Dynamical Black Holes in Topologically Massive Gravity

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Abstract

We investigate a dynamical metric that was discovered by Sachs [1] to be an exact vacuum solution of the field equations of Topologically Massive Gravity. After deriving the geodesic equations and solving them analytically for a special class of Killing geodesics, the metric is shown to describe a rotating non-stationary black hole with a timelike singularity in the causal structure of the spacetime. While its closed trapped surfaces can be obtained analytically in suitable coordinates, the inner and outer horizon can only be calculated numerically. We analyse the global structure of the corresponding spacetime in dependence of the topological mass of the gravity μ, and in particular investigate the evolution of the outer horizon showing that it may increase or decrease with time, depending on the value of the mass parameter. We calculate the time dependent entropy in TMG using an ansatz combining the approaches of Tachikawa and Wald and Iyer. We additionally show that the metric solves the equation of motion of New Massive Gravity and calculate the dynamic entropy of the black hole for the framework of this theory by using an ansatz proposed by Hayward. Furthermore, special attention is paid to the cases $\mu = \pm 1$, where we show that the metric reduces to previously known black hole solutions of the $2+1$ dimensional Einstein-Hilbert Gravity.

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Contents

Chapter 1

Introduction

Topologically Massive Gravity (TMG) was proposed by Deser, Jackiw and Tempelton in 1982 [2, 3] as a higher derivative model of gravity in $2+1$ dimensions, and the field equations of this theory are solved by the black hole metrics discovered by Bañados, Teitelboim and Zanellli [4, 5]. In a recent publication, an exact solution to TMG was presented that seems to describe a BTZ black hole perturbed by a pp-wave [1]. Initially, apart from the fact that this metric is a solution of TMG and some other properties, not much was known about this spacetime. In this thesis, the new solution is investigated in detail, and many interesting new findings are presented.

The structure of this work is as follows: In chapter 2 some basic properties of black holes in arbitrary dimensions will be reviewed, with an emphasis on topics that will be of interest in the later analysis. Afterwards, in section 3, we will review several models of gravity in $2+1$ and comment on their respective properties. In section 4 we review the BTZ black holes and their linear perturbations that led to the discovery of [1]. Chapter 5 is denoted to discussing the metric discovered in [1] and its local properties, such as Killing symmetries and lightcones, while chapter 6 focuses on properties that characterize the global structure of the spacetime, such as the singularity or event and trapping horizons. It will be argued that the metric under investigation indeed describes an exactly known dynamical vacuum black hole solution of TMG, in contrast to other exactly known dynamical black hole solutions (such as the Oppenheimer-Snyder metric [6] in four dimensions) that require the presence of matter fields. The metric at hand therefore naturally lends itself as a testing ground for competing definitions of dynamical black hole properties. In sections 7 and 8 we will therefore apply two competing definitions of black hole entropy to the dynamical black holes, unfortunately in the framework of two different models of gravity in three dimensions. We will close with a chapter on open questions and possibilities for further research, and with an extensive appendix.

In the entire thesis, we will use units in which $\hbar = c = k = 1$ and the convention that spacetime indices in d dimensions take values μ, ν in {1,2,...,d}.

Part I

Theoretical Preliminaries

Chapter 2

Physical Properties of Black Holes

2.1 Overview and History

Shortly after Albert Einstein proposed the theory of General Relativity (GR) as a theory of gravity, Karl Schwarzschild found the exact, static, spherically symmetric vacuum solution nowadays known as the Schwarzschild metric in 1916 (see e.g. [7]). It was later verified that this metric can be used to approximately describe the gravitational field outside of a gravitating object as it reproduces the correct Newtonian gravitational potential for large distances [7–9] and predicts experimentally accessible phenomena such as the perihelion shift of mercury [8, 9] and the deflection of light by the sun's gravitational force [8]. Initially, it was widely believed that the metric "ended" at the coordinate singularity and the interior of the Schwarzschild solution had no physical meaning [9]. For a full understanding of Schwarzschild's solution it took an extensive study of geodesics (and their completeness) and the advent of the Painlevé-Gullstrand coordinates in 1921, the Eddington-Finkelstein coordinates in 1924 and the Kruskal-Szekeres coordinates in 1960 [7, 9]. Even the word "black hole" that nowadays found its way from scientific language into pop culture was not used before 1967 [7].

Starting from Schwarzschild's insights, research proceeded in different directions: His stationary metric was generalized by Reissner and Nordström in 1916 respectively 1918 and by Kerr in 1963 and Newman in 1965 (see e.g. [7]). Other researchers realized that the "eternal black holes" found by Schwarzschild and others can at best be used as approximations to real astronomical objects, and it was investigated whether black holes could be formed by the collapse of matter, resulting in dynamical black hole metrics such as the Oppenheimer-Snyder solution [6]. Aside from such theoretical achievements, it was a great success for Einsteins theory of gravity when astronomers finally discovered objects such as Cygnus X-1 that could be best described by the mathematical model of black holes [8]. In the near future, interesting new observations might be made on the central black hole of our Galaxy, Sgr A* [10, 11]. A major breakthrough in gravitational and black hole physics came in the seventies with the formulation of Hawking radiation, black hole entropy and black hole thermodynamics (see [7] and references therein). It is this previously unexpected connection between gravitational physics, quantum physics and statistical physics that makes black holes such an interesting research object among theoretical physicists.

In this section, we will review important definitions in the context of black hole physics, many of which will be important throughout this thesis.

2.2 Event Horizons

Before dealing with strict mathematical definitions, it is worthwhile to first discuss some general ideas. The notion of a black hole is that it is a part of spacetime from which it is not possible to escape to the outside [7,8]. From this basic idea, it is already clear that the important thing about a black hole will be its boundary, and not its deep interior[1]. It is also obvious from these thoughts that the definition of a black hole will require a precise notion of "outside of" or "far away from" the black hole. One possible definition reads:

Definition [7,8]: An asymptotically flat spacetime M is said to contain a *black hole region* B if the causal past of the future null infinity, $I^-(\mathcal{I}^+)$, does not cover the entire spacetime M. B is then defined as the complement of $I^-(\mathcal{I}^+)$ in M. The boundary of B is called the *event horizon*.

There are some features about this definition that will be of importance in this thesis and should be pointed out here. First of all, the above definition technically only holds for asymptotically flat spacetimes. Of course, similar definitions can be set

[1]Although, of course, in some cases the existence of a curvature singularity in the center of the black hole can be inferred by using the singularity theorems [12]. Also, it is widely believed that while event horizons actually exist in nature, black hole metrics are not trustworthy in the deep interior of the black hole due to quantum effects of gravity and even classical instabilities such as mass inflation, see [7].

up whenever one has a clear notion of "asymptotic infinity" as for example in the case of asymptotically AdS spaces [8]. This corresponds to the necessity of finding a precise notion of "outside" as mentioned above. However, there exist spacetime-asymptotics in which it is not possible to globally define black holes following the line of thought of above, the $k = +1$ Robertson-Walker universe being an example [8].

Secondly, it can be proven mathematically [7], and in some sense this is already clear from the above definition, that event horizons will always be generated by null geodesics. But how can one find out *which* null geodesics generate the event horizon? This obviously requires knowledge of the entire future evolution of the spacetime. When analytically studying a metric which was obtained as a solution to Einsteins field equations this is not necessarily a problem, but for example in numerical relativity, where the evolution of a spacelike slice of spacetime is calculated step by step into the future, it is not possible to determine the event horizon. As the definition makes reference to the asymptotic infinity (and future), the definition of black holes is intrinsically nonlocal which is called the *teleological nature* of event horizons [7]. Researchers have therefore tried to come up with alternative definitions of black hole boundaries, and one of them will be discussed in the following section.

2.3 Trapping Horizons

Again, before giving the precise mathematical definition, we will discuss the principal idea first, see [12]. Suppose that there is a sphere in flat spacetime which emits flashes of coherent light from time to time. The light from a certain flash spreads out and forms, according to Huygens' principle, wavefronts of spherical form. The inward-bound wavefront will initially be shrinking while the outward-bound wavefront will be expanding. Now suppose that a point mass is placed in the center of the sphere, turning the flat space into the (asymptotically flat) Schwarzschild spacetime. If the radius of the sphere is large, there will be no qualitative change, but if the radius of the sphere is smaller than the Schwarzschild radius, both wavefronts will be pulled towards the center of the sphere, and will therefore be shrinking with time. If the radius of the sphere is exactly equal to the Schwarzschild radius, the inner wavefront will be shrinking while the outer will have constant size and define the event horizon. This will be called the trapping horizon and as this example already shows, event and trapping-horizons coincide for stationary black holes [13, 14]. For dynamic black holes, however, these two definitions of black hole boundaries do in general not agree, and it is not clear which one of both will be the "legitimate" description of black holes.

Definition [7,15,16]: In a d dimensional spacetime, the *expansion* θ of a null geodesic vector field u^α is defined to be

$$\theta = \frac{1}{d-2} u^\alpha_{;\alpha} \tag{2.1}$$

In $d \geq 3$ dimensional spacetimes, there are actually three so-called *optical scalars*: *expansion*, *shear* and *twist*. The latter two trivially vanish in three dimensional spacetimes [15,16]. This definition allows us to mathematically formalize the ideas presented above in the following definitions:

Definition [13,17]: Within a d dimensional spacetime, a *trapped surface* is a $(d-2)$ dimensional, closed, compact, spacelike surface S such that for the expansions of the two families of future pointing null geodesics orthogonal to S, θ_+ and θ_-, $\theta_+\theta_- > 0$ holds everywhere on S. The surface is called *past trapped* or *anti trapped* when $\theta_\pm > 0$ everywhere on S, and *future trapped* when $\theta_\pm < 0$ everywhere on S.

Past trapped surfaces are typical for the interiors of white holes while future trapped surfaces are typical for black hole interiors. In order to describe black and white hole *boundaries*, this definition has to be refined in the following way:

Definition [13]: A *marginal surface* is a $(d-2)$ dimensional, closed, compact spacelike surface S such that either θ_+ or θ_- (but not both) vanish on S.

How can such marginal surfaces be calculated? Suppose that the closed spacelike surface S is embedded into a $(d-1)$ dimensional spacelike slice Σ of the space time $\mathcal{M}$. We denote by n^μ the future pointing timelike normal vector to Σ (with $n_\mu n^\mu = -1$), and by s^μ the outward pointing spacelike vector normal to S in Σ[2] with $s_\mu s^\mu = 1$. Furthermore, $g_{\mu\nu}$ is the metric on $\mathcal{M}$ while $\mathfrak{g}_{ij}$ is the induced metric on Σ ($i, j \in \{2, ...d\}$) and K_{ij} is the extrinsic curvature of Σ in $\mathcal{M}$. Then, for the outward-pointing null vector $k^\mu = n^\mu + s^\mu$ orthogonal to S, assuming affine parametrization $k^\nu k^\mu_{;\nu} = 0$, the condition $k^\mu_{;\mu} = 0$ everywhere on S is equivalent to [7,18]

$$s^i_{:i} + K_{ij} s^i s^j - \mathfrak{g}^{ij} K_{ij} = 0 \tag{2.2}$$

everywhere on S. Here $s^i_{:i}$ denotes the covariant derivative with respect to the metric $\mathfrak{g}_{ij}$. A sign change of s^i has the same effect as a sign change in the definition of K_{ij},

[2] If a proper notion of infinity is available, it can always be determined which direction is "outward", otherwise the distinction between outward and inward pointing vectors is an arbitrary choice [7]

and makes it possible to check whether the expansion of the ingoing null vector field orthogonal to S vanishes. In section 2.5 we will learn that in spacetimes with the symmetry of a $(d-2)$ dimensional sphere, there is an easier way to calculate marginal surfaces.

Finally, we propose a definition of black (and white) hole boundaries that will be used later on as an alternative to event horizons:

Definition [13, 19]: A *trapping horizon* $\bar{H}$ is the closure of a $(d-1)$ dimensional surface H foliated by marginal surfaces with $\theta_a \neq 0$ and $\mathcal{L}_a \theta_b \neq 0$ everywhere on H. Here, we use the notation $a, b \in \{+, -\}$ and $\mathcal{L}_\pm$ denotes the Lie-derivative with respect to the out- or ingoing null geodesic vector field orthogonal to the marginal surfaces.

2.4 Properties of Black Hole Boundaries

There are still other possible notions of black hole boundaries, for example so-called *apparent horizons* [7], but these follow basically the same line of thought as the definition of trapping horizons and differ from them only in mathematical details, see [20,21] for an overview. In this thesis we will therefore restrict ourselves to the study of event and trapping horizons, see sections 6.2 and 6.3. Before proceeding, let us compare the properties of event and trapping horizons.

Event horizons are by definition null surfaces. Their definition is a global one and requires knowledge of the entire future of the spacetime as well as a precise notion of asymptotic infinity. Thus, the definition captures the idea that a black hole is a region in spacetime from which (classically) nothing will *ever* be able to escape. There are important theorems describing the dynamics of event horizons, such as the Hawking area theorem [7, 12].

In contrast, trapping and apparent horizons may be hypersurfaces of any signature, but they will always be null or spacelike when certain conditions hold [7, 13]. Furthermore, it can be proven that when the weak energy condition holds, the apparent horizon either lies inside of or coincides with an event horizon [7]. This means that in many cases, the trapping and apparent horizons are quantities that may be determined locally, but not by an outside observer. Such a case will for example be discussed in section 6.2. Apparent horizons may also have discontinuities and jumps [7, 20] and an example of possible unphysical behaviour of a trapping horizon will be discussed in section 6.2. There exist versions of the area law for trapping horizons [13], and the existence of trapped surfaces is an often used assumption in theorems, for example in the singularity theorems [12, 22]. Another important point is that the existence of trapping horizons is not necessary for the existence of event horizons, as discussed in [7].

Because of these facts, it seems that event horizons are the more natural definition of black hole boundaries. Of course, different geometrical objects might be useful for investigating different questions. As one key questions of this thesis will be the entropy of dynamical black holes, it should be noted that there is research that seems to indicate that in fact trapping or apparent horizons are the objects that entropy should be associated with, and not event horizons. See [23, 24] and [14, 25, 26] for two different approaches to dynamic black hole entropy that both favour trapping or apparent horizons over event horizons.

2.5 The Kodama Vector

In 1980, Kodama investigated four dimensional black hole spacetimes with spherical symmetry. He found that in this case a vector field can be defined which coincides with the timelike Killing vector in the stationary case at least up to normalization and thereby offers a possible generalization of the timelike Killing vector to dynamic spacetimes [14, 25–29]. We will now present a generalization of this approach to dimensions $d \geq 3$[3]:

Suppose we have a d dimensional spacetime $\mathcal{M}$ which has the symmetry of a $(d-2)$ dimensional (hyper-)sphere S^{d-2} with all corresponding Killing vectors being spacelike. Starting from any point $\mathcal{P}$ in the spacetime and following the flows of the Killing vectors of this symmetry will generate a $(d-2)$-sphere as spacelike submanifold. This sphere will be a geometrical invariant, therefore its $(d-2)$-volume $\mathcal{V}$ and its thereby defined aerial radius $r = \left(\mathcal{V}\frac{\Gamma((d-1)/2))}{2\pi^{(d-1)/2}}\right)^{\frac{1}{d-2}}$ will be coordinate invariant scalar quantities defined at every point in the spacetime. Because of this, $\nabla_\mu r = \partial_\mu r$ will fix a well-defined one-form. This one form can now be contracted with the binormal[4] $\epsilon_{\mu\nu}$ of the 2 dimensional space orthogonal to the $(d-2)$-sphere at $\mathcal{P}$ to yield the Kodama

[3] While the sources used in this thesis [14, 25–29] restrict their discussion to four dimensional spacetimes, a generalization of the Kodama vector to d dimensions has been discussed in [30, 31]. Nevertheless these authors assume that the coordinate system can be brought to a block diagonal (or *warped product*) form $ds^2 = g_{\alpha\beta}dy^\alpha dy^\beta + r^2(y)\gamma_{ij}(z)dz^i dz^j$ (with $\alpha, \beta \in \{1, 2\}$, $i, j \in \{3, ..., d\}$), which is not necessarily possible for a three dimensional metric with rotational symmetry.

[4] We define the binormal to a closed spacelike surface S as $\epsilon^{\mu\nu} = l^\mu n^\nu - l^\mu n^\nu$ where l^μ is the ingoing and n^μ is the outgoing null vector field orthogonal to S with $l^\mu n_\mu = -1$ [32]. It obviously follows $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$.

vector

$$k^\mu = \epsilon^{\mu\nu}\partial_\nu r \tag{2.3}$$

as it was defined for $d = 4$ in [14,26]. For four dimensional spacetimes, many interesting properties can be proven about the Kodama vector, some of which will now be stated here:

The geometrical meaning of the Kodama vector field is that it is tangent to constant r-hypersurfaces, as obviously $k(r) = k^\mu\partial_\mu r = \epsilon^{\mu\nu}\partial_\nu r\partial_\mu r = 0$ due to the antisymmetry of $\epsilon^{\mu\nu}$ [28]. Therefore, the Kodama vector field is spacelike in trapped regions, null on trapping horizons and timelike otherwise [28]. Therefore, it is easy to calculate trapping horizons when the Kodama vector field is known. This is an important point, as from the definitions above it seems that trapping horizons might be foliation dependent quantities. But as the Kodama vector naturally behaves well under coordinate transformations, we see that in cases where the Kodama vector can be defined the trapping horizons will certainly not be foliation dependent. Furthermore, the Kodama vector is conserved: $k^\mu_{;\mu} = 0$ [28]. The charge associated with the flow of the Kodama vector field is the aerial volume $\frac{4}{3}\pi r^3$ of the 2-spheres [28].

Kodama and Killing vector agree in stationary, spherically symmetric spacetimes if the vector fields k^μ and $g^{\mu\nu}\partial_\nu r$ commute [29].

Whether these conditions also hold in the 3 dimensional spacetimes investigated in this work will be discussed in section 8.2.

2.6 Black Hole Entropy

There is a very simple argument that shows why black holes must have entropy if they are supposed to be consistent with classical thermodynamics. If a solid body with non-vanishing entropy falls into a black hole, and if there is no entropy assigned to the black hole, then entropy seems to vanish from the universe [7]. The Hawking area theorem proves that in general relativity, as long as the null energy condition holds, the event horizon area $\mathcal{A}$ cannot decrease as a function of time (see [12]), similarly to the entropy according to the second law of thermodynamics. This similarity led Bekenstein to propose that the entropy of a black hole should be proportional to its horizon area [33], but the proportionality constant was at first unknown. It was established in 1974, when Hawking proved [34, 35] that black holes emit radiation at the so-called *Hawking temperature* $T_H = \frac{\kappa}{2\pi}$, where κ is referred to as *surface gravity.*

With this knowledge, a comparison of the first law of thermodynamics

$$dE = TdS$$

with the first law of (for simplicity uncharged and non-rotating) black hole thermodynamics

$$dM = \frac{\kappa}{8\pi} d\mathcal{A}$$

yields the Bekenstein-Hawking entropy $S = \frac{\mathcal{A}}{4G_N}$ (for Einstein-Hilbert gravity) when $T = T_H$ is assumed (see e.g. [7]).

In [36] and the subsequent papers [37, 38] an algorithm was presented which made it possible to calculate the entropy $\mathcal{S}$ of a stationary black hole in the framework of an arbitrary covariant theory of gravity as the Noether charge corresponding to a certain Killing vector field. A d dimensional covariant theory is meant to be a theory with a Lagrangian d-form L that transforms under diffeomorphisms generated by the vector field ξ as $\delta_\xi L = \mathcal{L}_\xi L$. According to this definition, Lagrangians that involve so-called Chern-Simons terms (or short CS) terms, as the one stated in section 3.2, are not covariant as they transform as[5]

$$\delta_\xi L = \mathcal{L}_\xi L + d\Xi_\xi \tag{2.4}$$

with the $(d-1)$-form Ξ_ξ [32]. It was shown in [32,39] how the Noether charge approach to black hole entropy can be generalized to such theories, and in this subsection we will present these generalized findings, restricting ourselves to so-called *pure gravitational CS terms* [39].

In the following, we will assume a d dimensional theory with a Lagrangian d-form

$$L = L_{cov} + L_{CS} \tag{2.5}$$

that can be divided into a covariant part and a part containing a CS term L_{CS}. Taking now the first-order variation δ of (2.5) yields [36, 39]

$$\delta L(g_{\mu\nu}) = E^{\mu\nu}\delta g_{\mu\nu} + d\Theta(g_{\mu\nu}, \delta g_{\mu\nu})$$

with the equations of motion d-form $E^{\mu\nu}$ and the *symplectic potential* $(d-1)$-form Θ.

[5] Here it is understood that the Lie-derivative $\mathcal{L}$ acts on objects such as the Christoffel symbols as if their indices where tensorial [39].

In the following, we will drop the spacetime-indices of the metric. We will also define the quantities [38, 39]:

$$\omega(g, \delta_1 g, \delta_2 g) = \delta_1 \Theta(g, \delta_2 g) - \delta_2 \Theta(g, \delta_1 g)$$
$$\Omega(g, \delta_1 g, \delta_2 g) = \int_{\mathcal{C}} \omega(g, \delta_1 g, \delta_2 g)$$

where $\mathcal{C}$ is assumed to be a Cauchy surface.

As the diffeomorphism generated by ξ is a symmetry, there is a (on-shell) conserved Noether current $(d-1)$-form [32, 36, 38, 39]

$$J_\xi = \Theta_\xi - \iota_\xi L - \Xi_\xi$$

with a $(d-2)$-form Q_ξ such that

$$J_\xi \approx dQ_\xi$$

Here, $\approx$ indicates that this is only true on-shell, i.e. when the equations of motion are fulfilled. It is this *Noether charge* $(d-2)$*-form* that is crucial for calculating conserved quantities. One can now show [32, 39] that there must exist a $(d-2)$-form Σ_ξ such that

$$\delta_\xi \Theta - \mathcal{L}_\xi \Theta - \delta \Xi_\xi \approx d\Sigma_\xi \tag{2.6}$$

For a pure gravitational Chern-Simons term it was proven in [39] that Σ_ξ vanishes in three dimensions. For the covariant part of the action it vanishes by definition. Thus, we will on omit Σ_ξ in the following.

Using (2.6) it is possible to show that [32, 36, 39]

$$\delta J_\xi \approx \omega(g, \delta g, \delta_\xi g) + d(\iota_\xi \Theta) \tag{2.7}$$

If it was now possible to find a C_ξ such that $\delta C_\xi = \iota_\xi \Theta$, one could define the quantity $Q'_\xi = Q_\xi - C_\xi$ for which $\delta dQ'_\xi \approx \omega(g, \delta g, \delta_\xi g)$ [39].

We will now discuss how these calculations yield the first law of black hole thermodynamics and the black hole entropy $\mathcal{S}$ for stationary black holes with a bifurcate Killing horizon that is generated by a vector field ξ which a is Killing field and hence $\delta_\xi g = \mathcal{L}_\xi g = 0$ [38]. Therefore, $\omega(g, \delta g, \delta_\xi g) = 0$ and assuming that δg fulfills the linearized equations of motion, (2.7) yields [38]

$$d\delta Q - d(\iota_\xi \Theta) \approx 0 \tag{2.8}$$

Integration of (2.8) on a Cauchy surface $\mathcal{C}$ yields [38]

$$\int_{\partial\mathcal{C}} \delta Q - \iota_\xi \Theta \approx \delta \int_{\partial\mathcal{C}} Q' \approx 0 \tag{2.9}$$

Assuming now $\xi = \partial_t + \Omega\partial_\phi$, and as $\mathcal{C}$ has boundaries at infinity and at the bifurcation surface Σ, this results in [38]

$$\delta \int_\Sigma Q'_\xi \approx \delta \int_\infty Q'_{\partial_t} + \Omega\, \delta \int_\infty Q'_{\partial_\phi}$$

The right-hand side can be identified with the expression $\delta M - \Omega\delta J$, where M and J are mass and angular momentum of the black hole, respectively, while the left-hand side can be identified with $\frac{\kappa}{2\pi}\delta\mathcal{S}$ [38]. On the bifurcation surface Σ we can assume [36] that dQ'_ξ is only a function of ξ $\nabla_\mu \xi_\nu$ with $\xi = 0$ and $\nabla_\mu \xi_\nu = \kappa\epsilon_{\mu\nu}$ on the bifurcation surface where we introduced the *surface gravity* κ and the *binormal* $\epsilon_{\mu\nu}$. Using this, the first law of black hole thermodynamics can finally be derived as

$$\frac{\kappa}{2\pi}\delta\mathcal{S} \approx \delta M - \Omega\delta J$$

by using the definition [32, 38]

$$\mathcal{S} \equiv 2\pi \int_\Sigma Q'_\xi\big|_{\xi\to 0, \nabla_\mu\xi_\nu \to \epsilon_{\mu\nu}}$$

It should be noted that $C_\xi \to 0$ as $\xi \to 0$, and therefore $Q' = Q$ on the bifurcation surface Σ' [39].

In the case of an action of the form $S = \frac{1}{16\pi G_N}\int dx^d\sqrt{-g}L$ with covariant Lagrangian $L = L(g_{\mu\nu}, R_{\alpha\beta\gamma\delta}, \nabla_\mu R_{\alpha\beta\gamma\delta}, ...)$, the expression for the entropy simplifies to [38]

$$\mathcal{S} = \frac{-2\pi}{16\pi G_N}\int_\Sigma X^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} \tag{2.10}$$

with $X^{\alpha\beta\gamma\delta} \equiv \frac{\partial L}{\partial R_{\alpha\beta\gamma\delta}}$. An extensive discussion of the Noether charge approach and its subtleties can be found in the references [32, 36–39].

Chapter 3

Models of Gravity in three dimensions

3.1 Einstein-Hilbert Gravity

When trying to find a model for gravity in three dimensional spacetime, the natural approach is to use the three dimensional version of the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_N} \int d^3x \sqrt{-g}\,(R - 2\Lambda) \tag{3.1}$$

with Newtons constant G_N and, in general, a cosmological constant Λ. In this thesis we will often be interested in asymptotically AdS-black holes which require a negative cosmological constant, so we will often set $\Lambda = -\frac{1}{l^2}$ with $l > 0$ in the following. The equations of motion following from (3.1) are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \tag{3.2}$$

where we defined the modified Einstein tensor $G_{\mu\nu}$. For $\Lambda < 0$, these equations are for example solved by the three dimensional Anti-de Sitter spacetime, short AdS_3.

Unfortunately, in contrast to the four dimensional case, the theory (3.1) has no propagating bulk degrees freedom. In the linearized theory of d dimensional Einstein-Hilbert gravity (with $d \geq 3$), there are only $\frac{1}{2}d(d-3)$ transverse-traceless components of the metric, yielding zero in the case considered here [3][1]. A more detailed analysis of the full nonlinear theory can be done by an ADM like decomposition of the action, yielding for $\Lambda = 0$ [40]

$$S \propto \int d^3x \left(\pi^{ij}\dot{g}_{ij} - NH - N_i P^i\right)$$

with spacial indices $i, j \in \{2, 3\}$. Here, H and the $P_i's$ are the three (first class) Hamiltonian- and momentum-constraints, and the (due to symmetry) three g_{ij} are the generalized coordinates with conjugate momenta π^{ij}. Therefore, there are zero bulk degrees of freedom[2] [40]. Because of this feature, it is tempting to examine actions were higher derivative terms are added to Einstein-Hilbert gravity in order to enhance the dynamical content of the theory. Two such models will be discussed in the following subsections.

According to the AdS_3/CFT_2-correspondence, a dual CFT to three dimensional Einstein-Hilbert Gravity can be conjectured to have left- and right-moving central charges [41]

$$c_L = c_R = c = \frac{3l}{2G_N}$$

where l is the AdS-radius defined above.

3.2 Topologically Massive Gravity

One possibility to introduce dynamical degrees of freedom to three dimensional gravity is to add a so-called gravitational Chern-Simons term to the Einstein-Hilbert action (3.1), yielding *topologically massive gravity* (or short TMG), proposed by Deser, Jackiw and Tempelton in 1982 [2, 3, 42]:

$$S_{TMG} = \frac{1}{16\pi G_N} \int d^3x \sqrt{-g} \left[R + \frac{2}{l^2} + \frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu \Gamma^{\sigma}_{\nu\rho} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right)\right] \tag{3.3}$$

[1]Another feature of gravity in three dimensions is that the Weyl tensor $C_{\alpha\beta\gamma\delta}$ vanishes identically, and that therefore, Ricci tensor and Riemann tensor are equivalent [3]. In some sense the Cotton-tensor (3.5) is the 3 dimensional analogue of the Weyl tensor [3].

