

Applications of ℓ^∞ -cohomology

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Statement of authorship

I declare that this thesis is my original work and that all sources and contributions of others have been fully acknowledged. This thesis has not been submitted, in whole or in part, for any other degree or qualification.

Signature:

Date:

Introduction

Bounded cohomology theories have long been used in geometric group theory as tools for relating algebraic invariants of groups to their large-scale geometry. One such theory is ℓ^∞ -cohomology, introduced by Gersten in a series of papers in the 1990s. Among its applications are characterization of hyperbolic groups [Ger96a, Theorem 5.1]:

Theorem. *A finitely presented group G is hyperbolic if and only if $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$.*

and characterization of undistorted subgroups [Ger98, Corollary 10.3]:

Theorem. *A finitely generated subgroup H of an hyperbolic group G is quasi-isometrically embedded in G if and only if the restriction homomorphism $H_{(\infty)}^1(G; \mathbb{R}) \rightarrow H_{(\infty)}^1(H; \mathbb{R})$ is surjective.*

The theory of ℓ^∞ -cohomology and the proof of these characterizations are spread across several papers by Gersten, some of which were never published and are available only as preprints. The main goal of this thesis is to give a unified and readable account of Gersten's ℓ^∞ -cohomology theory and its application to the characterizations mentioned above. The emphasis is expository rather than original: we collect the relevant definitions, constructions, and results, reorganize them into a single narrative, and present the arguments using modern terminology and notation. Where possible, we give more precise statements and clear references to the original sources, while avoiding unnecessary technicalities.

The thesis begins with the definition of ℓ^∞ -cohomology for groups. We then introduce several related tools that are used in the proof of the hyperbolicity characterization, including filling functions and combinatorial complexes. These ideas are developed step by step and eventually combined to prove that, for finitely presented groups, the vanishing of the second ℓ^∞ -cohomology is equivalent to hyperbolicity. At the end we also give a ℓ^∞ -cohomological characterization of quasi-convex subgroups of hyperbolic groups. While many arguments closely follow Gersten's original work, the presentation aims to make the overall structure of the theory more transparent.

One motivating question behind this work on our part comes from the coarse geometry of group extensions. Given a short exact sequence of finitely generated groups

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$$

one can ask whether there exists a quasi-isometric section $\sigma: Q \rightarrow G$, i.e., a right-inverse to the quotient map that is also a quasi-isometric embedding at the level of word metrics. Partial results in this direction are known: for example, Mosher showed that if the kernel K is non-elementary hyperbolic, then such a quasi-isometric section exists [Mos96]. A particularly interesting test case comes from low-dimensional topology: the handlebody group \mathcal{H}_g , i.e. the mapping class group of a handlebody of genus g V_g surjects onto the outer automorphism group of the free group on g generators $\text{Out}(F_g)$ via the induced action of homeomorphisms of V_g on the fundamental group $\pi_1(V_g) = F_g$. This projection has an infinitely generated kernel [McC85] and is known to be geometrically complicated, and it remains open whether one can find a quasi-isometric section in this case. Although it is known by the works of Bridson and Vogtmann that $\text{Out}(F_g)$ has exponential Dehn function for $g \geq 3$ [BV95, BV12, HV96]; and by Hamenstädt and Hensel that the same is true for \mathcal{H}_g for $g \geq 3$ [HH12, HH13, HH21], these results relies on detailed, case-specific geometric constructions. If such a section existed, one could, for instance, pull back known geometric invariants such as the Dehn function. From a

broader perspective, it would be desirable to have cohomological or coarse-geometric tools that detect when quasi-isometric sections can or cannot exist, thereby explaining such phenomena in a more systematic way. Since ℓ^∞ -cohomology is sensitive to filling properties and large-scale geometry, it appears to be a natural candidate framework for studying these questions.

Although no new results concerning the existence of quasi-isometric sections are obtained in this thesis, the exposition presented here provides a systematic introduction to ℓ^∞ -cohomology and some of its applications. It is hoped that this material may serve as a useful reference for further work in this direction.

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1. Preliminaries

In this chapter, we collect the preliminary notions and results that will be used throughout this thesis. The focus is on providing sufficient background to make the subsequent discussion of ℓ^∞ -cohomology accessible, while avoiding unnecessary technical details that can be found in standard references. We assume the reader is familiar with basic group theory and standard algebraic notions; basic topology and in particular CW-complexes. A reference for the former is [Löh17, Chapter 2] and the latter is [Hat02]. On the other hand, we give a more detailed treatment of combinatorial complexes, combinatorial maps between them and local operators defined over them, as these concepts are less widely known and play a central role in our approach. Additionally, we briefly recall the relevant aspects of graph theory, group cohomology and CW-complexes, stating key definitions and results without proofs, and refer to the literature for further details.

1.1. Graph theory

We mainly follow the definitions and notations of [Löh17, Chapter 3.1].

Definition 1.1. A *graph* is a pair $X = (V, E)$ such that

$$E \subset \{e \subset V : |e| = 2\}$$

The elements of V are called vertices and the elements of E are called edges. If $e = \{v_1, v_2\} \in E$, e is said to be an edge between v_1 and v_2 . One can also further specify a notion of direction for each edge to get a directed graph.

We will mainly be working with undirected graphs.

Definition 1.2. Let $X = (V, E)$ be a graph.

- Let $n \in \mathbb{N} \cup \{\infty\}$. A *path of length n in X* is a sequence of distinct vertices v_0, \dots, v_n with $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, \dots, n-1\}$. If $n < \infty$, then it is said that the path connects v_0 and v_n .
- X is said to be *connected* if for every pair of its vertices there exists a path connecting them.
- Let $n \in \mathbb{N}_{>2}$. A *cycle of length n in X* is a path of length n v_0, \dots, v_{n-1} such that $\{v_{n-1}, v_0\} \in E$.

Definition 1.3. A *tree* is a connected graph that does not contain any cycles.

Definition 1.4. Let $X = (V, E)$ be a connected graph. The map

$$d: V \times V \rightarrow \mathbb{R}_{\geq 0}$$

$$(v, w) \mapsto \min\{n \in \mathbb{N} \mid \exists \text{ a path of length } n \text{ in } X \text{ connecting } v \text{ and } w\}$$

defines a metric on V , called the *graph metric* on X .

From now on we consider graphs as metric spaces equipped with the graph metric.

Lemma 1.5. Let $X = (V, E)$ be a connected graph. For any pair of vertices $(v, w) \in V \times V$, there exists a path of length $d(v, w)$ in X connecting v and w .

Proof. Since X is connected, the set

$$S = \{n \in \mathbb{N} \mid \exists \text{ a path of length } n \text{ in } X \text{ connecting } v \text{ and } w\}$$

is non-empty. Furthermore, since $S \subset \mathbb{N}$ is discrete and is bounded below by 0, it follows that there exists a path connecting v and w whose length realizes the minimum of S . \square

Definition 1.6. Let $X = (V, E)$ be a connected graph and $v, w \in V$. A *geodesic* in X from v to w is a path of length $d(v, w)$ connecting v and w . The set of all geodesics from v to w is denoted by $[v, w]$.

We will denote a choice of a geodesic from v to w also as $[v, w]$ as a slight abuse of notation.

Remark 1.7. The geodesics on a graph as we have defined here does not actually yield geodesics in the usual sense but rather $(1, 1)$ -quasi-geodesics. See Chapter 5.3 of [Löh17] for relevant definitions and in particular the Example 5.3.5 for a discussion of this. This does not however present much of a problem for our purposes. So, with this technicality aside, we will consider connected graphs endowed with the graph metric as geodesic metric spaces where geodesics are length realizing.

1.2. Geometric group theory

We mainly follow the definitions and notations of [Löh17].

Definition 1.8. (Cayley graph) Let G be a group and let $S \subset G$ be a generating set of G . The *Cayley graph of G with respect to the generating set S* is the graph $\text{Cay}(G, S)$ with

- vertices as G , and
- edges as the set

$$\{\{g, g \cdot s\} \mid g \in G, s \in (S \cup S^{-1}) \setminus \{e\}\}$$

Definition 1.9. Let $f: X \rightarrow Y$ be a map between metric spaces (X, d_X) and (Y, d_Y) .

- The map f is a (λ, μ) -*quasi-isometric embedding* if there are constants $\lambda \in \mathbb{R}_{\geq 1}$ and $\mu \in \mathbb{R}_{\geq 0}$ such that

$$\frac{1}{\lambda}d_X(x, x') - \mu \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + \mu$$

for all $x, x' \in X$.

- A map $f': X \rightarrow Y$ has *finite distance from f* if there is a $\nu \in \mathbb{R}_{\geq 0}$ such that

$$d_Y(f(x), f'(x)) \leq \nu$$

for all $x \in X$.

- The map $f: X \rightarrow Y$ is a *quasi-isometry* if it is a quasi-isometric embedding for which there is a *quasi-inverse* quasi-isometric embedding, i.e., if there is a quasi-isometric embedding $g: Y \rightarrow X$ such that $g \circ f$ has finite distance from id_X and $f \circ g$ has finite distance from id_Y .
- The metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $X \rightarrow Y$, then we write $X \sim_{\text{QI}} Y$.

Proposition 1.10. [Löh17, Proposition 5.2.5] *Let G be a finitely generated group and let S, S' be finite generating sets of G . Then the Cayley graphs $\text{Cay}(G, S)$ and $\text{Cay}(G, S')$ are quasi-isometric.*

Remark 1.11. This means that the notions invariant under quasi-isometry are independent of the choice of a generating set. A very important example for us is the notion of hyperbolicity which is a quasi-isometry invariant for geodesic metric spaces [Löh17, Corollary 7.2.13].

Definition 1.12. A geodesic triangle in a metric space (X, d) is said to be δ -*slim* for some $\delta \in \mathbb{R}_{\geq 0}$ if each side is contained in the δ -neighbourhood of the union of the other two sides.

Definition 1.13. A geodesic metric space (X, d) is said to be δ -*hyperbolic* if there exists a $\delta \in \mathbb{R}_{\geq 0}$ such that all geodesic triangles in X are δ -slim.

Definition 1.14. A finitely generated group G is said to be *hyperbolic* if for some finite generating set S of G the Cayley graph $\text{Cay}(G, S)$ is δ -hyperbolic for some $\delta \in \mathbb{R}_{\geq 0}$.

1.3. Combinatorial complexes and maps

We mostly follow the definitions and notations of [BRS07].

Definition 1.15. [BKS21, Definition 2.4] Combinatorial complexes and combinatorial maps between them are defined recursively on dimension. Zero dimensional cell complexes are defined to be combinatorial as are arbitrary maps between them. In general, a continuous map between CW-complexes is said to be combinatorial if it maps each open cell of the domain homeomorphically to an open cell of the target. Having defined combinatorial k -complexes and combinatorial maps between combinatorial k -complexes, one defines a combinatorial $(k + 1)$ -complex to be one whose k -skeleton is a combinatorial complex and whose attaching maps ϕ_e of $(k + 1)$ -cells e are combinatorial maps for suitable combinatorial structures on $\partial D_e^{k+1} = S^k$.

Lemma 1.16. *Combinatorial maps are 1-Lipschitz with respect to the graph metric on the 1-skeletons of respective combinatorial complexes.*

Proof. Let $k \in \mathbb{N}_{\geq 1}$ and let X, Y be combinatorial k -complexes with a combinatorial map $f: X \rightarrow Y$ between them. Endow the 1-skeletons with the graph metric, more precisely we endow the graphs $(X^{(0)}, X^{(1)})$, $(Y^{(0)}, Y^{(1)})$ with their respective graph metrics (we take the distance to be ∞ for vertices in different connected components). We want to show that for any $u, v \in X^{(0)}$

$$d_Y(f(u), f(v)) \leq d_X(u, v).$$

First we consider the case when two vertices $u, v \in X^{(0)}$ are in different connected components of X . Then $d_X(u, v) = \infty$ and it is therefore vacuously true that $d_Y(f(u), f(v)) \leq d_X(u, v)$. So suppose that u and v are in the same connected component of X . We have that f maps each vertex of X to a vertex of Y and similarly it maps each edge of X homeomorphically onto an edge of Y . Moreover, f being continuous, we must have that the endpoints of an edge e map to the endpoints of $f(e)$, tail to tail, head to head. Hence, for any path p in $X^{(1)}$ of length n connecting u and v , the image path $f(p)$ formed by the images of each edge p must be a path of length n in $Y^{(1)}$ connecting $f(u)$ and $f(v)$. From this, the desired inequality follows and it is not necessarily an equality as there might be shorter paths in $Y^{(1)}$ connecting $f(u)$ and $f(v)$. \square

Definition 1.17. Let G be a group and let $\mathcal{P} = \langle s_1, \dots, s_m \mid r_1, \dots, r_n \rangle$ be a presentation for G . The *presentation 2-complex of \mathcal{P}* is the combinatorial 2-complex formed by attaching m oriented edges labeled by the generators to a single vertex and then attaching 2-cells C_1, \dots, C_n where C_i has $\ell(r_i)$ edges, by identifying the boundary of C_i with the edge-path that reads r_i in the current 1-complex.

The *Cayley 2-complex of \mathcal{P}* , $\text{Cay}^2(\mathcal{P})$, is the universal cover of the presentation 2-complex of \mathcal{P} .

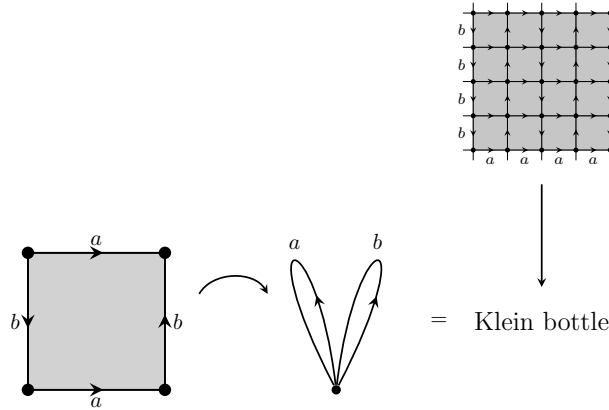


FIGURE 1. The presentation and the Cayley 2-complexes for $\langle a, b \mid aba^{-1} = b^{-1} \rangle$, the fundamental group of the Klein bottle. Influenced by [BRS07, p. 90, Figure 1.2].

Lemma 1.18. *Given a group G and a presentation \mathcal{P} of G , 1-skeleton of the Cayley 2-complex $\text{Cay}^2(\mathcal{P})$ of G is the Cayley graph $\text{Cay}(G, S)$. In particular, the 0-skeleton corresponds to G .*

Remark 1.19. For any choice of a presentation \mathcal{P} for a group G , the presentation 2-complex of \mathcal{P} gives a model for the classifying space BG of G .

Definition 1.20. Given a word w in the alphabet $S \cup S^{-1}$, let η_w denote the path in $\text{Cay}^2(\mathcal{P})^{(1)}$ connecting e and the element of G represented by w .

Definition 1.21. A word w in the alphabet $S \cup S^{-1}$ is said to be *null-homotopic* if it represents the identity. The name is justified by the fact that η_w being a loop in that case.

Definition 1.22. Let G be a group and let $\mathcal{P} = \langle S \mid R \rangle$ be a presentation of G . Let w be a null-homotopic word in the alphabet $S \cup S^{-1}$. A \mathcal{P} -van Kampen diagram for w is a combinatorial map $f : D \rightarrow \text{Cay}^2(\mathcal{P})$ where $D = S^2 \setminus e_\infty$ for some combinatorial 2-complex S^2 homeomorphic to the 2-sphere and some open 2-cell e_∞ of S^2 , so that the edge-circuit $f|_{\partial D}$, based at vertex \star on ∂D , agrees with $\eta_w \subset \text{Cay}^2(\mathcal{P})^{(1)}$.

Lemma 1.23 (van Kampen's lemma). *A word w in the alphabet $S \cup S^{-1}$ is null-homotopic if and only if it admits a \mathcal{P} -van Kampen diagram.*

Definition 1.24. Let G be a finitely presented group and let \mathcal{P} be a finite presentation of G .

- The area of a \mathcal{P} -van Kampen diagram D is the number of 2-cells in D , denoted by $\text{Area}(D)$.
- The area of a null-homotopic word w is the minimal area among all \mathcal{P} -van Kampen diagrams for w , denoted by $\text{Area}(w)$, i.e.,

$$\text{Area}(w) = \min\{\text{Area}(D) \mid D \text{ a } \mathcal{P}\text{-van Kampen diagram for } w\}.$$

- The Dehn function of \mathcal{P} is a map $\text{Dehn} : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$\text{Dehn}(n) = \max\{\text{Area}(w) \mid \text{null-homotopic words } w \text{ with } \ell(w) \leq n\}.$$

- An isoperimetric function for \mathcal{P} is any map $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{Dehn}(n) \leq f(n)$ for all $n \in \mathbb{N}$.

We now define some local operators that come in handy when working with combinatorial complexes. In what follows we denote by X a combinatorial CW-complex and by S a collection of open cells of X .

