
\end{aligned} \tag{5.9}$$

On the right-hand side, the sign factor and the Grassmann element in front of the L_∞ relation on $\mathcal{V}$ are non-zero. The L_∞ relations on $\mathcal{H}$ are satisfied if the left-hand side vanishes. This is the case if and only if the right-hand side vanishes, i.e. if the L_∞ relations on $\mathcal{V}$ are satisfied. This completes the proof.

B Computations from Section 4.3

Nilpotency of the QED Differential

In the following, we demonstrate the computation that the QED differential (equation (4.27)) squares to zero. We did prove above that the L_∞ relations on the full space $\mathcal{H}$ are satisfied if they are satisfied on the commuting-functions-only field space $\mathcal{V}$. Computations on $\mathcal{V}$ involve fewer degree-related signs and are therefore simpler to perform. Hence, we will show that the differential $\mathrm{d}^{\mathcal{V}}$ on $\mathcal{V}$ squares to zero. We do this component-wise, i.e we show that each term in the linear combination

$$\mathrm{d}^{\mathcal{V}}\left(\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\right) = \sum_{\boldsymbol{\Phi}} \left[\mathrm{d}^{\mathcal{V}}\left(\mathrm{d}^{\mathcal{V}}(\boldsymbol{y})\right)\right]_{\Phi} \boldsymbol{e}_{\Phi} \tag{5.10}$$

vanishes for arbitrary input vector $\boldsymbol{y} = \sum_{\boldsymbol{\Phi}} y_{\Phi} \boldsymbol{e}_{\Phi}$.

We compute:

$$
\begin{aligned}
\boldsymbol{\Phi} = \boldsymbol{c} \quad &: \quad \text{vanishes trivially, since } \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_c = 0 \\
\boldsymbol{\Phi} = \boldsymbol{A} \quad &: \quad \partial_\mu \, \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_c = 0 \\
\boldsymbol{\Phi} = \boldsymbol{\psi} \quad &: \quad \text{vanishes trivially, since } \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_\psi = 0 \\
\boldsymbol{\Phi} = \bar{\boldsymbol{\psi}} \quad &: \quad \text{vanishes trivially, since } \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_{\bar{\psi}} = 0 \\
\boldsymbol{\Phi} = \bar{\boldsymbol{\psi}}^* &: \quad i\partial\!\!\!/ \, \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_\psi = 0 \\
\boldsymbol{\Phi} = \boldsymbol{\psi}^* &: \quad i \, \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_{\bar{\psi}} \overleftarrow{\partial\!\!\!/} = 0 \\
\boldsymbol{\Phi} = \boldsymbol{A}^* &: \quad \Box \, \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_A{}^\nu - \partial^\nu \partial_\mu \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_A{}^\mu = \partial_\mu \partial^\mu (\partial^\nu y_c) - \partial^\nu \partial_\mu (\partial^\mu y_c) = 0 \\
\boldsymbol{\Phi} = \boldsymbol{c}^* \quad &: \quad -\partial_\nu \, \mathrm{d}^{\mathcal{V}}(y)_{A^*}{}^\nu = -\partial_\nu (\partial_\mu \partial^\mu y_{A^*}{}^\nu - \partial^\nu \partial_\mu y_{A^*}{}^\mu) = 0 \quad .
\end{aligned}
$$

The above computations were of course very simple. We showed them here in full detail to introduce the notation in which we present the higher L_∞ relations for QED below.