[2]In four dimensions, the ADM composition would look the same with $i, j \in 2, 3, 4$. This would result in six conjugate pairs $\pi^{ij}\dot{g}_{ij}$ contrasted by one Hamiltonian- and three momentum-constraints, leading to two degrees of freedom.

As will become clear upon analysis of the linearized equations of motion, the coupling parameter μ between the Einstein-Hilbert part of the action and the Chern-Simons term is called the *mass parameter* [2]. The equations of motion following from (3.3) are [2, 3]

$$G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \tag{3.4}$$

where we used the modified Einstein-tensor (3.2) and the Cotton tensor

$$C_{\mu\nu} = \varepsilon_\mu{}^{\kappa\sigma} \nabla_\kappa \big(R_{\sigma\nu} - \frac{1}{4} g_{\sigma\nu} R \big) \tag{3.5}$$

The Cotton tensor is identically covariantly conserved, symmetric, and traceless [3]. Therefore, taking the trace of (3.4) yields $R = 6\Lambda$ exactly as in the case of pure Einstein-Hilbert gravity (3.2).

The equations (3.4) linearized around an AdS_3 background $\bar{g}_{\mu\nu}$ with curvature radius $l = 1$ using transverse traceless gauge ($\bar{\nabla}_\mu h^{\mu\nu} = 0 = h_\mu{}^\mu$) are the third order linear partial differential equations [43]

$$\big(\mathcal{D}^L \mathcal{D}^R \mathcal{D}^M h\big)_{\mu\nu} = 0 \tag{3.6}$$

$$\text{with} \quad \Big(\mathcal{D}^{L/R}\Big)_\mu{}^\nu = \delta_\mu{}^\nu \pm \epsilon_\mu{}^{\alpha\nu} \bar{\nabla}_\alpha \tag{3.7}$$

$$\text{and} \quad \big(\mathcal{D}^M\big)_\mu{}^\nu = \delta_\mu{}^\nu + \frac{1}{\mu} \epsilon_\mu{}^{\alpha\nu} \bar{\nabla}_\alpha \tag{3.8}$$

As these operators are mutually commuting [43], it suffices for many applications to solve the equation

$$\big(\mathcal{D}^M h\big)_{\mu\nu} = h_{\mu\nu} + \frac{1}{\mu} \epsilon_\mu{}^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu} = 0 \tag{3.9}$$

as it describes the massive graviton modes [43–45]. Solutions to this equation will be presented in section 4.3.

Since its proposal in 1982, TMG has been examined in detail (see e.g. [43–52]), and some of the interesting properties that where found will now briefly be stated.

The most interesting feature of TMG is of course that, in contrast to Einstein-Hilbert gravity, the theory contains one propagating degree of freedom with spin either $+2$ or -2, the sign of the spin being dependent on that of μ, which is another sign of parity violation [2, 3]. This was shown in [2, 3] for the case of vanishing cosmological constant $\Lambda = 0$ by an analysis of the linearized equations of motion. Also for $\Lambda = 0$, the existence of one degree of freedom in the full non-linear theory was argued in [40] based on an ADM like approach: Due to the higher derivative nature of TMG, there are four pairs of generalized coordinates and their respective conjugated momenta in contrast to the three such pairs of Einstein-Hilbert gravity. As there are three first class constraints, this leaves one propagating degree of freedom[3]. In [46, 47, 49] it was finally shown by analysing the constraint algebra of the theory that even in the presence of a negative cosmological constant $\Lambda < 0$ TMG exhibits one propagating degree of freedom independently of the value of μ.

TMG is parity violating due to the presence of the factor $\epsilon^{\lambda\mu\nu}$ in the action. When using a coordinate system with coordinates x, y, z one has to fix an orientation, or equivalently make a sign choice $\epsilon^{xyz} > 0$ or $\epsilon^{xyz} < 0$. As in [44, 45] we use the convention $\epsilon^{xyz} = \frac{\pm 1}{\sqrt{-g}}$.

Additionally, under diffeomorphisms generated by ξ the Lagrangian d-form L in (3.3) transforms as (2.4), which means that the theory is only diffeomorphism invariant up to boundary terms [2,32,39]. In the framework of $\mathrm{AdS}_3/\mathrm{CFT}_2$-correspondence, this means that the dual field theory will have a diffeomorphism anomaly indicated by non-equal left and right central charges [50, 53, 54]

$$c_L = \frac{3l}{2G_N}\left(1 - \frac{1}{\mu l}\right) \ , \ c_L = \frac{3l}{2G_N}\left(1 + \frac{1}{\mu l}\right) \tag{3.10}$$

This means that positivity of the central charges and therefore unitarity of the dual CFT require $\mu l \geq 1$ [50]. In fact, there is much ongoing research on how TMG might fit into the $\mathrm{AdS}_3/\mathrm{CFT}_2$-conjecture, see for example [50–52].

In [43] it was shown that non-negativity of the energy of massive graviton modes solving (3.9) requires $\mu l \leq 1$. Non-negativity of gravitons and BTZ black holes is therefore only possible at the so-called *chiral point* $\mu l = 1$. At this point, $c_L = 0$ and $c_L = \frac{3l}{G_N}$ and TMG is a chiral theory [43]. This point has been subject to intensive investigation (see e.g. [43,47,51,52]) and will also play a very special role in this work, together with the "opposite" point $\mu l = -1$, see section 6.4.

It is well known that higher dimensional gravitational Chern-Simons terms sometimes appear in low-energy effective actions due to compactifications of string the-

[3]If the Chern-Simons term would be considered alone without the Einstein-Hilbert contribution, there would be four pairs of independent variables, but also four constraints as there is an additional constraint due to the conformal symmetry of the Chern-Simons term that is broken by the Einstein-Hilbert term in (3.3) [40]. Therefore, pure Chern-Simons gravity would also be trivial.

ory [39, 55–57]. However, whether TMG may also be motivated this way remains an open question.

3.3 New Massive Gravity

Another model of three dimensional higher derivative gravity, the so-called *new massive gravity*, or short NMG, was proposed by Bergshoeff, Hohm and Townsend in 2009 [41, 58]. Their action can be written in the form[4] [15, 41, 59]

$$S_{NMG} = \frac{\sigma}{16\pi G_N} \int d^3x\sqrt{-g}\left(R - 2\lambda - \frac{1}{m^2}K\right) \tag{3.11}$$

where λ is the cosmological constant, $\sigma = \pm 1$ is the overall sign of the action that is irrelevant for the equations of motion but relevant for conserved charges, and $K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2$ is the trace of the tensor [58]

$$\begin{aligned} K_{\mu\nu} = {} & 2\nabla^2 R_{\mu\nu} - \frac{1}{2}\left(\nabla_\mu\nabla_\nu R + g_{\mu\nu}\nabla^2 R\right) - 8R_\mu{}^\alpha R_{\alpha\nu} + \frac{9}{2}RR_{\mu\nu} \\ & + \left(3R^{\mu\nu}R_{\mu\nu} - \frac{13}{8}R^2\right) g_{\mu\nu} \end{aligned} \tag{3.12}$$

It should be noted that the parameter m^2 will be allowed to have positive as well as negative values [41]. The equations of motion read [15, 58]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} = 0 \tag{3.13}$$

and taking the trace obviously yields

$$R - 6\lambda + \frac{1}{m^2}K = 0 \tag{3.14}$$

This means that in contrast to Einstein-Hilbert gravity and TMG, in NMG the Ricci scalar R is not fixed by the cosmological constant. For a maximally symmetric spacetime (such as AdS_3) with $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ and therefore $R = 6\Lambda$ the expressions containing ∇ in (3.12) will automatically vanish yielding $K_{\mu\nu} = -\frac{1}{2}\Lambda^2 g_{\mu\nu}$ and consequently $K = -\frac{3}{2}\Lambda^2$. Upon inserting these expressions, the equations of motion (3.13) reduce

[4] Unfortunately, there seem to be competing conventions on how to present this action in the literature. The form employed in [41, 58, 59] has the integrand $\sigma' R - 2\lambda' m'^2 + \frac{1}{m'^2}K$. The dictionary for comparing results obtained with the two actions reads: $\sigma = \sigma'$, $\lambda = \lambda' m'^2/\sigma'$ or $\lambda' = -\lambda/m^2$ and $\sigma' m'^2 = -m^2$

to

$$-\Lambda g_{\mu\nu} + \lambda g_{\mu\nu} + \frac{\Lambda^2}{4m^2} g_{\mu\nu} = 0$$

Evidently, for a maximally symmetric spacetime with curvature Λ to be a solution of NMG the parameters have to fulfill[5] [41]

$$\Lambda = 2m^2 \left(1 \pm \sqrt{1 - \frac{\lambda}{m^2}}\right) \tag{3.15}$$

For $\frac{\lambda}{m^2} > 1$, maximally symmetric solutions are obviously not possible.

The equations (3.13) linearized around an AdS$_3$background $\bar{g}_{\mu\nu}$ with curvature radius $l = 1$ using transverse traceless gauge are the fourth order linear partial differential equations [59–61]

$$(\mathcal{D}^L \mathcal{D}^R \mathcal{D}^{M_+} \mathcal{D}^{M_-} h)_{\mu\nu} = 0 \tag{3.16}$$

with the mutually commuting operators $\mathcal{D}^{L/R}$ as in (3.7) and

$$\left(\mathcal{D}^{M_\pm}\right)_\mu{}^\nu = \delta_\mu{}^\nu \pm \frac{1}{\sqrt{m^2 + \frac{1}{2}}} \epsilon_\mu{}^{\alpha\nu} \bar{\nabla}_\alpha \tag{3.17}$$

It is furthermore possible to find a relation between (3.17) and (3.8) by setting $m^2 = \mu^2 - \frac{1}{2}$, and in fact exactly this relation has to be satisfied for a nontrivial (i.e. massive graviton) solution of (3.6) to be also a nontrivial solution of (3.16). Because of these similarities on the linearized level, TMG is sometimes described as "square root" of NMG [58]. Indeed, similarities between TMG and NMG are not restricted to the linearized solutions as discussed for example in [62] and section 5.2.

NMG has been acknowledged by the scientific community as having very interesting features [63]. Some noteworthy issues will now be briefly discussed.

In contrast to TMG, NMG is a parity preserving theory [58].

When the theory is linearized around a maximally symmetric background metric satisfying $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ (for example AdS$_3$), it can be proven [41] that NMG is ghost-

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[5] For our conventions of signs and prefactors in (3.11), this is equivalent to the condition presented in (2) of [59] and in (1.11) of [41] which relates the AdS-radius l ($\Lambda = -\frac{1}{l^2}$) of possible AdS solutions of NMG to the parameters of the theory.

free when the condition

$$\frac{m^2}{\sigma}\left(\Lambda + 2m^2\right) < 0 \tag{3.18}$$

is satisfied. Together with (3.15) and the Breitenloher-Freedman bound [41]

$$2m^2 \geq \Lambda \tag{3.19}$$

there are several inequalities that restrict the physically acceptable sets of parameters σ, λ and m^2 for which linearization about an AdS background yields a unitary, ghost free theory [41]. We will come back to these issues in section 5.2.

NMG has two propagating bulk degrees of freedom corresponding to massive graviton modes with spin ± 2, except for $-\frac{\lambda}{m^2} = -1$ and $-\frac{\lambda}{m^2} = 3, \Lambda = -2m^2$ [41]. In the first exceptional case there appears a so-called single partially massless mode [41]. The second exceptional case, where $-\frac{\lambda}{m^2} = 3$, was shown to be a very special situation. There, the linearized Lagrangian equals the Proca Lagrangian for a spin 1 field with squared mass $8m^2$. As in this case unitarity requires $m^2\sigma < 0$, the spin 1 modes are Tachyons for $\sigma = 1$ but physical for $\sigma = -1$ [41].

When the parameters of NMG are chosen in order to allow AdS-vacua with $R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu}$ $(l > 0)$, then a dual CFT can be conjectured to exist according to the AdS_3/CFT_2-correspondence, having left- and right-moving central charges [41]

$$c_L = c_R = c = \frac{3l\sigma}{2G_N}\left(1 - \frac{1}{2m^2 l^2}\right) \tag{3.20}$$

The sign of the central charges obviously depends on σ and changes when $m^2 = \frac{1}{2l^2}$. Positivity of the central charge is required as well for unitarity of the CFT as for positivity of entropy and mass of the BTZ black hole, presented in (4.9) and (4.11) [41]. Unfortunately, as realized in [41] the conditions on the parameter space arising from the requirement $c \geq 0$ are inconsistent with the requirements arising from the desire to have unitary positive-energy modes apart from the special case $-\frac{\lambda}{m^2} = 3$ where $c = 0$. As noted in [41] this situation is quite similar to the one discussed for TMG in the previous subsection.

Chapter 4

The BTZ Black Holes

4.1 The Metric

Although Einstein-Hilbert gravity in three dimensions is trivial, rotating black hole solutions where found by Bañados, Teitelboim and Zanellli for negative cosmological constants in [4,5], see [64] for a review. This was possible because these black holes are locally equivalent to the maximally symmetric AdS-space which solves the equations of motion, but are distinguished from it by global identifications.

The metric of the general BTZ black hole is [4,5]

$$\begin{aligned} ds^2 = &\left(8G_N M - \frac{r^2}{l^2}\right) dt^2 + \left(-8G_N M + \frac{r^2}{l^2} + \frac{(8G_N J)^2}{4r^2}\right)^{-1} dr^2 \\ &- 8G_N J dt d\phi + r^2 d\phi^2 \end{aligned} \tag{4.1}$$

with $t \in]-\infty, +\infty[$, $r \in]0, +\infty[$ and the angular coordinate $\phi \in [0, 2\pi[$ with the identification $\phi \sim \phi + 2\pi$. As this metric is a solution of ordinary Einstein-Hilbert gravity, taking the trace of the equations of motion 3.2 yields $R = 6\Lambda$ and therefore $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$. In the coordinates used above the metric looks quite similar to the rotating Kerr black holes known from four dimensional gravity, and indeed many properties of these two kinds of black holes are similar. The BTZ-metric has globally well defined Killing vectors ∂_t and ∂_ϕ and is therefore stationary and axially symmetric [5]. While the parameter l appearing in the metric is fixed by the cosmological constant $\Lambda = -\frac{1}{l^2}$, the parameters M and J are ADM mass and angular momentum of the black hole in the framework of Einstein-Hilbert gravity [5]. Event and trapping horizons coincide and inner and outer horizons are located at the coordinate singularities

$$r_{\pm}^2 = 4G_N M l^2 \left(1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right) \tag{4.2}$$

where the rr-component of the metric diverges [5]. At this point we encounter a little caveat that the reader should be aware of: Based on (4.2), in the literature (see e.g. [64] it is sometimes made use of the relations

$$8G_N M = \frac{r_+^2 + r_-^2}{l^2} \ , \ 8G_N J = \frac{2r_+ r_-}{l} \tag{4.3}$$

Inserting this into (4.1) yields a form of the metric which depends on r_+ and r_- instead of M and J, and which was for example used for calculations in [32]. As we want to allow for positive as well as negative values of J, (4.3) can only be true when either r_+, r_- or l may be negative. At first glance the logical choice would be to choose $l < 0$ whenever $J < 0$, but as l appears in the original expressions (4.1) and (4.2) only squared there is no real reason to do so. In fact, for example in [43] the convention $\text{sign}(r_-) = \text{sign}(J)$ was adopted, and we will follow the same convention in this thesis.

As can be proven from the Noether charge approach, the entropy of a stationary black hole in d dimensional Einstein-Hilbert gravity will always be proportional to the $d-2$ dimensional area of the event horizon, yielding in the case of the BTZ-black hole (see [64] and references therein for a discussion of the calculation of entropy using the euclidean path integral)

$$\mathcal{S}_{EH} = \frac{\pi r_+}{2G_N} \tag{4.4}$$

Obviously, for horizons to exist at all and therefore to ensure the cosmic censorship principle, the bound $|J| \leq Ml$ has to be introduced [5]. This is another similarity between the rotating black holes in three and four dimensions, but there is one important difference which concerns the nature of the central singularity, located at $r = 0$. As the BTZ metric is locally identical to AdS_3, all curvature scalars such as the Ricci scalar R or the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ of both metrics are identical. Hence, in contrast to the four dimensional black holes, the BTZ black holes do not have a curvature singularity indicated by a diverging curvature scalar [5]. Instead, they have what Bañados, Teitelboim and Zanelli called a *singularity in the causal structure of the spacetime*, due to the emergence of closed causal curves [5]. As stated above, the

BTZ black holes can be derived by imposing identifications on the AdS_3-space. This "wraps up" the previously non-compact direction ∂_ϕ. The problem is that in different regions of the pure AdS_3-space, the vector ∂_ϕ was spacelike, null and timelike, leading to a quotient space in which closed causal curves are present in some regions. According to Bañados, Teitelboim and Zanelli this part of the new spacetime should be considered to be unphysical and therefore be "cut off" from the physical part of the spacetime, rendering the BTZ black hole geodesically incomplete [5]. In fact, based on the properties of the BTZ black hole as a quotient space, Bañados, Teitelboim and Zanelli where able to prove in [5] that the non-rotating black hole ($J = 0, M > 0$) fails to be a Hausdorff manifold at the central point $r = 0$ if it is not cut away [5].

As the BTZ-metric is locally isometric to AdS_3, it "inherits" [43,45] a local $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ symmetry algebra with Killing vector fields L_k and $\bar{L}_k$ ($k \in \{-1, 0, 1\}$) obeying

$$\begin{aligned} [L_0, L_{\pm 1}] &= \mp L_{\pm 1}\ ,\ [L_1, L_{-1}] = 2L_0 \\ [\bar{L}_0, \bar{L}_{\pm 1}] &= \mp \bar{L}_{\pm 1}\ ,\ [\bar{L}_1, \bar{L}_{-1}] = 2\bar{L}_0 \\ [L_k, \bar{L}_{k'}] &= 0 \end{aligned} \tag{4.5}$$

Most of these vector fields are not globally well defined as their components will in general be non-periodic functions of the coordinate ϕ. There are only two linear independent globally well defined Killing vectors for BTZ-black holes [5], $\partial_t = -L_0 - \bar{L}_0$ and $\partial_\phi = -L_0 + \bar{L}_0$ [45]. Some useful properties of these Killing vector fields will be presented in Table 10.1, appendix B.

4.2 The BTZ metric in TMG and NMG

As already discussed, for a cosmological constant $\Lambda = -\frac{1}{l^2}$, the BTZ metrics have a vanishing modified Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R - \frac{1}{l^2}\, g_{\mu\nu} = 0$ and are therefore solutions to ordinary Einstein-Hilbert gravity. Thus, the Cotton tensor (3.5) vanishes automatically

$$C_{\mu\nu} = \varepsilon_\mu{}^{\kappa\sigma} \nabla_\kappa \left(R_{\sigma\nu} - \frac{1}{4}\, g_{\sigma\nu} R \right) = \varepsilon_\mu{}^{\kappa\sigma} \nabla_\kappa \left(G_{\sigma\nu} + const(\Lambda) \cdot g_{\sigma\nu} \right) = 0$$

as $\nabla_\alpha g_{\beta\gamma} = 0$. This proves that every solution to Einstein-Hilbert gravity in three dimensions is also a solution to TMG, independent of the value of μ.

In the ADM- and Noether-charge approaches conserved quantities such as mass, angular momentum and entropy depend not only on the metric, but also on the action to which the metric is a solution. In higher derivative models the values of these conserved quantities will therefore not be equivalent to the parameters M and J appearing in the metric anymore, but will be corrected by terms involving the parameters of the theory, such as μ in the case of TMG. In [32, 50, 53, 65–67] it was found:

$$\text{mass:} \qquad lM_{TMG} = lM + \frac{J}{\mu l} \tag{4.6}$$

$$\text{angular momentum:} \qquad J_{TMG} = J + \frac{M}{\mu} \tag{4.7}$$

$$\text{entropy:} \qquad \mathcal{S}_{TMG} = \frac{\pi}{2G_N}\left(r_+ + \frac{1}{\mu l}r_-\right) \tag{4.8}$$

Obviously, even when the cosmic censorship bound $lM \geq |J|$ is fulfilled, non-negativity of the BTZ-energy (4.6) is only guaranteed when $\mu l \geq 1$ which as discussed in section 3.2 is also the necessary condition for positive central charges (3.10) [43].

Let us now have a look on the BTZ metric in the framework of NMG: As the BTZ metric is locally isometric to AdS_3, the metric (4.1) will be a solution to NMG if and only if relation (3.15) holds with $\Lambda = -\frac{1}{l^2}$. In the framework of NMG (3.11), the conserved quantities of (4.1) are [68–70]

$$\text{mass:} \qquad M_{NMG} = \left(1 - \frac{1}{2m^2l^2}\right)\sigma M \tag{4.9}$$

$$\text{angular momentum:} \qquad J_{NMG} = \left(1 - \frac{1}{2m^2l^2}\right)\sigma J \tag{4.10}$$

$$\text{entropy:} \qquad \mathcal{S}_{NMG} = \left(1 - \frac{1}{2m^2l^2}\right)\frac{\sigma\pi r_+}{2G_N} \tag{4.11}$$

4.3 Stability and quasi-normal Modes in TMG

From now on, we will consider the BTZ black hole with parameters $M = 1$, $J = 0$ and $l = 1$, setting $8G_N = 1$ as in [4, 5]. Starting from (4.1) and introducing the new coordinate $\rho = \text{arcosh}(r)$ (such that the event horizon will be at $\rho = 0$), the line element takes the simple form [44, 45]

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -\sinh^2(\rho)dt^2 + \cosh^2(\rho)d\phi^2 + d\rho^2 \tag{4.12}$$

For this background, in [44,45] linear perturbations of the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where found by solving equation (3.9). As noted in [45], the operator $\mathcal{D}^M$ commutes with the operator $L_{-1}\bar{L}_{-1}$ constructed from the Killing fields of the background spacetime. For this reason, infinitely high towers $h^{(n)}_{\mu\nu} = \left(L_{-1}\bar{L}_{-1}\right)^n h^{(0)}_{\mu\nu}$ (or $H^{(n)}_{\mu\nu} = \left(L_{+1}\bar{L}_{+1}\right)^n H^{(0)}_{\mu\nu}$) of solutions can be generated when one "lowest mode" $h^{(0)}_{\mu\nu}$ or $H^{(0)}_{\mu\nu}$ is known. In the coordinate system where $x^1 \equiv u = t+\phi$, $x^2 \equiv v = t-\phi$, $x^3 \equiv \rho$ and $\epsilon^{uv\rho} = \frac{+1}{\sqrt{-g}}$, the lowest modes found in [44, 45] read:

$$h^R_{\mu\nu} = e^{(1-\mu)t+ikv}\sinh(\rho)^{1-\mu}\tanh(\rho)^{ik}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh(2\rho)} \\ 0 & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix} \tag{4.13}$$

$$H^R_{\mu\nu} = e^{(\mu-1)t+ikv}\sinh(\rho)^{1-\mu}\tanh(\rho)^{-ik}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{\sinh(2\rho)} \\ 0 & -\frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix} \tag{4.14}$$

$$h^L_{\mu\nu} = e^{(1+\mu)t-iku}\sinh(\rho)^{1+\mu}\tanh(\rho)^{-ik}\begin{pmatrix} 1 & 0 & \frac{2}{\sinh(2\rho)} \\ 0 & 0 & 0 \\ \frac{2}{\sinh(2\rho)} & 0 & \frac{4}{\sinh^2(2\rho)} \end{pmatrix} \tag{4.15}$$

$$H^L_{\mu\nu} = e^{(-1-\mu)t-iku}\sinh(\rho)^{1+\mu}\tanh(\rho)^{ik}\begin{pmatrix} 1 & 0 & -\frac{2}{\sinh(2\rho)} \\ 0 & 0 & 0 \\ -\frac{2}{\sinh(2\rho)} & 0 & \frac{4}{\sinh^2(2\rho)} \end{pmatrix} \tag{4.16}$$

The modes with a superscript R/L are right- and left-moving, respectively, while the modes denoted by h/H are called ingoing and outgoing respectively [44]. In [44, 45] it was discussed that these solution are true lowest modes as they fulfill the *chiral highest weight* conditions

$$L_1 h^R_{\mu\nu} = 0\ ,\ \bar{L}_1 h^L_{\mu\nu} = 0\ ,\ L_{-1}H^R_{\mu\nu} = 0 \text{ and } \bar{L}_{-1}H^L_{\mu\nu} = 0 \tag{4.17}$$

In [44, 45] where the primary goal was to establish the stability of the BTZ black hole (4.12), many of these modes where discarded as being unphysical due to violation of boundary conditions at infinity or at the horizon, but as we will see in the next section 5, some of these metrics might be interesting on their own as describing exact nonlinear solutions to TMG.

Part II

Global Structure and Entropy of the Sachs Metric

Chapter 5

The Sachs Metric

5.1 Structure

In [1] it was observed that the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ with $\bar{g}_{\mu\nu}$ from (4.12) and $h_{\mu\nu}$ from (4.15) with $k = 0$ indeed presents a solution to the full equations of motion (3.4) of TMG. In the coordinates $x^1 = t$, $x^2 = \phi$ and $x^3 = \rho$ ($\epsilon^{t\rho\phi} = \frac{+1}{\sqrt{-g}}$) that were used in (4.12) to describe the exterior of the background black hole-metric, this new solution reads:

$$g_{\mu\nu} = \begin{pmatrix} -\sinh^2(\rho) & 0 & 0 \\ 0 & \cosh^2(\rho) & 0 \\ 0 & 0 & 1 \end{pmatrix} + (e^t \sinh\rho)^{1+\mu} \begin{pmatrix} 1 & 1 & \frac{2}{\sinh(2\rho)} \\ 1 & 1 & \frac{2}{\sinh(2\rho)} \\ \frac{2}{\sinh(2\rho)} & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix} \tag{5.1}$$

Of course, a solution of the *linearized* equations describing also an exact solution of the full *non-nonlinear* equations of a theory is a highly unusual thing, but before discussing why this metric solves the equations of motion (3.4), we will first have a look on the structure of the solution.

First of all, we should remember that in the coordinate system we are using $t \in]-\infty, +\infty[$, $\rho \in]0, +\infty[$ and $\phi \in [0, 2\pi[$ with $\phi \sim \phi + 2\pi$. Obviously, the metric (5.1) has the structure "background plus distortion". Due to the overall factor $(e^t \sinh\rho)^{1+\mu}$, depending on the sign of $1 + \mu$, in limits as for example $t \to \pm\infty$ the distortion $h_{\mu\nu}$ might become small or dominatingly large compared to the background $\bar{g}_{\mu\nu}$ (4.12). One might at first fear that in certain limits of the coordinates the metric (5.1) might lose its $\{-,+,+\}$ signature due to an ever increasing contribution from $h_{\mu\nu}$, but this is not the case. Indeed, as already realized in [1], $h_{\mu\nu}$ can be expressed by a (with respect to $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ as well) null vector l_μ via the expression

$$h_{\mu\nu} = l_\mu l_\nu \tag{5.2}$$

which means that $h_{\mu\nu}$ is, when viewed as a square matrix, positive semi-definite with one positive and two zero eigenvalues. Thus, it is possible to prove that there will always be a timelike direction at every point in the spacetime (5.1). In fact, a simple calculation of the determinant shows after some algebra that $\det(g_{\mu\nu}) \equiv g = \bar{g} = -\cosh^2(\rho)\sinh^2(\rho) < 0$ everywhere in the spacetime and independently of μ.