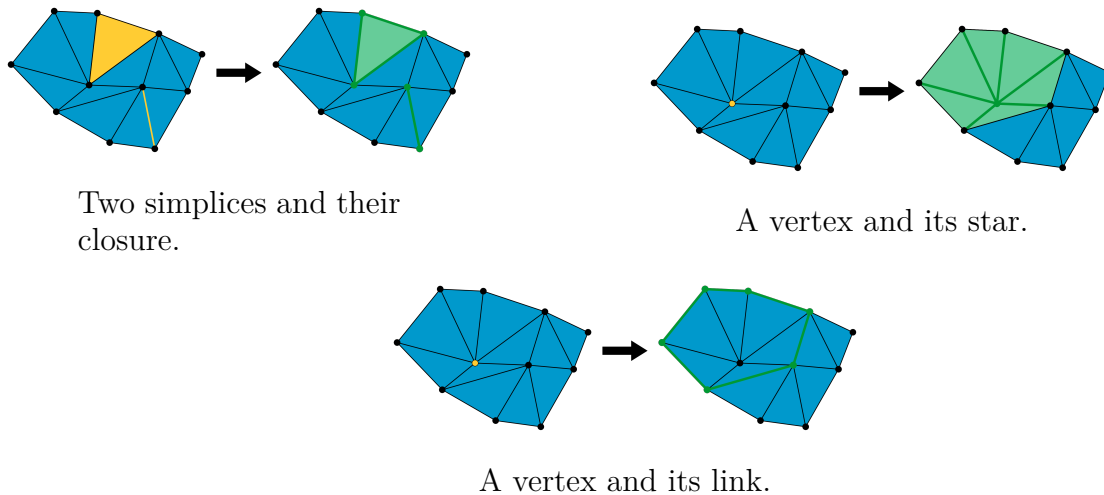


FIGURE 2. Depictions of Cl , St , Lk operators, taken from [con26].

Definition 1.25. For open cells σ, τ of X , we say that τ is a *face* of σ if $\tau \subset \bar{\sigma}$ and write $\tau \prec \sigma$.

Definition 1.26. Closure of an open cell σ of X $\text{Cl}(\sigma)$ is the subcomplex

$$\text{Cl}(\sigma) = \bigcup_{\tau \prec \sigma} \tau.$$

Closure of S in X $\text{Cl}(S)$ is the union of the closures of each open cell in S .

Remark 1.27. It is clear that $\text{Cl}(\sigma) = \bar{\sigma}$. We however keep the notation to signify the context: $\bar{\sigma}$ emphasizes a topological operation whereas $\text{Cl}(\sigma)$ is the downward closure in the face poset which is a combinatorial operation.

Definition 1.28. The *star* of an open cell σ of X is

$$\text{St}(\sigma) = \bigcup_{\sigma \prec \eta} \eta.$$

That is, star consists of the open cells that has σ as a face. Similarly, the star of S is defined as the union of the star of each open cell in S .

Definition 1.29. Link of an open cell σ of X is

$$\text{Lk}(\sigma) = \{\tau \mid \tau \prec \eta, \sigma \prec \eta, \tau \not\prec \sigma\}.$$

That is, link consists of all open cells which are not a face of σ but are faces of the open cells that has σ as a face. Equivalently, $\text{Lk}(\sigma) = \text{Cl}(\text{St}(\sigma)) \setminus \text{St}(\text{Cl}(\sigma))$.

1.4. Group cohomology

We follow the discussion of [Fri17]. In what follows let G be a discrete group and R a commutative ring with identity.

Definition 1.30. We denote by $R[G]$ the group ring associated to R and G , i.e., the abelian group $\bigoplus_G R$ endowed with the multiplication

$$\left(\sum_{g \in G} a_g g \right) \cdot \left(\sum_{g \in G} b_g g \right) = \sum_{g \in G} \left(\sum_{h \in H} a_{gh} b_{h^{-1}} \right) g$$

Definition 1.31. An $R[G]$ -module V is an R -module endowed with an action of G by R -linear maps and an $R[G]$ -linear map between $R[G]$ -modules is an R -linear map that commutes with the actions of G on either side.

Remark 1.32. In the case of $R = \mathbb{Z}$, \mathbb{Z} -modules are abelian groups. This shows the equivalence of notions between $\mathbb{Z}[G]$ -modules and G -modules which are abelian groups endowed with an action of G compatible with the abelian group structure and with G -equivariant group homomorphisms between them as morphisms.

Remark 1.33. When R is a field, $R[G]$ -modules become commutative R -algebras.

Example 1.34. For $n \in \mathbb{N}_{>0}$, let C_n be the cyclic group of order n with generator a and identity element 1. Then the elements of the group ring $\mathbb{C}[C_n]$ associated to \mathbb{C} and C_n can be written as

$$z_0 1 + z_1 a + \cdots + z_n a^n$$

where $z_0, \dots, z_n \in \mathbb{C}$. We then have that

$$\mathbb{C}[C_n] \cong_{\text{CAlg}_{\mathbb{C}}} \mathbb{C}[x]/(x^n - 1)$$

Definition 1.35. Let V be an $R[G]$ -module. We denote by V^G the R -submodule of G -invariants, that is

$$V^G := \{v \in V \mid g \cdot v = v \ \forall g \in G\}.$$

Note that we stress here that V^G is not necessarily an $R[G]$ -module.

Example 1.36. Let $R \in \{\mathbb{R}, \mathbb{Z}\}$ and let $\ell^\infty(G, R) := \{f: G \rightarrow R \mid \sup_{g \in G} \|f(g)\| < \infty\}$. Then $\ell^\infty(G, R)$ is a R -module with respect to pointwise addition and scalar multiplication. Moreover, the following action of G equips it with the structure of a $R[G]$ -module:

$$\begin{aligned} G \times \ell^\infty(G, R) &\longrightarrow \ell^\infty(G, R) \\ (g, f) &\longmapsto (h \mapsto f(hg)). \end{aligned}$$

We then have that $g \cdot f = f$ for all $g \in G$ only when f is constant, so it follows that

$$\ell^\infty(G, R)^G \cong_{R\text{-Mod}} R$$

Definition 1.37. Let V be an $R[G]$ -module. We define the \mathbb{N} -indexed homogeneous cochain complex $(C^\bullet(G, V), \delta^\bullet)$ associated to G and V in the following way:

- For $n \in \mathbb{N}$, we consider the $R[G]$ -modules

$$C^n(G, V) := \{f: G^{n+1} \rightarrow V\}$$

where the G -action is defined as:

$$(g \cdot f)(g_0, \dots, g_n) = g(f(g^{-1}g_0, \dots, g^{-1}g_n))$$

for every $g \in G, f \in C^n(G, V)$.

- For all $n \in \mathbb{N}$, we define the coboundary operator $\delta^n: C^n(G, V) \rightarrow C^{n+1}(G, V)$ as:

$$(\delta^n f)(g_0, \dots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \dots, \widehat{g}_i, \dots, g_{n+1}).$$

It can be checked that δ^n is $R[G]$ -linear and $\delta^{n+1} \circ \delta^n = 0$ for every $n \in \mathbb{N}$.

Definition 1.38. We define the cohomology of G with coefficients in V , $H^\bullet(G, V)$, in the following way:

- Consider the subcomplex $(C^\bullet(G, V)^G, \delta^\bullet) \subset (C^\bullet(G, V), \delta^\bullet)$ where we use the same symbol the restriction of the coboundary maps to the R -submodule of G -invariants (it being a subcomplex follows from the fact that coboundary maps are $R[G]$ -linear).
- For $n \in \mathbb{N}$, we define

$$Z^n(G, V) = C^n(G, V)^G \cap \ker \delta^n.$$

called *cocycles*.

- For $n \in \mathbb{N}_{>0}$, we define

$$B^n(G, V) = \delta^{n-1}(C^{n-1}(G, V)^G)$$

and we set $B^0(G, V) = 0$, called *coboundaries*.

- For $n \in \mathbb{N}$, we define

$$H^n(G, V) = Z^n(G, V)/B^n(G, V).$$

Example 1.39. We have that $C^0(G, V) = \{f: G \rightarrow V\}$ and

$$(\delta^0 f)(g_0, g_1) = f(g_1) - f(g_0) = 0$$

for all $f \in C^0(G, V)$. Therefore, $f \in \ker \delta^0$ if and only if f is constant. On the other hand, any $f \in C^0(G, V)^G$ satisfies

$$(g \cdot f)(g_0) = g(f(g^{-1}g_0)) = f(g_0)$$

for all $g, g_0 \in G$ and hence

$$f(gh) = gf(h)$$

for all $g, h \in G$.

Let $f \in Z^0(G, V)$. Then f is constant and there exists a $v \in V$ such that $f(g) = v$ for all $g \in G$. Furthermore, f being G -invariant, we have

$$f(gh) = v = gv = gf(h)$$

for all $g, h \in G$. Hence $v \in V^G$ and it follows that $H^0(G, V) = V^G$.

Example 1.40. Let V be an $\mathbb{Z}[G]$ -module with the trivial G -action. Then, $f \in C^1(G, V)$ if

$$f(gg_0, gg_1) = f(g_0, g_1)$$

for all $g \in G$ and $f \in \ker \delta^1$ if

$$(\delta^1 f)(g_0, g_1, g_2) = f(g_1, g_2) - f(g_0, g_2) + f(g_0, g_1) = 0$$

for all $g_0, g_1, g_2 \in G$. Therefore for any $f \in Z^1(G, V)$ we can rewrite the cocycle condition using the translation invariance:

$$f(1, g_1^{-1}g_2) - f(1, g_0^{-1}g_2) + f(1, g_0^{-1}g_1) = 0$$

for all $g_0, g_1, g_2 \in G$. Letting $g = g_0^{-1}g_1$, $h = g_1^{-1}g_2$, we obtain

$$f(1, g) + f(1, h) = f(1, gh)$$

for all $g, h \in G$. Finally, defining $\varphi: G \rightarrow V$ by $\varphi(g) = f(1, g)$ for all $g \in G$, we see φ is a group homomorphism with respect to the group structure on G and the abelian group structure on V . Hence, we get

$$H^1(G, V) \cong_{\mathbb{Z}} \text{Hom}_{\text{Group}}(G, V)$$

Remark 1.41. For any CW-complex X of type $K(G, 1)$, we have that $H^n(X, V)$ and $H^n(G, V)$ are canonically isomorphic where $H^n(X, V)$ denotes the singular cohomology of X , see [Fri17, Section 4.4]. In fact, if we let \widetilde{X} be the universal cover of X , we have that $C^n(X, V) \cong C^n(\widetilde{X}, V)^G$. This follows from the fact that each singular simplex $\sigma: \Delta^n \rightarrow X$ lifts to a simplex $\tilde{\sigma}: \Delta^n \rightarrow \widetilde{X}$ and two lifts differ by the action of G ($G = \pi_1(X)$ acts on \widetilde{X} by deck transformations). Therefore, a V -valued singular cochain on X is equivalent to a V -valued G -equivariant singular cochain on \widetilde{X} .

1.5. Bounded cohomology

In what follows, we assume that $R \in \{\mathbb{Z}, \mathbb{R}\}$ and we refer to the absolute value norm when considering a norm on the elements of R .

Definition 1.42. A *normed $R[G]$ -module* V is an $R[G]$ -module endowed with an invariant norm, i.e., a map $\|\cdot\|: V \rightarrow \mathbb{R}$ such that:

- $\|v\| = 0$ if and only if $v = 0$,
- $\|r \cdot v\| \leq |r| \cdot \|v\|$ for every $r \in R, v \in V$,
- $\|v + w\| \leq \|v\| + \|w\|$ for every $v, w \in V$,
- $\|g \cdot v\| = \|v\|$ for every $g \in G, v \in V$.

A morphism of normed $R[G]$ -modules is an $R[G]$ -linear map between the underlying $R[G]$ -modules that is bounded with respect to the norms.

Remark 1.43. A normed $\mathbb{R}[G]$ -module is a normed real vector space with an action of G by linear isometries.

Example 1.44. Let V be a normed $R[G]$ -module and consider the $R[G]$ -modules $C^n(G, V)$. For every $f \in C^n(G, V)$, we consider the ℓ^∞ -norm:

$$\|f\|_\infty := \sup\{\|f(g_0, \dots, g_n)\| \mid (g_0, \dots, g_n) \in G^{n+1}\} \in [0, +\infty]$$

and we define

$$C_b^n(G, V) := \{f \in C^n(G, V) \mid \|f\|_\infty < \infty\}.$$

It follows that $C_b^n(G, V) \subset C^n(G, V)$ is an $R[G]$ -submodule and hence is a normed $R[G]$ -module. Furthermore, the coboundary maps restrict to bounded $R[G]$ -linear maps $\delta^n: C_b^n(G, V) \rightarrow C_b^{n+1}(G, V)$.

Definition 1.45. We define the *bounded cohomology of G with coefficients in V* as

$$H_b^n(G, V) := Z_b^n(G, V) / B_b^n(G, V)$$

where

$$Z_b^n(G, V) = C_b^n(G, V)^G \cap \ker \delta^n$$

for $n \in \mathbb{N}$,

$$B_b^n(G, V) = \delta^{n-1}(C_b^{n-1}(G, V)^G)$$

for $n \in \mathbb{N}_{>0}$ and $B_b^0(G, V) = 0$.

Example 1.46. The same argument as in Example 1.40 shows that

$$H_b^1(G; \mathbb{R}) \cong_{\mathbb{R}} \{\varphi \in \text{Hom}_{\text{Group}}(G, \mathbb{R}) \mid \sup_{g \in G} |\varphi(g)| < \infty\} \cong_{\mathbb{R}} 0$$

as \mathbb{R} has no non-trivial bounded subgroups.

2. ℓ^∞ -cohomology

In this chapter we first define Gersten's ℓ^∞ -cohomology of groups and spaces. The cohomology theory was first introduced in his preprint [Ger91] together with the proof of its quasi-isometry invariance but since this preprint is now hard to come by, we refer to [Ger98] where it is summarised.

The main motivation of Gersten in developing the ℓ^∞ -cohomology was to give lower bounds for isoperimetric functions and Dehn functions consequently, this will be explored later on.

Gersten's definition requires certain finiteness conditions, these requirements were then removed by Milizia [Mil23] via a definition that recovers Gersten's definition. We then give a generalized version for graphs following [MPSV25].

2.1. Gersten's definition

Let X be a CW-complex with finite n -skeleton for some $n \in \mathbb{N}$ and let \widetilde{X} be its universal cover endowed with the pullback cell structure via the covering projection. Let us denote a k -cell of X and \widetilde{X} by $e^{(k)}$ and $\tilde{e}^{(k)}$. Also, denote by $\mathcal{C}_k(X)$ and $\mathcal{C}_k(\widetilde{X})$ the set of k -cells of X and \widetilde{X} , respectively. Let A be a normed abelian group and V a normed $\mathbb{R}[\pi_1(X)]$ -module. We consider the abelian groups $C_\bullet^{\text{cell}}(\widetilde{X}; A)$ of cellular chains and the $\mathbb{R}[\pi_1(X)]$ -modules of $C_{\text{cell}}^\bullet(\widetilde{X}; V)$ cellular cochains on \widetilde{X} .

We endow the cellular chain complex $C_\bullet^{\text{cell}}(\widetilde{X}; A)$ with the ℓ^1 -norm with respect to the cellular basis, that is, given a cellular k -chain $c = \sum_{i \in \mathcal{I}} a_i \tilde{e}_i \in C_k^{\text{cell}}(\widetilde{X}; A)$ where \mathcal{I} is a finite index set and $a_i \in A$, $\tilde{e}_i \in \mathcal{C}_k(\widetilde{X})$ for each $i \in \mathcal{I}$, we set

$$\|c\|_1 = \sum_{i \in \mathcal{I}} \|a_i\|_A.$$

We then endow the cellular cochain complex $C_{\text{cell}}^\bullet(\widetilde{X}; V)$ with the ℓ^∞ -norm:

$$\|f\|_\infty = \sup_{\tilde{e}^{(k)} \in \mathcal{C}_k(\widetilde{X})} \|f(\tilde{e}^{(k)})\|_V \in [0, +\infty]$$

Then, we define for every $k \in \mathbb{N}$

$$C_{(\infty)}^k(\widetilde{X}; V) := \{f \in C_{\text{cell}}^k(\widetilde{X}; V) : \|f\|_\infty < \infty\}$$

to be the normed $\pi_1(X)$ -module of V -valued bounded cellular k -cochains. Note that, here the boundedness refers to a uniform bound on the norm of values taken by the cochain on the k -cells with respect to the ℓ_1 -norm, equivalently, a k -cochain f is said to be bounded if there exists a $K_f \in \mathbb{R}_{>0}$ such that $\|f(\tilde{e}^{(k)})\|_V \leq K_f$ for all k -cells $\tilde{e}^{(k)} \in \mathcal{C}_k(\widetilde{X})$. The infimum of the set of all such constants is equal to $\|f\|_\infty$.

The finiteness of the n -skeleton of X guarantees that the usual cellular coboundary sends bounded k -cochains to bounded $(k+1)$ -cochains for $k < n$. To sketch a proof of this statement: given a bounded k -cochain f , one has that $(\delta f)(\tilde{e}^{(k+1)}) = f(\partial \tilde{e}^{(k+1)})$ for any $(k+1)$ -cell $\tilde{e}^{(k+1)}$. Since there are finitely many attaching maps for $(k+1)$ -cells of X for $k+1 \leq n$ and hence for their lifts in \widetilde{X} , one then gets a uniform bound. This gives us well-defined bounded cellular coboundaries $B_{(\infty)}^k(\widetilde{X}; V)$ for all degrees $k \leq n$. Furthermore, since the condition $\delta f = 0$ does not rely on any finiteness conditions on X , the bounded cellular cocycles $Z_{(\infty)}^\bullet(\widetilde{X}; V)$

are well-defined in each degree. Therefore, $C_{(\infty)}^{\leq n}(\widetilde{X}; V)$ is a subcomplex of the full cellular cochain complex in degrees $\leq n$ and we define the ℓ^∞ -cohomology of \widetilde{X} with coefficients in V , $H_{(\infty)}^{\leq n}(\widetilde{X}; V)$, as the cohomology of this subcomplex for degrees $\leq n$.