Second L_∞ relation for QED

The second L_∞ relation is

$$
\mathrm{m}_1\left(\mathrm{m}_2(\boldsymbol{y}_1, \boldsymbol{y}_2)\right) + \sum_{\sigma \in \mathcal{S}_{1,1}} \epsilon(\sigma) \, \mathrm{m}_2\left(\mathrm{m}_1(\boldsymbol{y}_{\sigma(1)}), \boldsymbol{y}_{\sigma(2)}\right) = 0 \, . \tag{5.11}
$$

In the following computations, to safe space, we will drop the summation symbol and the Koszul sign $\epsilon(\sigma)$. Whenever a subscript $_{\sigma(\dots)}$ appears in an equation, we implicitly sum over unshuffles weighted by the respective Koszul sign. Furthermore, we use the following shorthand notation:

$$
y_\Phi y_\Psi \begin{Bmatrix} +12 \\ -21 \end{Bmatrix} := y_{1\Phi} y_{2\Psi} - y_{2\Phi} y_{1\Psi} \tag{5.12}
$$

The sign in this shorthand notation is the Koszul sign of the respective unshuffle. Here we have assumed the fields $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ to be of odd degree so that they anti-commute. Obviously, this need not be the case. As long as we have only two inputs, keeping track of signs is simple. Already when we get to three input vectors, getting the Koszul sign right can be confusing. One should always keep in mind that the Koszul sign depends both on the permutation and the specific vectors, which are permuted.

As before, we compute the L_∞ relation component-wise:

$$\boldsymbol{\Phi}=\boldsymbol{c} \quad : \quad \text{vanishes trivially, since } \mathrm{d}^{\mathcal{V}}(\boldsymbol{y})_c=0 \text{ and } \mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1,\boldsymbol{y}_2)_c=0$$

$$\boldsymbol{\Phi}=\boldsymbol{A} \quad : \quad \partial_\mu 0=0$$

$$\boldsymbol{\Phi}=\boldsymbol{\psi} \quad : \quad i\big(0\cdot y_{\sigma(2)\psi}-y_{\sigma(2)c}\cdot 0\big)=0$$

$$\boldsymbol{\Phi}=\bar{\boldsymbol{\psi}} \quad : \quad i\big(0\cdot y_{\sigma(2)c}-y_{\sigma(2)\bar\psi}\cdot 0\big)=0$$

$$\begin{aligned}\boldsymbol{\Phi}=\bar{\boldsymbol{\psi}}^* : \quad & i\partial\!\!\!/\Big[i\big(y_{\sigma(1)c}\,y_{\sigma(2)\psi}-y_{\sigma(2)c}\,y_{\sigma(1)\psi}\big)\Big]+\Big[(\partial_\mu y_{\sigma(1)c})\,\gamma^\mu\,y_{\sigma(2)\psi}+y_{\sigma(2)A,\mu}\,\gamma^\mu\cdot 0\Big]\\ & +i\Big[0\cdot y_{\sigma(2)\bar\psi^*}+y_{\sigma(2)c}\,(i\partial\!\!\!/ y_{\sigma(1)\psi})\Big]\\ & =\partial\!\!\!/(y_c y_\psi)\begin{Bmatrix}-12\\+21\end{Bmatrix}+(\partial\!\!\!/ y_c)y_\psi\begin{Bmatrix}+12\\-21\end{Bmatrix}+y_c(\partial\!\!\!/ y_\psi)\begin{Bmatrix}+12\\-21\end{Bmatrix}=0\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Phi}=\boldsymbol{\psi}^* : \quad & i\partial\!\!\!/\Big[i\big(y_{\sigma(1)\bar\psi}\,y_{\sigma(2)c}-y_{\sigma(2)\bar\psi}\,y_{\sigma(1)c}\big)\Big]-\Big[0\cdot\gamma^\mu\,y_{\sigma(2)A,\mu}+y_{\sigma(2)\bar\psi}(\partial_\mu y_{\sigma(1)c})\,\gamma^\mu\Big]\\ & -i\Big[(iy_{\sigma(1)\bar\psi}\overleftarrow{\partial\!\!\!/})y_{\sigma(2)c}+y_{\sigma(2)\psi^*}\cdot 0\Big]\\ & =(y_{\bar\psi}y_c)\overleftarrow{\partial\!\!\!/}\begin{Bmatrix}-12\\+21\end{Bmatrix}+y_{\bar\psi}(\partial\!\!\!/ y_c)\begin{Bmatrix}+12\\-21\end{Bmatrix}+(y_{\bar\psi}\overleftarrow{\partial\!\!\!/})y_c\begin{Bmatrix}+12\\-21\end{Bmatrix}=0\end{aligned}$$