We can now elucidate how this metric represents a solution to the equations of motion (3.4). The Riemann- and Einstein-tensor of (5.1) were already calculated in [1] and read

$$R_{\mu\nu} = \frac{R}{3} g_{\mu\nu} + \frac{\mu^2 - 1}{12} R h_{\mu\nu} \tag{5.3}$$

$$G_{\mu\nu} = \frac{1-\mu^2}{2} h_{\mu\nu} \tag{5.4}$$

where we made use of the Ricci-scalar $R = \bar{R} = -6$ which for vacuum solutions is fixed by the trace of the equations of motion (3.4). See appendix C for Christoffel symbols and some comments on curvature tensors. Inserting this into (3.4) yields the expression [1]

$$G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = \frac{1-\mu^2}{2} (h_{\mu\nu} + \frac{1}{\mu} \varepsilon_\mu{}^{\kappa\sigma} \nabla_\kappa h_{\sigma\nu}) \tag{5.5}$$

which quite closely resembles the linearized equations of motion (3.9) of which $h_{\mu\nu}$ will by definition be a solution, but with one important difference: While in (3.9) the covariant derivative $\bar{\nabla}$ with respect to the background metric $\bar{g}_{\mu\nu}$ appears, the derivative ∇ with respect to the full metric $g_{\mu\nu}$ appears in (5.5). Therefore, the reason that $g_{\mu\nu}$ is a solution of TMG is simply due to the fact that

$$\bar{\nabla}_\alpha h_{\mu\nu} = \nabla_\alpha h_{\mu\nu} \tag{5.6}$$

Using $g_{\mu\nu} = \bar{g}_{\mu\nu} + l_\mu l_\nu$ (and therefore $g^{\mu\nu} = \bar{g}^{\mu\nu} - l^\mu l^\nu$ [1]) in the calculation of the

Christoffel symbols, it is easy to find that (5.6) reduces to the equation

$$l^{\lambda}\left(\partial_{\alpha} l_{\lambda}-\partial_{\lambda} l_{\alpha}\right)=0 \tag{5.7}$$

It is most easy to check the validity of this equation in the global coordinates introduced in section 5.4. This explanation of why the linearized solution $h_{\mu\nu}$ also describes an exact solution is on a quite superficial mathematical level, and it is not excluded that there is some "deeper" reason for this seeming mathematical coincidence. In fact, not only do all solutions (4.13-4.16)[1] describe similar exact solutions to TMG, but we will see in section 5.2 that the same phenomenon also occurs for NMG: These and similar solutions will not only be solutions of the linearized equations of motion, but also to the full nonlinear equations of motion (3.13) for a suitable choice of parameters λ and m^2. Indeed, it has already been proven in [15] that some solutions of the linearized equations of motion may also describe exact solutions of the exact equations of motion for TMG and NMG. However, it can be proven (see section A) that there is no coordinate transformation that brings the metric 5.1 to the form of metrics used in [15, 16]. Some ideas for future research will be presented in section 9.

Having justified that (5.1) represents an exact vacuum-solution of the full non-linear equations of motion (3.4) of TMG, we can ask: What kind of spacetime does this metric describe? In [1] the spacetime was already classified as an AdS pp-wave spacetime [1, 71] of Petrov type N (see [71]) and Kundt-CSI type (see [15, 16]). Apart from these facts, as the metric (5.1) was derived from a black hole background and indeed asymptotes to this background in certain limits, we can already conjecture that this metric might describe a dynamical black hole spacetime.

To gain further insights into the nature of this spacetime, we will proceed as follows: In section 5.3, we will investigate the symmetries of our solution, in section 5.4, we will discuss different coordinate systems that can be used to describe the spacetime. For our purposes, a desirable coordinate system should yield a simple form of the metric and cover as much of the spacetime manifold as possible. In fact, having a suitable coordinate system at one's disposal is crucial for the understanding of a black hole spacetime and finding one is a nontrivial task, as probably already became clear in our overview of the history of black hole research in section 2.1. In section 5.5 we will have a look on geodesics and lightcones of our spacetime, but first we will prove that (5.1) is indeed a solution to the equations of motion (3.13) of NMG.

[1]Of course one will have to set $k = 0$ in these solutions as, in contrast to a solution of the linearized equations of motion, a solution of the non-linear equations of motion must not be complex.

5.2 New Massive Gravity

In the last subsection we showed that the metric (5.1) is a solution to the full non-linear vacuum equations of TMG. In section 3.3 we saw that a solution (such as (4.15)) of the linearized equations of motion of TMG (3.9) will also be a solution of the linearized equations of motion of NMG (3.16) if we set $m^2 = \mu^2 - 1/2$. We can now ask whether we will have the same effect for NMG as for TMG, i.e. whether the metric (5.1) is also a solution of the full non-linear equations of motion (3.13) of NMG. In order to answer this question, it is advisable to first consider the trace of the equations of motion of NMG, (3.14). Making use of $R = -6$ and (5.3) it is easy to find $K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2 = -\frac{3}{2}$ independently of μ, which is also the case for the background metric $\bar{g}_{\mu\nu}$. (3.14) then reads

$$\begin{aligned} 6 + 6\lambda + \frac{3}{2m^2} &= 0 \\ \Rightarrow \lambda = \frac{-4m^2 - 1}{4m^2} &\text{ or } m^2 = -\frac{1}{4(\lambda+1)} \end{aligned} \tag{5.8}$$

Inserting now (5.1) in (3.13) using $m^2 = -\frac{1}{4(\lambda+1)}$ yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} = \frac{1}{2}\left(1-\mu^2\right)\left(4(\lambda+1)\mu^2 - 2\lambda - 1\right)h_{\mu\nu}$$

The right-hand side is obviously zero for $\mu = \pm 1$ or $\mu = \pm\sqrt{\frac{2\lambda+1}{4\lambda+4}}$ or equivalently $\lambda = \frac{1-4\mu^2}{2(2\mu^2-1)}$. This means that for the correct choices of the parameters μ, m^2 and λ, (5.1) is also a solution to NMG.

Inserting the relation between μ and λ into the relation (5.8) between λ and m^2 yields the expression $m^2 = \mu^2 - \frac{1}{2}$ which is exactly the condition that we where expecting from the comparison between the linearized equations of motion of TMG and NMG, see section 3.3.

Making use of $m^2 = \mu^2 - \frac{1}{2}$ and $\Lambda = -\frac{1}{l^2} = -1$, we find that (3.19) is automatically satisfied while (3.18) yields

$$\frac{2}{\sigma}\left(\mu^2 - 1\right)\left(\mu^2 - \frac{1}{2}\right) < 0 \tag{5.9}$$

This can always be fulfilled with the choice of σ that yields negative central charges (3.20) and negative BTZ entropy (4.11). The special cases $-\frac{\lambda}{m^2} = -1$ and $-\frac{\lambda}{m^2} = 3, \Lambda = -2m^2$ (see section 3.3) correspond to $\mu = 0$ and $\mu = \pm 1$.

5.3 Killing Fields and their Geodesics

As already mentioned in section 4.1, the background metric $\bar{g}_{\mu\nu}$, being locally isometric to AdS_3, has a local symmetry algebra generated by six linearly independent Killing vector fields of which only the two ∂_t and ∂_ϕ are globally well defined. The (outside of the black hole) timelike vector field ∂_t describes time-translations and is the reason why $\bar{g}_{\mu\nu}$ is stationary, while the (in the physical part of the spacetime[2]) spacelike vector field ∂_ϕ describes the rotational (or axial) symmetry of the spacetime.

It is a well known fact that when a metric $g_{\mu\nu}$ is formulated with respect to coordinates x^i, than the vector field ∂_{x^j} will be a Killing vector field if and only if the metric is not a function of x^j, i.e. $\partial_{x^j} g_{\mu\nu} = 0$ [9]. From this it is already easy to see that for (5.1) there will still be a rotational symmetry described by the Killing vector field ∂_ϕ, but no time translational symmetry as ∂_t is not a Killing vector field anymore. This confirms the conjecture that the metric (5.1) will describe a dynamical process, at which's beginning or end (depending on μ) there will be a non-rotating BTZ black hole.

One can also investigate whether any of the other (local) Killing fields mentioned in section 4.1 "survives" the transition from $\bar{g}_{\mu\nu}$ to $g_{\mu\nu}$, and as the results presented in table 10.1 show, this is indeed the case for $-\bar{L}_1$ [3] which in t,ϕ,ρ coordinates reads [1,45]

$$\xi = -e^{t-\phi}\left(-\coth(\rho)\partial_t + \tanh(\rho)\partial_\phi + \partial_\rho\right) \tag{5.10}$$

Because of the prefactor $e^{t-\phi}$, ξ is mapped to $e^{-2\pi}\xi$ when following the vector field ∂_ϕ once around the black hole. In this sense that we call this vector field *not globally well defined*. Nevertheless, as we will see, the vector field ξ is extremely useful for understanding the spacetime under investigation in this thesis. We will now discuss some of its properties:

First of all, ξ is lightlike as well for the background metric as for the full metric:

$$0 = \bar{g}_{\mu\nu}\xi^\mu\xi^\nu = g_{\mu\nu}\xi^\mu\xi^\nu$$

This already proves that $h_{\mu\nu}\xi^\mu\xi^\nu = 0$, but in fact it was shown in [1] that $h_{\mu\nu}$ cannot

[2]As already explained in section 4.1, any part of the spacetime where ∂_ϕ is causal contains closed causal curves and is therefore by definition considered to be unphysical.

[3]We included an overall minus-sign so that ξ will be future pointing.

only be written in the form (5.2) but also in the form

$$h_{\mu\nu} = f(t, \phi, \rho)\xi_\mu \xi_\nu \tag{5.11}$$

This indicates that ξ takes the role of the wave-vector of the AdS pp-wave that disturbs the background metric $\bar{g}_{\mu\nu}$.

It should be noted that ξ fulfills the geodesic equation $\nabla_\xi \xi = 0$, therefore, there exists a family of null geodesics with ξ as tangent vector field. These geodesics can be computed analytically as follows: We set

$$\begin{pmatrix} \dot{t} \\ \dot{\phi} \\ \dot{\rho} \end{pmatrix} = -e^{t-\phi} \begin{pmatrix} -\coth(\rho) \\ \tanh(\rho) \\ 1 \end{pmatrix} \tag{5.12}$$

where $\dot{(...)} = \frac{d}{d\tau}(...)$ denotes the differentiation with respect to the affine parameter τ. Furthermore, we make use of the fact that the scalar product of a Killing vector field $\tilde{\xi}$ and the tangent of a geodesic $\gamma^\mu = \dot{x}^\mu$ is constant along a geodesic: $g_{\mu\nu}\tilde{\xi}^\mu \gamma^\nu = C$ [8]. Using the Killing vector ∂_ϕ, we find:

$$C_\phi = g_{\mu\nu}\xi^\mu (\partial_\phi)^\nu = -e^{t-\phi}\cosh(\rho)\sinh(\rho) \tag{5.13}$$

which obviously requires $C_\phi < 0$. We can use this to eliminate the prefactor $-e^{t-\phi}$ from (5.12) and obtain from the third component:

$$\dot{\rho} = \frac{C_\phi}{\cosh(\rho)\sinh(\rho)} \Rightarrow \rho(\tau') = \frac{1}{2}\text{arcosh}(-4\tau' + A) \tag{5.14}$$

where we used $\tau' = |C_\phi|(\tau - \tau_0)^4$ and $A = \cosh(2\rho_0) > 1$. We can insert this into the equations

$$\dot{t} = \frac{-C_\phi \coth(\rho)}{\cosh(\rho)\sinh(\rho)} \ , \ \dot{\phi} = \frac{C_\phi \tanh(\rho)}{\cosh(\rho)\sinh(\rho)}$$

[4] The fact that $|C_\phi|$ can be absorbed into the affine parameter is understandable when one considers what this constant means: The shift $\phi \to \phi + \Delta\phi$ is a symmetry transformation and should not have any physical effect. In (5.12) this transformation results in the multiplication of the tangent vector $\dot{x}^\mu$ with a constant that can always be absorbed by a redefinition of the affine parameter. See also the discussion of (5.29) in section 5.5.

and obtain upon integration

$$t(\tau') = t_0 - \frac{1}{2}\ln\left[\frac{A-1-4\tau'}{A-1}\right] \;,\; \phi(\tau') = \phi_0 + \frac{1}{2}\ln\left[\frac{A+1-4\tau'}{A+1}\right] \tag{5.15}$$

We see from these solutions that $\rho(\tau')$ is monotonously decreasing as a function of τ' until at the maximal value $\tau'_{\max} = \frac{1}{4}(A-1) > 0$ $\rho = 0$ is reached. In the same way, for $\tau' \to \tau'_{max}$ we find $t(\tau') \to +\infty$ while the function $\phi(\tau')$ is well behaved under this limit. It should be noted that the vector field ξ is defined independently of the value of μ or the presence of the distortion $h_{\mu\nu}$, ξ is a Killing field for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ as well. The only step where the metric was used in our calculation of the geodesics was (5.13), but as $h_{\mu\nu}\xi^\nu = 0$ due to (5.11) this is again insensitive to the presence of $h_{\mu\nu}$ or the exact value of μ. Therefore, the geodesics computed above "do not feel" the distortion $h_{\mu\nu}$ and are exactly the same for $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$. This makes it easy to interpret the behaviour of the geodesics that was unraveled above in the light of $\bar{g}_{\mu\nu}$: the limit $\rho \to 0$ means that the light rays enter the (background metric) event horizon, which happens for a finite affine parameter $\tau'_{max} < +\infty$ but takes infinitely long ($t \to +\infty$) in the eyes of an outside observer. For the same geodesics moving in the spacetime $g_{\mu\nu}$, there is no such easy interpretation as in this metric the coordinate t will not be the time-coordinate of an asymptotic observer, and the event horizon will not be located at $\rho = 0$. Nevertheless, the behaviour of these geodesics gives a strong motivation to find coordinate systems in which they can be extended beyond τ'_{max}. In fact, when moving along the integral lines of a Killing field the metric does "not change" (i.e. one can move along such a path and never encounter a diverging curvature scalar) and it would be very helpful for an understanding of the black hole interior to find coordinate systems where τ' can be extended to the full range $\tau' \in]-\infty, +\infty[$[5]. This will be achieved in the next section.

To end the discussion of these geodesics, it should be noted that they have vanishing expansion as $\theta = \xi^\mu_{;\mu} = 0$.

We know that (see table 10.1) ξ and ϕ are the only Killing vector fields that are present in the background metric $\bar{g}_{\mu\nu}$ and the metric $g_{\mu\nu}$ as well, but of course this does not prove that there might not be new Killing fields emerging in $g_{\mu\nu}$ at least for certain values of μ. In fact, we will find in section 6.4 that for $\mu = \pm 1$ the metric $g_{\mu\nu}$ can be shown to describe a BTZ black hole by suitable coordinate transformations,

[5]For spacetimes in which coordinates x^μ can be defined such that ∂_{x^1} is a Killing vector field, there seem to be two possibilities for the allowed range of the coordinate x^1: Either $x^1 \in]-\infty, +\infty[$ (as is the case for t in (4.1)) or $x^1 \in]0, x^1_{max}[$ with the identification $x^1 \sim x^1 + x^1_{max}$ imposed (as is the case for ϕ in (4.1)).

thereby proving the necessary existence of further Killing vector fields in these cases. To find Killing vector fields one could make the general ansatz

$$\tilde{\xi} = f_1(t,\phi,\rho)\partial_t + f_2(t,\phi,\rho)\partial_\phi + f_3(t,\phi,\rho)\partial_\rho$$

and solve the Killing equation [8]:

$$\nabla_\mu \tilde{\xi}_\nu + \nabla_\nu \tilde{\xi}_\mu = 0 \tag{5.16}$$

This is a system of six coupled partial linear differential equations for the three unknown functions $f_i(t,\phi,\rho)$. Due to the rather complicated form of the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ involving both the coordinates t and ρ, these equations are hard to solve for arbitrary μ. In fact, except for the special cases $\mu = \pm 1$ no additional Killing vectors are currently known. One candidate case where a closer inspection might be worthwhile is $\mu = \pm 3$ as there the solution might exhibit *Schrödinger symmetry* [62]. Having already an idea of which symmetry to search for might be a considerable advantage. If one has a certain algebra in mind, one could make assumptions on how the already known Killing vectors ξ and ∂_ϕ fit into this algebra. Then, one could possibly avoid solving the equations (5.16) and instead use the commutation relations of this algebra as defining equations for an unknown Killing field $\tilde{\xi}$.

5.4 Coordinate Systems

The history of black hole physics summarized in section 2.1 shows that suitable coordinate systems are crucial for the understanding of black hole spacetimes. Before presenting different coordinate systems that can be used to investigate the spacetime described by (5.1), we will comment on what kind of coordinate systems are possible.

Of course, the ideal coordinate system should cover as much of the spacetime as possible and yield a simple form of the line element. One effective way to achieve simplicity is to take the symmetries of the spacetime into account. There are two Killing vector fields and a priori it seems desirable to construct a coordinate system with coordinates x^μ such that $\partial_{x^2} = \xi$ and $\partial_{x^3} = \partial_\phi$. Then, as discussed in section 5.3, the metric $g_{\mu\nu}$ in these coordinates would be a function of only x^1. Unfortunately, using $\xi = -\bar{L}_1$, $\partial_\phi = -L_0 + \bar{L}_0$ and the algebra (4.5) it is easy to show that $[\xi, \partial_\phi] = \xi$ which means that such a coordinate system cannot exist. Therefore, in any coordinate system the metric $g_{\mu\nu}$ will at least be dependent on two of the three coordinates, and the coordinate systems where a Killing vector field is also a coordinate vector field will be divided into two classes: Those (like (5.1)) in which ∂_ϕ is the coordinate vector and those where $\xi \equiv \partial_\lambda$ is the coordinate vector.

One coordinate system of the latter category is the one already introduced in [1]. It is related to the coordinate system used in section 5.1 by the coordinate transformations[6]

$$\begin{aligned} t + \phi &= -\log[\tanh(\rho)] + 2U \\ t - \phi &= -\log[\sinh(2\rho)] + 2V \\ \rho &= \frac{1}{2}\text{arcosh}(e^{2V}(-\lambda)) \end{aligned}$$

yielding the line element

$$ds^2 = -e^{2V} d\lambda dU + dV^2 + \left(1 + 2^{\frac{3-\mu}{2}} e^{(1+\mu)(U+V)}\right) dU^2 \tag{5.17}$$

We choose $x^1 = V$, $x^2 = U$, $x^3 = \lambda$ and $\epsilon^{VU\lambda} = \frac{+1}{\sqrt{-g}}$. In this coordinate system

$$\partial_\phi = \frac{1}{2}\partial_U - \frac{1}{2}\partial_V + \lambda\partial_\lambda \ , \ \xi = \partial_\lambda$$

and the null-geodesics discussed in section 5.3 take the simple form $U(\tilde{\tau}) = const.$, $V(\tilde{\tau}) = const.$ and $\lambda(\tilde{\tau}) = \tilde{\tau} + \lambda_0$ where $\tilde{\tau}$ can a priori take the full range $\tilde{\tau} \in]-\infty, +\infty[$. Therefore, the Killing geodesics described by ξ are geodesically complete in this coordinate system which evidently covers a larger part of the spacetime than the coordinates used in section 5.1.

The identifications introduced by $\phi \sim \phi + 2\pi$ onto the coordinates U, V and especially λ are quite complicated, and thus we would like to have a coordinate system which covers a larger part of the spacetime than the one used in section 5.1 but which has ∂_ϕ as a coordinate vector.

One first attempt to achieve this goal would be to use the Kruskal coordinates that were presented in [5]. It should be noted that these are the Kruskal coordinates for the background metric $\bar{g}_{\mu\nu}$ as for genuine Kruskal coordinates for $g_{\mu\nu}$ we would need to know exact solutions for in- and outgoing null geodesics, see also section 5.5 for a discussion. For the background metric $\bar{g}_{\mu\nu}$ it is known that these coordinates cover the

[6] In contrast to the transformation given in [1] we introduce here the additional transformation $\lambda \to -\lambda$ in order to render the Killing vector field ∂_λ future pointing. For the same reason we choose $\xi = -\bar{L}_1$ in section 5.3 instead of $\xi = +\bar{L}_1$ as was done in [1].

whole physical part of the spacetime [5]. The coordinate transformation that relates these coordinates to those in section 5.1 reads [5, 44]:

$$U = \tanh\left(\frac{\rho}{2}\right)\cosh(t) \qquad \Leftrightarrow \qquad \rho = 2\,\mathrm{arctanh}\left(\sqrt{U^2 - V^2}\right) \tag{5.18}$$

$$V = \tanh\left(\frac{\rho}{2}\right)\sinh(t) \qquad \Leftrightarrow \qquad t = \ln\left(\frac{U+V}{\sqrt{U^2-V^2}}\right) \tag{5.19}$$

$$\phi' = \phi \tag{5.20}$$

In these coordinates, the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ takes the form (choosing $x^1 = V$, $x^2 = U$, $x^3 = \phi' = \phi$ and $\epsilon^{VU\phi'} = \frac{+1}{\sqrt{-g}}$)

$$g_{\mu\nu} = \begin{pmatrix} \frac{-4}{(1-U^2+V^2)^2} & 0 & 0 \\ 0 & \frac{4}{(1-U^2+V^2)^2} & 0 \\ 0 & 0 & \frac{(1+U^2-V^2)^2}{(1-U^2+V^2)^2} \end{pmatrix} + \left[\frac{2(U+V)}{1+V^2-U^2}\right]^{1+\mu}$$
$$\times \begin{pmatrix} \frac{\left(1+(U+V)^2\right)^2}{(U+V)^2(1+U^2-V^2)^2} & \frac{\left(1-(U+V)^2\right)\left(1+(U+V)^2\right)}{(U+V)^2(1+U^2-V^2)^2} & \frac{1+(U+V)^2}{(U+V)(1+U^2-V^2)} \\ \frac{\left(1-(U+V)^2\right)\left(1+(U+V)^2\right)}{(U+V)^2(1+U^2-V^2)^2} & \frac{\left(1-(U+V)^2\right)^2}{(U+V)^2(1+U^2-V^2)^2} & \frac{1-(U+V)^2}{(U+V)(1+U^2-V^2)} \\ \frac{1+(U+V)^2}{(U+V)(1+U^2-V^2)} & \frac{1-(U+V)^2}{(U+V)(1+U^2-V^2)} & 1 \end{pmatrix}$$

This metric looks complicated, but some algebraic expressions appear over and over again like building blocks, and an analysis of these few blocks gives already some insight in the global structure.

One of these expressions is $1 + U^2 - V^2$ which appears in the $\bar{g}_{\phi\phi}$ component of the background part of the metric. As discussed in section 4.1 the singularity of the BTZ black hole is by definition the hypersurface where ∂_ϕ becomes a null vector, and for the background metric this is obviously indicated by the vanishing of $1 + U^2 - V^2$. In fact, $\bar{g}_{\phi\phi} = \frac{(1+U^2-V^2)^2}{(1-U^2+V^2)^2} = r^2$ where r is the function defined in section 2.5 by using the orbits of the Killing field ∂_ϕ and also simply the radial coordinate used in (4.1). Obviously, because of the addition of the distortion $h_{\mu\nu}$, we find

$$g_{\phi\phi} = \frac{(1+U^2-V^2)^2}{(1-U^2+V^2)^2} + \underbrace{\left[\frac{2(U+V)}{1+V^2-U^2}\right]^{1+\mu} \times 1}_{>0}$$

indicating that at the hypersurface where the singularity occurred for the background part, ∂_ϕ is still spacelike in the disturbed metric $g_{\mu\nu}$. This means that the singularity in the new spacetime is either not present at all or shifted backwards. Unfortunately,

$1+U^2-V^2$ appears in some denominators in $h_{\mu\nu}$. Therefore there is a *coordinate singularity* at the hypersurface where $1+U^2-V^2=0$.

Another important expression is $1-U^2+V^2 = 1-\tanh\left(\frac{\rho}{2}\right)$ which appears in many denominators as well in the background part as in the overall prefactor of the distortion. This expression is positive in the bulk of the spacetime and goes to zero in the limit $\rho\to\infty$ which describes spacelike infinity. Points with $1-U^2+V^2<0$ cannot be considered as they would be "beyond" infinity. The prefactor of the distortion reads $\left[\frac{2(U+V)}{1+V^2-U^2}\right]^{1+\mu}$. As we do not restrict μ to integer values, the expression in the brackets needs to be positive, and if we assume $1+V^2-U^2>0$ then we have to impose $U+V>0$. This restriction has to be imposed only for the full metric $g_{\mu\nu}$ and is not required for the background metric $\bar{g}_{\mu\nu}$. Considering the background metric, requiring $U+V>0$ means cutting away the white hole and the parallel universe present in the lower left corner of the Carter-Penrose diagram of the non-rotating BTZ black hole, see [5].

In this new coordinate system, our beloved vector field ξ reads

$$\xi = e^{-\phi}\left[\frac{1+(U+V)^2}{2}\partial_V + \frac{-1+(U+V)^2}{2}\partial_U - \frac{2(U+V)}{1-V^2+U^2}\partial_\phi\right]$$

The form of the Killing geodesics discussed in section 5.3 for this coordinate system can easily be calculated by inserting the expressions for $t(\tau')$ (5.15) and $\rho(\tau')$ (5.14) into (5.18) and (5.19). One then finds that $1+U(\tau')^2-V(\tau')^2\to 0$ as $\tau'\to\frac{1+A}{4}$. This is actually the same limit in which $\phi(\tau')$ as stated in (5.15) also diverges. This shows that the geodesics reach the coordinate singularity where for the background metric the causal singularity was within finite affine parameter. In order to investigate the behaviour of the geodesics beyond this point, a new coordinate system is needed.

Before presenting two more coordinate systems, we will introduce a short definition that clarifies what kind of coordinate systems are preferable, and what properties they have.

Definition A coordinate transformation from a coordinate system x^1, x^2, x^3 where the vector ∂_{x^3} is a Killing vector to another coordinate system $y^1(x^\mu)$, $y^2(x^\mu)$, $y^3(x^\mu)$ is said to *respect the symmetry* associated with the Killing field ∂_{x^3} if in the new coordinate system there is an $i\in\{1,2,3\}$ such that $\partial_{y^i}=\partial_{x^3}$. Let us for simplicity choose $i=3$. Then

$$\partial_{x^3} = \frac{\partial y^1}{\partial x^3}\partial_{y^1} + \frac{\partial y^2}{\partial x^3}\partial_{y^2} + \frac{\partial y^3}{\partial x^3}\partial_{y^3} \equiv \partial_{y^3}$$

from which follows

$$y^1 = f^1(x^1, x^2)\ ,\ y^2 = f^2(x^1, x^2)\ ,\ y^3 = x^3 + f^3(x^1, x^2)$$

and equivalently for the back-transformation

$$x^1 = f'^1(y^1, y^2)\ ,\ x^2 = f'^2(y^1, y^2)\ ,\ x^3 = y^3 + f'^3(y^1, y^2)$$

We want to respect the symmetry generated by ∂_ϕ, thus we will restrict ourselves to this kind of coordinate transformations. As the transformation law for the metric $g_{\mu\nu}$ reads

$$\tilde{g}_{\mu\nu}(y^i) = g_{\alpha\beta}(x^i)\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} \tag{5.21}$$

we find $\tilde{g}_{33}(y^i) = g_{\alpha\beta}(x^i)\frac{\partial x^\alpha}{\partial y^3}\frac{\partial x^\beta}{\partial y^3} = g_{33}(x^i)$ where in the last expression the x^i have to be seen as functions of the y^i. Obviously, under coordinate transformations that respect the symmetry generated by the Killing field ∂_ϕ, the value of the metric-component $g_{\phi\phi}$ is invariant at a certain point. In fact, the (spacelike) one dimensional subspace that is generated by the integral line of the vector field ∂_ϕ through a certain point $\mathcal{P}$ has volume (or better circumference) $\mathcal{C} = \int_0^{2\pi} \sqrt{g_{\phi\phi}} \equiv 2\pi r$. Here, r is exactly the invariant geometrical quantity used for the definition of the Kodama vector in section 2.5.