A connected topological space X is said to be of type $K(G, n)$ for some group G and a positive integer n if it has n th homotopy group $\pi_n(X) \cong G$ and all other homotopy groups trivial.

A group G is said to be of type F_n for $n \in \mathbb{N}_+$ if there exists a CW-complex of type $K(G, 1)$ with finite n -skeleton. A group G is said to be of type F_∞ if it is of type F_n for all $n \in \mathbb{N}_+$.

The ℓ^∞ -cohomology of a group G of type F_n in degrees $\leq n$, $H_{(\infty)}^{\leq n}(\widetilde{X}; V)$ is defined as the ℓ^∞ -cohomology of \widetilde{X} where X is a CW-complex of type $K(G, 1)$ with a finite n -skeleton and \widetilde{X} is the universal cover of X endowed with the pullback cell structure via the covering projection. This is well-defined as it was already shown that the property of being of type F_n and $H_{(\infty)}^{\leq n}(-; V)$ are quasi-isometry invariants ([Gro93, Alo94]). That is, the definition is independent up to canonical isomorphism of the choice of a classifying space X .

Remark 2.1. A group is of type F_2 if and only if it is finitely presented. Therefore, for a finitely presented group G , $H_{(\infty)}^2(G; V)$ is always well-defined. This will be the main object of interest for our discussion.

Remark 2.2. In this setting, \widetilde{X} is the universal cover of a CW-complex X of type $K(G, 1)$. This means, that $\pi_1(X) \cong G$ and $\pi_n(X) = 0$ for all $n \geq 2$. Since covering maps induce isomorphisms on the higher homotopy groups we have that $\pi_n(\widetilde{X}) = \pi_n(X) = 0$ for $n \geq 2$ and furthermore, since \widetilde{X} is a universal cover, $\pi_1(\widetilde{X}) = 0$ as well. Hence, \widetilde{X} is contractible. It follows that $Z_k^{\text{cell}}(\widetilde{X}; A) = B_k^{\text{cell}}(\widetilde{X}; A)$ for any $k \geq 1$.

Remark 2.3. The difference between ℓ^∞ -cohomology and the bounded cohomology is that in bounded cohomology one considers the bounded cochains on X which is the same as bounded G -invariant cochains on \widetilde{X} (see Remark 1.41). Namely, the cochain complex

$$C_b^\bullet(X; A) := \{f \in C_{(\infty)}^\bullet(\widetilde{X}; A) : \forall g \in G \quad g \cdot f = f\}$$

gives rise to Gromov's bounded cohomology in this context. See the following for a discussion of the G -action mentioned here.

Another way of describing the ℓ^∞ -cohomology of G is through the natural cellular action of G on \widetilde{X} : We have that $G = \pi_1(X)$ and that \widetilde{X} is the universal cover of X . This means that G acts on \widetilde{X} freely and cocompactly via covering transformations and $\widetilde{X}/G = X$. This action can also be extended to cochains by

$$(g \cdot f)(\tilde{e}^{(k)}) = f(g^{-1} \cdot \tilde{e}^{(k)}), \quad f \in C_{\text{cell}}^k(\widetilde{X}; A), \quad \tilde{e}^{(k)} \in \mathcal{C}_k(\widetilde{X}).$$

Since X has finite n -skeleton, G acts on $\mathcal{C}_k(X)$ with finitely many orbits for $k \leq n$. Let us choose for every k -cell $e^{(k)}$ of X a lift $\tilde{e}^{(k)}$, which are basically the orbit representatives, that is, $q(\tilde{e}^{(k)}) = e^{(k)}$ where $q: \widetilde{X} \rightarrow \widetilde{X}/G$ is the covering map. Let \mathcal{I}_k index the k -cells of X . We can then enumerate all k -cells of \widetilde{X} by:

$$\tilde{e}_{g,i}^{(k)} := g \cdot \tilde{e}_i^{(k)}, \quad g \in G, \quad i \in \mathcal{I}_k$$

where $\tilde{e}_i^{(k)}$ is the chosen lift of $e_i^{(k)}$. This gives us the following factorization of a k -cochain through $G \times \mathcal{I}_k$:

$$\begin{aligned} f: \quad G \times \mathcal{I}_k &\longrightarrow V \\ (g, i) &\longmapsto f(\tilde{e}_{g,i}^{(k)}) \end{aligned}$$

We can then express its boundedness as

$$\sup_{\tilde{e}^{(k)} \in \mathcal{C}_k(\tilde{X})} \|f(\tilde{e}^{(k)})\|_V = \sup_{g \in G, i \in \mathcal{I}_k} \|f(g, i)\|_V < +\infty.$$

Interpreted this way, one can think of a bounded cellular k -cochain on \tilde{X} as an attachment of bounded V -valued functions on G to each k -cell of X . This gives the following correspondence

$$\begin{aligned} \Psi: \quad C_{(\infty)}^k(\tilde{X}; V) &\longrightarrow C_{\text{cell}}^k(X; \ell^\infty(G, V)) \\ \varphi &\longmapsto (e^{(k)} \mapsto (g \mapsto \varphi(g^{-1} \cdot \tilde{e}^{(k)}))) \\ \Phi: \quad C_{\text{cell}}^k(X; \ell^\infty(G, V)) &\longrightarrow C_{(\infty)}^k(\tilde{X}; V) \\ \psi &\longmapsto (\tilde{e}^{(k)} \mapsto \psi(e^{(k)})(g)) \end{aligned}$$

where

$$\ell^\infty(G, V) = \{f : G \rightarrow V \mid f(G) \subseteq V \text{ is bounded}\}$$

is the normed $\mathbb{R}[G]$ -module of bounded V -valued functions on G , endowed with the supremum norm with respect to the norm on V and equipped with the G -action: $(g \cdot f)(h) = g \cdot (f(g^{-1}h))$ where $f \in \ell^\infty(G, V)$, $g, h \in G$.

This is a fleshed out discussion of the proof of the following theorem [Wie12, Proposition 5.2], [Bla16, Theorem 6.3.5]:

Theorem 2.4. *Let $n \in \mathbb{N}$ and G be a group of type F_n . Let V be a normed $\mathbb{R}[G]$ -module. Then there is a canonical isomorphism*

$$H_{(\infty)}^k(G; V) \cong H^k(G; \ell^\infty(G, V))$$

for $k \leq n$.

Remark 2.5. We have that $H^k(G; \ell^\infty(G; V))$ is well-defined in each degree and one can extend the correspondence for degrees beyond n by generalizing Gersten's definition via removing the finite n -skeleton condition and taking cellular k -cochains on \tilde{X} that are bounded on each G -orbit of k -cells of \tilde{X} but not necessarily uniformly bounded over all k -cells. This definition recovers Gersten's original definition as in the case of X having finite n -skeleton, there are only finitely many orbits of k -cells of \tilde{X} and being bounded on every G -orbit of k -cells is equivalent to being uniformly bounded over all k -cells [Mil23, p. 2].

Corollary 2.6. *Let $n \in \mathbb{N}$ and G be a group. If X and Y are two distinct CW-complexes of type $K(G, 1)$ with finite n -skeleta and universal covers \tilde{X}, \tilde{Y} , respectively, then*

$$H_{(\infty)}^k(\tilde{X}; V) \cong H_{(\infty)}^k(\tilde{Y}; V)$$

for all $k \in \mathbb{N}_{<n}$.

2.2. Comparison maps

We consider the inclusion map $\iota : V \rightarrow \ell^\infty(G, V)$ defined as

$$\iota(v)(g) = v$$

for every $v \in V$ and $g \in G$. Then ι is a morphism of normed $\mathbb{R}[G]$ -modules because it is bounded as $\|\iota(v)\|_\infty = \|v\|$ and it is G -equivariant as

$$(g \cdot \iota(v))(h) = g \cdot (\iota(v)(g^{-1}h)) = g \cdot v = g \cdot \iota(v)(h)$$

for every $v \in V$, $g, h \in G$. It therefore induces a change of coefficients map:

$$\begin{aligned} \iota^k: C_{\text{cell}}^k(X; V) &\longrightarrow C_{\text{cell}}^k(X; \ell^\infty(G, V)) \\ f &\longmapsto \iota \circ f \end{aligned}$$

which then descends to cohomology $\iota^k: H^k(X; V) \rightarrow H^k(X, \ell^\infty(G, V))$. Then by Theorem 2.4, we get a map $\iota^k: H^k(X; V) \rightarrow H_{(\infty)}^k(X; V)$. We also denote by $\iota^k: H^k(G; V) \rightarrow H_{(\infty)}^k(G; V)$ the map induced by ι in the group cohomology.

The following theorem was first proved by Gersten in [Ger91, Proposition 10.3], and other proofs can be found in [Wie12, Proposition 5.1], [Bla16, Proposition 6.3.9], [Mil23, Corollary 3.4]. We follow the former's exposition:

Theorem 2.7. *The composition*

$$H_b^k(X; V) \xrightarrow{c^k} H^k(X; V) \xrightarrow{\iota^k} H_{(\infty)}^k(X; V)$$

is zero for all $k \in \mathbb{N}_{\geq 1}$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} H_b^k(X; V) & \xrightarrow{c^k} & H^k(X; V) & \xrightarrow{\iota^k} & H_{(\infty)}^k(X; V) \\ & \searrow \iota^k & & & \uparrow \cong \\ & & H_b^k(X; \ell^\infty(G, V)) & \xrightarrow{c^k} & H^k(X; \ell^\infty(G, V)) \end{array}$$

where the map $H_b^k(X; V) \xrightarrow{\iota^k} H_b^k(X; \ell^\infty(G, V))$ is the map induced by the restriction of ι^k to the bounded cochains and the isomorphism on the right is the one from Theorem 2.4. We will show that $H_b^k(X; \ell^\infty(G, V)) \cong H_b^k(\widetilde{X}; V)$ for all $k \in \mathbb{N}_{\geq 1}$ and from this it will follow that it vanishes since $H_b^\bullet(\widetilde{X}; V)$ is isomorphic to the bounded cohomology of the trivial group.

First recall that $C^k(X; V) \cong C^k(\widetilde{X}; V)^G$ by Remark 1.41. Define $\hat{\cdot}: C_b^k(\widetilde{X}; \ell^\infty(G, V))^G \rightarrow C_b^k(\widetilde{X}; V)$ by $\hat{\alpha}(c) = \alpha(c)(1_G)$ for every $\alpha \in C_b^k(\widetilde{X}; \ell^\infty(G, V))$ where $c \in C_{\text{cell}}^k(\widetilde{X})$. We will show that this map induces a chain isomorphism between the two cochain complexes.

Let $\alpha \in C_b^k(\widetilde{X}; \ell^\infty(G, V))^G$. We have that

$$\begin{aligned} (\alpha(c))(g) &= ((g \cdot \alpha)(c))(g) \\ &= (g \cdot \alpha(g^{-1} \cdot c))(g) \\ &= g \cdot (\alpha(g^{-1} \cdot c)(1_G)) \\ &= g \cdot \hat{\alpha}(g^{-1} \cdot c) \end{aligned}$$

for every $c \in C_{\text{cell}}^k(\widetilde{X}; V)$ and $g \in G$. This shows that $\hat{\cdot}$ is injective. Moreover, given $\beta \in C_b^k(\widetilde{X}; V)$, we have that $\beta = \hat{\alpha}$ where α is defined as $(\alpha(c))(g) = g \cdot \beta(g^{-1} \cdot c)$. It can be verified easily that the α defined via this formula is G -invariant and that $\alpha(c) \in \ell^\infty(G, V)$ for every cellular k -chain c on \widetilde{X} . Finally, the map $\hat{\cdot}$ is \mathbb{R} -linear and $\delta \hat{\alpha} = \widehat{\delta \alpha}$ as:

$$(\delta \hat{\alpha})(c) = \hat{\alpha}(\partial c) = (\alpha(\partial c))(1_G) = ((\delta \alpha)(c))(1_G) = \widehat{(\delta \alpha)}(c)$$

for every $c \in C_{\text{cell}}^{k+1}(\widetilde{X}; V)$. This finishes the proof. \square

3. Characterization of hyperbolic groups

We now give an overview of Gersten's proof of the characterization of hyperbolic groups in terms of his ℓ^∞ -cohomology. The main result is [Ger96a, Theorem 5.1]

Theorem. *A finitely presented group G is hyperbolic if and only if $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$.*

Let X be a CW-complex of type $K(G, 1)$ with a finite 2-skeleton and let \widetilde{X} denote its universal cover, endowed with the pull-back cell structure via the covering projection $\pi: \widetilde{X} \rightarrow X$. Let A be a normed abelian group. We will mainly be concerned with $A = \mathbb{Z}$ or $A = \mathbb{R}$ in this section. We begin by endowing the cellular 1-boundaries $B_1^{\text{cell}}(\widetilde{X}; A)$ with two different norms, which are generally inequivalent. Let b be a cellular 1-boundary. The filling norm is given by:

$$\|b\|_F := \inf\{\|c\|_1 : c \in C_2^{\text{cell}}(\widetilde{X}; A), \partial c = b\}.$$

The ℓ_1 -norm is just the restriction of the ℓ_1 -norm on $C_1^{\text{cell}}(\widetilde{X}; A)$ to $B_1^{\text{cell}}(\widetilde{X}; A)$ with respect to the basis of k -cells.

Remark 3.1. The filling norm can be seen as the quotient norm induced by the boundary map. Assuming the norm on A is a norm, the filling norm is indeed a norm and not a seminorm: since X has a finite 2-skeleton and the covering projection is a cellular map, we have that the boundary map $\partial: C_2^{\text{cell}}(\widetilde{X}; A) \rightarrow C_1^{\text{cell}}(\widetilde{X}; A)$ is bounded with respect to the ℓ_1 -norm and there exists an $M \in \mathbb{R}_+$ such that $\|\partial c\|_1 \leq M\|c\|_1$ for all $c \in C_2^{\text{cell}}(\widetilde{X}; A)$. Hence, it follows that

$$\|b\|_1 \leq M\|b\|_F \quad (*)$$

This shows that if $\|b\|_F = 0$, then $\|b\|_1 = 0$ and $b = 0$. This is the part where the assumption of the norm on A being a norm is used, precisely to say that $b = 0$ if $\|b\|_1 = 0$, see the § 3.3 on filling norms for more details. Also note that the finiteness of $X^{(2)}$ is essential to having the inequality (*). For cellular boundaries on an arbitrary CW-complex, it is not necessarily true that the filling norm dominates the ℓ_1 -norm.

The main result we cited above follows from the characterizations of hyperbolicity of a group [Ger98, Theorem A.1] and the vanishing of the ℓ^∞ -cohomology [Ger96a, Theorem 4.3] in terms of the equivalences of the two norms we defined above under different coefficients:

Theorem. *Let G be a finitely presented group. Then*

- (1) *G is hyperbolic if and only if the filling norm and the ℓ_1 -norm on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ are equivalent.*
- (2) *$H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$ if and only if the filling norm and the ℓ_1 -norm on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ are equivalent.*

Note that the main theorem doesn't directly follow from these two results but we will bridge the gap between the two.

3.1. Lower bounds for isoperimetric functions

To motivate the role played by these norms: in this setting, the 1-skeleton of \widetilde{X} corresponds to the Cayley graph of G and therefore, one can study the hyperbolicity of G through the behaviour of loops in $\widetilde{X}^{(1)}$, namely through their isoperimetric behaviour. As is known, being

δ -hyperbolic is equivalent to satisfying a linear isoperimetric inequality for finitely presented groups [Gro87]. Here, in broader terms, the linear isoperimetric inequality refers to a linear relationship between the length and the enclosed area of a closed curve. The connection between the length and the area enclosed by a closed loop in $\widetilde{X}^{(1)}$ is strongly related to the norms we introduced above.

We begin by expanding on this dictionary between $\widetilde{X}^{(1)}$ and the Cayley graph of G . An oriented edge corresponds to an open 1-cell of \widetilde{X} endowed with an orientation, which then defines the head and the tail of the oriented edge. Note here that the head and tail are defined as usual: Every open 1-cell e of X is attached via a continuous map $\varphi_e: [0, 1] \rightarrow X^{(1)}$ such that the endpoints are mapped into $X^{(0)}$, so we take $\varphi_e(0)$ to be the tail and $\varphi_e(1)$ to be the head of e . Then, under lifting, this orientation is preserved and the corresponding endpoints are called the tail and the head of the open 1-cell \tilde{e} of \widetilde{X} . For brevity, we use the notation $\nu(e)$ and $\tau(e)$ for the head and tail of an oriented edge, respectively.

A path p is a finite sequence of oriented edges $(e_i)_{1 \leq i \leq n}$ such that the head of e_i is equal to the tail of e_{i+1} for $i = 1, \dots, n-1$ and $n \in \mathbb{Z}_+$ is the length of the path, denoted by $\text{Len}(p)$. A path p is said to be closed if the head of its final edge is equal to the tail of its initial edge.