$$\boldsymbol{\Phi}=\boldsymbol{A}^* : \quad \Box 0-\partial^\nu\partial_\mu 0-\big(0\cdot\gamma^\nu\,y_{2\psi}-y_{2\bar\psi}\,\gamma^\nu\cdot 0\big)=0$$

$$\begin{aligned}\boldsymbol{\Phi}=\boldsymbol{c}^* \quad : \quad & -\partial_\nu\Big[-\big(y_{\sigma(1)\bar\psi}\,\gamma^\nu\,y_{\sigma(2)\psi}-y_{\sigma(2)\bar\psi}\,\gamma^\nu\,y_{\sigma(1)\psi}\big)\Big]\\ & +i\Big[(iy_{\sigma(1)\bar\psi}\overleftarrow{\partial\!\!\!/})\,y_{\sigma(2)\psi}+y_{\sigma(2)\psi^*}\cdot 0\Big]-i\Big[0\cdot y_{\sigma(2)\bar\psi^*}+y_{\sigma(2)\bar\psi}\,(i\partial\!\!\!/ y_{\sigma(1)\psi})\Big]\\ & =\partial_\nu(y_{\bar\psi}\,\gamma^\nu\,y_\psi)\begin{Bmatrix}+12\\-21\end{Bmatrix}+y_{\bar\psi}(\partial\!\!\!/ y_\psi)\begin{Bmatrix}-12\\+21\end{Bmatrix}+(y_{\bar\psi}\overleftarrow{\partial\!\!\!/})y_\psi\begin{Bmatrix}-12\\+21\end{Bmatrix}=0 \quad .\end{aligned}$$

For two inputs, all components of the L_∞ relation vanish and the relation is satisfied as claimed in the main text.

Third L_∞ relation for QED

We only need to check one more L_∞ relation: the one with three inputs. For QED, all higher products than the 2-inputs product vanish. Thus all higher relations are satisfied trivially. The third L_∞ relation for QED is

$$\sum_{\sigma\in\mathcal{S}_{2,1}}\epsilon(\sigma)\ \mathrm{m}_2\left(\mathrm{m}_2(\boldsymbol{y}_{\sigma(1)},\boldsymbol{y}_{\sigma(2)}),\boldsymbol{y}_{\sigma(3)}\right)=0\ . \tag{5.13}$$

We employ the short-hand notation introduced above, and summation over unshuffles is again implicit. In the computations for various components, we use that the field configurations are all commuting. This is the reason why sometimes we freely exchange the indices of identical field configurations from one step to the next. No other computational tricks are used. All the signs follow from direct computation and from the Koszul sign of the respective unshuffle.

We compute the L_∞ relation component-wise:

$$\boldsymbol{\Phi} = \boldsymbol{c} \quad : \quad \text{vanishes trivially, since } \mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1, \boldsymbol{y}_2)_c = 0$$

$$\boldsymbol{\Phi} = \boldsymbol{A} \quad : \quad \text{vanishes trivially, since } \mathrm{m}_2^{\mathcal{V}}(\boldsymbol{y}_1, \boldsymbol{y}_2)_A = 0$$