One coordinate system that respects the symmetry generated by ∂_ϕ and does transform the metric into a simpler form is defined by the relations:

$$\begin{aligned}
z_+ &= e^t \sinh(\rho) &&\Leftrightarrow\quad \rho = \operatorname{arcsinh}(-z_+ z_-)\\
z_- &= -e^{-t}\sinh(\rho) &&\Leftrightarrow\quad t = \frac{1}{2}\ln\left(-\frac{z_+}{z_-}\right)\\
\phi_+ &= \phi - \ln(\cosh(\rho)) &&\Leftrightarrow\quad \phi = \phi_+ + \frac{1}{2}\ln(1 - z_+ z_-)
\end{aligned}$$

The metric then takes the form (choosing $x^1 = z_+$, $x^2 = z_-$, $x^3 = \phi_+$, $\epsilon^{z_+ z_- \phi_+} = \frac{-1}{\sqrt{-g}}$)

$$g_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{z_-}{2} \\ -\frac{1}{2} & 0 & -\frac{z_+}{2} \\ -\frac{z_-}{2} & -\frac{z_+}{2} & 1 - z_- z_+ \end{pmatrix} + z_+^{1+\mu}\begin{pmatrix} \frac{1}{z_+^2} & 0 & \frac{1}{z_+} \\ 0 & 0 & 0 \\ \frac{1}{z_+} & 0 & 1 \end{pmatrix}$$

This coordinate system obviously brings $\bar{g}_{\mu\nu}$ into a lightcone-coordinate form, but for the full metric $g_{\mu\nu}$ only ∂_{z_-} remains a null vector. The Killing vectors in these coordinates read $\partial_\phi = \partial_{\phi_+}$ and $\xi = 2e^{-\phi_+}\partial_{z_-}$.

The coordinates that proved most useful for discussing properties of the metric $g_{\mu\nu}$ are defined by

$$\begin{aligned}
z &= e^{-t}\frac{1}{\sinh(\rho)} &&\Leftrightarrow\quad \rho = \operatorname{arcosh}\left(\frac{\sqrt{R}}{z}\right)\\
R &= e^{-2t}\coth^2(\rho) &&\Leftrightarrow\quad t = -\ln\left(\sqrt{R-z^2}\right)\\
y &= t+\phi+\log\left(\tanh(\rho)\right) &&\Leftrightarrow\quad \phi = y+\frac{1}{2}\ln(R)
\end{aligned} \tag{5.22}$$

where we choose $x^1 = z$, $x^2 = y$, $x^3 = R$ and $\epsilon^{zyR} = \frac{+1}{\sqrt{-g}}$. In these coordinates, the Killing vector ∂_ϕ is equal to ∂_y while $\xi = -2e^{-y}\partial_R$. The line element of the metric (5.1) takes the very simple form

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dydR + Rdy^2\right) + \frac{1}{z^{1+\mu}}dy^2 \tag{5.23}$$

Here, the second term on the right-hand side corresponds to the perturbation $h_{\mu\nu}$ while the first term corresponds to the background metric $\bar{g}_{\mu\nu}$ (4.12). Because of the factor $z^{-\mu}$, we need to restrict z to positive values for general μ. Apart from this we can set $y \in [0, 2\pi[$ with $y \sim y + 2\pi$ and $R \in]-\infty, +\infty[$ which means that in this coordinate system, the Killing geodesics discussed in section 5.3 are geodesically complete as they are simply lines with

$$z = const.\ ,\ y = const.\ \text{and}\ R(\tau) = const.\tau + R_0 \tag{5.24}$$

This coordinate system thus covers a much larger part of the spacetime than it was the case for the Schwarzschild-like coordinates used in (5.1) or the Kruskal coordinates discussed above. Therefore, we call these very special coordinates the *global coordinates.*

The spacetime's structure is much easier to understand in these new coordinates. In [5] it was pointed out that the singularity of BTZ black holes is not a curvature singularity but merely a singularity in the causal structure of the spacetime, evoked by the presence of closed causal curves. In order to find out whether there is a similar singularity present in the family of metrics given by equation (5.1), we note that because of the periodicity in the coordinate ϕ and (5.22), the point (z, R, y) is identified with the point $(z, R, y+2\pi)$. As closed causal curves therefore appear where ∂_y is null or timelike, we have to restrict the physical part of the spacetime to the region where $R > -z^{1-\mu}$, with the equation $R = -z^{1-\mu}$ determining the singularity. This will be discussed in more detail in section 6.1.

5.5 Analysis of Geodesics and Causal Curves

In this section, we will mostly work in the global coordinates defined in (5.22) as these cover a large part of the spacetime and present the metric in the very simple form (choosing $x^1 = z$, $x^2 = y$, $x^3 = R$)

$$g_{\mu\nu} = \frac{1}{z^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & R + z^{1-\mu} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \tag{5.25}$$

In this section we will (if not explicitly stated otherwise) restrict our analysis to the physical part of the spacetime defined by the condition $R + z^{1-\mu} > 0$. Let us first discuss the lightcones in this spacetime, obtained by setting $ds^2 = 0$ in (5.23). Due to the absence of $dzdy$ and $dzdR$ terms, the lightcones will be symmetric under inversion with respect to the y-R-plane at every spacetime point. For the background metric they are even independent of z as for $d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu$ the coordinate z appears only in a conformal factor. Figure 5.1 shows the plot of a lightcone at a representative point of the spacetime for $\mu = 0.5$.

Furthermore, in (5.10) ξ^t was strictly positive which means that the vector ξ is future pointing. For this reason and with $\xi = -2e^{-y}\partial_R$, the coordinate R will have to decrease along the geodesics with ξ as tangent vector field. Indeed, the coordinate R has to decrease along every future pointing causal geodesic in the physical part of the spacetime, and can hence be used as a measure of time. On the light cone we have

$$0 \equiv ds^2 = \underbrace{dz^2}_{\geq 0} + dydR + \underbrace{\left(R + z^{1-\mu}\right) dy^2}_{\geq 0} \tag{5.26}$$

which requires $dydR \leq 0$. This is possible for either $dy \geq 0$ and $dR \leq 0$ or $dy \leq 0$ and $dR \geq 0$. As we already chose for one family of geodesics (having $dy = 0$) to be future pointing when $dR < 0$, consistency requires that a causal curve is called future pointing when $dy \geq 0$ and $dR \leq 0$. The situation $dR = 0$ occurs for causal geodesics only in the unphysical part of the spacetime, since then (5.26) with $ds^2 \leq 0$ would

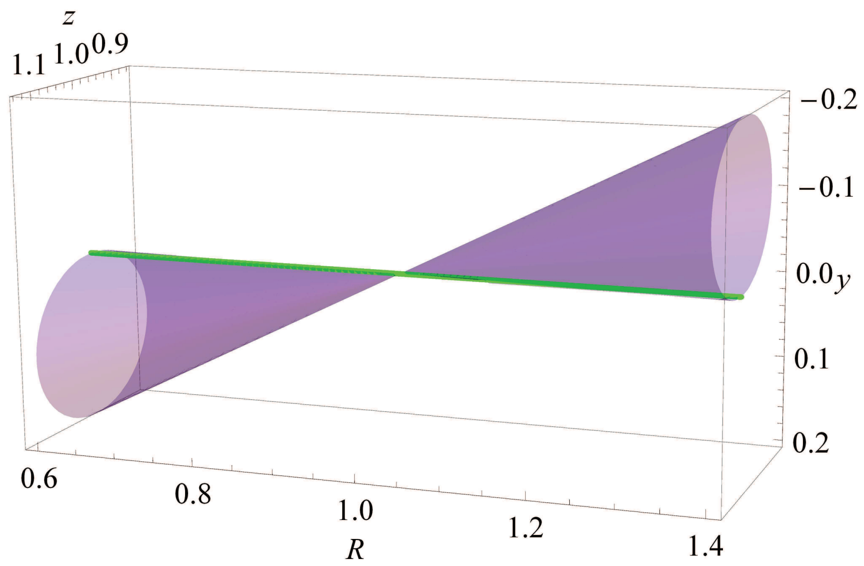

Fig. 5.1: Lightcone for $\mu = 0.5$ at $z = 1, R = 1, y = 0$. The solid green line represents one of the geodesics defined by $\xi = -2e^{-y}\partial_R$ [72].

require $R + z^{1-\mu} \leq 0$ as setting $dR = dz = dy$ would be a trivial solution. The only possible causal curves with $y = const.$ are those null geodesics described by the tangent vector field ξ, as $ds^2 \leq 0$ and $dy = 0$ in (5.26) necessarily require $dz = 0$ and $ds^2 = 0$.

In order to find out whether there are other analytically solvable geodesics besides the ones found in (5.14-5.15), we have to study the geodesic equations [8]

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \tag{5.27}$$

where for global coordinates the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ are listed in appendix C.

First of all we can easily verify that the Killing geodesics (5.24) are indeed geodesics. From (5.27) it follows for these curves that Γ^μ_{RR} has to vanish which is true as can be seen from appendix C. Unfortunately, most of the other Christoffel symbols are not vanishing and hence exact solutions of (5.27) apart from the Killing geodesics are hard to find[7]. Nevertheless, there is another way to calculate geodesics: Along every

[7] In fact, there is a family of analytically obtainable spacelike geodesics associated with the vector field $\bar{L}_0$, see table 10.1. In this thesis, however, we will only be interested in causal geodesics and

geodesic with tangent vector $\dot{x}^\mu$ the quantity $g_{\mu\nu}\dot{x}^\mu\tilde{\xi}^\nu = C_{\tilde{\xi}}$ is conserved if $\tilde{\xi}$ is a Killing vector [8]. Since the causal nature $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \in \{-1, 0, 1\}$ of an affinely parametrized geodesic is conserved [8], we can set up the following system of three coupled first order differential equations for the three unknown functions $z(\tau)$, $R(\tau)$ and $y(\tau)$:

$$C_y = g_{\mu\nu}\dot{x}^\mu(\partial_y)^\nu = \frac{1}{2z^2}\left(\dot{R} + 2\dot{y}\left(R + z^{1-\mu}\right)\right) \tag{5.28}$$

$$C_\xi = g_{\mu\nu}\dot{x}^\mu\xi^\nu = \frac{-e^{-y}\dot{y}}{z^2} \tag{5.29}$$

$$s = \frac{1}{z^2}\left(\dot{z}^2 + \dot{y}\left(\dot{R} + \dot{y}\left(R + z^{1-\mu}\right)\right)\right) \tag{5.30}$$

with s being 0 for lightlike and -1 for timelike geodesics.

Let us briefly discuss the meaning of the constants C_y and C_ξ. It is known that the quantity C_y can be interpreted as the angular momentum of the geodesic [8], but what meaning does the quantity C_ξ have? To answer this question, we should note that the transformation $y \to y + \Delta y$ is a symmetry transformation that should not have any physical importance. Due to the prefactor e^{-y} in ξ this would result in a positive prefactor which can be absorbed into the value of C_ξ. Instead of a conserved quantity with a meaningful continuum of values like C_y (or also C_t for geodesics of the stationary background $\bar{g}_{\mu\nu}$) we should regard equation (5.29) as defining three equivalence classes of geodesics for which C_ξ is either positive, zero or negative. It should be noted that while the three differential equations (5.28-5.30) are independent, there are nevertheless certain connections between the values of C_y, C_ξ and s for future pointing geodesics.

For example if $C_\xi > 0$, then necessarily $\dot{y} < 0$. We already saw that for future pointing causal geodesics $\dot{y} < 0$ is not possible, hence such a geodesic would have to be either spacelike or causal and past pointing. Setting $C_\xi = 0$ requires $\dot{y} = 0$, but as discussed above, the only causal geodesics with $\dot{y} = 0$ are the Killing geodesics themselves. Assuming $C_y > 0$ requires, using $R + z^{1-\mu} > 0$ and $\dot{R} < 0$, that $\dot{y} > 0$ which then requires $C_\xi < 0$.

The reader should be aware of the fact that the differential equations (5.28-5.30) are not fully equivalent to the equations (5.27). In fact, the former allow solutions that are not solutions of the latter in general. We will discuss these "fake solutions" (and the question when they are also solutions of (5.27)) now as they are quite interesting. We already discussed that the singularity of the spacetime is by definition generated by closed null curves, and it seems natural to ask whether these curves are also geodesics.

curves.

One could thus make the ansatz $z(\tau) = const. \equiv z_0$ and $R(\tau) = const. \equiv R_0$ such that $R_0 + z_0^{1-\mu} = 0$. Inserting this into (5.28-5.30) yields the three equations:

$$C_y = 0 \ , \ s = 0 \ , \ C_\xi = \frac{-e^{-y(\tau)}\dot{y}(\tau)}{z_0^2}$$

For future pointing geodesics we still require $C_\xi < 0 \Leftrightarrow \dot{y} < 0$. The last equation can then easily be solved by separation of variables which gives (with initial conditions $y(0) \equiv 0$) the result $y(\tau) = -\log\left(C_\xi z_0^2 \tau + 1\right)$. This obviously diverges when $\tau \to \tau_{max} = \frac{-1}{C_\xi z_0^2}$. It is easy to see that in general these curves are not solutions of (5.27). The conditions $z(\tau) = const.$ and $R(\tau) = const.$ obviously require $\dot{z} = \dot{R} = \ddot{z} = \ddot{R} = 0$. Inserting this into (5.27) yields

$$0 = \Gamma^z_{yy}\big|_{R+z^{1-\mu}=0} = \Gamma^R_{yy}\big|_{R+z^{1-\mu}=0}$$

as $\dot{y} \neq 0$. Unfortunately, as can be seen from appendix C, $\Gamma^z_{yy}\big|_{R+z^{1-\mu}=0} = \frac{1}{2}(\mu - 1)z^{-\mu}$ which only vanishes in the special case $\mu = +1$. We will briefly return to the possible meaning of these geodesics in section 6.1.

Before closing this section on geodesics and causal curves, we should discuss two quite important families of geodesics that can (so far) not be analytically computed in general, but which can at least be identified. We start with the fact that any null geodesic has to satisfy (5.30) with $s = 0$. Multiplying with $z^2 \neq 0$ and solving for $\frac{\dot{z}}{\dot{R}}$ results in

$$\frac{\dot{z}}{\dot{R}} = \pm\sqrt{-\frac{\dot{y}}{\dot{R}}\left(1 + \frac{\dot{y}}{\dot{R}}a\right)}$$

where we used $a = R + z^{1-\mu}$, which is a measure of how far the spacetime point is away from the singularity. Obviously, the value of $\frac{\dot{z}}{\dot{R}}$ depends for given z, y, R on the value of $\frac{\dot{y}}{\dot{R}}$. For $\frac{\dot{y}}{\dot{R}} = \frac{-1}{2a}$ the extremal values

$$\left(\frac{\dot{z}}{\dot{R}}\right)_{max/min} = \pm\frac{1}{2\sqrt{a}} \tag{5.31}$$

are obtained. For this reason, all causal curves have to have slopes between $\frac{dz}{dR} = \frac{+1}{2\sqrt{a}}$ and $\frac{-1}{2\sqrt{a}}$ when projected to the z-R-plane. Interestingly, the two families of null-curves with maximal and minimal slope, i.e. the curves tangent to the vector fields

$$i/o = \pm\sqrt{R + z^{1-\mu}}\partial_z + \partial_y - 2\left(R + z^{1-\mu}\right)\partial_R \tag{5.32}$$

are actually families of null geodesics with zero angular momentum. Inserting $-2a\dot{y} = \dot{R}$ and $\pm 2\sqrt{a}\dot{z} = \dot{R}$ into (5.28) and (5.30) trivially leads to $C_y = 0$ and $s = 0$ while (5.29) cannot be considerably simplified. It can be shown that $\nabla_i i = f_1(z, R)i$ and $\nabla_o o = f_2(z, R)o$, which means that the approaches $i^\mu/o^\mu = \dot{x}^\mu_{(1/2)}$ would yield geodesics that are not affinely parametrized. Nevertheless, affinely parametrized geodesics which are tangent to the vector fields i and o[8] do exist. In the following, we will refer to these two families of geodesics as *ingoers* (tangent to i) and *outgoers* (tangent to o). In the discussion of the global structure of the spacetime, we will often be only interested in the projections of these geodesics to the z-R-plane. Following from (5.31), these curves will be solutions of the differential equations $\frac{dR}{dz} = \pm 2\sqrt{(R + z^{1-\mu})}$. As will be discussed in section 6.3, these curves play an important role in the determination of the event horizons of the spacetime.

[8]Just like the Killing geodesics are tangent to both the vector fields $\xi = -2e^{-y}\partial_R$ and $-\partial_R$, but because of the prefactors, they are only the integral curves of the latter.

Chapter 6

Discussion of the Global Structure

6.1 Properties of the Singularity

Following [5], we define the *singularity in the causal structure of the spacetime*, or short singularity in this thesis, as the boundary between the (unphysical) part of the spacetime where closed causal curves are possible and the physical part of the spacetime where causality is not violated. The singularity is by definition a hypersurface in which closed null curves are embedded. As we employ the identification $y \sim y + 2\pi$, closed null curves emerge where ∂_y becomes a null vector.

As can be seen from the line element (5.23), the singularity is for general μ the two dimensional hypersurface defined by the relations $R(z) = -z^{1-\mu}$ and y arbitrary. From the form $R(z) = -z^{1-\mu}$ it is obvious that $R(z) \leq 0$ for general μ, with $R(z) \to 0$ for $z \to 0$ and $R(z) \to -\infty$ for $z \to \infty$ for $\mu < 1$. For $\mu = 1$ we find $R(z) = -1$ (being equivalent to the inner trapping horizon in this case), and for $\mu > 1$ it is obvious that $R(z) \to -\infty$ for $z \to 0$ and $R(z) \to 0$ for $z \to \infty$. Using $dR = (\mu - 1)z^{-\mu}dz$, we can calculate the line element of the induced metric $\mathfrak{g}_{\mu\nu}$ of this hypersurface to be

$$ds^2_{SING} = \mathfrak{g}_{\mu\nu}dx^\mu dx^\nu = \frac{1}{z^2}\left(dz^2 + (\mu + 1)z^{-\mu}dzdy\right)$$

Calculating the determinant of $\mathfrak{g}_{\mu\nu}$ yields $\mathfrak{g} = -\frac{1}{4}(\mu - 1)^2 z^{-2(\mu+2)}$. Obviously, the singularity is a $(1 + 1)$ dimensional, i.e. timelike, hypersurface except for the case

$\mu = +1$ where it contains many spacelike and one lightlike, but no timelike directions. This means that for $\mu \neq 1$ an observer outside of the singularity will in principle be able to obtain information coming out of the singularity on causal curves before crossing the singularity himself. This will require the existence of an *inner* or *Cauchy horizon* which is defined to be the border between points in the spacetime from which the singularity is visible and points from which it is not visible.

It was discussed in section 4 that the BTZ black holes are derived from the AdS_3-metric as a quotient space. In [5] this quotient space construction was used to prove that BTZ black holes fail to be Hausdorff manifolds at their singularity ($r = 0$) when $J = 0$, and the similarity between the singularity of the non-rotating BTZ black hole and the Taub-NUT space (see e.g. [12]) was noticed. For the Taub-NUT space it was pointed out in [73] that there are closed causal geodesics on which one cannot extend the geodesic to infinite values of the affine parameter. Here the "fake geodesics" discussed in section 5.5 become interesting because exactly the same behaviour was observed for these: they are closed causal geodesics, nevertheless the solution cannot be extended to values $\tau \geq \tau_{max}$. This problem cannot be cured by a coordinate transformation that keeps intact the S^1-topology of the compact dimension in ∂_y direction. As we also discussed in section 5.5, these curves are only geodesics for $\mu = +1$, and we will see in section 6.4 that for this value, the metric (5.1) indeed describes a non-rotating BTZ black hole. Therefore, it seems that the pathological nature of the singularity of the non-rotating BTZ black hole can be eased by introducing either a non-vanishing angular momentum $J \neq 0$ or a distortion as $h_{\mu\nu}$.

6.2 Calculation of Trapping horizons

It was stated in section 2.3 that it is possible to use (2.2) for computing closed trapped surfaces, and hence trapping horizons. In order to do so, one has to first define spacelike slices Σ and then calculate the extrinsic curvature K_{ij} of these slices. The global coordinates defined in section 5.4 are perfect for this task. The slices with $R = const.$ and hence $dR = 0$ have induced metric $\mathfrak{g}_{ij} = \frac{1}{z^2}\text{diag}(1, R + z^{1-\mu})$ which in the physical part of the spacetime has positive determinant, and can be used as spacelike slices Σ. We can now decompose our metric in the following way [9, 74] (this time using $x^1 = R$,

$x^2 = z$, $x^3 = y$ for convenience):

$$g_{\mu\nu} = \frac{1}{z^2}\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & R + z^{1-\mu} \end{pmatrix} \equiv \begin{pmatrix} N_k N^k - N^2 & N_j \\ N_i & \mathfrak{g}_{ij} \end{pmatrix}$$

with $i, j, k \in \{2, 3\}$. Here, besides the induced metric $\mathfrak{g}_{ij}$ on Σ, we introduced the shift N_k and the lapse N. All Latin indices are raised and lowered with the induced metric $\mathfrak{g}_{ij}$ or its inverse $\mathfrak{g}^{ij}$. One finds

$$\mathfrak{g}^{ij} = \begin{pmatrix} z^2 & 0 \\ 0 & \frac{z^{\mu+2}}{Rz^\mu + z} \end{pmatrix}, \; N_k = \begin{pmatrix} 0 \\ \frac{1}{2z^2} \end{pmatrix}, \; N^k = \begin{pmatrix} 0 \\ \frac{z^\mu}{2(Rz^\mu + z)} \end{pmatrix}, \; N = \frac{1}{2z\sqrt{R + z^{1-\mu}}}$$

Using $K_{ij} = \frac{1}{2N}\left(N_{j:i} + N_{i:j} - \partial_{x^1}\mathfrak{g}_{ij}\right)$ [9] ($(...)_{:j}$ denoting the covariant derivative with respect to the euclidean metric $\mathfrak{g}_{ij}$) it follows

$$K_{ij} = \begin{pmatrix} 0 & \frac{1}{2}(1-\mu)z^{-1-\mu}\frac{1}{\sqrt{R+z^{1-\mu}}} \\ \frac{1}{2}(1-\mu)z^{-1-\mu}\frac{1}{\sqrt{R+z^{1-\mu}}} & \frac{\sqrt{R+z^{1-\mu}}}{z} \end{pmatrix}$$

In a given slice Σ, we can search for marginal surfaces S defined by fixed values of R and z and arbitrary values of y (due to symmetry). To do so one uses (2.2) with $s^i = (\mp z, 0)^T$ (chosen such that $\mathfrak{g}_{ij}s^i s^j = 1$, $\mathfrak{g}_{ij}s^i(\partial_y)^j = 0$) and gets:

$$1 - z^2\sqrt{\frac{z^{\mu-2}}{Rz^\mu + z}} + \frac{(\mu-1)z}{2\left(Rz^\mu + z\right)} = 0 \text{ for } s^i = \begin{pmatrix} -z \\ 0 \end{pmatrix} \tag{6.1}$$

$$-1 - z^2\sqrt{\frac{z^{\mu-2}}{Rz^\mu + z}} - \frac{(\mu-1)z}{2\left(Rz^\mu + z\right)} = 0 \text{ for } s^i = \begin{pmatrix} +z \\ 0 \end{pmatrix} \tag{6.2}$$

The equations (6.1) and (6.2) determine marginal surfaces on which out- and ingoing orthogonal light rays have zero expansion. The solutions to these equations will determine the trapping horizons that we call *outer* and *inner*. They are given by the curves

$$R_{outer}(z) = \frac{1}{2}z^{-2\mu}\left((-\mu-1)z^{\mu+1} + z^{2\mu+2} + \sqrt{z^{3\mu+3}\left(z^{\mu+1} - 2\mu + 2\right)}\right) \tag{6.3}$$

$$R_{inner}(z) = \frac{1}{2}z^{-2\mu}\left((-\mu-1)z^{\mu+1} + z^{2\mu+2} - \sqrt{z^{3\mu+3}\left(z^{\mu+1} - 2\mu + 2\right)}\right) \tag{6.4}$$

Let us shortly discuss the properties of the hypersurfaces described by these curves.

For $\mu = +1$, the equations (6.3) and (6.4) simplify to $R_{inner}(z) = -1$, $R_{outer}(z) = z^2 - 1$, while for $\mu = -1$ we find $R_{inner}(z) = \frac{1}{2}\left(1 - \sqrt{5}\right) z^2$, $R_{outer}(z) = \frac{1}{2}\left(1 + \sqrt{5}\right) z^2$. These special values will become important in section 6.4.

For general $\mu < 1$ it is easy to show that both $R_{outer}(z) \to 0$ and $R_{inner}(z) \to 0$ in the limit $z \to 0$. This means that in global coordinates, both trapping horizons and the singularity meet at $R = 0 = z$. Furthermore, one can show that $R_{inner}(z) \geq -z^{1-\mu}$ for any μ with equality for $\mu < 1$ only for $z = 0$ or the limit $z \to +\infty$, which means that the inner trapping horizon (and also the outer one as $R_{outer}(z) \geq R_{inner}(z)$) will always be in the physical part of the spacetime. In the limit $z \to +\infty$ it is easy to see that $R_{outer}(z) \to +\infty$ for any μ while in the same limit $R_{inner}(z) \to -\infty$ for $-1 \leq \mu < 1$ and $R_{inner}(z) \to +\infty$ for $\mu < -1$.

Another interesting feature is that while for $\mu \leq -1$, $R_{outer}(z)$ is a monotonous function of z, for $|\mu| < 1$ the function $R_{outer}(z)$ initially decreases, attains a minimum and then increases again with z. This "bow" of the outer trapping horizon is quite unphysical if we want the trapping horizon to be a description of the black hole boundary. This means that there are points in the spacetime which are outside of the outer trapping horizon but which have a coordinate $R < 0$, and from which it is not possible to escape the singularity as R has to decrease along every future pointing causal curve in the physical part of the spacetime. See figure 6.2 in section **??** for a plot of the trapping horizons for $\mu = \frac{1}{2}$ and $\mu = -\frac{3}{2}$.

Using $dR \equiv \frac{\partial R_{outer}(z)}{\partial z} dz$ in (5.23) it is possible to calculate the induced metric $\mathfrak{g}_{\mu\nu}$ ($\mu, \nu \in \{z, y\}$) on the trapping horizon. For the determinant $\mathfrak{g}$ of this metric one finds a rather complicated expression which contains one important physical information: its sign. For $\mu = \pm 1$ it follows that $\mathfrak{g}(z) = 0$ which means that the outer trapping horizon is a null-surface in these cases. For $\mu < -1$ we find $\mathfrak{g}(z) < 0$ for any z which means that in these cases the outer trapping horizon is a timelike hypersurface with signature $(-1, +1)$. For $|\mu| < 1$ nevertheless, $\mathfrak{g}(z) < 0$ for small z and $\mathfrak{g}(z) > 0$ for large z, indicating that due to the bow discussed above and shown in figure 6.2, the outer trapping horizon switches from a spacelike to a timelike hypersurface for some value of z. We called the trapping horizon calculated with $s = (-z, 0)^T$ "outer" as s points outward, towards infinity (at least for the background metric $\bar{g}_{\mu\nu}$). Hayward [13] used the deviating terminology that a trapping horizon is *outer* when the expansion of the family of null geodesics that vanishes on the horizon *shrinks* while passing through the horizon following the other family of null geodesics (with non-vanishing expansion) and *inner* when it *grows*. In this sense, what we called the outer trapping horizon changes from being an outer trapping horizon to being an inner trapping horizon when $z \to 0$.

A similar analysis of the induced metric and its determinant for the inner trapping horizon shows that these hypersurfaces are null for $\mu = \pm 1$, spacelike for $\mu < -1$ and timelike for $|\mu| < 1$.

For $\mu > +1$ there is a minimal value of z that can be attained for trapping horizons due to the square-root in (6.3) and (6.4). At this value both these curves are solutions to (6.1) (but not to (6.2)) and turn out to be only two branches of one single outer trapping horizon. Additionally, the singularity is completely outside of this trapping horizon, and not inside of it as for $\mu < +1$, see figure 6.1.

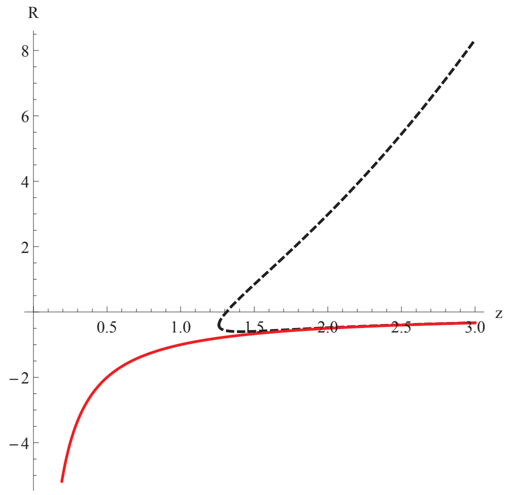

Fig. 6.1: Trapping horizon drawn as dashed black line for $\mu = 3$, singularity drawn as solid red line.