Given a path p , we obtain a unique chain in $C_1^{\text{cell}}(\widetilde{X}; A)$ associated with it by summing the 1-cells corresponding to its oriented edges; this is denoted by $\langle p \rangle$. It is seen that $\|\langle p \rangle\|_1 = \text{Len}(p)$.

Let $\mathcal{P} = \langle S | R \rangle$ be a finite presentation for the group G and let $w \in F_{\text{red}}(S)$ be a reduced word representing the trivial element in G . Then we must have that

$$w = \prod_{i=1}^n a_i r_i a_i^{-1} \in F(S) \quad (1)$$

for some $a_1, \dots, a_n \in F(S)$ and $r_1, \dots, r_n \in R \cup R^{-1}$. The area of w with respect to the presentation \mathcal{P} is defined as the minimum $n \in \mathbb{N}$ for which there exists an expression as in (1), denoted $\text{Area}_{\mathcal{P}}(w)$. The function $f_{\mathcal{P}}: \mathbb{N} \rightarrow \mathbb{N}$ is an isoperimetric function for \mathcal{P} if for all reduced words $w \in F_{\text{red}}(S)$ representing the trivial element in G with $l(w) \leq n$, we have $\text{Area}_{\mathcal{P}}(w) \leq f_{\mathcal{P}}(n)$. If there exists a linear isoperimetric function, \mathcal{P} is said to satisfy the linear isoperimetric inequality. We will omit the presentation from the notation when it is clear from the context.

There is another geometric-topological construction that also gives the same notion of area and another perspective on isoperimetric functions. We first note that the 2-skeleton of our CW-complex X is a presentation complex for our group G . So, a choice of CW-complex X of type $K(G, 1)$ with a finite 2-skeleton already fixes a finite presentation \mathcal{P} for G . Recall that a van Kampen diagram for a reduced word w representing the identity is a combinatorial map $D \rightarrow X^{(2)}$ where D is a topological disc and the image of the boundary circle of D lifts to the closed path p in $\widetilde{X}^{(1)}$ corresponding to the element w (since D is simply connected, the whole map can be lifted to \widetilde{X}). The minimum area among all such diagrams is denoted by $\text{Area}_{\mathcal{P}}(w)$. The diagram also determines a 2-chain $\langle D \rangle \in C_2^{\text{cell}}(\widetilde{X}; A)$.

Let us now briefly explore how this theory comes in handy for getting lower bounds of isoperimetric functions. Let w be a reduced word that represents the trivial element in G and let p denote the closed path in $\widetilde{X}^{(1)}$ corresponding to w . Let $f \in Z_{(\infty)}^2(\widetilde{X}; V)$ and let $k: D \rightarrow \widetilde{X}$ be a singular disc diagram filling of p . Let $\langle D \rangle$ denote the cellular 2-chain associated with D in $C_2(\widetilde{X}; V)$. We can evaluate f on $\langle D \rangle$ via the Kronecker pairing. We first note that since f is a cocycle, the value of this evaluation is independent of the choice of van Kampen diagram D for w . Indeed, if D_1 and D_2 are two different van Kampen diagrams for w , we have that

$$\partial \langle D_1 \rangle = \partial \langle D_2 \rangle = \langle p \rangle$$

and $\langle D_1 \rangle, \langle D_2 \rangle$ must differ by a 2-cycle in $C_2(\widetilde{X})$. So, we get:

$$f(\langle D_1 \rangle) - f(\langle D_2 \rangle) = f(\langle D_1 \rangle - \langle D_2 \rangle) = f(\partial C) = \delta f(C) = 0$$

where C is any 3-chain with $\partial C = \langle D_1 \rangle - \langle D_2 \rangle$. Therefore, we can choose the filling to be minimal, which then corresponds to $\text{Area}_{\mathcal{P}}(w)$ many 2-cells. As f is bounded above by $\|f\|_{\infty}$, we obtain the following inequality:

$$\|f(\langle D \rangle)\| \leq \|f\|_{\infty} \text{Area}_{\mathcal{P}}(w)$$

Furthermore, if we have that $f = \delta h$ for some $h \in C_{(\infty)}^1(\widetilde{X}; V)$, we get:

$$f(\langle D \rangle) = \delta h(\langle D \rangle) = h(\partial \langle D \rangle) = h(\langle p \rangle)$$

and

$$\|f(\langle D \rangle)\| \leq \|h\|_{\infty} \ell(w).$$

So, we see that cocycles provide lower bounds for the isoperimetric function of a finite presentation and that this bound is linear for coboundaries.

3.2. Weak linear isoperimetric inequality

This is a weakening of the isoperimetric functions that follows from a generalization of the geometric notion of area, called weak-area. This was first introduced by Gersten in [Ger96b]. It helps to connect the filling norm directly into the picture.

Definition 3.2. Let X, \widetilde{X}, G, V be as in § 2.1. Let \mathcal{P} be the finite presentation of G corresponding to the presentation complex $X^{(2)}$ and let w be a reduced word representing the trivial element in G . Let p denote the closed edge path in $\widetilde{X}^{(1)}$ corresponding to w . An orientable surface diagram for w is a combinatorial map $S \rightarrow \widetilde{X}^{(2)}$ mapping the boundary circle of S to $p \in \widetilde{X}^{(1)}$ where S is a singular orientable surface with one boundary component. The number of 2-cells in S is referred to as the $\text{Area}(S)$ and the minimum area among such diagrams for w is defined as the weak-area of w , denoted $\text{w-Area}_{\mathcal{P}}(w)$. This determines a 2-chain $\langle S \rangle \in C_2^{\text{cell}}(\widetilde{X}; A)$.

Remark 3.3. Notice that in the case of van Kampen diagrams since the domain is simply connected, it didn't matter whether we mapped into X or \widetilde{X} as every map would lift to \widetilde{X} anyway but for orientable surface diagrams this is not the case. We have to explicitly map into \widetilde{X} .

So, this is indeed a generalization of the notion of area we described above via van Kampen diagrams for a word representing the identity where instead of considering only disc diagrams, diagrams of higher genus singular orientable surfaces with one boundary component are also allowed. Weak isoperimetric functions are also defined analogously:

Definition 3.4. A function $f_{\mathcal{P}}: \mathbb{N} \rightarrow \mathbb{N}$ is a weak isoperimetric function for the finite presentation \mathcal{P} if for all words w representing the identity element with $l(w) \leq n$, we have

$$\text{w-Area}(w) \leq f(n)$$

Similarly, if there exists a linear weak isoperimetric function for \mathcal{P} , \mathcal{P} is said to satisfy the weak linear isoperimetric inequality. Gersten has shown that [Ger96b, Theorem 3.1]:

Theorem 3.5. *Let \mathcal{P} be a finite presentation for a group G . The following are equivalent:*

- (1) *The group G is δ -hyperbolic.*

- (2) \mathcal{P} satisfies the linear isoperimetric inequality.
- (3) \mathcal{P} satisfies the weak linear isoperimetric inequality.

Proof. The equivalence of (1) and (2) is already well-established. We also have that disc diagrams are orientable surface diagrams, therefore linear isoperimetric inequality implies weak linear isoperimetric inequality. So we only need to show that weak linear isoperimetric inequality implies hyperbolicity. We refer the reader to the proof of linear isoperimetric inequality impling hyperbolicity at [GS91, Appendix 2], in particular to the middle paragraph of the page 331: the only property of a disc diagram D being used is the fact each 1-cell which is not contained in ∂D must be contained in the boundary of 2-cells in D . This is true for orientable surface diagrams as well and the result follows. \square

Now we see how the notion of weak area interacts with the filling norm and how this helps to construct a finite presentation that achieves the weak area.

Proposition 3.6. [Ger96b, Proposition 3.2] *Let \mathcal{P} be a finite presentation for G and let w be a reduced word representing the identity element in G . Let p denote the closed path in $\widetilde{X}^{(1)}$ corresponding to w . Then, $\langle p \rangle \in B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ satisfies:*

$$\text{w-Area}(p) = \|\langle p \rangle\|_F$$

Proof. Let S be an orientable surface diagram filling of p . If there are no cancelling terms in $\langle S \rangle$, then $\text{Area}(S) = \|\langle S \rangle\|_1$, since each face of S must appear in $\langle S \rangle$ with a coefficient ± 1 . If however there is a pair of terms cancelling in $\langle S \rangle$, they must come from the same face R in S mapping to the same 2-cell of \widetilde{X} and appear in $\langle S \rangle$ with opposite signs, hence we get $\|\langle S \rangle\|_1 < \text{Area}(S)$. We make surgery on S over the two faces to create a handle: remove the interiors of the faces from S and glue the two boundaries.

This results in an orientable surface diagram S_1 with $\text{Area}(S_1) = \text{Area}(S) - 2$, since we removed 2 faces of S . Also, $\|\langle S_1 \rangle\|_1 = \|\langle S \rangle\|_1$ since the removed faces had no contribution to the ℓ_1 -norm in the image in \widetilde{X} and the rest of the faces were left untouched. Continuing with the same process iteratively for each pair of faces cancelling in the induced chain, the process must terminate at some finite step $k \in \mathbb{N}$ since we have finitely many terms in the 2-chain $\langle S \rangle$. After all the cancelling terms are eliminated, one obtains an orientable surface diagram S_k for p with $\text{Area}(S_k) = \|\langle S_k \rangle\|_1 = \|\langle S \rangle\|_1$.

If we choose another orientable surface diagram S' for p to begin with, we have that $\langle S' \rangle = \langle S \rangle + z$ for some 2-cycle $z \in Z_2^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ since $\partial \langle S \rangle = \partial \langle S' \rangle = \langle p \rangle$. Furthermore, since \widetilde{X} is simply connected, $\pi_2(\widetilde{X}) \cong H_2^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ by Hurewicz theorem and hence, the homology class represented by any 2-cycle z must come from a sphere $S^2 \rightarrow \widetilde{X}$. Therefore, for any 2-cycle $z \in Z_2^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ the 2-chain $\langle S \rangle + z$ corresponds to an orientable surface diagram. This gives:

$$\text{w-Area}(p) = \inf_{S' \in \mathcal{S}(p)} \text{Area}(S') = \inf_{z \in Z_2^{\text{cell}}(\widetilde{X}; \mathbb{Z})} \|\langle S \rangle + z\|_1$$

where $\mathcal{S}(p)$ denotes the set of all orientable surface diagrams for p .

On the other hand, for any $c, c' \in C_2^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ with $\partial c = \partial c' = \langle p \rangle$, we have that $c - c' = z$ for some $z \in Z_2^{\text{cell}}(\widetilde{X}; \mathbb{Z})$. Since $\langle S \rangle \in C_2^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ with $\partial \langle S \rangle = \langle p \rangle$, we get:

$$\|\langle p \rangle\|_F = \inf\{\|c\|_1 : c \in C_2^{\text{cell}}(\widetilde{X}; \mathbb{Z}), \partial c = \langle p \rangle\} = \inf_{z \in Z_2^{\text{cell}}(\widetilde{X}; \mathbb{Z})} \|\langle S \rangle + z\|_1 = \text{w-Area}(p)$$

\square

We are now ready to prove the first half our main theorem:

Theorem 3.7. *If a group G is hyperbolic, then the filling norm $\|\cdot\|_F$ and the ℓ_1 -norm $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ are equivalent.*

Proof. Assume first that the two norms are equivalent. Let p be a closed path in $\widetilde{X}^{(1)}$. It follows from the equivalence of norms that there exists a $M \in \mathbb{R}_+$ such that $\text{w-Area}(p) = \|\langle p \rangle\|_F \leq M \|\langle p \rangle\|_1 \leq K \text{Len}(p)$. This means that G satisfies the weak linear isoperimetric inequality and by Theorem 3.5 it follows that G is hyperbolic.

Now, suppose that G is hyperbolic. As we have remarked in Remark 3.1, we already have that there exists a $M \in \mathbb{R}_+$ such that for all $b \in B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$, $\|b\|_1 \leq M \|b\|_F$. We need to show the opposite inequality for some constant. As G is hyperbolic, it satisfies a linear isoperimetric inequality: there exists $C \in \mathbb{R}_+$ such that for all closed paths p in $\widetilde{X}^{(1)}$, $\text{Area}(p) \leq C \text{Len}(p)$. We need the following lemma:

Lemma 3.8. *For each $z \in Z_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$, there exists an $n \in \mathbb{N}$, positive coefficients $c_1, \dots, c_n \in \mathbb{Z}_+$ and closed paths $p_1, \dots, p_n \in \widetilde{X}^{(1)}$ such that*

- (1) $z = \sum_{k=1}^n c_k \langle p_k \rangle$
- (2) $\|z\|_1 = \sum_{k=1}^n c_k \text{Len}(z_k)$

Proof. If $z = 0$ there is nothing to show, so assume that $z \neq 0$. Then, choose an orientation on the 1-cells of \widetilde{X} so that $z = \sum_{i \in \mathcal{I}} c_i e_i$ for some finite index set \mathcal{I} and $c_i \in \mathbb{Z}_+$, 1-cells e_i for $i \in \mathcal{I}$. Since z is a cellular 1-cycle, we have

$$\partial z = \sum_{i \in \mathcal{I}} c_i \partial e_i = \sum_{i \in \mathcal{I}} c_i (\nu(e_i) - \tau(e_i)) = 0.$$

Therefore, for each $i \in \mathcal{I}$, there exists a $j \neq i \in \mathcal{I}$ such that

$$c_i \nu(e_i) = c_j \tau(e_j). \quad (*)$$

This is only possible if $c_i = c_j$ and $\nu(e_i) = \tau(e_j)$. Pick an $i_1 \in \mathcal{I}$ and choose the $j \neq i$ from \mathcal{I} satisfying (*). If we apply this process iteratively, since \mathcal{I} is finite, this process must lead to a permutation σ of \mathcal{I} of order p with $2 \leq p \leq |\mathcal{I}|$. Let $\mathcal{I}_1 = \{i_1, \sigma(i_1), \dots, \sigma^{p-1}(i_1)\}$. We repeat the same process over $\mathcal{I} - \mathcal{I}_1$ to get \mathcal{I}_2 and so on. As \mathcal{I} is finite, the process will terminate for some $n \in \mathbb{N}$ and we will have the following disjoint union

$$\mathcal{I} = \coprod_{k=1}^n \mathcal{I}_k.$$

Let $p_k = (e_i)_{i \in \mathcal{I}_k}$ for $1 \leq k \leq n$. As we noted before, $c_i = c_j$ for each pair $i, j \in \mathcal{I}_k$ for all $1 \leq k \leq n$. So, let c_k denote the coefficient corresponding to \mathcal{I}_k . We then have

$$z = \sum_{k=1}^n c_k \langle p_k \rangle$$

and

$$\|z\|_1 = \sum_{i \in \mathcal{I}} \|c_i\| = \sum_{k=1}^n \sum_{i \in \mathcal{I}_k} \|c_i\| = \sum_{k=1}^n c_k \text{Len}(z_k)$$

□

Let $n \in \mathbb{N}$ and p_1, \dots, p_n be as in the lemma. Then, for each $1 \leq i \leq n$, we have

$$\|\langle p_i \rangle\|_F = \text{w-Area}(p_i) \leq \text{Area}(p_i) \leq C \text{Len}(p_i) = C \|\langle p_i \rangle\|_1$$

and

$$\|z\|_F = \left\| \sum_{i=1}^n \langle p_i \rangle \right\|_F \leq \sum_{i=1}^n \|\langle p_i \rangle\|_F \leq C \sum_{i=1}^n \|\langle p_i \rangle\|_1 = C \|z\|_1.$$

Hence, the result follows. \square

3.3. Filling seminorms

In this section we closely follow the § 3 of [Ger96a]. Since it was a preprint that never appeared in publishing, we restate the relevant facts (and make improvements wherever possible) to make the content more accessible.

We will explore how filling norm behaves under different coefficients. Let Y be a CW-complex. We endow the cellular chains with ℓ_1 -norm with respect to the norm on the coefficients for some normed abelian group as before and extend the previous definition of filling norm on 1-boundaries to all k -boundaries for $k \in \mathbb{N}$:

Definition 3.9. The filling seminorm on $B_k^{\text{cell}}(Y; A)$ is defined as

$$\|b\|_F = \inf\{\|c\|_1 : c \in C_{k+1}^{\text{cell}}(Y; A), \partial c = b\}$$

for all $b \in B_k^{\text{cell}}(Y; A)$.

Note that we emphasize that it's a seminorm in general. When we first introduced it, we were already working with a CW-complex with lots of constraints which guaranteed that its a norm. We will later introduce a sufficient condition on a CW-complex to have a filling norm.

We will include the coefficient group A in the notation $\|\cdot\|_F^A$ to denote the filling norm associated to the k -boundaries $B_k^{\text{cell}}(Y; A)$ when its not clear from the context. We also use the notation ma with $m \in \mathbb{Z}$ and $a \in A$ to mean repeated addition. We begin with some technical definitions and results. In what follows, A denotes a seminormed abelian group equipped with the seminorm $\|\cdot\|: A \rightarrow \mathbb{R}_{\geq 0}$ unless otherwise stated.