$$\begin{aligned}
\boldsymbol{\Phi} = \boldsymbol{\psi} \quad : \quad & i\Big[0 \cdot y_{\sigma(3)\psi} - y_{\sigma(3)c}\, i\big(y_{\sigma(1)c}\, y_{\sigma(2)\psi} - y_{\sigma(2)c}\, y_{\sigma(1)\psi}\big)\Big] \\
& = y_c y_c y_\psi \begin{Bmatrix} +312 - 321 \\ -213 + 231 \\ +123 - 132 \end{Bmatrix} = y_c y_c y_\psi \begin{Bmatrix} +312 - 231 \\ -123 + 231 \\ +123 - 312 \end{Bmatrix} = 0
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Phi} = \bar{\boldsymbol{\psi}} \quad : \quad & i\Big[i\big(y_{\sigma(1)\bar\psi}\, y_{\sigma(2)c} - y_{\sigma(2)\bar\psi}\, y_{\sigma(1)c}\big)\, y_{\sigma(3)c} - y_{\sigma(3)\bar\psi} \cdot 0\Big] \\
& = y_{\bar\psi} y_c y_c \begin{Bmatrix} -123 + 213 \\ +132 - 312 \\ -231 + 321 \end{Bmatrix} = y_{\bar\psi} y_c y_c \begin{Bmatrix} -132 + 231 \\ +132 - 321 \\ -231 + 321 \end{Bmatrix} = 0
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Phi} = \bar{\boldsymbol{\psi}}^{\boldsymbol{*}} : \quad & \Big[(0 \cdot \gamma^\mu\, y_{\sigma(3)\psi} + y_{\sigma(3)A,\mu}\, \gamma^\mu\, i\big(y_{\sigma(1)c}\, y_{\sigma(2)\psi} - y_{\sigma(2)c}\, y_{\sigma(1)\psi}\big)\Big] \\
& + i\Big[0 \cdot y_{\sigma(3)\bar\psi^*} + y_{\sigma(3)c}\, \big(y_{\sigma(1)A,\mu}\, \gamma^\mu\, y_{\sigma(2)\psi} + y_{\sigma(2)A,\mu}\, \gamma^\mu\, y_{\sigma(1)\psi}\big) \\
& \qquad + y_{\sigma(3)c}\, i\big(y_{\sigma(1)c}\, y_{\sigma(2)\bar\psi^*} + y_{\sigma(2)c}\, y_{\sigma(1)\bar\psi^*}\big)\Big] \\
& = i\, y_{A,\mu} \gamma^\mu y_c y_\Psi \begin{Bmatrix} +312 - 321 + 132 + 231 \\ +213 - 231 - 123 + 321 \\ +123 - 132 - 213 - 312 \end{Bmatrix} + y_c y_c y_{\bar\psi^*} \begin{Bmatrix} -312 - 321 \\ -213 + 231 \\ +123 + 132 \end{Bmatrix} \\
& = 0 + y_c y_c y_{\bar\psi^*} \begin{Bmatrix} -312 - 231 \\ -123 + 231 \\ +123 + 312 \end{Bmatrix} = 0
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Phi} = \boldsymbol{\psi}^{\boldsymbol{*}} : \quad & -\Big[i\big(y_{\sigma(1)\bar\psi}\, y_{\sigma(2)c} - y_{\sigma(2)\bar\psi}\, y_{\sigma(1)c}\big)\, \gamma^\mu\, y_{\sigma(3)A,\mu} + y_{\sigma(3)\bar\psi} \cdot 0 \cdot \gamma^\mu\Big] \\
& - i\Big[-\big(y_{\sigma(1)\bar\psi}\, y_{\sigma(2)A,\mu}\, \gamma^\mu + y_{\sigma(2)\bar\psi}\, y_{\sigma(1)A,\mu}\, \gamma^\mu\big)\, y_{\sigma(3)c} - i\big(y_{\sigma(1)\psi^*}\, y_{\sigma(2)c} \\
& \qquad + y_{\sigma(2)\psi^*}\, y_{\sigma(1)c}\big)\, y_{\sigma(3)c} + y_{\sigma(3)\psi^*} \cdot 0\Big] \\
& = i\, y_{\bar\psi} y_c \gamma^\mu y_{A,\mu} \begin{Bmatrix} -123 + 213 + 132 + 231 \\ -132 + 312 + 123 - 321 \\ -231 + 321 - 213 - 312 \end{Bmatrix} + y_{\psi^*} y_c y_c \begin{Bmatrix} -123 - 213 \\ +132 - 312 \\ +231 + 321 \end{Bmatrix} \\