6.3 Event Horizons

Having proven the existence of a singularity in section 6.1 and of trapping horizons in section 6.2, it is natural to ask about the existence of event horizons as defined in section 2.2. For simplicity, we will limit our investigation to the cases where $\mu \leq 1$. Now, the previously stated global nature of the definition of event horizons becomes a problem, especially as the asymptotics of our spacetime at infinity are not necessarily AdS-like for general μ. The limit $\rho \to \infty$ and $t = const.$ corresponds to $z \to 0$ and $R \to const.$ in global coordinates. We therefore adopt the viewpoint that in these coordinates, $z = 0$, $R > \lim_{z\to 0} \left(-z^{1-\mu}\right)$ and y being arbitrary describes "infinity", and that the (outer) event horizon of the spacetime will be described by the boundary of its causal past. This ansatz is far from perfect, the possible problems of such an approach were discussed in [75]. We will nevertheless pursue this approach for three reasons: Firstly, it reproduces the correct event horizon in the cases $\mu = \pm 1$ as we will see in section 6.4. Secondly, for $\mu < -1$ the asymptotics for $\rho \to \infty$ are the same as in the BTZ-case as $(e^t \sinh(\rho))^{1+\mu} \to 0$ in this limit. Thirdly, using this definition for $\mu < 1$, in a spacetime diagram such as figure 6.2 event and trapping horizons approach the same point $z = 0 = R$ in the limit $z \to 0$.

In the following, we will show how to numerically determine the horizons. As the singularity contains a timelike direction for $\mu \neq 1$, there will in general be an outer as well as an inner horizon as discussed in section 6.1.

Due to the definition of the outer and inner event horizons as boundaries between points from which a certain limit or hypersurface can be reached on causal curves[1] and points from which this is not possible, the event horizons will be generated by null geodesics of maximal and minimal slope in the z-R-plane defined in (5.31) and the following discussion in section 5.5. Therefore, the outer horizon is now for $\mu < 1$ defined to be the solution of the differential equation

$$\frac{dR}{dz} = 2\sqrt{(R + z^{1-\mu})} \text{ with the initial condition } R(0) = 0. \tag{6.5}$$

Similarly, the inner horizon is defined to be the solution of

$$\frac{dR}{dz} = -2\sqrt{(R + z^{1-\mu})} \text{ with } R(0) = 0. \tag{6.6}$$

Unfortunately, there is no closed-form expression for the solutions of these equations for $|\mu| \neq 1$ (see section 6.4 for the discussion of the case $|\mu| = 1$), but numerical solutions can be calculated. They are shown for $\mu = \frac{1}{2}$ and $\mu = -\frac{3}{2}$ together with the trapping horizons and the singularity in figure 6.2. In contrast to the trapping horizons calculated in section 6.2, the event horizons calculated by (6.5) and (6.6) are by definition monotonous functions $R(z)$. The unphysical behaviour of the outer trapping horizon for $|\mu| < 1$ is therefore not mirrored by the outer event horizon.

Next we want to investigate the properties of the outer event horizon. The topology of the spacetime at hand is $\mathbb{R}^2 \times S^1$ and in the physical part of the spacetime the radius

[1] We define the inner event horizon to be the boundary between points in the physical part of the spacetime from which the singularity can be reached on past-pointing causal geodesics and such points in the physical part of the spacetime from which this is not possible, see section 6.1.

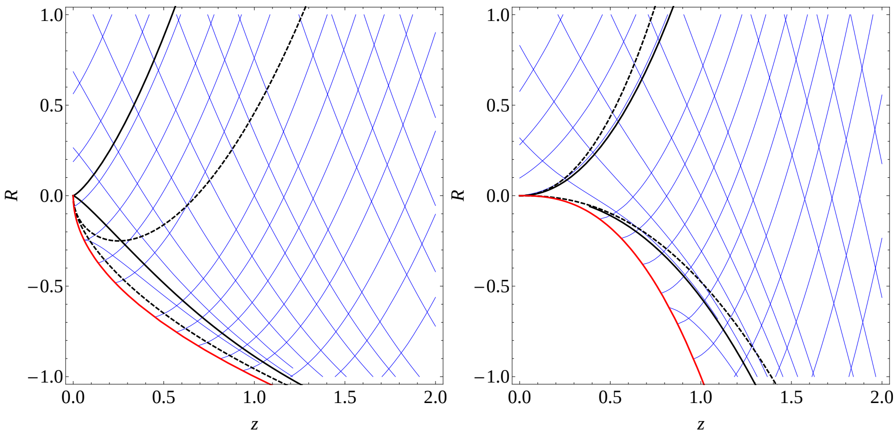

Fig. 6.2: Diagrams for $\mu = 1/2$ on the left, and $\mu = -3/2$ on the right. Event horizons are depicted as solid black lines, trapping horizons as dashed black lines and the singularity as red line. The projections of several in- and outgoing null geodesics to the z-R-plane are drawn as thin blue lines.

of the compact dimension is

$$r(z, R) = \sqrt{g_{yy}} = \frac{\sqrt{R + z^{1-\mu}}}{z} \tag{6.7}$$

We can numerically compute the radius r of the outer event horizon as a function of z. As the outer event horizon is always defined by a monotonous function $R(z)$, z can be used as a measure of time instead of R, with large values of z corresponding to early times and small values of z corresponding to late times. Figure 6.3 shows the evolution of the outer event horizons as functions of z for $\mu = 0.5$ (solid) and $\mu = -1.5$ (dashed).

While for $1 > \mu > -1$ the horizon-circumference generally increases towards small z, it generally decreases for $-1 > \mu$. This is not particularly surprising for two reasons: First of all, the radius r will be large when the distortion $z^{-1-\mu}dy^2$ is large. Where this distortion becomes small, the metric (5.23) approximates the metric of the non-rotating BTZ-black hole and it is not surprising if there the horizon-radius approaches the value $r = 1$ which is also the radius of the event horizon in the unperturbed background-metric. Secondly, the famous Hawking area theorem is derived under the

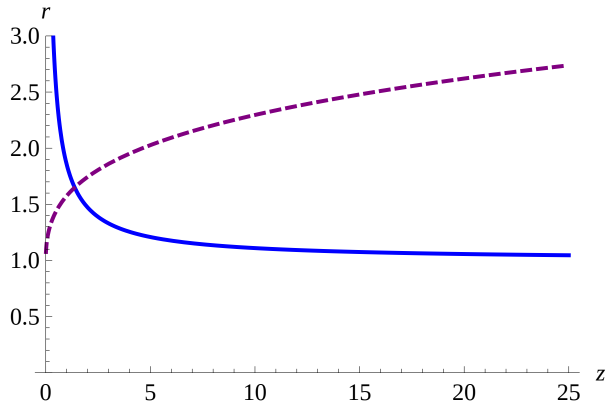

Fig. 6.3: Radii $r = \sqrt{g_{yy}}$ of the outer event horizon as a function of z for $\mu = \frac{1}{2}$ (solid) and $\mu = -\frac{3}{2}$ (dashed). Smaller values of z correspond to later times.

assumption that [12]:

$$R_{\mu\nu}k^\mu k^\nu \geq 0 \text{ for every null vector } k^\mu$$

which is often related to the null or weak energy condition by use of the Einstein-field-equations, see e.g. [7, 12]. Here, we have using (3.4)

$$R_{\mu\nu} = \frac{-1}{\mu}C_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} + \frac{1}{l^2}g_{\mu\nu} = \frac{1-\mu^2}{2}h_{\mu\nu} - 2g_{\mu\nu}$$

where in the last step we used $R = -6$, $l = 1$ and (5.4). Obviously, it follows that for any null vector k^μ

$$R_{\mu\nu}k^\mu k^\nu = \frac{1-\mu^2}{2}h_{\mu\nu}k^\mu k^\nu$$

Since we can write $h_{\mu\nu} = l_\mu l_\nu$ (see (5.2) and below) it follows that $h_{\mu\nu}k^\mu k^\nu = (l_\mu k^\mu)^2 \geq 0$ for any vector k^μ. For this reason, the area-theorem is only applicable for $|\mu| \leq 1$.

6.4 Special Values: $\mu = \pm 1$

From (5.4) it follows that the metric (5.1) is not only a solution of topologically massive gravity, but also a solution of ordinary Einstein gravity in the special cases $\mu = \pm 1$. These special cases shall be investigated in more details in the following section.

Let us again state the line element of (5.1) in the global coordinates defined in section 5.4 and identify the background- and distortion parts:

$$ds^2 = \underbrace{\frac{1}{z^2}\left(dz^2 + dydR + Rdy^2\right)}_{\bar{g}_{\mu\nu}dx^\mu dx^\nu} + \underbrace{\frac{1}{z^{1+\mu}}dy^2}_{h_{\mu\nu}dx^\mu dx^\nu} \tag{6.8}$$

It can easily be seen that in the case $\mu = +1$ the line element (6.8) is equivalent to the line element of the undisturbed BTZ black hole for $M = 1, J = 0$ which can be verified by a simple coordinate shift $R' = R + 1$.

Before moving on to the investigation of the case $\mu = -1$, we will now comment on a detail of the metric (5.1) that was not addressed so far. In section 5.1 it was described how the solution (4.15) of the linearized equations of motion around the background $\bar{g}_{\mu\nu}$ describes the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ which is a solution to the full equations of motion. But solutions to the linearized equations of motion can have arbitrary prefactors, and in general, we could have multiplied the solutions in (4.13-4.16) with an arbitrary prefactor Ξ. It can be seen from (5.5) that even if we had chosen to do so in section 5.1, this would not have affected the fact that (5.1) fulfills the non-linear equations of motion. For (5.1), such a prefactor Ξ can obviously always be absorbed *up to sign* by a shift in the coordinate t, except for the case where $\mu = -1$, as there the prefactor becomes $(e^t \sinh\rho)^{1+\mu} = 1$. Therefore, $h_{\mu\nu}$ actually describes *two distinct*[2] one-parameter families of exact solutions of TMG, $g_{\mu\nu}(\mu) = \bar{g}_{\mu\nu} + h_{\mu\nu}(\mu)$ and $g'_{\mu\nu}(\mu) = \bar{g}_{\mu\nu} - h_{\mu\nu}(\mu)$, which at the point $\mu = -1$ are connected by a continuum of non-isometric metrics $g^{\Xi}_{\mu\nu} = \bar{g}_{\mu\nu} + \Xi h_{\mu\nu}(-1)$.

Let us now come back to the metric $g_{\mu\nu}$ with $\mu = -1$ and $\Xi = 1$. In this case, the singularity still contains a timelike direction as discussed in section 6.1, and there are still two horizons, an outer and an inner one as discussed in sections 6.2 and 6.3. The metric can therefore not be globally equivalent to the background metric $\bar{g}_{\mu\nu}$ as was the case for $\mu = +1$, but might describe a rotating black hole with parameters $M \neq 1$, $J \neq 0$. It is indeed easy to show that in the case $\mu = -1$ the event horizons agree with the trapping horizons (6.3,6.4) that take the very simple forms $R_{outer/inner}(z) = \frac{1}{2}\left(1 \pm \sqrt{5}\right) z^2$, i.e. that these functions are solutions to the differential equations (6.5,6.6) for $\mu = -1$. This equivalence of trapping and event horizons should be expected in the case of stationary BTZ black holes. We can now calculate

[2] Of course, the choice $\Xi = 0$ would lead to the trivial solution $g^0_{\mu\nu} = \bar{g}_{\mu\nu}$. We nevertheless do not explicitly exclude the possibility $\Xi = 0$ as for the continuum of solutions at $\mu = -1$ this value will be important, too.

the radii of the event horizons as in section 6.3 and find the results

$$r_{outer/inner} = \frac{\sqrt{R_{outer/inner}(z) + z^2}}{z} = \sqrt{\frac{1}{2}\left(3 \pm \sqrt{5}\right)}$$

which are constant as expected for a BTZ black hole. Using the relations (4.3) (with $l = 1$, $\text{sign}(r_-) = \text{sign}(J)$ and for simplicity $8G_N = 1$ as in [4, 5]) and inserting $r_+ = r_{outer}$ and $|r_-| = r_{inner}$ yields $M = 3$, $|J| = 2$. Motivated by this result, we can now search for a coordinate transformation that maps the metric (see (5.1))

$$g_{\mu\nu} = \begin{pmatrix} -\sinh^2(\rho) & 0 & 0 \\ 0 & \cosh^2(\rho) & 0 \\ 0 & 0 & 1 \end{pmatrix} + \Xi \begin{pmatrix} 1 & 1 & \frac{2}{\sinh(2\rho)} \\ 1 & 1 & \frac{2}{\sinh(2\rho)} \\ \frac{2}{\sinh(2\rho)} & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix} \tag{6.9}$$

to the BTZ metric (4.1) for $l = 1$ (with $x^1 = t'$, $x^2 = \phi'$, $x^3 = r$):

$$g_{BTZ\mu\nu} = \begin{pmatrix} M - r^2 & -\frac{J}{2} & 0 \\ -\frac{J}{2} & r^2 & 0 \\ 0 & 0 & \frac{1}{\frac{J^2}{4r^2} + r^2 - M} \end{pmatrix} \tag{6.10}$$

with parameters M and J that will certainly depend on Ξ. Such a coordinate transformation can easily be found and reads:

$$t = t' + \frac{1}{4}\left(-2\log\left(r^2 - 1 - \Xi\right) + \frac{2\text{arctanh}\left(\frac{2\Xi - 2r^2 + 1}{\sqrt{4\Xi+1}}\right)}{\sqrt{4\Xi + 1}} + \log\left(\Xi^2 + r^4 - (2\Xi + 1)r^2\right)\right)$$

$$\phi = \phi' + \frac{1}{4}\left(-\log\left(\left(r^2 - \Xi\right)^2 - r^2\right) + 2\log\left(r^2 - \Xi\right) - \frac{2\text{arctanh}\left(\frac{-2\Xi + 2r^2 - 1}{\sqrt{4\Xi+1}}\right)}{\sqrt{4\Xi + 1}}\right)$$

$$\rho = \cosh^{-1}\left(\sqrt{r^2 - \Xi}\right)$$

This transformation maps the metric (6.9) to the metric (6.10) with parameters $M = 1 + 2\Xi$ and $J = -2\Xi$. It should be noted that the BTZ metric with M and J can always be transformed to the BTZ metric with $M' = M$ and $J' = -J$ by a reversal of time, $t \to t' = -t$. As discussed in section 3.2, TMG behaves odd under such transformations and this is why above we restricted ourselves to transformations that do not reverse time. It should also be noted that in the form written down above this coordinate transformation is only valid for $r > \frac{1}{2}\left(\sqrt{4\Xi + 1} + 1\right)$, and this lower bound can indeed be shown to be the radius r_+ of the outer event horizon of the black hole with $M = 1 + 2\Xi$, $J = -2\Xi$. Interestingly, the cosmic censorship bound $Ml \geq |J|$ is only fulfilled for $\Xi \geq -\frac{1}{4}$.

Obviously, these results tell us that the family of metrics $g_{\mu\nu}(\mu) = \bar{g}_{\mu\nu} + \Xi h_{\mu\nu}(\mu)$ evaluated at the point $\mu = -1$ single out a one dimensional subset of the two dimensional parameter space of BTZ metrics with $l = 1$. We can now ask what is so special about this subset of BTZ black holes. In order to answer this question, it is useful to calculate mass, angular momentum and entropy of these black holes in the framework of TMG, using the equations (4.6-4.8) with $l = 1$ and $\mu = -1$. One obtains the results $M_{TMG} = 1 + 4\Xi$, $J_{TMG} = -1 - 4\Xi$ and $\mathcal{S}_{TMG} = \frac{\pi}{2G_N}$ independently of Ξ[3], which is also the value of the entropy of the background metric independently of μ.

This is a very fascinating result: While M and J take arbitrary values depending on Ξ, the entropy $\mathcal{S}_{TMG}$ seems to be a meaningful quantity for describing the whole family of exact solutions $g_{\mu\nu}(\mu) = \bar{g}_{\mu\nu} + \Xi h_{\mu\nu}(\mu)$. So far, we know the entropy only for the two cases $\mu = \pm 1$ and in these cases it always takes the same value, robustly even against changes of the parameter Ξ that affect the angular momentum, which should be a well defined quantity due to the Killing symmetry generated by ∂_ϕ. One might therefore set up the conjecture that independently of the value of μ, whenever an event horizon exists in the metric $g_{\mu\nu}(\mu)$, the entropy can be defined and will be a constant in time of the same value as in the entropy of the background metric. Unfortunately, we will see in section 7.3 that this seems not to be true.

6.5 Towards a Conformal Diagram

It is now time to discuss the global structure of the spacetime described by the solution (5.1). In [5] the conformal diagrams for non-extremal BTZ black holes where presented. In such a diagram, the spacetime is mapped to a compact region in such a way that at every point of the diagram the lines of slope $\pm 45°$ represent in- and outgoing null-geodesics. It would be extremely helpful if such diagrams could be drawn for all spacetimes with general μ. To do so, the first step would be to map the entire spacetime to a compact set. In global coordinates, this can be achieved by introducing coordinates $\tilde{R} = \arctan(R)$ and $\tilde{z} = \arctan(z)$. The compactified analogue of figure 6.2 is shown in figure 6.4.

[3]But of course only for $\Xi \geq -\frac{1}{4}$, as for smaller Ξ there are no event horizons at all, and hence entropy is not well defined for these metrics.

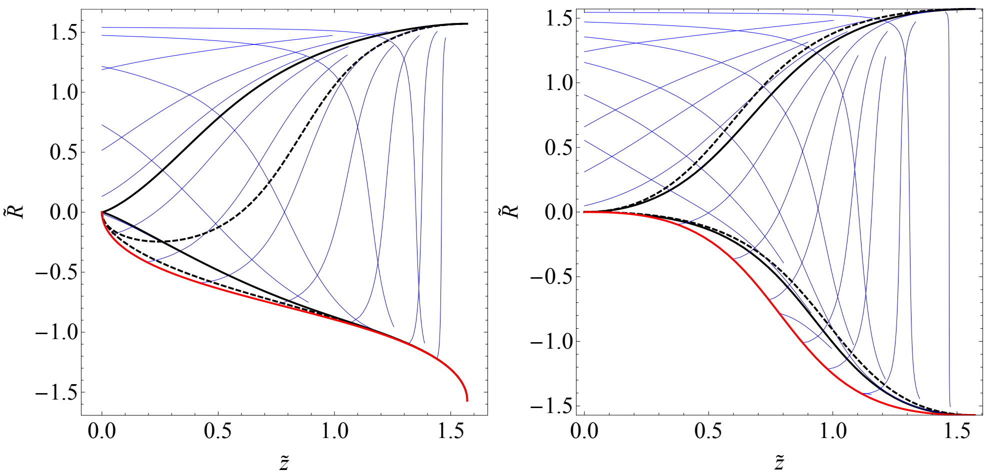

Fig. 6.4: Diagrams for $\mu = 1/2$ on the left, and $\mu = -3/2$ on the right. Event horizons are depicted as solid black lines, trapping horizons as dashed black lines and the singularity as red line. The projections of several in- and outgoing null geodesics to the z-R-plane are drawn as thin blue lines.

The in- and outgoer null-geodesics were defined to be the causal curves with maximal and minimal slopes when projected down to the z-R-plane in section 5.5. This means that where two blue lines in figure 6.4 cross, the angle at which this happens shows how wide the lightcone is at this point in spacetime. In order to to draw a conformal diagram, one would therefore have to find a coordinate system in which these lines always cross with an angle of 90°, i.e. coordinates $t(z, R)$, $x(z, R)$ such that

$$\frac{dR}{dz} = \pm 2\sqrt{R + z^{1-\mu}} \Rightarrow \frac{dt}{dx} = \pm 1$$

This ansatz yields the system

$$\frac{\partial x}{\partial z} = 2\frac{\partial t}{\partial R}\sqrt{R + z^{1-\mu}} \ , \ \frac{\partial t}{\partial z} = 2\frac{\partial x}{\partial R}\sqrt{R + z^{1-\mu}}$$

which unfortunately cannot be solved analytically for general μ. It is nevertheless possible to make statements about the global structure of the spacetime, as we will demonstrate in this section.

We will start with the case $\mu = -1$ for which we already know how the conformal diagram looks like, see [5]. Apart from the fact that event and trapping-horizons coincide for $\mu = -1$, the compactified diagram for this case looks very similar to the compactified diagram for $\mu = -\frac{3}{2}$ depicted in figure 6.4. Firstly, we note that infinity and singularity as well are timelike hypersurfaces, which will therefore have slopes $> 45°$ in a conformal diagram. For simplicity we will draw them as vertical lines. Secondly, the outgoing null geodesics can be categorized into three groups: Those going to infinity ($z \to 0, R > 0$), those generating the outer event horizon ($z \to 0, R \to 0$) and those ending up at the singularity. In the same way there are three kinds of ingoers: those starting at infinity, those generating the inner horizon and those starting at the singularity. It is obvious that all ingoers seem to asymptote towards the singularity in the limit $z \to \infty, R \to -\infty$[4]. In a true conformal diagram all ingoers would be parallel lines, which means that this point in the lower right corner of the compactified diagram would be "blown up" to a (diagonal) line. Similarly, the "point" $z \to \infty, R \to \infty$ would be blown up to a line by the need to draw the outgoers as parallel lines, while the lines $z \to \infty$, $|R| < \infty$ and $R \to \infty$, $z < \infty$ would shrink to a point.

How can we find out whether the ingoers finally fall into the singularity in the limit $z \to \infty, R \to -\infty$? One way to answer this question is to numerically calculate the curve $R(z)$ for a certain ingoer and determine, by following this specific ingoer in the limit $z \to \infty$[5], whether the function $r(z) = \sqrt{g_{yy}}$ decreases to zero or not. If this was the case, this would be an indication that the ingoer approaches and finally reaches the singularity which is defined by the condition $\sqrt{g_{yy}} = 0$. In the same way, we can investigate where the outgoers "come from" by studying the function $r(z)$ when following an outgoer in the limit $z \to \infty, R \to \infty$. The results are depicted in figure 6.5. In the limits $z \to \infty, R \to \pm\infty$ we find that $r(z)$ approaches the constant values $r \approx 1.618$ and $r \approx 0.618$ which are just the radii of outer and inner event horizon. This indicates that the null geodesics eventually approach copies of the inner and outer event horizons. In this way, the diagram shown in figure 6.5 fits nicely into the conformal diagram for the rotating BTZ black hole depicted in [5] , covering one copy of the regions I, II and III each.

A similar analysis can be done for the cases $|\mu| < 1$ and $\mu < -1$, with results shown in figure 6.5 in the middle and on the right. Let us discuss the situation $|\mu| < 1$ first. Here, we find that the following the outgoers back in time one reaches a region where $r \approx 1$ which is the value expected for the past horizon of the background metric $\bar{g}_{\mu\nu}$ as the distortion $h_{\mu\nu}$ goes to zero in this limit. Following the ingoers one approaches the limit $r \to 0$ indicating that the ingoers finally fall into the singularity. For $\mu < -1$ one

[4]The ingoers cannot cross the singularity for finite z, as for $R = -z^{1-\mu}$ their slope would be $dR/dz = 0$. Therefore ingoers can only emerge at the singularity for finite z.

[5]One might worry how it is possible to investigate *numerically* the behaviour of these geodesics in this limit. It is beneficial for this task that this limit is reached for finite affine parameter.

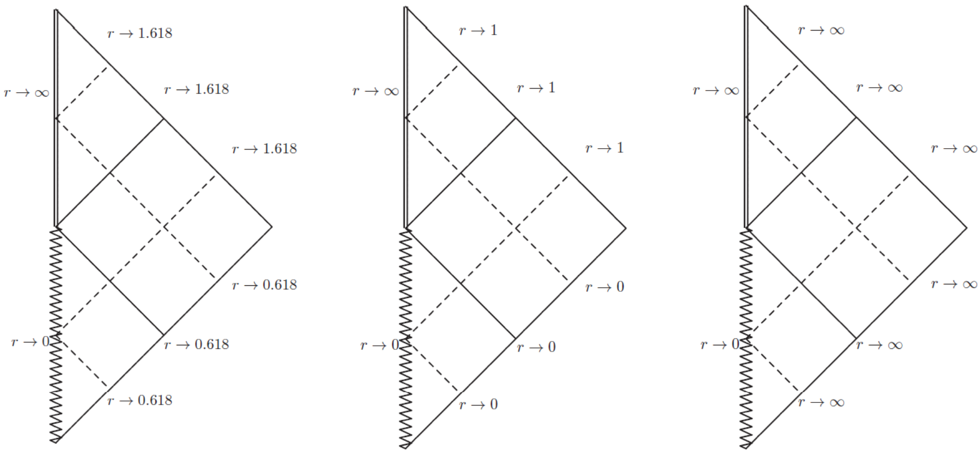

Fig. 6.5: Diagrams for $\mu = -1$ (left), $|\mu| < 1$ (middle) and $\mu < -1$ (right). In- and outgoers are depicted as dashed lines, event horizons as solid lines. Infinity is drawn as double line while the singularity is drawn as zigzag line. At several points of the diagram, we indicated what limit the function $r(z, R)$ approaches when following null geodesics towards this point. As in figures 6.2 and 6.4, time evolves from top to bottom.

finds that $r \to \infty$ when following the outgoers back in time as well when following the ingoers forward in time. It should be noted that the limits discussed here and above are all reached for finite affine parameter, indicating that for global coordinates the physical part of the spacetime is geodesically incomplete even for some geodesics that do not fall into or emerge from the singularity. Unfortunately for $\mu \neq \pm 1$ there are no known coordinate systems that allow to extend the geodesics beyond these limits.

It should be noted that the diagrams shown in figure 6.5 are not usual conformal diagrams, as we only dealt with projections of null geodesics to the z-R-plane in figures 6.2 and 6.4. These geodesics are not confined to this plane and, even more, a "net" of geodesics as depicted in figures 6.4 and 6.5 can never be embedded in one plane. Suppose that from a point with $z = z_0$, $R = R_0$ and $y = y_0 = 0$ two photons are emitted, one on an ingoer trajectory, one on an outgoer trajectory. After a while, these photons hit mirrors such that ingoers are reflected to outgoers and vice versa. For $\mu \neq +1$, the projections of the trajectories of these two photons will then meet again at a point with $z = z_1$ and $R = R_1 < R_0$ in a compactified diagram as figure 6.4, but their y-coordinates will differ by $\Delta y \neq 0$.

Chapter 7

Iyer-Wald approach to Dynamic Entropy

7.1 Idea

Immediately after the discovery that black hole entropy can be calculated via the Noether charge approach in [36] (see section 2.6) ideas were presented in [36–38] how these results could be used to generalize the definition of black hole entropy to the non-stationary case. In this section, we will make use of the prescription for defining dynamical black hole entropy that was put forward by Vivek Iyer and Robert Wald in [38], and which we will call the Iyer-Wald approach.

As discussed in section 2.6, the entropy of a black hole can be calculated by an integral of the form (see (2.10) and [38])

$$\mathcal{S}(\Sigma') = 2\pi \int_{\Sigma'} X^{\gamma\delta} \epsilon'_{\gamma\delta} \tag{7.1}$$

where Σ' is a spacelike slice of the horizon and ϵ' is the binormal to Σ'. It was shown in [37] that in the stationary case the value of (7.1) is independent of the choice of the slice Σ' and that we can consequently choose Σ' to be the bifurcation surface Σ. In the dynamic case the entropy will be a function of time by definition. Thus, if an expression of the form (7.1) is still valid in the dynamical case, the choice of spacelike slice Σ' corresponds to the choice of time at which the entropy is to be computed. What is now needed for a definition of dynamical black hole entropy is a generalization of the integrand $X^{\gamma\delta}$ to the dynamical case [38].