3.3.1. Homogenization

Definition 3.10. The seminorm $\|\cdot\|$ is said to be homogeneous if $\|na\| = n\|a\|$ for all $a \in A$ and $n \in \mathbb{Z}_+$. We also define the homogenization ${}^H\|\cdot\|: A \rightarrow \mathbb{R}_{\geq 0}$ of a seminorm as

$${}^H\|a\| = \lim_{n \rightarrow \infty} \frac{\|na\|}{n}$$

Lemma 3.11. Homogenization ${}^H\|\cdot\|$ of a seminorm $\|\cdot\|$ is a homogeneous seminorm.

Proof. Let $a \in A$. The limit of $\frac{\|na\|}{n}$ exists: by triangle inequality, the sequence $(\|na\|)_n$ is sub-additive and therefore it's equal to $\inf_{n \in \mathbb{N}_+} \frac{\|na\|}{n}$ by Fekete's lemma. Since each term in the sequence is non-negative, the infimum is finite. From this, all the properties of a norm follow but positive definiteness:

$${}^H\|a\| = \inf_{n \in \mathbb{N}_+} \frac{\|na\|}{n} = 0 \not\Rightarrow a = 0.$$

We also have that

$${}^H\|ma\| = \inf_{n \in \mathbb{N}_+} \frac{\|n(ma)\|}{n} = m \inf_{n \in \mathbb{N}_+} \frac{\|(nm)a\|}{nm} = m {}^H\|a\|.$$

Hence, ${}^H\|\cdot\|$ is a homogeneous seminorm. \square

Lemma 3.12. The seminorm $\|\cdot\|$ is homogeneous if and only if ${}^H\|\cdot\| = \|\cdot\|$. Furthermore, ${}^H\|a\| \leq \|a\|$ for all $a \in A$.

Proof. If $\|\cdot\|$ is homogeneous, then it follows trivially that ${}^H\|\cdot\| = \|\cdot\|$. Conversely, suppose that $\|\cdot\|$ is not homogeneous. Then, $\|ka\| < k\|a\|$ for some $k \in \mathbb{N}_+$ and $a \in A$ and

$${}^H\|a\| = \inf_{n \in \mathbb{N}_+} \frac{\|na\|}{n} \leq \frac{\|ka\|}{k} < \|a\|$$

□

From this, it is straightforward to see that one cannot get a norm by homogenizing a proper seminorm.

Corollary 3.13. *Homogenization of a proper seminorm (non positive-definite) is a proper seminorm.*

Lemma 3.14. *Let A be a normed abelian group and let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on A such that there exists an $M \in \mathbb{R}_+$ satisfying $\|a\|_1 \leq M\|a\|_2$ for all $a \in A$. Then, ${}^H\|a\|_1 \leq M{}^H\|a\|_2$ for all $a \in A$.*

Proof. We have

$${}^H\|a\|_1 = \inf_{n \in \mathbb{N}_+} \frac{\|na\|_1}{n} \leq \inf_{m \in \mathbb{N}_+} \frac{M\|na\|_2}{n} = M{}^H\|a\|_2.$$

□

We now move onto how the ℓ_1 -norm and the filling norms are affected by the norm on the coefficient group. We will be working with normed abelian groups as coefficients. In this case, it is a straightforward calculation to see that ℓ_1 is a norm on $C_k^{\text{cell}}(Y; A)$ for any $k \in \mathbb{N}$, we focus on the filling seminorms and the homogenizations of the ℓ_1 -norm and the filling seminorms. In what follows, A denotes a normed abelian group equipped with a norm $\|\cdot\|_A: A \rightarrow \mathbb{R}_{\geq 0}$.

Lemma 3.15. *Let A be a normed abelian group and let $\|\cdot\|_A$ denote the norm on A . We have:*

- (1) *The norm $\|\cdot\|_1$ is homogeneous if and only if $\|\cdot\|_A$ is homogeneous.*
- (2) *The seminorm ${}^H\|\cdot\|_1$ on $C_{\bullet}^{\text{cell}}(Y; A)$ is a norm if and only if ${}^H\|\cdot\|_A$ is a norm on A .*

Proof. (1) : Suppose $\|\cdot\|_A$ is homogeneous and let $m \in \mathbb{N}_+$, then

$$\|mc\|_1 = \sum_{i \in \mathcal{I}} \|ma_i\|_A = \sum_{i \in \mathcal{I}} m\|a_i\|_A = m\|c\|_1.$$

Conversely, suppose $\|\cdot\|_1$ is homogeneous. For any nonzero $a \in A$, there exists a chain $c = ae$ supported on a single basis element e . We get:

$$\|mc\|_1 = \|ma\|_A = m\|a\|_A = m\|c\|_1.$$

(2) : Let $k \in \mathbb{N}$ and let $c \in C_k^{\text{cell}}(Y; A)$. Then $\|c\|_1 = \sum_i \|a_i\|_A$ and

$${}^H\|c\|_1 = \inf_{n \in \mathbb{N}_+} \frac{\|nc\|_1}{n} = \inf_{n \in \mathbb{N}_+} \frac{\sum_{i \in \mathcal{I}} \|na_i\|_A}{n} = \sum_{i \in \mathcal{I}} \inf_{n \in \mathbb{N}_+} \frac{\|na_i\|_A}{n} = \sum_{i \in \mathcal{I}} {}^H\|a_i\|_A.$$

Now, suppose ${}^H\|\cdot\|_1$ is a norm. If there existed a nonzero $a \in A$ with ${}^H\|a\|_A = 0$, the chain $c = ae$ supported on a single basis element e , would have had ${}^H\|c\|_1 = {}^H\|a\|_A = 0$ while $c \neq 0$, contradicting the positive definiteness of ${}^H\|\cdot\|_1$.

Conversely, suppose ${}^H\|\cdot\|_A$ is a norm. If c is nonzero, then $a_j \neq 0$ for some j and ${}^H\|c\|_1 \geq {}^H\|a_j\|_A > 0$. Thus, ${}^H\|\cdot\|_1$ is positive definite and hence, a norm. □

The situation is more complex regarding the filling seminorms. We will see that in general one cannot guarantee a filling norm without any further assumptions on the CW-complex Y or even say much about how it compares against the ℓ_1 -norm. Therefore, we impose the following constraint:

Definition 3.16. A CW-complex Y is said to have bounded geometry in dimension $k \in \mathbb{N}$ if there exists an $M \in \mathbb{R}_+$ such that $\|\partial e^{(k)}\|_1 \leq M$ for all k -cells $e^{(k)}$ of Y . Equivalently, the boundary map $\partial : C_k^{\text{cell}}(Y; A) \rightarrow C_{k-1}^{\text{cell}}(Y; A)$ is bounded with respect to the ℓ_1 -norm: for any $c \in C_k^{\text{cell}}(Y; A)$, $\|\partial c\|_1 \leq M\|c\|_1$.

We in fact have the following equivalences:

Proposition 3.17. *Let Y be a CW-complex, A a normed abelian group and $k \in \mathbb{N}$. The following statements are equivalent:*

- (1) Y has bounded geometry in dimension $k + 1$.
- (2) The boundary map $\partial : C_{k+1}^{\text{cell}}(Y; A) \rightarrow C_k^{\text{cell}}(Y; A)$ is bounded with respect to the ℓ_1 -norm.
- (3) There exists an $M \in \mathbb{R}_+$ such that $\|b\|_1 \leq M\|b\|_F$ for all $b \in B_k^{\text{cell}}(Y; A)$.

Proof. The equivalence (1) and (2) follows from the definition of bounded geometry. We show the rest:

(2) \implies (3) : Let $c \in C_{k+1}^{\text{cell}}(Y; A)$ be such that $\partial c = b$ and let $M \in \mathbb{R}_+$ be as in the definition of bounded geometry. Then

$$\|b\|_1 = \|\partial c\|_1 \leq M\|c\|_1.$$

Taking the infimum over all such $(k+1)$ -chains, we get

$$\|b\|_1 \leq M\|b\|_F.$$

(3) \implies (2) : Let $c \in C_{k+1}^{\text{cell}}(Y; A)$. Then

$$\|\partial c\|_1 \leq M\|\partial c\|_F \leq M\|c\|_1.$$

□

Corollary 3.18. *If Y has bounded geometry in dimension $k + 1$, then the filling seminorm $\|\cdot\|_F$ on $B_k^{\text{cell}}(Y; A)$ is a norm.*

The converse doesn't hold in general: one can construct CW-complexes with with positive definite filling seminorm and unbounded geometry. We give an example for $k = 1$ but it applies for each $k \in \mathbb{N}$:

Example 3.19. For each positive integer $m \in \mathbb{Z}_+$ let S_m be a circle subdivided into m 1-cells,

$$S_m = e_1^{(1)} \cup \dots \cup e_m^{(1)}$$

Let Y be a CW-complex with 1-skeleton as the disjoint union of circles S_m , $\bigsqcup_{m \in \mathbb{Z}_+} S_m$ and a 2-cell $e_m^{(2)}$ attached to each circle S_m . Let $b_n = e_1$. Consider cellular chains on Y with \mathbb{Z} -coefficients, $C_{\bullet}^{\text{cell}}(Y; \mathbb{Z})$. Then $\|\partial e_m^{(2)}\|_1 = m$ for each $m \in \mathbb{Z}_+$ and Y has unbounded geometry in dimension 2. On the other hand, since the boundary map when restricted to 2-cells is injective, it is injective over the 2-chains. Therefore, the filling seminorm on $B_1^{\text{cell}}(Y; \mathbb{Z})$ is a norm since it is the quotient norm induced by the boundary map.

Lemma 3.20. *Let Y be a CW-complex with bounded geometry in dimension $k + 1$ and A a normed abelian group. Then*

- (1) *The filling norm $\|\cdot\|_F$ on $B_k^{\text{cell}}(Y; A)$ is homogeneous if $\|\cdot\|_A$ on A is homogeneous and A is divisible (i.e. for every $m \in \mathbb{N}_+$ the map $A \rightarrow A, a \mapsto ma$ is surjective).*

(2) *The homogenization of the filling norm ${}^H\|\cdot\|_F$ on $B_k^{\text{cell}}(Y; A)$ is a norm if the homogenization of the norm on A , ${}^H\|\cdot\|_A$ is a norm on A .*

Proof. (1) : Suppose $\|\cdot\|_A$ is homogeneous and A is divisible. Then $\|\cdot\|_1$ is homogeneous and for any $m \in \mathbb{N}_+$, $c' \in C_{k+1}^{\text{cell}}(Y; A)$ with $\partial c' = mb$, there exists $c \in C_{k+1}^{\text{cell}}(Y; A)$ such that $c' = mc$. Therefore

$$\begin{aligned} \|mb\|_F &= \inf\{\|c'\|_1 : c' \in C_{k+1}^{\text{cell}}(Y; A), \partial c' = mb\} \\ &= \inf\{\|mc\|_1 : c \in C_{k+1}^{\text{cell}}(Y; A), \partial c = b\} \\ &= m \inf\{\|c\|_1 : c \in C_{k+1}^{\text{cell}}(Y; A), \partial c = b\} \\ &= m\|b\|_F. \end{aligned}$$

(2) : Let $b \in B_k^{\text{cell}}(\widetilde{X}; A)$. We have

$$\begin{aligned} {}^H\|b\|_F &= \inf_{n \in \mathbb{N}_+} \frac{\|nb\|_F}{n} = \inf_{n \in \mathbb{N}_+} \frac{\inf\{\|nc\|_1 : c \in C_{k+1}^{\text{cell}}(\widetilde{X}; A), \partial c = b\}}{n} \\ &= \inf \left\{ \inf_{n \in \mathbb{N}_+} \frac{\|nc\|_1}{n} : c \in C_{k+1}^{\text{cell}}(\widetilde{X}; A), \partial c = b \right\} \\ &= \inf\{{}^H\|c\|_1 : c \in C_{k+1}^{\text{cell}}(\widetilde{X}; A), \partial c = b\} \end{aligned}$$

Since Y has bounded geometry in dimension k , $\|\cdot\|_F$ dominates $\|\cdot\|_1$ and it follows from Lemma 3.14 that ${}^H\|\cdot\|_F$ dominates ${}^H\|\cdot\|_1$. Then, if ${}^H\|\cdot\|_A$ is a norm, ${}^H\|\cdot\|_1$ is a norm by Lemma 3.15 and hence ${}^H\|\cdot\|_F$ is a norm as well.

For the converse direction, suppose ${}^H\|\cdot\|_A$ is not a norm on A . Then, there exists a nonzero $a \in A$ with ${}^H\|a\|_A = 0$. Consider the 2-chain $c = ae$ where e is not a 2-cycle, so that $\partial c \neq 0$. Note that such a choice is possible because otherwise we would have had $\partial_2 = 0$ and $H_2^{\text{cell}}(\widetilde{X}; A) = C_2^{\text{cell}}(\widetilde{X}; A) \neq 0$ but this contradicts contractibility of \widetilde{X} . Then, we have

$${}^H\|\partial c\|_F \leq {}^H\|c\|_1 = {}^H\|a\|_A = 0.$$

Therefore, ${}^H\|\cdot\|_F$ is also not a norm. This completes the proof. \square

3.3.2. \mathbb{Z} , \mathbb{Q} and \mathbb{R} -coefficients

We now restrict our attention to the cases of $A = \mathbb{Z}$, $A = \mathbb{Q}$ or $A = \mathbb{R}$: we equip each with the relevant absolute value norm which is a homogeneous norm. Therefore, each ℓ_1 -norm and the filling norms corresponding to \mathbb{Q} or \mathbb{R} -coefficients are homogeneous norms. Let Y be a CW-complex without any further constraints and fix a $k \in \mathbb{N}$. We start with the following useful lemma:

Lemma 3.21. *Let A_1, A_2 be normed abelian groups with $A_1 \subseteq A_2$ and let $i : C_{\bullet}^{\text{cell}}(Y; A_1) \hookrightarrow C_{\bullet}^{\text{cell}}(Y; A_2)$ be the canonical inclusion map. Then, for any $b \in B_1^{\text{cell}}(\widetilde{X}, A_1)$, we have*

$$\|i(b)\|_F^{A_2} \leq \|b\|_F^{A_1}.$$

Proof. We have that $\|i(c)\|_1 = \|c\|_1$ for any $c \in C_*^{\text{cell}}(\widetilde{X}; A_1)$ and

$$\{i(c) \in C_2^{\text{cell}}(\widetilde{X}; A_2) : c \in C_2^{\text{cell}}(\widetilde{X}; A_1), \partial c = b\} \subseteq \{d \in C_2^{\text{cell}}(\widetilde{X}; A_2) : \partial d = i(b)\}.$$

Therefore,

$$\begin{aligned}
\|i(b)\|_F^{A_2} &= \inf\{\|d\|_1 : d \in C_2^{\text{cell}}(\widetilde{X}; A_2), \partial d = i(b)\} \\
&\leq \inf\{\|i(c)\|_1 : c \in C_2^{\text{cell}}(\widetilde{X}; A_1), \partial c = b\} \\
&= \inf\{\|c\|_1 : c \in C_2^{\text{cell}}(\widetilde{X}; A_1), \partial c = b\} \\
&= \|b\|_F^{A_1}.
\end{aligned}$$

□

We will omit the inclusion and use the same letter for its inclusion as well.

Proposition 3.22. *Restriction of $\|\cdot\|_F^{\mathbb{Q}}$ to $B_k^{\text{cell}}(Y; \mathbb{Z}) \subset B_k^{\text{cell}}(Y; \mathbb{Q})$ coincides with ${}^H\|\cdot\|_F^{\mathbb{Z}}$.*

Proof. Let $b \in B_k^{\text{cell}}(Y; \mathbb{Z})$. We have that $\|b\|_F^{\mathbb{Q}} \leq \|b\|_F^{\mathbb{Z}}$ by Lemma 3.21 and ${}^H\|b\|_F^{\mathbb{Q}} \leq {}^H\|b\|_F^{\mathbb{Z}}$ by Lemma 3.14. Since \mathbb{Q} is divisible, $\|\cdot\|_F^{\mathbb{Q}}$ is homogeneous and therefore $\|b\|_F^{\mathbb{Q}} = {}^H\|b\|_F^{\mathbb{Q}} \leq {}^H\|b\|_F^{\mathbb{Z}}$.

For the opposite inequality, let $b \in B_k^{\text{cell}}(Y; \mathbb{Q})$. We have that

$$\|b\|_F^{\mathbb{Q}} = \inf\{\|c\|_1 : c \in C_{k+1}^{\text{cell}}(Y; \mathbb{Q}), \partial c = b\}$$

Then, for every $\varepsilon \in \mathbb{R}_+$ there exists a $c \in C_{k+1}^{\text{cell}}(Y; \mathbb{Q})$ with $\partial c = b$ such that $\|c\|_1 \leq \|b\|_F^{\mathbb{Q}} + \varepsilon$. Let $N \in \mathbb{Z}_+$ such that $kNc \in C_{k+1}^{\text{cell}}(Y; \mathbb{Z})$ for all $k \in \mathbb{N}_+$. This can be achieved by taking N as the least common ancestor of the denominators of the coefficients appearing in c . Then we have that $\partial(kNc) = kNb$ and $\|kNb\|_F^{\mathbb{Z}} \leq \|kNc\|_1$. Then by homogeneity of $\|\cdot\|_1$, we get

$$\frac{\|kNb\|_F^{\mathbb{Z}}}{kN} \leq \|c\|_1 \leq \|b\|_F^{\mathbb{Q}} + \varepsilon.$$

Taking the limit $k \rightarrow \infty$, it follows that ${}^H\|b\|_F^{\mathbb{Z}} \leq \|b\|_F^{\mathbb{Q}} + \varepsilon$. For any other choice of ε , the 2-chain c changes but the final inequality holds. Therefore, ${}^H\|b\|_F^{\mathbb{Z}} \leq \|b\|_F^{\mathbb{Q}}$. □

Proposition 3.23. *Restriction of $\|\cdot\|_F^{\mathbb{R}}$ to $B_k^{\text{cell}}(Y; \mathbb{Q}) \subset B_k^{\text{cell}}(Y; \mathbb{R})$ coincides with $\|\cdot\|_F^{\mathbb{Q}}$.*

Proof. Let $b \in B_k^{\text{cell}}(Y; \mathbb{Q})$. We have that $\|b\|_F^{\mathbb{R}} \leq \|b\|_F^{\mathbb{Q}}$ by Lemma 3.21.