& = 0 + y_{\psi^*} y_c y_c \begin{Bmatrix} -132 - 231 \\ +132 - 321 \\ +231 + 321 \end{Bmatrix} = 0
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\Phi} = \boldsymbol{A^*} : \quad & - \Big[i \big(y_{\sigma(1)\bar{\psi}}\, y_{\sigma(2)c} - y_{\sigma(2)\bar{\psi}}\, y_{\sigma(1)c} \big)\, \gamma^{\nu}\, y_{\sigma(3)\psi} \\
& \qquad - y_{\sigma(3)\bar{\psi}}\, \gamma^{\nu}\, i \big(y_{\sigma(1)c}\, y_{\sigma(2)\psi} - y_{\sigma(2)c}\, y_{\sigma(1)\psi} \big) \Big] \\
& = i\, y_{\bar{\psi}} y_c y_{\psi} \begin{Bmatrix} -123 + 213 + 312 - 321 \\ +132 - 312 - 213 + 231 \\ -231 + 321 + 123 - 132 \end{Bmatrix} = 0
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Phi} = \boldsymbol{c^*} \quad : \quad & i \Big[- \big(y_{\sigma(1)\bar{\psi}}\, y_{\sigma(2)A,\mu}\, \gamma^{\mu} + y_{\sigma(2)\bar{\psi}}\, y_{\sigma(1)A,\mu}\, \gamma^{\mu} \big)\, y_{\sigma(3)\psi} \\
& \quad - i \big(y_{\sigma(1)\psi^*}\, y_{\sigma(2)c} + y_{\sigma(2)\psi^*}\, y_{\sigma(1)c} \big)\, y_{\sigma(3)\psi} \\
& \quad + y_{\sigma(3)\psi^*}\, i \big(y_{\sigma(1)c}\, y_{\sigma(2)\psi} - y_{\sigma(2)c}\, y_{\sigma(1)\psi} \big) \Big] \\
& - i \Big[i \big(y_{\sigma(1)\bar{\psi}}\, y_{\sigma(2)c} - y_{\sigma(2)\bar{\psi}}\, y_{\sigma(1)c} \big)\, y_{\sigma(3)\bar{\psi}^*} \\
& \qquad + y_{\sigma(3)\bar{\psi}}\, \big(y_{\sigma(1)A,\mu}\, \gamma^{\mu}\, y_{\sigma(2)\psi} + y_{\sigma(2)A,\mu}\, \gamma^{\mu}\, y_{\sigma(1)\psi} \big) \\
& \qquad + y_{\sigma(3)\bar{\psi}}\, i \big(y_{\sigma(1)c}\, y_{\sigma(2)\bar{\psi}^*} + y_{\sigma(2)c}\, y_{\sigma(1)\bar{\psi}^*} \big) \Big] \\
& = i\, y_{\bar{\psi}} y_{A,\mu} \gamma^{\mu} y_{\psi} \begin{Bmatrix} -123 - 213 - 312 - 321 \\ -132 + 312 + 213 - 231 \\ +231 + 321 + 123 + 132 \end{Bmatrix} + y_{\psi^*} y_c y_{\Psi} \begin{Bmatrix} +123 + 213 - 312 + 321 \\ -132 + 312 - 213 + 231 \\ -231 - 321 - 123 + 132 \end{Bmatrix} \\
& \quad + y_{\bar{\psi}} y_c \gamma^{\mu} y_{A,\mu} \begin{Bmatrix} +123 - 213 + 312 + 321 \\ +132 - 312 + 213 - 231 \\ +231 - 321 - 123 - 132 \end{Bmatrix} = 0 \quad .
\end{aligned}
$$

All components vanish. Hence, the third L_∞ relation is satisfied for QED. As mentioned, all higher relations vanish trivially. Thus, the m_s products we had computed for QED indeed define a homotopy Lie algebra.

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Declaration of Authorship

I hereby declare that I have written this thesis independently and that I have not used any sources and aids other than those indicated in the thesis.

Munich, 22.07.2022 Sebastian Albrecht