The Iyer-Wald approach is based on the following idea [38]: Consider a spacetime with metric $g_{\mu\nu}$ with a dynamical outer event horizon, and take a spacelike slice Σ' of this horizon corresponding to a certain time. Then apply a transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$ that generates an entirely new metric in which the horizon slice Σ' is embedded as the bifurcation surface of a stationary black hole. The entropy $\tilde{\mathcal{S}}(\Sigma')$ of this black hole can readily be calculated using the appropriate formula for the stationary case (7.1) and is set to be equal to the dynamic black hole entropy $\mathcal{S}(\Sigma')$. This embedding of the horizon slice does obviously not change the horizon area. Therefore, for dynamical black holes in Einstein-Hilbert gravity the entropy calculated using the Iyer-Wald approach is proportional to the horizon surface. Due to the area theorem [7, 12] this means that for Einstein-Hilbert gravity a second law can be inferred for the dynamic entropy following from the Iyer-Wald approach [38].

In the following, we will give the definition of the transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$ which Wald and Iyer proposed in [38] in order to calculate dynamic black hole entropy.

Definition [38]: Let Σ' be a $(d-2)$ dimensional spacelike surface with a field $M^{\alpha_1,\alpha_2,\cdots}{}_{\beta_1,\beta_2,\ldots}$ defined on it. $M^{\alpha_1,\cdots}{}_{\beta_1,\ldots}$ will be called *boost invariant on* Σ' if for every point $\mathcal{P}$ on Σ', $M^{\alpha_1,\cdots}{}_{\beta_1,\ldots}$ is invariant under Lorentz boosts in the tangent space at $\mathcal{P}$ in the $(1+1)$ dimensional plane orthogonal to Σ'. When at the point $\mathcal{P}$ one chooses a set of orthogonal spacelike vectors s_i^μ $(i \in \{1, ...d-2\})$ tangent to Σ and l^μ and n^μ as independent null vectors orthogonal to Σ', then these vectors can be used to define a tetrad $e^\mu{}_a$[1]. One can then expand M in this basis:

$$M^{\alpha_1,\alpha_2,\cdots}{}_{\beta_1,\beta_2,\ldots} = \tilde{M}^{a_1 a_2,\cdots}{}_{b_1,b_2,\ldots} e^{\alpha_1}{}_{a_1} e^{\alpha_2}{}_{a_2} e_{\beta_1}{}^{b_1} e_{\beta_2}{}^{b_2} \ldots \tag{7.2}$$

The tensor M is boost invariant if and only if the basis expansion coefficients $\tilde{M}^{a_1,\cdots}{}_{b_1,\ldots}$ are only non-vanishing for terms with equal numbers of l^μ's and n^μ's.

In order to illustrate this definition and obtain an important result, we will now prove for $d = 3$ that the metric is always boost invariant on Σ' [38]: Let us choose the tetrad $e^\mu{}_0 = l^\mu$, $e^\mu{}_1 = n^\mu$ and $e^\mu{}_2 = s^\mu$. The tetrad expansion (7.2) of the metric can easily be found as the relation $\eta_{mn} = g_{\mu\nu} e^\mu{}_m e^\nu{}_n$ holds [8]. This relation defines the three dimensional Minkowski metric in lightcone coordinates, as we are working with a null tetrad. We can read off $\eta_{00} = l_\mu l^\nu = 0$, $\eta_{11} = n_\mu n^\nu = 0$ and $\eta_{02} = l_\mu s^\mu = 0 = \eta_{20} =$

[1] Technically, the term *tetrad* is only correct in the four dimensional case. The general term is *frame field*.

$\eta_{12} = \eta_{21}$ due to orthogonality. Therefore, the inverse relation $g_{\mu\nu} = \eta_{mn} e_\mu{}^m e_\nu{}^n$ yields the expression $g_{\mu\nu} = \eta_{01} l_\mu n_\nu + \eta_{10} l_\nu n_\mu + \eta_{22} s_\mu s_\nu$ where in each term the number of l^μ's equals the number of n^μ's. Therefore, the metric is always boost invariant on Σ'.

Definition [38]: Let Σ', $M^{\alpha_1,\cdots}{}_{\beta_1,\ldots}$ and the tetrad be defined as in the previous definition. When $M^{\alpha_1,\cdots}{}_{\beta_1,\ldots}$ is not boost invariant, then we can extract the *boost invariant part* $\widehat{M^{\alpha_1,\cdots}{}_{\beta_1,\ldots}}$ of $M^{\alpha_1,\cdots}{}_{\beta_1,\ldots}$ by defining it to be the field on Σ' that is obtained when in the tetrad expansion (7.2) only the terms with equal numbers of l^μ's and n^μ's are kept.

It should be noted that this definition of the boost invariant part is independent of the choice of the tetrad [38]. Although the metric itself is always boost invariant this does not hold for objects containing derivatives of the metric, such as Christoffel symbols and curvature tensors. Hence, it is useful to define a metric $g^I_{\mu\nu}$ which is boost invariant and also yields boost invariant curvature tensors [38]. In order to achieve this goal, Iyer and Wald proposed to define a certain coordinate system in the neighbourhood of Σ' in the following approach [38][2]: On Σ' we define again a null-tetrad with vectors l^μ, n^μ and s^μ such as in the definitions above. Furthermore, we require the normalization $l_\mu n^\mu = -1$. The neighbourhood around Σ' that we are going to investigate is assumed to be small enough that every point $\mathcal{P}'$ lies on a unique geodesic orthogonal to Σ'. This geodesic is assumed to be (affinely) parametrized in such a way that $\mathcal{P}'$ is at unit affine distance from Σ', and γ^μ is then assumed to be the tangent of the geodesic at the intersection point $\mathcal{P}$ with Σ'. The coordinates of $\mathcal{P}'$ are now defined to be U, V and s where U and V are the components of γ^μ along l^μ and n^μ respectively and s is the coordinate of $\mathcal{P}$ on Σ'.

In these coordinates the Taylor expansion of the metric $g_{\mu\nu}$ around Σ' (being defined by $U = 0 = V$, s being arbitrary) reads [38]:

$$g_{\alpha\beta} = \sum_{n,m=0}^{\infty} \frac{U^m V^n}{m!n!} \left(\frac{\partial^{m+n} g_{\alpha\beta}(U,V,s)}{\partial U^m \partial V^n} \right) \Bigg|_{U=V=0}$$

In an arbitrary coordinate system this equation reads

$$g_{ab} = \sum_{n,m=0}^{\infty} \frac{U^m V^n}{m!n!} \left(l^{c_1} \cdots l^{c_m} n^{c_{m+1}} \cdots n^{c_{m+n}} \partial_{c_1} \cdots \partial_{c_{m+n}} g_{ab} \right) \big|_{U=V=0} \tag{7.3}$$

[2]For simplicity, we will restrict the discussion to three dimensions in the following.

where U and V are to be understood as implicit functions of the new coordinates. It should be noted that in our three dimensional case the term $\left(l^{c_1}\cdots\partial_{c_{m+n}}g_{ab}\right)\big|_{U=V=0}$ is a constant as U and V are set to zero and as the metric does not depend on the remaining angular coordinate[3].

Wald and Iyer proposed [38] to define a new metric $g^{I_q}_{\mu\nu}$ by truncating the infinite series in (7.3) at the level $n+m=q$ and replacing each of the expressions $\partial_{c_1}\cdots g_{\alpha\beta}$ by its boost invariant part. They realized [38] that the metric $g^{I_\infty}_{\mu\nu}$ has a Killing vector field $\xi = U\partial_U - V\partial_V$ which vanishes on the slice Σ' which is defined by $U{=}V{=}0$. Thus, this Killing vector field generates a Killing horizon with Σ' as bifurcation surface. The idea of Wald and Iyer to define dynamical black hole entropy with respect to a horizon slice Σ' was to construct the metric tensor $g^{I_q}_{\mu\nu}$ with q being larger than the highest derivative order appearing in the entropy formula and calculate the entropy of this new metric using the appropriate formula (2.10) for the stationary case [38].

7.2 Calculation

In order to apply the method described in the previous subsection it seems that we have to find the exact coordinate transformation $U = U(z,R)$, $V = V(z,R)$, $s = y+s'(z,R)$[4] that allows us to write the metric (5.25) with respect to these coordinates. However, for TMG there is an easier way to do this calculation.

For stationary black holes in TMG Tachikawa [32] found that the contribution of the Chern-Simons term to the entropy reads[5]

$$\mathcal{S}_{CS}(\Sigma') = \frac{1}{8G_N\mu}\int_{\Sigma'}\epsilon_{\alpha\beta}g^{\alpha\nu}g^{\beta\mu}\Gamma_{\mu\nu\rho}dx^\rho \qquad (7.4)$$

where $\epsilon_{\alpha\beta}$ denotes the binormal as defined in section 2.5. For the non-stationary case, according to Wald and Iyer one would have to calculate the Christoffel symbols $\Gamma_{\mu\nu\rho}(g^I_{\alpha\beta})$ with respect to the new metric. The construction of (7.3) is based on the substitution of the expressions $\partial_{c_1}\cdots g_{ab}$ by their boost invariant parts. Hence, one can ask if there is the possibility to calculate the boost invariant part of $\Gamma_{\mu\nu\rho}(g_{\alpha\beta})$ instead of $\Gamma_{\mu\nu\rho}(g^I_{\alpha\beta})$. For more general theories such as NMG we can furthermore ask whether instead of calculating for example the Ricci scalar $R(g^I_{\alpha\beta})$ we can write the

[3] For simplicity we always use slices of the horizon which are generated by the Killing vector ∂_ϕ.

[4] We assume a coordinate transformation that respects the Killing symmetry generated by ∂_y, see section 5.4.

[5] As mentioned above, the contribution from the Einstein-Hilbert term will still be proportional to the circumference of the horizon slice.

Ricci scalar as a function of the metric and it's derivatives $(R(g_{\alpha\beta}, \partial_c g_{\alpha\beta}, \partial_d\partial_c g_{\alpha\beta}))$ and subsequently substitute these expressions by their boost invariant parts. As we will see this is only possible for expressions with at most first derivative order of the metric.

As the metric is boost invariant it is obvious from (7.3) that on the horizon ($U = V = 0$)

$$g^I_{ab}\big|_{\Sigma'} = \widehat{g_{ab}}\big|_{\Sigma'} = g_{ab}\big|_{\Sigma'}$$

In addition, for the first derivative we find $\partial_y g^I_{ab}\big|_{\Sigma'} = 0 = \partial_y g_{ab}\big|_{\Sigma'}$ due to symmetry, and for $\partial_c g^I_{ab}\big|_{\Sigma'}$ with $c \neq y$:

$$\begin{aligned}
\partial_c g^I_{ab}\big|_{\Sigma'} &= \left(\sum_{n,m=0}^{\infty} \left[m\frac{U^{m-1}V^n}{m!n!}\partial_c U + n\frac{U^m V^{n-1}}{m!n!}\partial_c V \right] \left(l^{c_1} \cdots \partial_{c_{m+n}} g_{ab} \right)\Big|_{\Sigma'} \right)\Bigg|_{\Sigma'} \\
&= \left[\partial_c U l^{c_1} + \partial_c V n^{c_1}\right]\big|_{\Sigma'} \left(\widehat{\partial_{c_1} g_{ab}}\right)\Big|_{\Sigma'} \\
&= \delta^{c_1}_c \left(\widehat{\partial_{c_1} g_{ab}}\right)\Big|_{\Sigma'} \\
&= \widehat{\partial_c g_{ab}}\big|_{\Sigma'}
\end{aligned}$$

In this derivation we used that $\partial_U = l^\alpha \partial_\alpha$ and $\partial_V = n^\alpha \partial_\alpha$. From the coordinate relations it then follows that

$$\begin{aligned}
\partial_c &= \left(\frac{\partial U}{\partial x^c}\right)\partial_U + \left(\frac{\partial V}{\partial x^c}\right)\partial_V + \left(\frac{\partial s}{\partial x^c}\right)\partial_s \\
\Rightarrow \left((\partial_c U) l^\alpha + (\partial_c V) n^\alpha\right)\partial_\alpha &= \left(\delta^\alpha_c - (\partial_c s)\delta^\alpha_s\right)\partial_\alpha
\end{aligned}$$

Here the term containing δ^α_s can be omitted as the derivative of the metric with respect to the angular coordinate vanishes due to ∂_s being a Killing vector. It is therefore justified to substitute $\partial_c U l^{c_1} + \partial_c V n^{c_1}$ by $\delta^{c_1}_c$ in the above derivation. Using the same approach one can show that

$$\partial_d \partial_c g^I_{ab}\big|_{\Sigma'} \neq \widehat{\partial_d \partial_c g_{ab}}\big|_{\Sigma'}$$

due to terms involving expressions such as $(\partial_d \partial_c U)\big|_{\Sigma'}(l^{c_1}\widehat{\partial_{c_1} g_{ab})}\big|_{\Sigma'}$ that are not vanishing and that cannot be eliminated in a way similar to the one used above.

Therefore, we can calculate the dynamic entropy according to Iyer and Wald without knowing the exact coordinate transformation to the coordinate system U, V, s for

TMG, but not for NMG where higher derivatives of the metric are needed.

The detailed calculation can be found in appendix E, the results of these calculations will be discussed in the next subsection.

7.3 Discussion

Let us now discuss the results for TMG. The first consistency check of our calculations is that for $\mu = \pm 1$ we know (see section 6.4) that we need to find $\mathcal{S}_{TMG}(\Sigma') = 1$. This is indeed the case, but in some sense this is trivially the case for an unfortunate reason: While we have $\epsilon^{\mu\nu}\widehat{\Gamma_{\mu\nu\rho}} \neq \epsilon^{\mu\nu}\Gamma_{\mu\nu\rho}$ in general, we obtain $\epsilon^{\mu\nu}\widehat{\Gamma_{\mu\nu y}} = \epsilon^{\mu\nu}\Gamma_{\mu\nu y}$ which is the only part of the integrand that matters, as on the horizon $\int_{\Sigma'}(...)dx^\rho = \int_0^{2\pi}(...)\big|_{z=z',R=R'}dy$ in (7.4). This means that taking the boost invariant part does not give other results than the direct use of (7.4) would have given.

For $\mu < 1$ and $\mu \neq -1$ we find that the dynamic entropy $\mathcal{S}_{TMG}(\Sigma')$ will not be constant. As in section 6.2 it will be easiest to take spacelike slices of spacetime denoted by a certain value of $R \equiv R'$ which leads for the intersection with the horizon also to a certain value of $z \equiv z'$. As we wrote the horizons as functions $R(z)$ in section 6.3 for the event horizon and in section 6.2 for the trapping horizon, we can therefore also write the dynamic entropy as a function $\mathcal{S}(z)$. Due to monotonicity of the event horizons and for large enough z also of the trapping horizons, smaller values of z will correspond to the future and larger values of z will correspond to the past. Plots of the results for $\mathcal{S}(z)$ for several $\mu \leq 1$ can be found in figure D.1, appendix D. We find that when evaluated on the event horizon, $\mathcal{S}_{TMG}(z)$ is increasing (and actually diverging) in time for $\mu > 0$ and decreasing in time for $\mu < 0$, where as $z \to 0$ it diverges to $-\infty$ for $-1 < \mu < 0$ and limits to 1 for $\mu \leq -1$[6]. As expected, there is always a limit in which the entropy approaches the value $\mathcal{S}_{TMG}(z) \to +1$ which is the same limit in which the distortion $h_{\mu\nu}$ becomes small, i.e. $z \to +\infty$ for $|\mu| < 1$ and $z \to 0$ for $\mu < -1$.

The great advantage of the Iyer-Wald approach is that it is not intrinsically limited to slices of the event horizon. Indeed, there have been arguments that in the dynamic cases entropy should in fact be assigned to the trapping horizon rather than to the event horizon, see the discussion in section 2.4. We can therefore in our calculations of appendix E substitute the event horizon (6.5) with the trapping horizon (6.3) and calculate the dynamic entropy with respect to this quantity. It should be noted that for

[6]As the event horizon can only be studied numerically for $\mu \neq \pm 1$ there is always the risk that a certain behaviour at some limit is due to numerical problems.

$|\mu| < 1$ this might be problematic for small values of z due to the unphysical behaviour of the trapping horizon discussed in section 6.2. There, in our results for $\mathcal{S}_{TMG}(z)$ the variable z cannot be interpreted as a time variable anymore. As it turns out, the qualitative behaviour of $\mathcal{S}(z)$ calculated with respect to the trapping horizons is not different from the qualitative behaviour of the entropy when calculated with respect to the event horizon.

The results obtained using the Iyer-Wald approach are clearly not satisfying, as they indicate a decreasing entropy as a function of time for some parameters μ. This might be due to either the method we used for calculating the entropy or to the properties of TMG. On one hand, it was already pointed out in a note added to [38] that the entropy calculated using the Iyer-Wald approach is not invariant under field redefinitions, in contrast to what should be expected for physical reasons. On the other hand, it was discussed in section 3.2 that TMG has some unphysical properties for $l\mu \neq \pm 1$, making a possible violation of the second law of black hole thermodynamics less surprising.

It should also be noted that previously, we had to *choose* which part of the lightcones to denote as the *future lightcones*, and which part to denote as the *past lightcones*. As the solution (5.1) is a vacuum solution and hence there is no matter present, there is nothing that forbids such an arbitrary choice. If for example above we would have chosen the time direction as we did for $\mu > 0$ but the opposite way for $\mu < 0$, the entropy would always be a growing function of time.

Chapter 8

Hayward's approach to Dynamic Entropy

8.1 Idea

In section 2.6 it was explained how the entropy of a stationary black hole can be calculated via the Noether charge associated with a certain Killing field. In dynamic spacetimes such a Killing vector field does not exist, but it was suggested by Hayward and others [14, 25, 26] that one could use the Kodama vector defined in section 2.5 as a generalization of the Killing vector to dynamic spacetimes, and thereby assign entropy to the trapping horizon of a dynamical black hole via a Noether charge approach.

First, one has to define the dynamical surface gravity κ associated with the trapping horizon via [14, 26]

$$\kappa = \frac{1}{2}\epsilon^{\alpha\beta}\nabla_\alpha k_\beta \tag{8.1}$$

For a theory of the form $S = \frac{1}{16\pi G_N}\int dx^d\sqrt{-g}L(g_{\mu\nu}, R_{\alpha\beta\gamma\delta})$ the entropy of a spacelike slice Σ' of the trapping horizon is then proposed to be [26]

$$\mathcal{S} = \frac{1}{16G_N\kappa}\int_{\Sigma'} Q^{\mu\nu}\epsilon_{\mu\nu}\sqrt{\gamma}dy^{d-2} \tag{8.2}$$

where again $\epsilon_{\mu\nu}$ is the binormal defined in section 2.5 and $\sqrt{\gamma}dy^{d-2}$ is the volume element on Σ'. $Q^{\mu\nu}$ are the components of the Noether charge $(d-2)$-form corresponding to k^μ given by [26, 76, 77]

$$Q^{\alpha\beta} = 2\left[X^{\alpha\beta\mu\nu}\nabla_\mu k_\nu - 2k_\nu\nabla_\mu X^{\alpha\beta\mu\nu}\right] \tag{8.3}$$

with $X^{\alpha\beta\gamma\delta} \equiv \frac{\partial L}{\partial R_{\alpha\beta\gamma\delta}}$. We will call this proposal to dynamical entropy *Hayward's approach*. We will use this approach in the following subsections to calculate the dynamical entropy of the black holes given by (5.1) in the framework of NMG. There are formulas similar to (8.3) for TMG [39], but evaluating these on a dynamical trapping horizon does not give a coordinate invariant result. Therefore, we will not present any results of Hayward's approach applied to dynamical black holes in the framework of TMG.

For ordinary Einstein-Hilbert gravity the resulting dynamic entropy is just proportional to the area of the trapping horizon [25], but for theories with higher derivative terms the entropy might have a more complicated form.

It is also noteworthy that the proposal for dynamical entropy presented above makes use of the quantity Q which can easily be calculated for certain forms of Lagrangians [26, 76, 77], and not of the quantity Q' defined in section 2.6. There it was shown that on the bifurcation surface one can set $Q = Q'$, but this is not necessarily true in the dynamic case.

8.2 Calculation

For NMG, we see from (3.11) that the Lagrangian reads

$$L = \sigma\left(R - 2\lambda - \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)\right)$$

From now on, we will set $\sigma = +1$ for simplicity. In the following calculations this factor can be restored at any time as an overall factor. We can now calculate[1]

$$\begin{aligned} X^{\alpha\beta\gamma\delta} \equiv \frac{\partial L}{\partial R_{\alpha\beta\gamma\delta}} = & \left(\frac{1}{2} + \frac{3}{8m^2}R\right)\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ & - \frac{1}{2m^2}\left(g^{\alpha\gamma}R^{\beta\delta} - g^{\alpha\delta}R^{\beta\gamma} - g^{\beta\gamma}R^{\alpha\delta} + g^{\beta\delta}R^{\alpha\gamma}\right) \end{aligned}$$

[1] For the general BTZ black hole discussed in section 4.1, using $R = -\frac{6}{l^2}$ and $R_{\mu\nu} = -\frac{2}{l^2}g_{\mu\nu}$, this expression can be simplified to $X^{\alpha\beta\gamma\delta} = \frac{\sigma}{2}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right)\left(1 - \frac{1}{2m^2l^2}\right)$. Now, using (2.10) and $\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = -2$ it is easy to verify (4.11).

Inserting this into (8.3) yields:

$$\begin{aligned} Q^{\alpha\beta} = & \left(\frac{1}{2} + \frac{3}{8m^2}R\right)\left(\nabla^\alpha k^\beta - \nabla^\beta k^\alpha\right) \\ & - \frac{1}{2m^2}\left(\nabla^\alpha k^\nu R^\beta_\nu - \nabla^\mu k^\alpha R^\beta_\mu - \nabla^\beta k^\nu R^\alpha_\nu + \nabla^\mu k^\beta R^\alpha_\mu\right) \\ & + \frac{1}{m^2}\left(k_\nu \nabla^\alpha R^{\beta\nu} - k^\alpha \nabla_\mu R^{\beta\mu} - k_\nu \nabla^\beta R^{\alpha\nu} + k^\beta \nabla_\mu R^{\alpha\mu}\right) \end{aligned} \tag{8.4}$$

Using this and (8.2) one can calculate the dynamic entropy following Hayward's approach after calculating the Kodama vector (2.3) and the surface gravity (8.1). Using global coordinates, we find for the metric (5.1):

$$k^\mu = \epsilon^{\mu\nu}\partial_\nu r = \begin{pmatrix} -z \\ \frac{(\mu-1)z}{2(Rz^\mu+z)} + 1 \\ -(\mu+1)z^{1-\mu} - 2R \end{pmatrix}$$

It can be shown that this vector fulfills the Killing equation (5.16) if and only if $\mu = +1$. This is satisfying, because we know that in four dimensions, the Kodama vector equals the timelike Killing vector ∂_t in many stationary cases [29]. It is easy to show that this also happens in our three dimensional case for $\mu = +1$. What about the case $\mu = -1$? There, the black hole is stationary too, but obviously the Kodama vector calculated above does not equal a Killing vector of this spacetime. There is no obvious analogue to this in higher dimensions, as for example in four dimensions the definition of the Kodama vector requires spherical symmetry, which necessarily excludes rotating black holes. We will nevertheless see later that Hayward's approach yields the correct entropy for $\mu = +1$ and $\mu = -1$ as well.

Before going on with the calculation of the surface gravity κ and the dynamical entropy $\mathcal{S}$ we will briefly investigate which of the properties discussed for four dimensions in section 2.5 still hold in our three dimensional case. First of all, it is easy to see that $k^\mu\partial_\mu r = 0 = k^\mu_{;\mu}$. Secondly the Kodama vector has the norm

$$g_{\mu\nu}k^\mu k^\nu = 1 - \frac{\left(2Rz^\mu + \mu z + z\right)^2}{4z^{2\mu+2}\left(R + z^{1-\mu}\right)} \tag{8.5}$$

Taking now the limit to infinity $z \to 0$ (ensuring the physical spacetime property $R + z^{1-\mu} > 0$) we find $g_{\mu\nu}k^\mu k^\nu \to -\infty$. This means that at least far away from the black hole the Kodama vector is timelike[2]. In section 2.5 it was stated that the

[2]In an asymptotically flat spacetime, one would expect the norm to take the value -1 at asymptotic

hypersurfaces where k^μ has vanishing norm are exactly the trapping horizons. It is possible to verify that (8.5) indeed vanishes on the hypersurfaces defined by (6.3) and (6.4). Using $\mu \leq 1$ and the limit $z \to 0$ it is easy to see that the Kodama vector is future pointing outside of the outer trapping horizon.

We can calculate the dynamic surface gravity κ using the definition (8.1) proposed in [14] or alternatively using the definition $\pm\kappa k_\mu = k^\beta \nabla_{[\mu} k_{\beta]}$, $\kappa \geq 0$ proposed in [29]. It should be noted that these two definitions only coincide on the trapping horizon [29]. We find

$$\kappa = \frac{1}{2}\epsilon^{\alpha\beta}\nabla_\alpha k_\beta = \frac{\sqrt{R+z^{1-\mu}}}{z} + \frac{(\mu-1)\sqrt{R+z^{1-\mu}}\,(Rz^\mu + \mu R z^\mu + 2z)}{4\,(Rz^\mu + z)^2} \tag{8.6}$$

where we have to insert (6.3) for R in order to obtain κ on the outer trapping horizon. Some plots of $\kappa(z)$ are shown in figure D.2 (appendix D) for representative values of $\mu \leq 1$. The first thing that we should notice is that for $\mu = \pm 1$ κ is a constant in time and attains the correct values 1 and $\sqrt{\frac{10}{3+\sqrt{5}}}$ respectively. For $\mu < -1$ we find that κ is monotonously decreasing with z and approaches the value $\kappa = 1$ in the limit $z \to 0$, while for $z \to \infty$ we find $\kappa \to +\infty$. For $|\mu| < 1$ in contrast, we find $\kappa \to 1$ for $z \to \infty$ while for small κ a non-monotonous behaviour is possible. Starting from large values of z and taking the limit $z \to 0$ we find that at first κ decreases, only to attain a maximum for some $\kappa > 0$ and then diverge to $-\infty$. In general, it is obvious that κ attains the value $\kappa = 1$ of the background metric in limits where the distortion $h_{\mu\nu} \sim z^{1-\mu}$ becomes small and $g_{\mu\nu} \approx \bar{g}_{\mu\nu}$ while it shows a complicated behaviour where the distortion $h_{\mu\nu}$ is large. The values z_0 where $\kappa = 0$ for $|\mu| < 1$ are exactly the values where the outer trapping horizon switches from spacelike to timelike, as discussed in section 6.2. This is another reason why one might doubt the validity of the trapping horizons as black hole boundary at least for small z when $|\mu| < 1$.

We can now calculate the dynamic entropy according to Hayward's approach using (8.2). Some plots for $\mathcal{S}(z)$ for representative values of μ are given in figure D.3, appendix D. In these plots, z is used as a measure of time as the outer trapping horizons are monotonously increasing functions $R(z)$ at least for sufficiently large z, and as the coordinate R can be used as a measure of time, see section 5.5. Small values of z will then correspond to the future, while large values of z correspond to the past.

The results for the dynamical entropy will be discussed in the next subsection, they have been obtained with the Mathematica file presented in appendix E.

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infinity. The fact that here we encounter a diverging norm at infinity is not surprising as this is for example also the case for the norm of the vector ∂_t in the BTZ metric (4.1).

8.3 Discussion

When discussing the results obtained for the dynamical entropy we should be aware of the values that the entropy $\bar{\mathcal{S}}$ of the background metric $\bar{g}_{\mu\nu}$ would have as a function of μ. For TMG it is easy to see that due to (4.8) and $r_- = 0$ $\bar{\mathcal{S}}(\mu) = const.$, but for NMG we find with (4.11), $m^2 = \mu^2 - \frac{1}{2}$ (see section 5.2) and $l = 1$ that $\bar{\mathcal{S}}(\mu) = \frac{\sigma\pi}{2G_N}\left(1 + \frac{1}{1-2\mu^2}\right)$. This is plotted in figure 8.1 for $\sigma = 1$. It should be noted that for $\mu = \pm 1$ the entropy of all BTZ black holes vanishes regardless of their values for r_+ and r_-. This means that the family of black holes (with $M = 1 + 2\Xi$, $J = -2\Xi$, see section 6.4) singled out at $\mu = -1$ by (5.1) is not as special from the viewpoint of NMG as it is from the viewpoint of TMG.