For the other direction, let $b \in B_k^{\text{cell}}(Y; \mathbb{Q})$ and let $\varepsilon \in \mathbb{R}_+$. There exists a real 2-chain $c \in C_{k+1}^{\text{cell}}(Y; \mathbb{R})$ such that $\partial c = b$ and $\|c\|_1 \leq \|b\|_F^{\mathbb{R}} + \varepsilon$. Also b being a rational 1-boundary means that there exists a rational 2-chain $c_0 \in C_{k+1}^{\text{cell}}(Y; \mathbb{Q})$ with $\partial c_0 = b$. Since chains are defined as finite linear combinations of the basis cells, let \mathcal{S} denote the finite set of 2-cells appearing in c and c_0 . Also, let $\partial\mathcal{S}$ denote the finite set of 1-cells that occur as the boundaries of elements of \mathcal{S} . Let $V := \text{span}_{\mathbb{R}}\{e^{(2)} : e^{(2)} \in \mathcal{S}\} \subset C_{k+1}^{\text{cell}}(Y; \mathbb{R})$ be the finite dimensional real subspace spanned by the finitely many basis 2-cells in \mathcal{S} and let $W := \text{span}_{\mathbb{R}}\{e^{(1)} : e^{(1)} \in \partial\mathcal{S}\}$. Then the boundary operator $\partial: C_{k+1}^{\text{cell}}(Y; \mathbb{R}) \rightarrow C_k^{\text{cell}}(Y; \mathbb{R})$ restricts to a linear map

$$\partial|_V: V \rightarrow W.$$

Note that $c, c_0 \in V$ and $\partial|_V(c) = \partial|_V(c_0) = b$. We now use the following fact from linear algebra:

Lemma 3.24. *Let $M \in \mathbb{Z}^{m \times k}$ be an integer matrix and view it as a linear map*

$$M: \mathbb{R}^k \rightarrow \mathbb{R}^m.$$

Then the solution space of the homogeneous system $Mx = 0$, $\ker(M: \mathbb{R}^k \rightarrow \mathbb{R}^m)$ is a finite dimensional real vector space admitting a \mathbb{Q} -basis. Equivalently,

$$\ker(M: \mathbb{R}^k \rightarrow \mathbb{R}^m) = \ker(M: \mathbb{Q}^k \rightarrow \mathbb{Q}^m) \otimes_{\mathbb{Q}} \mathbb{R}.$$

Proof. Let $V_{\mathbb{Q}} := \ker(M: \mathbb{Q}^k \rightarrow \mathbb{Q}^m) \subset \mathbb{Q}^k$. Consider its scalar extension $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathbb{R}^k$. One always has

$$V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathbb{R}^k \subseteq \ker(M: \mathbb{R}^k \rightarrow \mathbb{R}^m).$$

Furthermore, since $M \in \mathbb{Z}^{m \times k}$, $\text{rank}_{\mathbb{Q}} M = \text{rank}_{\mathbb{R}} M$ and therefore, $\dim_{\mathbb{Q}} V_{\mathbb{Q}} = k - \text{rank}_{\mathbb{Q}} M = k - \text{rank}_{\mathbb{R}} M = \dim_{\mathbb{R}} \ker(M: \mathbb{R}^k \rightarrow \mathbb{R}^m)$. Hence,

$$V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} = \ker(M: \mathbb{R}^k \rightarrow \mathbb{R}^m).$$

So, the real kernel is the scalar extension of the rational kernel and in particular, admits a rational basis. \square

This means that $\ker(\partial|_V)$ admits a rational basis:

$$v_1, \dots, v_k \in \ker(\partial|_V) \cap C_2^{\text{cell}}(\widetilde{X}; \mathbb{Q})$$

where $k := \dim_{\mathbb{Q}} \ker(\partial|_V)$. Let $d := c - c_0$. Then, $d \in \ker(\partial|_V)$, so there exists $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ with

$$d = \sum_{i=1}^k \alpha_i v_i.$$

Let $K = \sum_{i=1}^k \|v_i\|_1$. Since the map $\mathbb{R}^k \rightarrow \mathbb{R}_{\geq 0}$, $(\alpha_1, \dots, \alpha_k) \mapsto \sum_{i=1}^k |\alpha_i|$ is continuous, there exists $q_1, \dots, q_k \in \mathbb{Q}$ such that

$$\sum_{i=1}^k |\alpha_i - q_i| < \frac{\varepsilon}{K}$$

and therefore

$$\|d - \sum_{i=1}^k q_i v_i\|_1 < \varepsilon.$$

Let $c' = c_0 + \sum_{i=1}^k q_i v_i$. Then $c' \in C_{k+1}^{\text{cell}}(Y; \mathbb{Q})$ since $c_0, v_1, \dots, v_k \in C_{k+1}^{\text{cell}}(Y; \mathbb{Q})$ and $q_1, \dots, q_k \in \mathbb{Q}$. Also,

$$\partial c' = \partial c_0 + \sum_{i=1}^k q_i \partial v_i = b$$

and

$$\|c - c'\|_1 = \|d - \sum_{i=1}^k q_i v_i\|_1 < \varepsilon.$$

Then, it follows that

$$\|b\|_F^{\mathbb{Q}} \leq \|c'\|_1 \leq \|c' - c\|_1 + \|c\|_1 \leq \|b\|_F^{\mathbb{R}} + 2\varepsilon.$$

Since the choice of ε was arbitrary, this holds for all $\varepsilon \in \mathbb{R}_+$ and hence $\|b\|_F^{\mathbb{Q}} \leq \|b\|_F^{\mathbb{R}}$. \square

Combining the Proposition 3.22 and Proposition 3.23, we get:

Corollary 3.25. *Restriction of $\|\cdot\|_F^{\mathbb{R}}$ to $B_k^{\text{cell}}(Y; \mathbb{Z}) \subset C_k^{\text{cell}}(Y; \mathbb{R})$ coincides with ${}^H\|\cdot\|_F^{\mathbb{Z}}$.*

We now state the main result of this section. So far, the statements hold for an arbitrary CW-complex but as we have seen, in order to be able to compare the filling seminorm with the ℓ_1 -norm, one needs to constrain the geometry.

Proposition 3.26. *Let $k \in \mathbb{N}$ and let Y be a CW-complex with bounded geometry in dimension $k + 1$. The following conditions are equivalent:*

- (1) *The norms ${}^H\|\cdot\|_F^{\mathbb{Z}}$ and $\|\cdot\|_1$ are equivalent on $B_k^{\text{cell}}(Y; \mathbb{Z})$.*
- (2) *The norms $\|\cdot\|_F^{\mathbb{Q}}$ and $\|\cdot\|_1$ are equivalent on $B_k^{\text{cell}}(Y; \mathbb{Q})$.*
- (3) *The norms $\|\cdot\|_F^{\mathbb{R}}$ and $\|\cdot\|_1$ are equivalent on $B_k^{\text{cell}}(Y; \mathbb{R})$.*

Proof. Since Y has bounded geometry in dimension $k + 1$, by Proposition 3.17, for any normed abelian group A , there exists an $M_A \in \mathbb{R}_+$ such that $\|b\|_1 \leq M_A \|b\|_F^A$ for all $b \in B_k^{\text{cell}}(Y; A)$. So for each statement, it is enough to show that the ℓ_1 -norm dominates the filling norm.

(1) \implies (2) : There exists a $K \in \mathbb{R}_+$ such that ${}^H \|b\|_F^{\mathbb{Z}} \leq K \|b\|_1$ for all $b \in B_k^{\text{cell}}(Y; \mathbb{Z})$, by assumption. Let $b \in B_k^{\text{cell}}(Y; \mathbb{Q})$ and let $N \in \mathbb{Z}_+$ so that $Nb \in B_k^{\text{cell}}(Y; \mathbb{Z})$. Then, $\|Nb\|_F^{\mathbb{Q}} = {}^H \|Nb\|_F^{\mathbb{Z}}$ by Proposition 3.22 and therefore by homogeneity of $\|\cdot\|_F^{\mathbb{Q}}$ and $\|\cdot\|_1$,

$$\|b\|_F^{\mathbb{Q}} = \frac{1}{N} \|Nb\|_F^{\mathbb{Q}} = \frac{1}{N} {}^H \|Nb\|_F^{\mathbb{Z}} \leq \frac{K}{N} \|Nb\|_1 \leq K \|b\|_1$$

(2) \implies (1) : If $\|\cdot\|_F^{\mathbb{Q}}$ and $\|\cdot\|_1$ are equivalent on $B_k^{\text{cell}}(Y; \mathbb{Q})$, then so are their restrictions to $B_k^{\text{cell}}(Y; \mathbb{Z})$. By Proposition 3.22, the restriction of $\|\cdot\|_F^{\mathbb{Q}}$ to $B_k^{\text{cell}}(Y; \mathbb{Z})$ is ${}^H \|\cdot\|_F^{\mathbb{Z}}$ and the result follows.

(2) \implies (3) : There exists a $K \in \mathbb{R}_+$ such that $\|b\|_F^{\mathbb{Q}} \leq K \|b\|_1$ for all $b \in B_k^{\text{cell}}(Y; \mathbb{Q})$, by assumption. Let $b \in B_k^{\text{cell}}(Y; \mathbb{R})$ and let $\varepsilon \in \mathbb{R}_+$. Let $c \in C_{k+1}^{\text{cell}}(Y; \mathbb{R})$ be such that $\partial c = b$ and $\|c\|_1 \leq \|b\|_F^{\mathbb{R}} + \varepsilon$. Let $c' \in C_{k+1}^{\text{cell}}(Y; \mathbb{Q})$ be such that $\|c - c'\|_1 \leq \varepsilon$. Then $b' := \partial c' \in B_k^{\text{cell}}(Y; \mathbb{Q})$ and $\|b - b'\|_F^{\mathbb{R}} \leq \|c - c'\|_1 \leq \varepsilon$. Therefore

$$\|b\|_F^{\mathbb{R}} \leq \|b - b'\|_F^{\mathbb{R}} + \|b'\|_F^{\mathbb{R}} \leq \varepsilon + \|b'\|_F^{\mathbb{Q}} \leq \varepsilon + K \|b'\|_1$$

and

$$\|b'\|_1 \leq \|b' - b\|_1 + \|b\|_1 \leq M_{\mathbb{R}} \|b' - b\|_F^{\mathbb{R}} + \|b\|_1 \leq M_{\mathbb{R}} \varepsilon + \|b\|_1$$

where $M_{\mathbb{R}}$ is as defined at the beginning of the proof. Substituting the second inequality in the first one, we get $\|b\|_F^{\mathbb{R}} \leq \varepsilon + M_{\mathbb{R}} K \varepsilon + \|b\|_1$. Since this holds for any $\varepsilon \in \mathbb{R}_+$, we have that $\|b\|_F^{\mathbb{R}} \leq K \|b\|_1$.

(3) \implies (2) : If $\|\cdot\|_F^{\mathbb{R}}$ and $\|\cdot\|_1$ are equivalent on $B_k^{\text{cell}}(Y; \mathbb{R})$, then so are their restrictions to $B_k^{\text{cell}}(Y; \mathbb{Q})$. By Proposition 3.23, the restriction of $\|\cdot\|_F^{\mathbb{R}}$ to $B_k^{\text{cell}}(Y; \mathbb{Q})$ is $\|\cdot\|_F^{\mathbb{Q}}$ and the result follows. \square

Definition 3.27. Let Y be a CW-complex with $H_k^{\text{cell}}(Y; \mathbb{Z}) = 0$ and A a normed abelian group. We define the k th homological Dehn function of Y , $\text{FA}_{Y;A}^k: \mathbb{N} \rightarrow A$ as

$$\text{FA}_{Y;A}^k(n) = \sup\{\|b\|_F^A : b \in B_k^{\text{cell}}(Y; A), \|b\|_1 \leq n\}$$

One can then formulate the equivalences of the norms $\|\cdot\|_F^A$ and $\|\cdot\|_1$ on $B_k^{\text{cell}}(Y; A)$ for a CW-complex Y with bounded geometry in dimension $k + 1$ in terms of the linearity of the k th homological Dehn function.

3.4. Vanishing of $H_{(\infty)}^2(\tilde{X}; V)$

One usually states the Hahn-Banach theorem in terms of subspaces:

Theorem 3.28 (Hahn-Banach Theorem, normed spaces). *Let V be normed real space and $W \subseteq V$ be a subspace of it. Then, every bounded linear functional $f \in W^*$ on W can be extended to a bounded linear functional $F \in V^*$ on V (i.e. $F|_W = f$) with $\|F\| \leq \|f\|$.*

One can formulate it as a two sided statement via the following definition:

Definition 3.29. A linear map $\phi: W \rightarrow V$ between real normed spaces is said to be an isomorphic embedding if its an isomorphism of real normed spaces onto its image. Equivalently, there exists a $C \in \mathbb{R}_+$ such that

$$\frac{1}{C} \|v\|_W \leq \|\phi(v)\|_V \leq C \|v\|_W.$$

for all $v \in W$.

Theorem 3.30 (Hahn-Banach Theorem, isomorphic embeddings). *Let $\phi: W \rightarrow V$ be a bounded linear map between real normed spaces. Then, ϕ is an isomorphic embedding if and only if there exists a $K \in \mathbb{R}_+$ such that for every bounded linear functional $f \in W^*$, there exists a bounded linear functional $F \in V^*$ with $f = F \circ \phi$ and $\|F\| \leq K\|f\|$. Furthermore, $\|\phi^{-1}\| \leq K$.*

Proof. The forward direction follows from the subspace version of the Hahn-Banach theorem: namely, assume that ϕ is an isomorphic embedding. We identify W with the subspace $\phi(W) \subset V$ and for each bounded linear functional $f \in W^*$ define $f_\phi \in \phi(W)^*$ as $f_\phi = f \circ \phi^{-1}$. This gives $\|f_\phi\| \leq \|f\|\|\phi^{-1}\|$. Then, f_ϕ extends to a bounded linear functional $F \in V^*$ with $\|F\| \leq \|f_\phi\|$ by Theorem 3.28, and hence

$$\|F\| \leq \|\phi^{-1}\|\|f\|.$$

Furthermore, since F agrees with f_ϕ over $\phi(W)$, we get

$$F \circ \phi = f_\phi \circ \phi = f.$$

For the converse direction, we first show the dual characterization of the norm (which is also equivalent to the real normed space version of the Hahn-Banach theorem):

Lemma 3.31 (Dual characterization of the norm). *Let V be a real normed space. Then, for any $v \in V$*

$$\|v\| = \sup\{\|f(v)\| : f \in V^*, \|f\| \leq 1\}.$$

Proof. " \geq " direction is straightforward: for any $f \in V^*$ with $\|f\| \leq 1$, we have that

$$\|f(v)\| \leq \|f\|\|v\| \leq \|v\|$$

For the " \leq " direction, let $v \in V$ with $v \neq 0$ and define $f_0: \text{span}\{v\} \rightarrow \mathbb{R}$ by $\alpha v \mapsto \alpha\|v\|$ for all $\alpha \in \mathbb{R}$. Then

$$\|f_0\| = \sup\{\|f_0(u)\| : u \in \text{span}\{v\}, \|u\| \leq 1\} = \sup_{\|\alpha\|\|v\| \leq 1} \|f_0(\alpha v)\| = 1$$

and by Theorem 3.28, $f_0 \in \text{span}\{v\}^*$ extends to an $f \in V^*$ with $\|f\| = 1$. Hence, $f(v) = \|v\|$ and the result follows. \square

We are now ready to prove the converse direction. We first show the injectivity of ϕ : let $v \in W$ with $v \neq 0$. Dual characterization of the norm implies that there exists an $f \in W^*$ with $f(v) \neq 0$ and therefore, for any $F \in V^*$ with $f = F \circ \phi$, we get

$$f(v) = F(\phi(v)) \neq 0.$$

By linearity of F , $\phi(v) \neq 0$ and it follows that ϕ is injective.

To show that ϕ preserves the norm, let $v \in W$ with $v \neq 0$ and $f \in W^*$ with $f(v) = \|v\|$ and $\|f\| = 1$. By assumption, there exists $F \in V^*$ with $f = F \circ \phi$ and $\|F\| \leq K$. Then

$$\|v\| = f(v) = F(\phi(v)) \leq \|F\|\|\phi(v)\| \leq K\|\phi(v)\|$$

and

$$\frac{1}{K}\|v\| \leq \|\phi(v)\| \leq \|\phi\|\|v\|.$$

Hence, it follows that $\phi: W \rightarrow V$ is an isomorphic embedding with $\|\phi^{-1}\| \leq K$. \square

Now, as a generalization of this extension property, one usually considers the injective objects of the category of Banach spaces with bounded linear maps, Ban:

Definition 3.32. A Banach space V is said to be 1-injective if for every bounded linear map $f: U \rightarrow V$, defined on a closed subspace of a Banach space W , there exists a bounded linear map $F: W \rightarrow V$ extending f such that $\|F\| \leq \|f\|$.

This also can be expressed naturally in terms of isomorphic embeddings:

Corollary 3.33. A Banach space V is 1-injective if and only if for every Banach space W , every isomorphic embedding $\phi: U \rightarrow W$ of Banach spaces and every bounded linear map $f: U \rightarrow V$, there exists a bounded linear map $F: W \rightarrow V$ such that $f = F \circ \phi$ and $\|F\| \leq \|\phi^{-1}\| \|f\|$.

Typical examples of 1-injective spaces are \mathbb{R} with the usual Euclidean norm or normed spaces $\ell^\infty(S, \mathbb{R})$ of bounded functions from a set S to \mathbb{R} , endowed with the supremum norm.

Recall that in what follows, we endow $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ with the filling norm $\|\cdot\|_F^{\mathbb{R}}$ and $C_k^{\text{cell}}(\widetilde{X}; \mathbb{R})$ with the ℓ_1 -norm for each $k \in \mathbb{N}$. Before we move on to the vanishing theorem, we first show the following observations:

Lemma 3.34. Let V be a real normed space. Then $Z_{(\infty)}^2(\widetilde{X}; V) \cong \mathcal{L}(B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}), V)$.

Proof. We first show that each bounded linear map $g: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ defines a 2-cocycle $f_g \in H_{(\infty)}^2(G; V)$. Define $f_g: C_2^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ by $f_g(c) := g(\partial c)$. Then

$$\|f_g(c)\|_V = \|g(\partial c)\|_V \leq \|g\| \|\partial c\|_F \leq \|g\| \|c\|_1$$

and f_g is a bounded linear map with $\|f_g\| \leq \|g\|$. Furthermore

$$(\delta f_g)(d) = f_g(\partial d) = g(\partial^2 d) = 0$$

for any $d \in C_4^{\text{cell}}(\widetilde{X}; \mathbb{R})$. So, $\delta f_g = 0$ and $f_g \in Z_{(\infty)}^2(\widetilde{X}; V)$.

Conversely, let $f \in Z_{(\infty)}^2(\widetilde{X}; V)$ be a bounded 2-cocycle. Then, f vanishes over 2-boundaries and since $Z_2^{\text{cell}}(\widetilde{X}; \mathbb{R}) = B_2^{\text{cell}}(\widetilde{X}; \mathbb{R})$ by Remark 2.2, f vanishes over 2-cycles as well. We can then define a bounded linear map $g_f: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ by $g_f(b) = f(c)$ for any $c \in C_2^{\text{cell}}(\widetilde{X}; \mathbb{R})$ with $\partial c = b$. This is well-defined as for any $c, c' \in C_2^{\text{cell}}(\widetilde{X}; \mathbb{R})$ with $\partial c = \partial c' = b$, we have that $\partial(c - c') = 0$ and $c - c' \in Z_2^{\text{cell}}(\widetilde{X}; \mathbb{R})$, so $f(c) = f(c')$. Linearity of g_f follows from the linearity of f and for its boundedness:

$$\|g_f(b)\|_V = \|f(c)\|_V \leq \|f\| \|c\|_1.$$

Taking the infimum over all such 2-chains c yields $\|g_f(b)\|_V \leq \|f\| \|b\|_F^{\mathbb{R}}$ and $\|g_f\| \leq \|f\|$.

We then have that

$$g_{f_g}(\partial c) = f_g(c) = g(\partial c)$$

and

$$f_{g_f}(c) = g_f(\partial c) = f(c)$$

for any $f \in Z_{(\infty)}^2(\widetilde{X}; V)$, $g \in \mathcal{L}(B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}), V)$ and $c \in C_2^{\text{cell}}(\widetilde{X}; \mathbb{R})$. \square

Now that we have this duality, we will see how this converts the problem of solving $w = \delta\alpha$ with bounded α to one of extending g_w to a V -valued bounded linear map on all of $C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$.

Proposition 3.35. Let V be a real normed space. $H_{(\infty)}^2(G; V) = 0$ if and only if every bounded linear map $g: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ extends to a bounded linear map $F: C_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow \mathbb{R}$. Furthermore, if V is Banach, then there exists a $K \in \mathbb{R}_+$ depending only on \widetilde{X} such that $\|F\| \leq K \|g\|$.

Proof. Suppose $H_{(\infty)}^2(G; V) = 0$. Let $g: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ be a bounded linear map and let $f_g \in Z_{(\infty)}^2(\widetilde{X}; V)$ be the corresponding bounded 2-cocycle as described in Lemma 3.34. There exists a $F \in C_{(\infty)}^1(\widetilde{X}; V)$ such that $\delta F = f_g$ by assumption. Then

$$F(\partial c) = (\delta F)(c) = f_g(c) = g(\partial c)$$

for any $c \in C_2^{\text{cell}}(\widetilde{X}; \mathbb{R})$ and F agrees with g over $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$. This gives us a bounded linear extension of g to $C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$.

As per the uniform norm control in the case of V being Banach: we have that $C_{(\infty)}^1(\widetilde{X}; V)$, $C_{(\infty)}^2(\widetilde{X}; V)$ are Banach and the coboundary map $\delta: C_{(\infty)}^1(\widetilde{X}; V) \rightarrow C_{(\infty)}^2(\widetilde{X}; V)$ is bounded because X has a finite 2-skeleton. Moreover, being the kernel of a bounded operator $Z_{(\infty)}^2(\widetilde{X}; V) = \delta(C_{(\infty)}^1(\widetilde{X}; V))$ is a closed subspace of $C_{(\infty)}^2(\widetilde{X}; V)$ and hence Banach. Therefore, the open mapping theorem applies and we get a bounded inverse for the coboundary map. It follows that there exists a $K \in \mathbb{R}_+$ such that

$$\|F\| = \|\delta^{-1}f_g\| \leq K\|f_g\| \leq K\|g\|.$$

For the converse direction, suppose that the extension property stated in the proposition holds for all bounded linear functionals on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$. Let $f \in Z_{(\infty)}^2(\widetilde{X}; V)$ be a 2-cocycle and let $g_f: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ be the corresponding bounded linear map as in Lemma 3.34. There exists a bounded linear map $F: C_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ agreeing with g_f over $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ by assumption. Then

$$(\delta F)(c) = F(\partial c) = g_f(\partial c) = f(c)$$

for any $c \in C_2^{\text{cell}}(\widetilde{X}; \mathbb{R})$ and $\delta F = f$. It follows that $H_{(\infty)}^2(G; V) = 0$. As we did not use anything about the norm of F , the case of V being Banach follows trivially. \square

Theorem 3.36. *Let G be a finitely presented group. The following conditions are equivalent:*

- (1) *The filling norm $\|\cdot\|_F^{\mathbb{R}}$ and the ℓ_1 -norm $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ are equivalent.*
- (2) *$H_{(\infty)}^2(G; V) = 0$ for all 1-injective Banach spaces V .*
- (3) *$H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$.*

Proof. Assume that the filling norm $\|\cdot\|_F^{\mathbb{R}}$ and the ℓ_1 -norm $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ are equivalent and let V be a 1-injective Banach space. This is equivalent to saying that the canonical inclusion $j: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \hookrightarrow C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ is an isomorphic embedding of normed real vector spaces where $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ is equipped with the real filling norm and $C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ is equipped with ℓ_1 -norm. In order to use the injectiveness of V , we first need to complete our normed real vector spaces to get Banach spaces. Let $\hat{B}_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ and $\hat{C}_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ denote the Banach completions of $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ and $C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$, respectively. Let $g: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ be a bounded linear map and let $\hat{g}: \hat{B}_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ denote its unique extension to the completion. Then, \hat{g} extends to a bounded linear map $\hat{F}: \hat{C}_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow V$ by 1-injectiveness of V and \hat{F} agrees with g over $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$. Therefore, $F := \hat{F}|_{C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})}$ gives our desired bounded linear extension of f to $C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ and it follows that $H_{(\infty)}^2(G; V) = 0$.

This shows (1) \implies (2) and (1) \implies (3). Since (2) \implies (3) is straightforward, we show (3) \implies (1).

Suppose $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$. Let S^* denote the unit sphere in $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})^*$, namely the set of all bounded linear maps $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow \mathbb{R}$ of norm 1. Consider the evaluation map

$$\begin{aligned} f: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) &\longrightarrow \ell^\infty(S^*, \mathbb{R}) \\ b &\longmapsto (g \mapsto g(b)) \end{aligned}$$

Then

$$\|f\| = \sup_{\|b\|_{\mathbb{R}}=1} \|f(b)\| = \sup_{g \in S^*} \|g\| = 1$$

and

$$f(b_1 + b_2)(g) = g(b_1 + b_2) = g(b_1) + g(b_2) = f(b_1)(g) + f(b_2)(g).$$

Hence f is a bounded linear map. From here on, one may continue with vanishing of $H_{(\infty)}^2(G; V)$ to extend f to a map on $C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ using Proposition 3.35 but since we want to show that it is enough to work with $\ell^\infty(\mathbb{N}, \mathbb{R})$ -coefficients, we will show that S^* is separable as a normed real vector space to get a countable dense subset. As we have noted towards the end of § 2.1, 2-cells of \widetilde{X} are enumerable via the elements of G and 2-cells of X : $C_2^{\text{cell}}(\widetilde{X}; \mathbb{R})$ is countably generated and hence it is separable as a normed real vector space. It follows that $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \cong C_2^{\text{cell}}(\widetilde{X}; \mathbb{R}) / \ker \partial$ is separable as well (we consider $\|\cdot\|_{\mathbb{R}}^{\mathbb{R}}$ on it, which is the induced quotient norm). Then, by Banach-Alaoglu theorem, the unit ball B^* in $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})^*$ is compact metrizable in weak- $*$ topology and hence separable. Being a closed subset, S^* is separable as well. Let $S_0^* \subset S^*$ be a countable dense subset in the weak- $*$ topology, then:

$$\sup_{g \in S^*} \|g\|_\infty = \sup_{g \in S_0^*} \|g\|_\infty = 1$$

So, we define $f_0: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow \ell^\infty(S_0^*, \mathbb{R})$ analogously to f and $\|f_0\| = 1$. Then, by Proposition 3.35 there exists a $K \in \mathbb{R}_+$ and a bounded linear map $F: C_2^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow \ell^\infty(S_0^*, \mathbb{R})$ with $\|F\| \leq K \|f_0\| = K$ and $F = f_0 \circ j$. Then, for each $g \in S_0^*$

$$\begin{aligned} F_g: C_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) &\longrightarrow \mathbb{R} \\ c &\longmapsto F(c)(g) \end{aligned}$$

provides a bounded linear extension to $C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ as

$$F_g(b) = F(b)(g) = f_0(b)(g) = g(b)$$

for all $b \in B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$. Moreover

$$\|F_g\| \leq \|F\| \leq K.$$

Given a bounded linear map $g: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \rightarrow \mathbb{R}$, let $g_0 := g/\|g\|$ be its normalization. Then $g_0 \in S_0^*$ and there exists a sequence of bounded linear maps $\{g_i\}_{i \in \mathbb{N}} \subset S_0^*$ converging to g_0 . Let $\{F_i\}_{i \in \mathbb{N}} \subset C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})^*$ denote the sequence of bounded linear maps with $g_i = F_i \circ j$ and $\|F_i\| \leq K$. Then the limit

$$\lim_{n \rightarrow \infty} F_n =: F$$

exists and satisfies $g = F \circ j$ and $\|F\| \leq K$. It follows from Theorem 3.30 that the inclusion $j: B_1^{\text{cell}}(\widetilde{X}; \mathbb{R}) \hookrightarrow C_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ is an isomorphic embedding and the norms $\|\cdot\|_{\mathbb{R}}^{\mathbb{R}}$ and $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ are equivalent. This finishes the proof. \square

3.5. Bridging the gap

We have so far shown that:

Theorem 3.37. *Let G be a finitely presented group, X a CW-complex of type $K(G, 1)$ with finite 2-skeleton and \widetilde{X} the universal cover of X . Then*

- (1) G is hyperbolic if and only if the norms $\|\cdot\|_F^{\mathbb{Z}}$ and $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ are equivalent.
- (2) $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$ if and only if the norms $\|\cdot\|_F^{\mathbb{R}}$ and $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ are equivalent.

We want to bridge the gap between the two statements and arrive at:

Theorem 3.38. *A finitely presented group G is hyperbolic if and only if $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$.*

We already have the one side of this from our investigation of filling seminorms under different coefficients (Proposition 3.26):

Corollary 3.39 (To Theorem 3.36). *If a finitely presented group G is hyperbolic, then $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$.*

Proof. Since G is hyperbolic, we have that the norms $\|\cdot\|_F^{\mathbb{Z}}$ and $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ are equivalent by Theorem 3.7. Then, it follows that ${}^H\|\cdot\|_F^{\mathbb{Z}}$ dominates ${}^H\|\cdot\|_1 = \|\cdot\|_1$ by Lemma 3.14 and homogeneity of the ℓ_1 -norm. On the other hand, since ${}^H\|\cdot\|_F^{\mathbb{Z}} \leq \|\cdot\|_F^{\mathbb{Z}}$, $\|\cdot\|_1$ dominating $\|\cdot\|_F^{\mathbb{Z}}$ implies the converse inequality. Hence, ${}^H\|\cdot\|_F^{\mathbb{Z}}$ and $\|\cdot\|_1$ are equivalent on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$. This is equivalent to the norms $\|\cdot\|_F^{\mathbb{R}}$ and $\|\cdot\|_1$ being equivalent on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ by Proposition 3.26 and the result follows. \square

For the converse direction, we will show that if G is not hyperbolic, then the norms ${}^H\|\cdot\|_F^{\mathbb{Z}}$ and $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ are not equivalent. This is equivalent to the norms $\|\cdot\|_F^{\mathbb{R}}$ and $\|\cdot\|_1$ on $B_1^{\text{cell}}(\widetilde{X}; \mathbb{R})$ not being equivalent by Proposition 3.26. So it would follow that $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) \neq 0$. In order to achieve this, we will construct a family of closed paths $\{p_t\}_{t \in T}$ in $\widetilde{X}^{(1)}$ where t ranges over an unbounded index set $T \subseteq \mathbb{N}$ such that for all p_t

- (1) $\ell(p_t) = \|\langle p_t \rangle\|_1 \asymp t$, in particular $\|\langle p_t \rangle\| \rightarrow \infty$,
- (2) $\|n\langle p_t \rangle\|_F^{\mathbb{Z}} \asymp nt^2$ for all $n \in \mathbb{N}$.

This implies that ${}^H\|p_t\|_F^{\mathbb{Z}} \asymp \|p_t\|_1^2$ for all $t \in T$ and hence the two norms are inequivalent.

We first introduce some definitions to work with

Definition 3.40. Let (Y, d) be a geodesic metric space and let P be a geodesic k -gon with vertices v_0, \dots, v_{k-1} in Y . Let $S_i = [v_i, v_{i+1}]$ with $i \in \{0, \dots, k-1\}$ denote the sides of P modulo k , that is $S_{k-1} = [v_{k-1}, v_0]$. The thickness $\tau(P)$ of P is defined as

$$\tau(P) := \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid \exists i, \exists q \in S_i, d(q, P \setminus S) \geq \lambda\}$$

P is said to be $\tau(P)$ -thick.

With this, the usual definition of δ -hyperbolicity through thin triangles can be formulated as: A metric space (Y, d) is δ -hyperbolic for some $\delta \in \mathbb{R}_{\geq 0}$ if and only if $\tau(\Delta) \leq \delta$ for all geodesic triangles Δ in Y . Papasoglu showed that for groups, it's enough to have all geodesic bigons (finite geodesics with common endpoints) in its Cayley graph be δ -thin for some $\delta \in \mathbb{R}_{\geq 0}$ [Pap95]. We emphasize that this characterization does not hold for geodesic metric spaces in general: for example in Euclidean spaces bigons are collinear and hence 0-thin, whereas geodesic triangles are of arbitrary thickness. Therefore, if a group is not hyperbolic, one can

find geodesic bigons of arbitrary thickness in its Cayley graph. We will mainly be working with this characterization to construct our family of closed paths with linearly growing length with respect to t . In [Ol'91, Lemma 5] uses a layering method to get a lower bound for the area of a van Kampen diagram of given thickness, we will modify it for orientable surface diagrams to get a quadratic lower bound for the growth of the filling norm of $\langle p_t \rangle$ with respect to t . Then apply it to p_t^n which is the closed path that traverses around p_t n -times, to get the lower bound with n . We begin with (1):

Lemma 3.41. *Let G be a non-hyperbolic finitely presented group. Then there exists a sequence of geodesic quadrilaterals $\{p_t\}_{t \in \mathbb{N}}$ in $\widetilde{X}^{(1)}$ such that*

- (1) $\tau(p_t) \geq t/2$,
- (2) $2t \leq \|\langle p_t \rangle\|_1 = \ell(p_t) \leq 10t$.