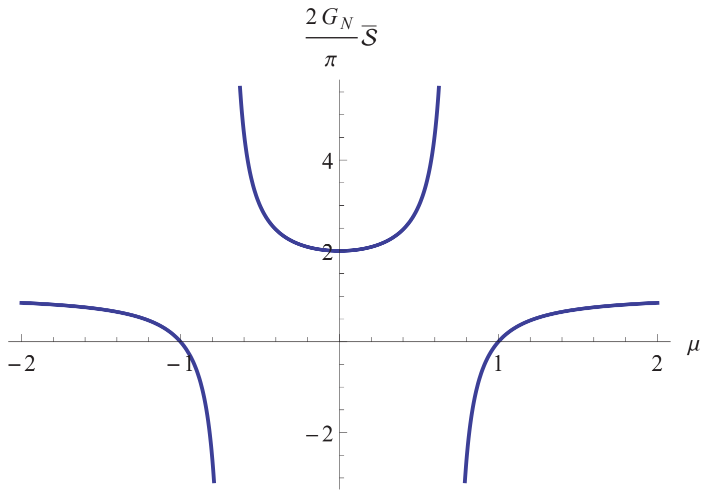


Fig. 8.1: $\bar{\mathcal{S}}(\mu)$ evaluated according to (4.11) with $\sigma = 1$.

Let us now first discuss the results for the entropy $\mathcal{S}(z)$ for the special values $\mu = \pm 1$. From figure D.3 we see that in these cases Hayward's approach reproduces the expected result $S(z) = 0 = const.$. For $\mu = +1$ this is not surprising as in this case the Kodama vector is a timelike Killing vector, but for $\mu = -1$ Hayward's approach yields the correct result for the entropy even though in this case the Kodama vector does not equal the Killing vector that would usually be used to calculate the entropy as discussed in section 2.6[3]. This is very interesting, as for rotating black holes the

[3] In fact the correct entropy is always reproduced by Hayward's approach for the metrics $g^{\Xi}_{\mu\nu} = \bar{g}_{\mu\nu} + \Xi h_{\mu\nu}\big|_{\mu=-1}$ with $\Xi > -\frac{1}{4}$.

Kodama vector can only be calculated in three dimensions, as only in three dimensions axial symmetry equals the symmetry of a $(d-2)$-sphere that is needed to define the Kodama vector.

For $\mu < -1$ $\mathcal{S}(z)$ is monotonously decreasing in time (i.e. increasing in z) for $\sigma = +1$ and approaching the value $\mathcal{S} = 1$ for $z \to 0$. The distortion $h_{\mu\nu} \sim z^{-1-\mu}$ becomes small in this limit and it is not surprising that $\mathcal{S} \to \bar{\mathcal{S}}$ as $g_{\mu\nu} \to \bar{g}_{\mu\nu}$. Furthermore, with $m^2 = \mu^2 - \frac{1}{2}$ the limit $\mu \to -\infty$ corresponds to the limit where the NMG-action (3.11) approaches the Einstein-Hilbert action (3.1), and thus the entropy becomes increasingly dominated by the horizon circumference which was shown to decrease with time in section 6.3. Choosing $\sigma = -1$ as required by (5.9) would result in an entropy $\mathcal{S}$ that is monotonously increasing from $-\infty$ for large z to -1 for $z \to 0$.

For $|\mu| < 1$ the behaviour of $\mathcal{S}(z)$ is more complicated. First of all, it should be noted again that due to the unphysical behaviour of the trapping horizon discussed in section 6.2 the coordinate z cannot be used as a time coordinate for arbitrarily small z. In the previous subsection we saw that the surface gravity κ vanishes at the z-value where the trapping horizon becomes timelike, and this leads to a divergence of $\mathcal{S}(z)$ at the same value of z. Secondly, for this range of μ $\mathcal{S}$ is generally not a monotonous function as can be seen in figure D.3. The behaviour for values $-1 < \mu < 1$ cannot be explained even quantitatively solely using the properties of NMG (such as unitarity, positivity of energy etc.) as the parameters of NMG, λ and m^2, only depend on μ^2 (see section 5.2). This means that for example the qualitative differences in $\mathcal{S}(z)$ for $\mu = \pm 0.2$ cannot be due to properties of the action.

The value $\mu = 0$ deserves special attention. For TMG this value has to be excluded due to the divergence in the action (3.3), but the metric (5.1) and the NMG action (3.11) are well defined for this value. In this special case one finds $\mathcal{S}(z) = \bar{\mathcal{S}} = const.$ although the metric is clearly not stationary as can be seen from the surface gravity $\kappa(z) \neq const.$ or the time dependent circumference of the event horizon. Interestingly, $\mu = 0$ corresponds to the special point $\frac{\lambda}{m^2} = 1$ discussed in section 3.3.

Chapter 9

Outlook and open Questions

9.1 Exact Solutions

In section 5 we stated that based on the linearized solution (4.15) the metric (5.1) can be constructed and turns out to be an exact solution of the vacuum equations of motion (3.4) of TMG, as was realized in [1]. We also presented the new insight that (5.1) is a solution of NMG in section 5.2. It is a very interesting open question what conditions have to be satisfied in the most general possible case for a solution $h_{\mu\nu}$ of the linearized equations of motion to also describe a solution of the exact equations of motion of a gravity theory. For Kundt-CSI spacetimes, some research on this issue has already been done in in [15], however as pointed out before, the metric (5.1) can by no coordinate change be transformed to the form of metrics investigated in this reference, see appendix A. Apart from this ansatz, what other approaches might be worthwhile? In fact all the lowest mode metrics (4.13-4.16) describe exact solutions of TMG, while the higher modes $h_{\mu\nu}^{L/R(n)} = \left(L_{-1}\bar{L}_{-1}\right)^n h_{\mu\nu}^{L/R}$ and $H_{\mu\nu}^{L/R(n)} = \left(L_{+1}\bar{L}_{+1}\right)^n H_{\mu\nu}^{L/R}$ do not describe exact solutions of TMG. One might therefore speculate whether it is possible to write the exact equations of motion of TMG or NMG for a metric of the form $g_{\mu\nu} = \bar{g}_{\mu\nu} \pm h_{\mu\nu}$ (possibly with the additional assumption $h_{\mu\nu} = l_\mu l_\nu$ as in (5.2)) in terms of the linearized equations of motion (3.6) or (3.16) and of the chiral highest weight conditions presented in (4.17). If this was possible, in would yield new methods of generating exact solutions by solving a set of linear equations.

An overview of known exact solutions derived from the linear solutions (4.13-4.16) is depicted in table 10.2, section B. Will all of these describe dynamical black holes? First of all, for the metric $\bar{g}_{\mu\nu} + \Xi h^L_{\mu\nu}(\mu)$ the factor Ξ can be absorbed up to sign for

$\mu \neq -1$ by a shift in the coordinate t. For $\Xi > 0$ this is just (5.1) studied extensively in this work. For $\Xi < 0$, the singularity is given by $R(z) = +|\Xi| z^{1-\mu}$ and bends upwards in a diagram such as figure 6.2. Therefore, one has to expect a naked singularity for some values of μ. Indeed, from our discussion in section 6.4 we know that this metric will have a naked singularity at least for $\mu = -1$, $\Xi < -\frac{1}{4}$. No event or trapping horizons can be defined in this case. The other metrics based on the solutions (4.13-4.14,4.16) are expected to describe similar dynamical spacetimes as (5.1). It should nevertheless be noted that for example in the metric $\bar{g}_{\mu\nu} + \Xi H^L_{\mu\nu}(\mu)$ the distortion has a prefactor $\sim e^{-(1+\mu)t}$ in contrast to the factor $e^{+(1+\mu)t}$ in (5.1). This means that for this metric, in the limits $\rho = const.$, $t \to \pm\infty$, the distortion will become large when the distortion in (5.1) becomes small and vice versa. This could have the effect that for certain values of μ, if an event horizon can be defined at all, the black hole described by $H^L_{\mu\nu}$ shrinks when the one based on $h^L_{\mu\nu}$ is growing and vice versa. It might also be worthwhile to investigate whether the solutions presented in table 10.2 fit into the general solutions discussed in [62].

9.2 Dynamic Black Hole Entropy

The metric (5.1) is an exact vacuum solution of (3.4) and (3.13) describing a dynamical black hole. As such, it is an ideal testing ground for competing proposals for calculating dynamical black hole entropy. In sections 7 and 8 we applied two different approaches to dynamical black hole entropy to the metric (5.1). Unfortunately, it was not possible to apply the Iyer-Wald approach to NMG as we do not know the exact coordinate transformation that yields the coordinates U, V and s used in [38]. Also, it was not possible to apply Hayward's approach to TMG. Therefore, it we could not compare these two approaches directly and thereby decide which one yields physically more acceptable results. But additional research might make it possible to apply the Iyer-Wald approach to NMG, and thereby enable direct comparison with Hayward's approach. Other ideas about dynamical black hole entropy that might be tested on the dynamical black hole metrics shown in table 10.2 are the ones published in [78] and the very recent paper [79].

Chapter 10

Appendix

A Kundt Spacetimes in $2+1$ Dimensions

A spacetime is called a *Kundt spacetime* [80] (see also [15, 16]) if it admits a geodesic null vector field with vanishing optical scalars, i.e. in three dimensions with vanishing expansion, see equation (2.1) and the discussion below it.

In [15, 16] it was claimed that the line element of every $2+1$ dimensional Kundt spacetime can be brought to the form

$$ds^2 = 2du\,(Hdu + dv + Wdx) + dx^2 \tag{10.1}$$

where u, v, and x are the new coordinates and there are two functions $H(u, v, x)$ and $W(u, v, x)$. In these coordinates, ∂_v is the geodesic null vector field that characterizes the spacetime as Kundt type.

Assume now a line element of the form

$$ds^2 = e^{2V} d\lambda dU + dV^2 + g(U, V)dU^2 \tag{10.2}$$

According to [1], this is AdS_3 in Poincaré coordinates for $g(U, V) = 0$, the non-rotating BTZ black hole (4.12) for $g(U, V) = 1$ and the metric (5.17) (with $\lambda \to -\lambda$) for $g(U, V) = 1 + 2^{\frac{3-\mu}{2}} e^{(1+\mu)(U+V)}$. These three line elements all describe Kundt spacetimes with the (Killing) vector field ∂_λ generating the defining null geodesics. Nevertheless, we will now prove that these line elements cannot be brought to the form (10.1) by a coordinate transformation. First of all, note that in the two line elements

(10.1) and (10.2) the vector fields ∂_v and ∂_λ take the same role. For this reason we restrict ourselves to the study of coordinate transformations that ensure $\partial_v = \partial_\lambda$, i.e. that respect this vector field in the sense of section 5.4. The most general coordinate transformation then reads:

$$\begin{aligned} U &= f_1(u,x) \\ V &= f_2(u,x) \\ \lambda &= v + f_3(u,x) \end{aligned} \tag{10.3}$$

Using this, the transformation law (5.21) and (10.2) we find for the new metric:

$$\begin{pmatrix} (...) & (...) & \frac{1}{2}e^{2f_2(u,x)}\partial_u f_1(u,x) \\ (...) & (...) & \frac{1}{2}e^{2f_2(u,x)}\partial_x f_1(u,x) \\ \frac{1}{2}e^{2f_2(u,x)}\partial_u f_1(u,x) & \frac{1}{2}e^{2f_2(u,x)}\partial_x f_1(u,x) & 0 \end{pmatrix}$$
$$\equiv \begin{pmatrix} 2H(u,x) & W(u,x) & 1 \\ W(u,x) & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where we substituted some rather longish expressions by (...). due to our ansatz (10.3), equality of the g_{vv} components is trivially ensured. For the g_{xv} component we find $\frac{1}{2}e^{2f_2(u,x)}\partial_x f_1(u,x) = 0$ which can only be satisfied by $f_1(u,x) = f_1(u)$. Inserting this, the equation for the g_{xx} component simplifies considerably to $(\partial_x f_2(u,x))^2 = 1$ demanding the ansatz $f_2(u,x) = \pm x + f_2'(u)$. Inserting this into the equation for the g_{uv} component yields $\frac{1}{2}\partial_u f_1(u)e^{2(f_2'(u)\pm x)} = 1$. This equation cannot be solved as the right-hand side is obviously constant while the left-hand side depends on the coordinate x. It is therefore not possible to bring the metrics (10.2) to the form (10.1) used in [15, 16].

B Tables

Killing vector fields

vector field	θ^{a1}	norm^{a2}	geodesic?^{a3}	Killing?^{b4}	θ^{b1}	norm^{b2}	geodesic?^{b3}
L_{-1}	0	0	yes	no	0	> 0	no
L_0	0	$\frac{1}{4}$	yes	no	0	> 0	no
L_1	0	0	yes	no	0	> 0	no
$\bar{L}_{-1}$	0	0	yes	no	0	> 0	no
$\bar{L}_0$	0	$\frac{1}{4}$	yes	no	0	$\frac{1}{4}$	yes
$\bar{L}_1$	0	0	yes	yes	0	0	yes

Table 10.1: Properties of the vector fields (4.5) with respect to the metrics (4.12) (^{a}) and (5.1) (^{b}). Statements with respect to (5.1) are to be understood as valid for general $\mu \neq \pm 1$.

[1] Expansion $\theta = u^{\alpha}{}_{;\alpha}$ as defined in 2.1.

[2] The norm is $u^{\alpha} u_{\alpha}$. The statement > 0 is supposed to mean that the norm is non-constant and positive at least outside of the black hole.

[3] The geodesic equation for a vector field u reads $\nabla_u u = 0$ [8].

[4] The Killing equation is given in (5.16).

Solutions of TMG and NMG

metric[1]	solution of TMG?[2]	solution of NMG?[3]
$\bar{g}_{\mu\nu} + \Xi h^L_{\mu\nu}(\mu)$	yes	yes
$\bar{g}_{\mu\nu} + \Xi h^R_{\mu\nu}(\mu)$	yes	yes
$\bar{g}_{\mu\nu} + \Xi H^L_{\mu\nu}(\mu)$	yes	yes
$\bar{g}_{\mu\nu} + \Xi H^R_{\mu\nu}(\mu)$	yes	yes
$\bar{g}_{\mu\nu} + \left(L_{-1}\bar{L}_{-1}\right) h^L_{\mu\nu}(\mu)$	no	no
$\bar{g}_{\mu\nu} + \Xi_A h^L_{\mu\nu}(\mu) + \Xi_B h^L_{\mu\nu}(-\mu)$	no[4]	yes

Table 10.2: Table of some metrics derived from the background metric $\bar{g}_{\mu\nu}$ (4.12) and the linear solutions (4.13-4.16) which where checked for being exact solutions of TMG and/or NMG.

[1] When referring to the metrics (4.13-4.16) we assume the parameter $k = 0$ in order to have completely real metrics.

[2] Entry will be "yes" if the metric solves the equations of motion (3.4) and "no" otherwise.

[3] Entry will be "yes" if the metric solves the equations of motion (3.13) with the choice of values $m^2(\mu)$ and $\lambda(\mu)$ as in section 5.2, "no" otherwise.

[4] Except for $\mu = \pm 1$.

C Christoffel Symbols

In this appendix we will state the Christoffel symbols and comment on the calculation of curvature tensors of the metric (5.1) in global coordinates (5.22). As in the whole thesis, we will employ the convention to use spacetime indices $\mu, \nu, ... \in \{1, 2, 3\}$. The metric reads $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ with (using $x^1 = z$, $x^2 = y$ and $x^3 = R$):

$$\bar{g}_{\mu\nu} = \frac{1}{z^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & R & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \,,\, h_{\mu\nu} = \left(\frac{1}{z}\right)^{1+\mu}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Christoffel symbols are computed using the definition [8]

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right)$$

and their components read (note that $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$):

$$\begin{array}{lclclclclcl}
\Gamma^1_{11} & = & -\frac{1}{z} & \Gamma^2_{11} & = & 0 & \Gamma^3_{11} & = & 0 \\
\Gamma^1_{22} & = & \frac{R}{z} + \frac{1}{2}(\mu+1)z^{-\mu} & \Gamma^2_{22} & = & -1 & \Gamma^3_{22} & = & 2\left(R + z^{1-\mu}\right) \\
\Gamma^1_{33} & = & 0 & \Gamma^2_{33} & = & 0 & \Gamma^3_{33} & = & 0 \\
\Gamma^1_{12} & = & 0 & \Gamma^2_{12} & = & -\frac{1}{z} & \Gamma^3_{12} & = & (1-\mu)z^{-\mu} \\
\Gamma^1_{13} & = & 0 & \Gamma^2_{13} & = & 0 & \Gamma^3_{13} & = & -\frac{1}{z} \\
\Gamma^1_{23} & = & \frac{1}{2z} & \Gamma^2_{23} & = & 0 & \Gamma^3_{23} & = & 1
\end{array}$$

The components of the Riemann tensor are defined by [8]

$$R^{\mu}{}_{\alpha\beta\gamma} = \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\mu}_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta}\Gamma^{\mu}_{\lambda\gamma}$$

We will not explicitly state these components here, as in three dimensions they can easily be computed from the metric $g_{\mu\nu}$, the Riemann tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and the Riemann scalar $R = R^{\mu}{}_{\mu}$ via the equation [3]

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}\tilde{R}_{\beta\delta} + g_{\beta\delta}\tilde{R}_{\alpha\gamma} - g_{\alpha\delta}\tilde{R}_{\beta\gamma} - g_{\beta\gamma}\tilde{R}_{\alpha\delta}$$

with $\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$. $R_{\mu\nu}$ fulfills the relations (5.3) and $R = -6$. The modified Einstein tensor can be computed by using (5.4) and from this the components of the Cotton tensor follow from the equations of motion (3.4).

D Figures

Results of section 7

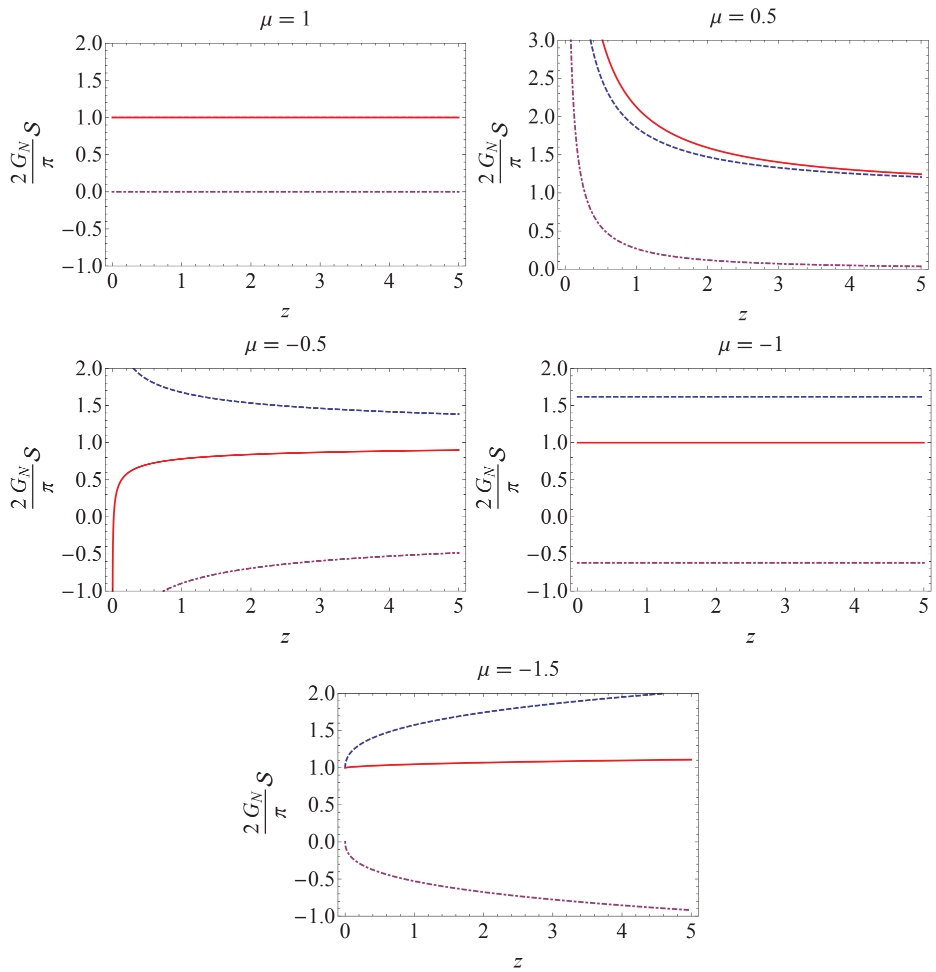


Fig. D.1: $\mathcal{S}(z)$ evaluated on the outer event horizon following the Iyer-Wald approach for different values of μ, see section 7. The dynamic entropy $\mathcal{S}(z)$ is shown as solid red line, the contribution from the Einstein-Hilbert term of the action (proportional to the horizon circumference) is shown as dashed blue line, the contribution from the Chern-Simons term is shown as dot-dashed purple line.

Results of section 8

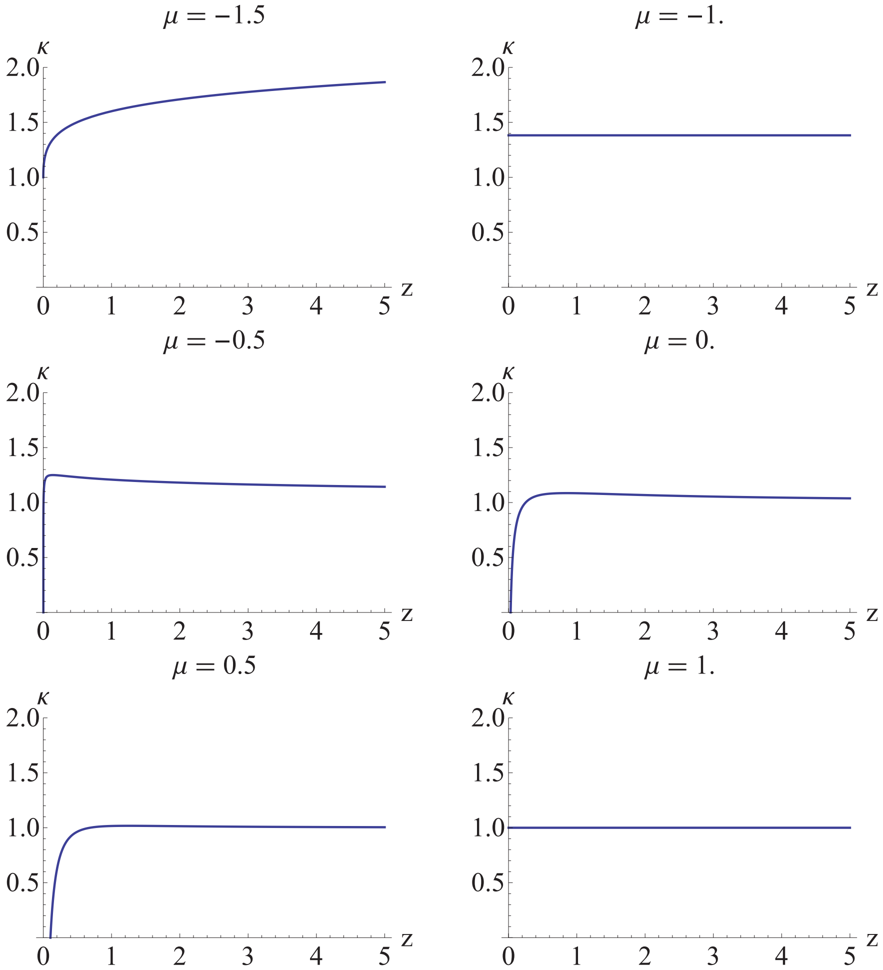


Fig. D.2: Dynamic surface gravity $\kappa(z)$ as described in (8.6) for various values of μ.

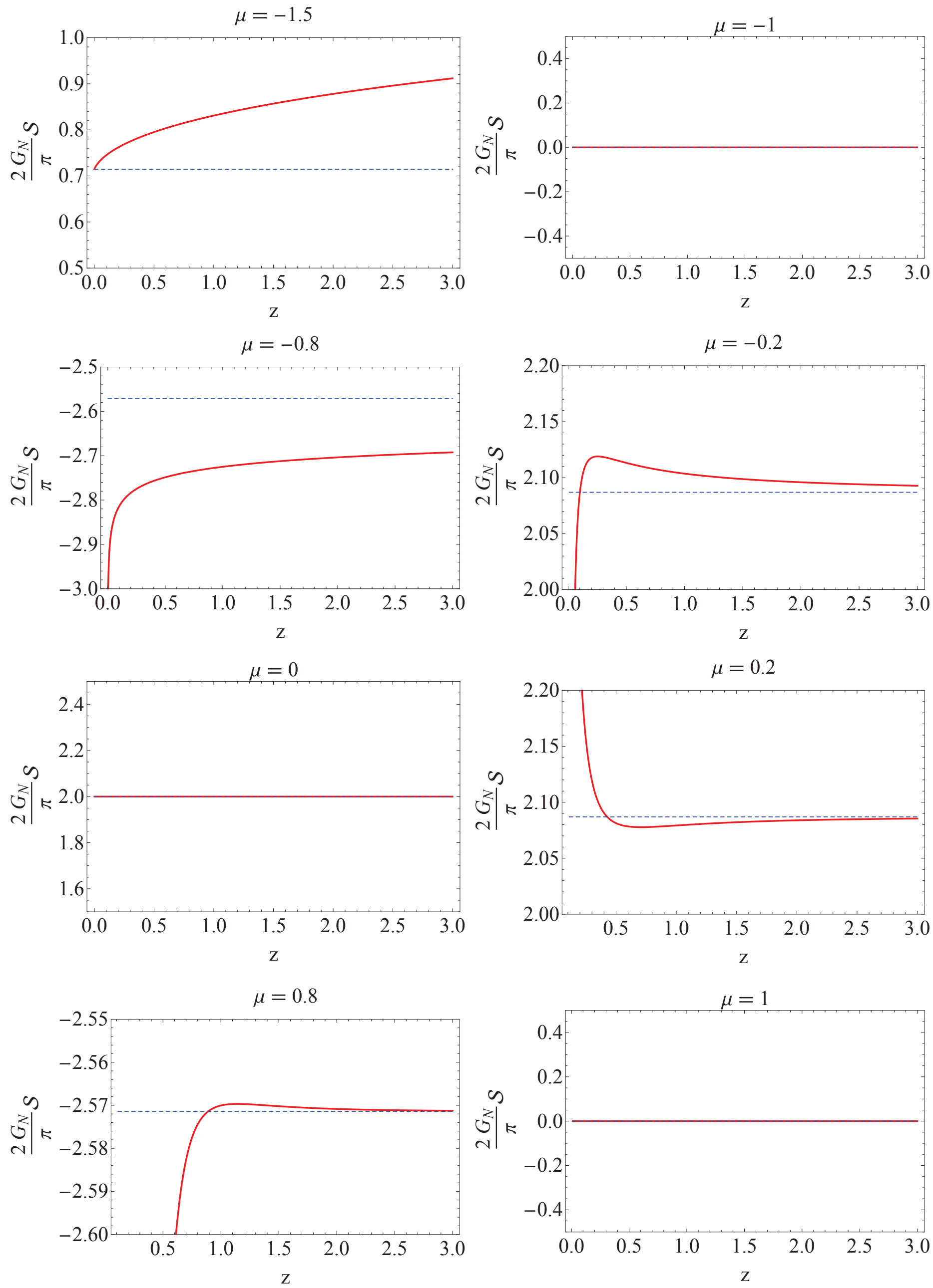


Fig. D.3: $\mathcal{S}(z)$ evaluated on the outer trapping horizon following Hayward's approach for different values of μ, see section 8.2 and (8.2). The dynamic entropy $\mathcal{S}(z)$ is shown as solid red line, the constant entropy value $\bar{\mathcal{S}}$ of the background metric $\bar{g}_{\mu\nu}$ for the respective value of $m^2 = \mu^2 - \frac{1}{2}$ is shown as dashed blue line. $\bar{\mathcal{S}}$ can be calculated from (4.11) with $r_+ = l = \sigma = 1$.

E Mathematica Files

Dynamic entropy for TMG, Wald Iyer approach

This file shows how for TMG, the dynamic entropy can be calculated following the Iyer-Wald approach, as discussed in section 7. We will use the variable m instead of μ, as there may frequently be summations over indices μ. First we set the value of m we want to investigate. Furthermore, we set the dimension equal to three.

$m = -1;$

dim = 3;

These assumptions will help the computer to simplify expressions later on:

$Assumptions = {$z > 0, R + z$^$(1 - m) > 0, \mathfrak{z} > 0$};

In this cell, the metric is defined. g is the metric with indices down, G is the metric with indices up.

$g = 1/z$^2 ∗ {{1, 0, 0}, {0, $R + z$^$(1 - m)$, 1/2}, {0, 1/2, 0}};

G = Inverse[g];

g//MatrixForm

$$\begin{pmatrix} \frac{1}{z^2} & 0 & 0 \\ 0 & \frac{R+z^2}{z^2} & \frac{1}{2z^2} \\ 0 & \frac{1}{2z^2} & 0 \end{pmatrix}$$

Here, the coordinates are defined:

$x_1 = z$; $x_2 = y$; $x_3 = R$;

This will be helpful for displaying tensor components:

perm[dim_, n_]:=Permutations[Sort[Flatten[(ConstantArray[a, n])/.a → Range[dim]]], {n}];

Christoffel symbols, defined with all three indices down:

Γ[β_, μ_, ν_]:=FullSimplify $\left[\frac{1}{2}\left(D\left[g[[\beta,\mu]],x_\nu\right]+D\left[g[[\beta,\nu]],x_\mu\right]-D\left[g[[\mu,\nu]],x_\beta\right]\right)\right]$

Now we define the Killing vector ∂_y as well as the in- and outgoer vector fields: (Indices are assumed to be up)

ϵ1 = {0, 1, 0};

$a = R + z$^$(1 - m)$;

i = {Sqrt[a], 1, -2 a};(*ingoer*)

o = {-Sqrt[a], 1, -2 a};(*outgoer*)

Check of the norms of these vectors:

$$\left\{\textbf{FullSimplify}\left[\sum_{\alpha=1}^{\textbf{dim}}\sum_{\beta=1}^{\textbf{dim}} g[[\alpha,\beta]]i[[\alpha]]i[[\beta]]\right] == 0,\right.$$
$$\textbf{FullSimplify}\left[\sum_{\alpha=1}^{\textbf{dim}}\sum_{\beta=1}^{\textbf{dim}} g[[\alpha,\beta]]o[[\alpha]]o[[\beta]]\right] == 0,$$
$$\left.\textbf{FullSimplify}\left[\sum_{\alpha=1}^{\textbf{dim}}\sum_{\beta=1}^{\textbf{dim}} g[[\alpha,\beta]]\epsilon 1[[\alpha]]\epsilon 1[[\beta]]\right] == g[[2,2]]\right\}$$

$\left\{\text{True}, \text{True}, 1+\frac{R}{z^2} == \frac{R+z^2}{z^2}\right\}$

Check of orthogonality:

$$\left\{\textbf{FullSimplify}\left[\sum_{\alpha=1}^{\textbf{dim}}\sum_{\beta=1}^{\textbf{dim}} g[[\alpha,\beta]]i[[\alpha]]\epsilon 1[[\beta]]\right] == 0,\right.$$
$$\left.\textbf{FullSimplify}\left[\sum_{\alpha=1}^{\textbf{dim}}\sum_{\beta=1}^{\textbf{dim}} g[[\alpha,\beta]]o[[\alpha]]\epsilon 1[[\beta]]\right] == 0\right\}$$

$\{\text{True}, \text{True}\}$

Here we determine the factor by which the vectors have to be normalized so that $i_\mu o^\mu$=-1.

$$\left\{\textbf{FullSimplify}\left[\sum_{\alpha=1}^{\textbf{dim}}\sum_{\beta=1}^{\textbf{dim}} g[[\alpha,\beta]]i[[\alpha]]o[[\beta]]\right],\right.$$
$$\left.\textbf{FullSimplify}\left[\left(\sum_{\alpha=1}^{\textbf{dim}}\sum_{\beta=1}^{\textbf{dim}} g[[\alpha,\beta]]i[[\alpha]]o[[\beta]]\right)\Big/(2*g[[2,2]])\right]\right\}$$

$\left\{-2-\frac{2R}{z^2}, -1\right\}$

Here we define the binormal (with indices up) as was done in section 2.5. Only the components $\neq$0 are shown.

$\boldsymbol{\epsilon[\mu\_,\nu\_]}$**:=FullSimplify[**$(i[[\mu]]o[[\nu]]-o[[\mu]]i[[\nu]])/(2*g[[2,2]])$**]**

Column[Select[({Superscript[ϵ**, #[[1]] * 10 + #[[2]]], Apply[**ϵ**, #]}&/@perm[dim, 2]),**

(!NumericQ[#[[2]]]||#[[2]] ≠ 0)&]]

$\left\{\epsilon^{12}, \frac{z^2}{\sqrt{R+z^2}}\right\}$
$\left\{\epsilon^{13}, -2z^2\sqrt{R+z^2}\right\}$
$\left\{\epsilon^{21}, -\frac{z^2}{\sqrt{R+z^2}}\right\}$
$\left\{\epsilon^{31}, 2z^2\sqrt{R+z^2}\right\}$

Now we define the tetrad ${e_a}^\mu$ with ${e_1}^\mu = o^\mu\mathcal{N}, {e_2}^\mu = s^\mu\mathcal{M}, {e_3}^\mu = i^\mu\mathcal{N}$,($\mathcal{N}$,$\mathcal{M}$

being normalization factors): the first index is the internal index (down), the second index is the spacetime index (up)

$\mathcal{N} = 1/\mathrm{Sqrt}[2g[[2,2]]]$;

$\mathcal{M} = 1/\mathrm{Sqrt}[g[[2,2]]]$;

$e[\mathrm{a\_}, \mu\_]$:=KroneckerDelta$[a,1] * o[[\mu]] * \mathcal{N}$ + KroneckerDelta$[a,2] * \epsilon 1[[\mu]] * \mathcal{M}$

+KroneckerDelta$[a,3] * i[[\mu]] * \mathcal{N}$

Inner metric: $\eta_{\mathrm{ab}} = g_{\mu\nu} e^{\mu}{}_{a} e^{\nu}{}_{b}$

ηcomponents[a_, b_]:=FullSimplify $\left[\left(\sum_{\mu=1}^{\mathrm{dim}} \sum_{\nu=1}^{\mathrm{dim}} (g[[\mu,\nu]] e[a,\mu] e[b,\nu])\right)\right]$

η = Table[ηcomponents$[\alpha,\beta], \{\alpha,1,3\}, \{\beta,1,3\}]$;

η//MatrixForm

Inverse$[\eta] == \eta$(*$\eta_{\mathrm{ab}} = \eta^{\mathrm{ab}}$*)

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

True

Inverse tetrad: $e^{b}{}_{\nu}$

$\tilde{e}[\mathrm{b\_}, \nu\_]$:=FullSimplify $\left[\sum_{\mu=1}^{\mathrm{dim}} \sum_{a=1}^{\mathrm{dim}} (\eta[[b,a]] * g[[\nu,\mu]] * e[a,\mu])\right]$

check: $g_{\mu\nu} = \eta_{\mathrm{ab}} e^{a}{}_{\mu} e^{b}{}_{\nu}$

htest$[\mu_, \nu_]$:=FullSimplify $\left[\sum_{a=1}^{\mathrm{dim}} \sum_{b=1}^{\mathrm{dim}} (\eta[[b,a]] \tilde{e}[a,\mu] \tilde{e}[b,\nu])\right]$

Table[htest$[\alpha,\beta], \{\alpha,1,3\}, \{\beta,1,3\}] == g$

$\left\{\left\{\frac{1}{z^2},0,0\right\},\left\{0,1+\frac{R}{z^2},\frac{1}{2z^2}\right\},\left\{0,\frac{1}{2z^2},0\right\}\right\} == \left\{\left\{\frac{1}{z^2},0,0\right\},\left\{0,\frac{R+z^2}{z^2},\frac{1}{2z^2}\right\},\left\{0,\frac{1}{2z^2},0\right\}\right\}$

check : $\delta^{a}_{b} = e^{a}{}_{\mu} e_{b}{}^{\mu}$ and $\delta^{\mu}_{\nu} = e^{a}{}_{\nu} e_{a}{}^{\mu}$

δtest1[a_, b_]:=FullSimplify $\left[\sum_{\mu=1}^{\mathrm{dim}} (\tilde{e}[a,\mu] * e[b,\mu])\right]$

δtest2$[\mu_, \nu_]$:=FullSimplify $\left[\sum_{a=1}^{\mathrm{dim}} (\tilde{e}[a,\nu] * e[a,\mu])\right]$

$\{\{\delta\mathrm{test1}[1,1], \delta\mathrm{test1}[2,2], \delta\mathrm{test1}[3,3], \delta\mathrm{test1}[1,2], \delta\mathrm{test1}[1,3], \delta\mathrm{test1}[2,3]\}==\{1,1,1,0,0,0\}$,

$\{\delta\mathrm{test2}[1,1], \delta\mathrm{test2}[2,2], \delta\mathrm{test2}[3,3], \delta\mathrm{test2}[1,2], \delta\mathrm{test2}[1,3], \delta\mathrm{test2}[2,3]\}==\{1,1,1,0,0,0\}\}$

{True, True}

In order to extract the boost invariant par, the Christoffel symbols are first expressed in the tetrad basis: (all indices down)

$$\tilde{\Gamma}[\text{a\_}, \text{b\_}, \text{c\_}] := \text{FullSimplify}\left[\sum_{\alpha=1}^{\text{dim}}\sum_{\beta=1}^{\text{dim}}\sum_{\gamma=1}^{\text{dim}} \Gamma[\alpha,\beta,\gamma] * e[a,\alpha] * e[b,\beta] * e[c,\gamma]\right]$$

As we choose $e_1{}^{\mu} = o^{\mu}\mathcal{N}, e_2{}^{\mu} = s^{\mu}\mathcal{M}, e_3{}^{\mu} = i^{\mu}\mathcal{N}$, these expressions can be used to project tensor components to their boost invariant parts.

p2[a1_, a2_]:=KroneckerDelta[0, a1 + a2 − 4]

p3[a1_, a2_, a3_]:=KroneckerDelta[0, a1 + a2 + a3 − 6]

Here we extract the boost invariant part of the Christoffel symbol and restore the spacetime indices afterwards.

$$\tilde{\Upsilon}[\text{a\_}, \text{b\_}, \text{c\_}] := \text{p3}[a,b,c] * \tilde{\Gamma}[a,b,c]$$

$$\Upsilon[\alpha\_, \beta\_, \gamma\_] := \text{FullSimplify}\left[\sum_{a=1}^{\text{dim}}\sum_{b=1}^{\text{dim}}\sum_{c=1}^{\text{dim}} \tilde{\Upsilon}[a,b,c] * \tilde{e}[a,\alpha] * \tilde{e}[b,\beta] * \tilde{e}[c,\gamma]\right]$$

This is the integrand of (7.4):

$$\text{contraction}[\alpha\_] := \text{FullSimplify}\left[\sum_{\mu=1}^{\text{dim}}\sum_{\nu=1}^{\text{dim}}(\epsilon[\nu,\mu] * \Upsilon[\mu,\nu,\alpha])\right]$$

{contraction[1], contraction[2], contraction[3]}

$$\left\{0, \frac{2z}{\sqrt{R+z^2}}, \frac{z}{(R+z^2)^{3/2}}\right\}$$

We can compare it to the expression we would have gotten if we would not have used the boost invariant part of the metric:

$$\text{contractionold}[\alpha\_] := \text{FullSimplify}\left[\sum_{\mu=1}^{\text{dim}}\sum_{\nu=1}^{\text{dim}}(\epsilon[\nu,\mu] * \Gamma[\mu,\nu,\alpha])\right]$$

{contractionold[1], contractionold[2], contractionold[3]}

$$\left\{0, \frac{2z}{\sqrt{R+z^2}}, -\frac{1}{z\sqrt{R+z^2}}\right\}$$

To calculate the entropy, we have to determine the event horizon. For $\mu \neq 1$ it is given by the following numerical solution. If one wants to use the trapping horizon instead of the event horizon, one has to use
$\text{R0}[\mathfrak{z}\_] = \left(\frac{1}{2}\mathfrak{z}^{-2m}\left(-(1+m)\mathfrak{z}^{1+m} + \mathfrak{z}^{2+2m} + \sqrt{\mathfrak{z}^{3+3m}\left(2 - 2m + \mathfrak{z}^{1+m}\right)}\right)\right)$.

NDS = NDSolve[{𝔑'[𝔷] == 2 ∗ Sqrt[𝔑[𝔷] + 𝔷^(1 − m)], 𝔑[0] == 0}, 𝔑, {𝔷, 0, 100},

WorkingPrecision → 30, AccuracyGoal → 20];

R0[𝔷_] = (𝔑[𝔷]/.NDS)[[1]];

The contribution from the Einstein-Hilbert term is proportional to the horizon circumference:

$\mathbf{S1}[\mathfrak{z}\_] = \mathbf{Sqrt}[\mathbf{R0}[\mathfrak{z}] + \mathfrak{z}\hat{}(1-m)]/\mathfrak{z};$

This is (in the same units as S1) the contribution from the Chern-Simons term:

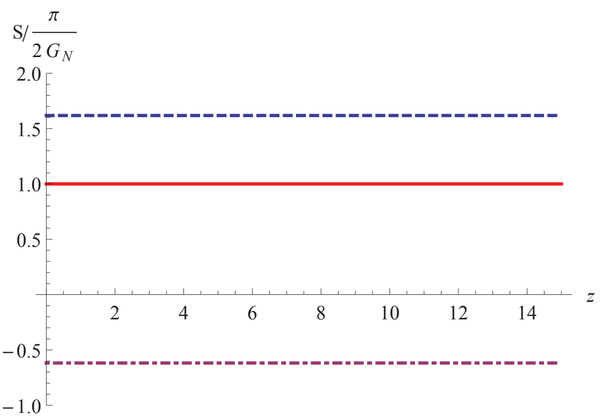
$\mathbf{S2}[\mathfrak{z}\_] = \frac{1}{2m} * \mathbf{contraction}[2]/.\{z \to \mathfrak{z}, R \to \mathbf{R0}[\mathfrak{z}]\};$

The full entropy reads:

$S[\mathfrak{z}\_] = \mathbf{S1}[\mathfrak{z}] + \mathbf{S2}[\mathfrak{z}];$

These are the results:

$\mathbf{Show}[\mathbf{Plot}[\{\mathbf{S1}[\mathfrak{z}], \mathbf{S2}[\mathfrak{z}]\}, \{\mathfrak{z}, 0, 15\}, \mathbf{PlotStyle} \to \{\{\mathbf{Dashed}, \mathbf{Thick}\}, \{\mathbf{DotDashed}, \mathbf{Thick}\}\},$
$\mathbf{PlotRange} \to \{-1, 2\}], \mathbf{Plot}[S[\mathfrak{z}], \{\mathfrak{z}, 0, 15\}, \mathbf{PlotStyle} \to \{\mathbf{Red}, \mathbf{Thick}\},$
$\mathbf{PlotRange} \to \{-1, 2\}], \mathbf{AxesLabel} \to \left\{z, \text{"S/}\frac{\pi}{2G_N}\text{"}\right\}, \mathbf{LabelStyle} \to \mathbf{Medium}\Big]$



Dynamic entropy for NMG, Hayward's approach

This file shows how for NMG, the dynamic entropy can be calculated following Hayward's approach, as discussed in section 8. The first few lines of the code are similar to the ones of the file presented in section E and will therefore not be reproduced here. The example of the file here is evaluated for $m = 1$ with $m = \mu$ in our notation.

This is how we calculate expressions like the Riemann and Ricci tensors etc.:

$R[\alpha\_, \beta\_, \gamma\_, \delta\_] :=$

$D[\Gamma[\alpha,\beta,\delta], x_\gamma] - D[\Gamma[\alpha,\gamma,\beta], x_\delta] + \sum_{\mu=1}^{\mathbf{dim}}(\Gamma[\mu,\beta,\delta]\Gamma[\alpha,\mu,\gamma] - \Gamma[\mu,\beta,\gamma]\Gamma[\alpha,\mu,\delta])$

$$\textbf{Ricci}[\alpha\_, \beta\_]\text{:=}\textbf{Simplify}\left[\sum_{\gamma=1}^{\mathbf{dim}} R[\gamma, \alpha, \gamma, \beta]\right]$$

$$\textbf{RS} = \textbf{Simplify}\left[\sum_{\alpha=1}^{\mathbf{dim}} \sum_{\beta=1}^{\mathbf{dim}} G[[\alpha, \beta]]\textbf{Ricci}[\beta, \alpha]\right];$$

$$\textbf{Riccio}[\alpha\_, \beta\_]\text{:=}\textbf{FullSimplify}\left[\sum_{\rho=1}^{\mathbf{dim}} \sum_{\lambda=1}^{\mathbf{dim}} (\textbf{Ricci}[\rho, \lambda] G[[\alpha, \rho]] G[[\beta, \lambda]])\right]$$

$$\textbf{R1u}[\mu\_, \alpha\_, \beta\_]\text{:=}\textbf{Simplify}[$$

$$\left(D\left[\textbf{Riccio}[\alpha, \beta], x_\mu\right] + \sum_{\rho=1}^{\mathbf{dim}} (\textbf{Riccio}[\rho, \alpha]\Gamma[\beta, \rho, \mu]) + \sum_{\rho=1}^{\mathbf{dim}} (\textbf{Riccio}[\rho, \beta]\Gamma[\alpha, \rho, \mu])\right)\right]$$

$$\textbf{R1o}[\nu\_, \alpha\_, \beta\_]\text{:=}\textbf{Simplify}\left[\sum_{\mu=1}^{\mathbf{dim}} (G[[\mu, \nu]] * \textbf{R1u}[\mu, \alpha, \beta])\right]$$

Now we can define the Kodama vector as in (2.3):

$$r = \sqrt{g[[2, 2]]};$$

$$\mathcal{K}[\alpha\_]\text{:=}\textbf{FullSimplify}\left[\sum_{\beta=1}^{\mathbf{dim}} \left(\epsilon[\alpha, \beta] D\left[r, x_\beta\right]\right)\right]$$

$$\epsilon\textbf{u} = \{\mathcal{K}[1], \mathcal{K}[2], \mathcal{K}[3]\};$$

$$\epsilon\textbf{u}//\textbf{MatrixForm}$$

$$\mathcal{K}\textbf{d}[\mu\_]\text{:=}\textbf{FullSimplify}\left[\sum_{\alpha=1}^{\mathbf{dim}} g[[\mu, \alpha]]\epsilon\textbf{u}[[\alpha]]\right]$$

$$\{\mathcal{K}\textbf{d}[1], \mathcal{K}\textbf{d}[2], \mathcal{K}\textbf{d}[3]\}//\textbf{MatrixForm};$$

$$\begin{pmatrix} -z \\ 1 \\ -2(1+R) \end{pmatrix}$$

We can now check whether the Kodama vector happens to be a Killing vector:

$$\textbf{Killing}[\alpha\_, \beta\_]\text{:=}\textbf{Simplify}\left[D\left[\mathcal{K}\textbf{d}[\alpha], x_\beta\right] + D\left[\mathcal{K}\textbf{d}[\beta], x_\alpha\right] - 2\sum_{\lambda=1}^{\mathbf{dim}} (\mathcal{K}\textbf{d}[\lambda]\Gamma[\lambda, \alpha, \beta])\right];$$

Column[Select[({Subscript[Killing, #[[1]] * 10 + #[[2]]],

Apply[Killing, #] == 0}&/@perm[dim, 2]), (!NumericQ[#[[2]]]||#[[2]] ≠ 0)&]]

$\{\text{Killing}_{11}, \text{True}\}$ $\{\text{Killing}_{12}, \text{True}\}$ $\{\text{Killing}_{13}, \text{True}\}$ $\{\text{Killing}_{21}, \text{True}\}$

$\{\text{Killing}_{22}, \text{True}\}$ $\{\text{Killing}_{23}, \text{True}\}$ $\{\text{Killing}_{31}, \text{True}\}$ $\{\text{Killing}_{32}, \text{True}\}$

$\{\text{Killing}_{33}, \text{True}\}$

Here we check whether the Kodama vector fulfills the expected properties:

$$\textbf{FullSimplify}\left[\sum_{\beta=1}^{\mathbf{dim}} \left(\mathcal{K}[\beta] D\left[r, x_\beta\right]\right)\right] == 0$$

$$\textbf{FullSimplify}\left[\sum_{\beta=1}^{\mathbf{dim}} \left(D\left[\mathcal{K}[\beta], x_\beta\right] + \sum_{\alpha=1}^{\mathbf{dim}} (\mathcal{K}[\alpha]\Gamma[\beta, \alpha, \beta])\right)\right] == 0$$

True

True

The norm of the Kodama vector reads:

$$\boldsymbol{n = \text{Simplify}\left[\sum_{\alpha=1}^{\text{dim}}\sum_{\beta=1}^{\text{dim}}(\mathcal{K}[\beta]\mathcal{K}[\alpha]g[[\alpha,\beta]])\right]}$$

$$\boldsymbol{\text{Assuming}[R>0,\text{Limit}[n,z\to 0]]}$$

$1-\frac{1+R}{z^2}$

$-\infty$

We now evaluate (8.1):

$$\boldsymbol{\mathcal{Q} = \text{Simplify}\left[1/2*\sum_{\alpha=1}^{\text{dim}}\sum_{\beta=1}^{\text{dim}}\left(\epsilon[\alpha,\beta]*\left(D\left[\mathcal{K}\text{d}[\beta],x_\alpha\right]-\sum_{\lambda=1}^{\text{dim}}(\mathcal{K}\text{d}[\lambda]\Gamma[\lambda,\alpha,\beta])\right)\right)\right]}$$

$\frac{\sqrt{1+R}}{z}$

This is the outer trapping horizon (6.3)

RCTS[z_]:=Simplify[

$$\boldsymbol{\left(\frac{1}{2}z^{-2m}\left(-(1+m)z^{1+m}+z^{2+2m}+\sqrt{z^{3+3m}\left(2-2m+z^{1+m}\right)}\right)\right)\Big]}$$

$$\boldsymbol{\text{RCTS}[z]}$$

$-1+z^2$

Now, the surface gravity κ can be evaluated on the trapping horizon

$$\boldsymbol{\kappa = \text{Simplify}[\mathcal{Q}/.\{R\to\text{RCTS}[z]\}]}$$

1

Here we plot the surface gravity and the background value $\bar{\kappa}=1$.

$$\boldsymbol{\text{Plot}[\{\kappa,1\},\{z,0,15\},\text{PlotRange}\to\{0,2\},}$$

$$\boldsymbol{\text{PlotStyle}\to\{\{\text{Red},\text{Thick}\},\{\text{Blue},\text{Thick},\text{Dashed}\}\}];}$$

The surface gravity vanishes when the trapping horizon becomes a timelike hypersurface:

$$\boldsymbol{\text{NSolve}[\kappa==0,z]}$$

{}

Here we investigate the determinant of the induced metric on the outer trapping horizon:

detCTS[z_]:=

$$-\frac{1}{8(2-2m+z^{1+m})}\left(-1+m^2\right)z^{-5-3m}\Big((-7+m)(-1+m)z^{2+2m}+4z^{3+3m}+$$
$$(-3+m)(-1+m)\sqrt{z^{3+3m}\left(2-2m+z^{1+m}\right)}$$
$$+z^{1+m}\left(-1+m+m^2-m^3+4\sqrt{z^{3+3m}\left(2-2m+z^{1+m}\right)}\right)\Big)$$

NSolve[detCTS[z] == 0, z]

{{}}

This is $\nabla^\alpha\xi^\nu$:

$$\textbf{grad}\mathcal{K}[\alpha\_,\nu\_]\textbf{:=Simplify}\left[\sum_{\mu=1}^{\mathbf{dim}}\left(G[[\alpha,\mu]]\left(D\left[\mathcal{K}[\nu],x_\mu\right]+\sum_{\lambda=1}^{\mathbf{dim}}(\mathcal{K}[\lambda]\Gamma[\nu,\lambda,\mu])\right)\right)\right]$$

These are the Noether charge (d-2)-form components according to (8.4):

$$Q[\alpha\_,\beta\_]\textbf{:=}2*\textbf{Simplify}\left[\left(\frac{1}{2}+\frac{3}{8*n}*\textbf{RS}\right)*(\textbf{grad}\mathcal{K}[\alpha,\beta]-\textbf{grad}\mathcal{K}[\beta,\alpha])\right.$$
$$-\frac{1}{2*n}*\Big($$
$$\sum_{\nu=1}^{\mathbf{dim}}\left(\textbf{grad}\mathcal{K}[\alpha,\nu]*\left(\sum_{\lambda=1}^{\mathbf{dim}}(G[[\beta,\lambda]]\textbf{Ricci}[\lambda,\nu])\right)\right)$$
$$-\sum_{\nu=1}^{\mathbf{dim}}\left(\textbf{grad}\mathcal{K}[\beta,\nu]*\left(\sum_{\lambda=1}^{\mathbf{dim}}(G[[\alpha,\lambda]]\textbf{Ricci}[\lambda,\nu])\right)\right)$$
$$-\sum_{\mu=1}^{\mathbf{dim}}\left(\textbf{grad}\mathcal{K}[\mu,\alpha]*\left(\sum_{\lambda=1}^{\mathbf{dim}}(G[[\beta,\lambda]]\textbf{Ricci}[\lambda,\mu])\right)\right)$$
$$+\sum_{\mu=1}^{\mathbf{dim}}\left(\textbf{grad}\mathcal{K}[\mu,\beta]*\left(\sum_{\lambda=1}^{\mathbf{dim}}(G[[\alpha,\lambda]]\textbf{Ricci}[\lambda,\mu])\right)\right)\Big)$$
$$+\frac{1}{n}*\left(\sum_{\nu=1}^{\mathbf{dim}}(\mathcal{K}\textbf{d}[\nu]*\textbf{R1o}[\alpha,\beta,\nu])-\sum_{\nu=1}^{\mathbf{dim}}(\mathcal{K}\textbf{d}[\nu]*\textbf{R1o}[\beta,\alpha,\nu])\right.$$
$$\left.\left.-\left(\mathcal{K}[\alpha]*\sum_{\mu=1}^{\mathbf{dim}}(\textbf{R1u}[\mu,\beta,\mu])\right)+\left(\mathcal{K}[\beta]*\sum_{\mu=1}^{\mathbf{dim}}(\textbf{R1u}[\mu,\alpha,\mu])\right)\right)\right]$$

In this cell, the dynamic entropy is calculated according to (8.2).

$$\mathcal{I}=\textbf{Simplify}[$$
$$\left(\frac{1}{8\mathfrak{G}*\kappa}*\textbf{Integrate}\left[\textbf{Simplify}\left[\sum_{\mu=1}^{\mathbf{dim}}\sum_{\nu=1}^{\mathbf{dim}}((Q[\mu,\nu])*\right.\right.\right.$$
$$\left.\left.\left.\left(\sum_{\alpha=1}^{\mathbf{dim}}\sum_{\beta=1}^{\mathbf{dim}}(\epsilon[\alpha,\beta]*g[[\mu,\alpha]]*g[[\nu,\beta]])\right)\right)\right]*\frac{1}{2}*\textbf{Sqrt}[g[[2,2]]],\{y,0,2\pi\}\right]\right)\Big];$$

$$S[\textbf{z\_}]\textbf{:=Simplify}\left[\frac{2\mathfrak{G}}{\pi}*(\mathcal{I}/.\{R\to\textbf{RCTS}[z]\})\right]$$

With this we plot the results as in figure D.3.

$$\textbf{Plot}\left[\left\{S[z]/.\{n\to m\textbf{\^{}}2-1/2\},1+\frac{1}{1-2m^2}\right\},\{z,0,3\},\right.$$

PlotStyle → {{Red, Thick}, {Blue, Dashed}}, Axes → False, Frame → True,

```
FrameLabel → {z, "(2G_N/π)𝒮"}, PlotLabel → Style[μ == m, Large],
LabelStyle->Directive[Black, Thick, Large]];
```

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Selbstständigkeitserklärung

Hiermit versichere ich, Mario Flory, die vorliegende Arbeit selbständig und lediglich unter Zuhilfenahme der genannten Quellen verfasst zu haben.

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Mario Flory

München, Februar 2013