Proof. As we have noted above, since G is not hyperbolic there exists geodesic bigons of arbitrary thickness in $\widetilde{X}^{(1)}$. Let $t \in \mathbb{N}$ and let P be a geodesic bigon with $\tau(P) = t$. Then there exists a point q on one of the sides S of P such that $d(q, S') = t$ where S' is the other side of P , by definition of the thickness of a geodesic polygon. Let a, b be the vertices of P . If $d(a, q) > 3t/2$, let c be the point on $[a, q] \subset S$ such that $d(c, q) = 3t/2$ and if $d(a, q) \leq 3t/2$, then let $c = a$. Then, if $c \neq a$, there exists a point $e \in S'$ such that $d(c, e) \leq t$ as otherwise P would have been of greater thickness than t . If $c = a$, let $e = a$ as well. Similarly, we choose $f \in [b, q]$ of distance $3t/2$ to q if possible or $f = b$ otherwise, and $g \in S'$ with $d(f, g) \leq t$. We then obtain the possibly degenerate geodesic quadrilateral $p_t := [c, g] \cup [g, f] \cup [f, e] \cup [e, c]$ (see Figure 3 in the next page).

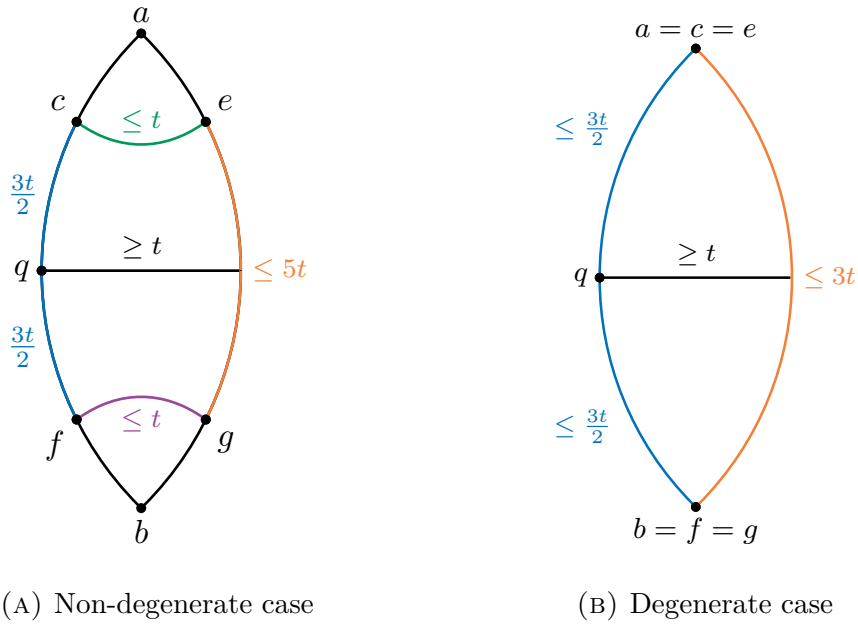


FIGURE 3. The geodesic quadrilateral p_t .

In the degenerate case, we have that $c = e = a \in S'$ and

$$d(q, [e, c]) = d(q, a) \geq d(q, S') = t$$

and the same holds for $d(q, [f, g])$ when $f = g = b \in S'$. In the non-degenerate case

$$d(q, [e, c]) = \inf_{x \in [e, c]} d(q, x) \geq d(q, c) - d(c, x) \geq \frac{t}{2}$$

and similarly for $d(q, [f, g])$. Therefore we always have that $d(q, [e, c]) \geq t/2$ and $d(q, [f, g]) \geq t/2$. Moreover, $d(q, [f, e]) \geq d(q, S') = t$ since $[f, e] \subseteq S'$. From this, it follows that $\tau(p_t) \geq t/2$ and (1) is established.

From the last argument, it follows in particular that $t \leq d(q, c) \leq 3t/2$ and $t \leq d(q, g) \leq 3t/2$ in either cases. As geodesics on a graph are length-minimizing with respect to the path metric, it follows that $2t \leq \ell([c, g]) = d(c, g) \leq 3t$. Therefore $\|\langle p_t \rangle\|_1 = \ell(p_t) \geq \ell([c, g]) \geq 2t$. Finally, we have that $d(f, e) \leq d(e, c) + d(c, g) + d(g, f) \leq 5t$ and $\ell(p_t) \leq 10t$ and (2) follows. \square

It remains to show that there exists a $C \in \mathbb{R}_+$ such that $\|n\langle p_t \rangle\|_F^{\mathbb{Z}} \geq Cnt^2$ for all sufficiently large $t \in \mathbb{N}$ and all $n \in \mathbb{N}$. In the proof of the following proposition, we make extensive use of properties of combinatorial maps between combinatorial complexes and local operations such as Cl, St and Lk. Relevant definitions and basic results can be found in the Preliminaries (§ 1).

Proposition 3.42. *Let G be a non-hyperbolic finitely presented group. There exists a $C \in \mathbb{R}_+$ so that for all sufficiently large $t \in \mathbb{N}$ and all geodesic polygons P in $\widetilde{X}^{(1)}$ with $\tau(P) \geq t$ one has*

$$\|\langle P^n \rangle\|_F^{\mathbb{Z}} = \|n\langle P \rangle\|_F^{\mathbb{Z}} \geq Cnt^2$$

for all $n \in \mathbb{N}_+$ where P^n denotes the closed path that traverses along P n -times.

Proof. We first show the case of $n = 1$. Note that the case of $n = 0$ is not included here to avoid having to define P^0 but the inequality holds regardless as both sides become 0 (recall that $\|\cdot\|_F^A$ is a norm on $B_1^{\text{cell}}(\widetilde{X}; A)$ for any normed abelian group A , Corollary 3.18). Let P be a geodesic polygon of thickness $\tau(P) = t$. Then there exists a side S' of P and a point $q \in S'$ such that $d(q, P \setminus S') = t$. Now, since the edge lengths are taken to be 1 on $\widetilde{X}^{(1)}$, there exists a vertex $v \in S'$ of P with $d(q, v) \leq 1$. Then for any $x \in P \setminus S'$, we have that

$$d(v, x) \geq d(q, x) - d(q, v) \geq t - 1$$

and

$$d(v, P \setminus S') = \inf_{x \in P \setminus S'} d(v, x) \geq t - 1.$$

Let $f : S \rightarrow \widetilde{X}^{(2)}$ be an orientable surface diagram for P of minimal area, that is $\text{Area}(S) = \text{w-Area}(P)$. As ∂S maps homeomorphically onto P , there exists vertices $y, a, b \in \partial S$ that map to $v, S'_-, S'_+ \in P$, where S'_-, S'_+ denote the endpoints of the side S' of P which is a geodesic segment in $\widetilde{X}^{(1)}$. Then, the segment $[a, b] \subset \partial S$ that contains the vertex y is a geodesic segment in S with respect to the path metric d_S that assigns edges with length 1. This is true because combinatorial maps are 1-Lipschitz with respect to d_S , see Lemma 1.16. Moreover

$$d_S(y, a) \geq d(v, S'_-) \geq t - 1, \quad d_S(y, b) \geq d(v, S'_+) \geq t - 1.$$

by the same reason.

We define Λ_1 as the subcomplex $\text{Cl}(\text{St}(y))$. This corresponds to the subcomplex formed by all the closed 2-cells whose boundaries contain the vertex y . This gives us the first layer. We also define $A_1 := \text{Lk}(y) \subset \Lambda_1$. We must have that $A_1 \cap \partial S \neq \emptyset$, in fact we have vertices $\{l_1, r_1\} \subset A_1 \cap \partial S$ with $d(l_1, y) = d(r_1, y) = 1$ and $l_1 \neq r_1$. The existence of such vertices within A_1 is guaranteed and we can prove it by contradiction: a vertex x is of distance 1 to a subcomplex \mathcal{A} if and only there is an edge contained in the star of one of the cells within \mathcal{A} and hence x is contained in $\text{Cl}(\text{St}(\mathcal{A}))$. Now, if $x \notin \text{Lk}(\mathcal{A})$, then we must have $x \in \text{St}(\text{Cl}(\mathcal{A})) = \text{St}(\mathcal{A})$ (\mathcal{A} is a subcomplex, so its equal to its closure) but this is not possible because a vertex can only be in the star of a set of cells if that vertex was in that set to begin with. Now that the existence of l_1 and r_1 is established, we have that $d(l_1, r_1) = 2$ as $[l_1, r_1] \subseteq [a, b] \subset \partial S$ which is a geodesic segment in S .

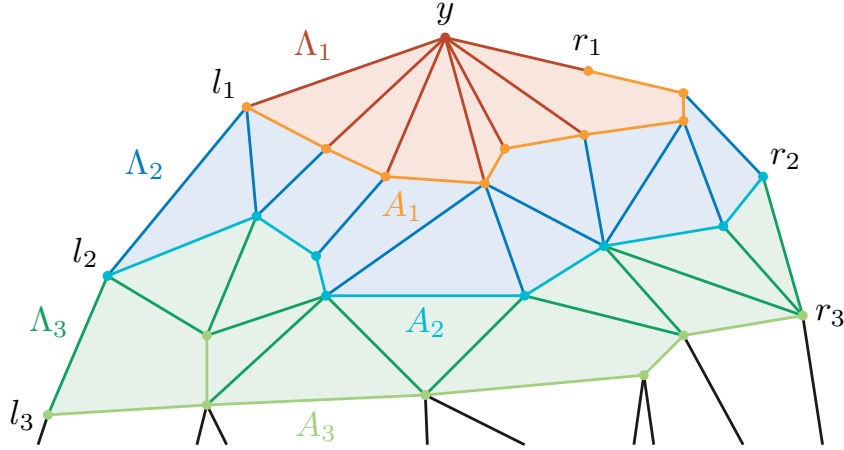


FIGURE 4. An example of the layer construction process.

We define the second layer similarly as $\Lambda_2 = \text{Cl}(\text{St}(A_1))$ and $A_2 = \text{Lk}(A_1)$, both within $S \setminus \text{St}(y)$. As before, we get $\{l_2, r_2\} \subset A_2 \cap \partial S$ with $d(l_2, A_1) = d(r_2, A_1) = 1$ and $l_2 \neq r_2$. We require further that $l_1 \in [l_2, y]$ and similarly for r_2 . This time however, we don't get an exact estimate of the distance between l_2 and r_2 but we can still put a lower bound on it since $d(y, l_2) = d(y, l_1) + d(l_1, l_2) \geq 1 + d(A_1, l_2) = 2$ and similarly $d(y, r_2) \geq 2$. Hence, we get $d(l_2, r_2) \geq 4$.

We continue inductively by letting $\Lambda_k = \text{Cl}(\text{St}(A_{k-1}))$, $A_k = \text{Lk}(A_{k-1})$ both within $S \setminus \bigcup_{i=0}^{k-1} \text{St}(A_i)$ where $A_0 = \{y\}$ and $\{l_k, r_k\} \subset A_k \cap \partial S$ with $d(l_k, A_k) = d(r_k, A_k) = 1$, $l_k \neq r_k$, $l_{k-1} \in [l_k, y]$, $r_{k-1} \in [r_k, y]$. Then our earlier calculations show that as long as $l_k, r_k \in [a, b]$, $d(l_k, r_k) \geq 2k$. This then gives us a lower bound on the number of edges $m(A_k)$ within A_k as it contains a path from l_k to r_k that does not contain y , hence $m(A_k) \geq d(l_2, r_2) \geq 2k$.

At each layer, we can get at most $\lfloor M/2 \rfloor$ away from y where M is the maximum number of edges among all 2-cells of $\widetilde{X}^{(2)}$. This is because combinatorial maps preserve incidence data and therefore any 2-cell of S can have at most M edges as well. Then we can safely continue this construction until at least for $N := t/\lfloor M/2 \rfloor - 1$ steps as then we would be at most $t - 1$ away from y . This means that along the boundary l_k, r_k stay within $[a, b]$ for any $1 \leq k \leq N$.

Now we can use $m(A_k)$ to estimate the number of 2-cells in Λ_k , $\text{Area}(\Lambda_k)$. As M bounds the perimeters of the 2-cells of S from above, we have that $\text{Area}(\Lambda_k) \geq m(A_k)/M$. Therefore

$$\text{Area}(S) \geq \sum_{k=1}^N \text{Area}(\Lambda_k) \geq \sum_{k=1}^N \frac{2k}{M} = \frac{t}{M \lfloor M/2 \rfloor} \left(t - \frac{1}{\lfloor M/2 \rfloor} \right)$$

which is larger than Ct^2 for sufficiently large $t \in \mathbb{N}$ and for some $C \in \mathbb{R}_+$ that depends only on M which in turn depends on \widetilde{X} . We have that $\text{Area}(S) = \text{w-Area}(P)$ as S was assumed to be of minimal area and $\text{w-Area}(P) = \|\langle P \rangle\|_F^{\mathbb{Z}}$ by Proposition 3.6. Thus the case of $n = 1$ is established.

We now generalize this method for $n \geq 1$. We consider the polygon P^n which is the closed path that traverses along P n -times. Let $f: S \rightarrow \widetilde{X}^{(2)}$ be an orientable surface diagram for P^n of minimal area as before. Now, we must have that $f^{-1}(v) = \{y_1, \dots, y_n\} \subset \partial S$, $f^{-1}(S'_-) = \{a_1, \dots, a_n\} \subset \partial S$ and $f^{-1}(S'_+) = \{b_1, \dots, b_n\} \subset \partial S$ with $y_i \in [a_i, b_i]$ as f wraps ∂S around P n -times. We also have that each $[a_i, b_i] \subset \partial S$ are geodesic segments and that $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ for all $i \neq j$. We run the layer construction simultaneously starting from each y_i for $1 \leq i \leq n$.

That is, we let $A_{i,0} = \{y_i\}$ and $\Lambda_{i,k} = \text{Cl}(\text{St}(A_{i,k-1}))$, $A_{i,k} = \text{Lk}(A_{i,k-1}) \subset \Lambda_{i,k}$ for $1 \leq i \leq n$ and $1 \leq k \leq N$. Then, similarly we get $\{l_{i,k}, r_{i,k}\} \subset A_{i,k} \cap \partial S$ with $d(l_{i,k}, A_{i,k}) = d(r_{i,k}, A_{i,k}) = 1$ and $d(l_{i,k}, r_{i,k}) \geq 2k$. Therefore, $m(A_{i,k}) \geq 2k$ for all $1 \leq i \leq n$ and $1 \leq k \leq N$ and $\text{Area}(\Lambda_{i,k}) \geq m(A_{i,k})/M \geq 2k/M$. So, we obtain

$$\text{Area}(S) \geq \sum_{i=1}^n \sum_{k=1}^N \text{Area}(\Lambda_{i,k}) \geq n \sum_{k=1}^N \frac{2k}{M}$$

which is larger than Cnt^2 for sufficiently large $t \in \mathbb{N}$ and for some $C \in \mathbb{R}_+$ that depends only on \tilde{X} . This establishes $\|\langle P^n \rangle\|_F^{\mathbb{Z}} \gtrsim nt^2$ for all $n \geq 1$.

We finally have that $\langle P^n \rangle = n\langle P \rangle$ as each edge of P appears in $\langle P^n \rangle$ with multiplicity n . \square

Together with Lemma 3.41, this establishes the following:

Proposition 3.43. *Let G be a non-hyperbolic finitely presented group. Then the norms $^H \|\cdot\|_F^{\mathbb{Z}}$ and $\|\cdot\|_1$ on $B_1^{\text{cell}}(\tilde{X}; \mathbb{Z})$ are not equivalent.*

So, we obtain the final piece of our puzzle:

Corollary 3.44. *A non-hyperbolic finitely presented group G has $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) \neq 0$.*

Recall the homological Dehn function (Definition 3.27): we note the following:

Corollary 3.45. *If $\text{FA}_{\tilde{X}; \mathbb{R}}^1$ grows subquadratically, then G is hyperbolic.*

4. Characterization of undistortion

Another application of ℓ^∞ -cohomology is the characterization of quasi-convex subgroups of hyperbolic groups. Recall that

Definition 4.1. Let X be a geodesic metric space and let $Y \subset X$ be a subspace of X . Y is said to be k -quasi convex if there exists a $k \in \mathbb{R}_{\geq 0}$ such that every geodesic between points of Y in X stays within the k -neighbourhood of Y .

Definition 4.2. Let G be a hyperbolic group with a finite generating set S and let $H < G$ be a finitely generated subgroup. H is said to be a quasi-convex subgroup of G if there exists a $k \in \mathbb{R}_{\geq 0}$ such that every geodesic between points of H in $\text{Cay}(G, S)$ stays within the k -neighbourhood of H .

Remark 4.3. Here the hyperbolicity of G is key to have a notion of quasi-convexity independent from the choice of a generating set. This has to do with the stability of quasi-geodesics in hyperbolic groups. As an example of dependence on the generating set when G is not hyperbolic, consider $G = \text{BS}(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle$ and $H = \langle b \rangle \cong \mathbb{Z} < G$. With the generating set $S = \{a, b\}$, the shortest word for b^{2^k} is $a^k b a^{-k}$ and it is seen that H is not quasi-convex with respect to S . If however one picks the generating set $S' = \{a, b^2\}$, then the shortest word for b^{2^k} is $(b^2)^k$ and it follows that H is quasi-convex with respect to S' .

Definition 4.4. Let H be a finitely generated subgroup of a finitely generated group G . Let S and S' be finite generating sets for G and H , respectively. The *distortion* $\Delta_H^G: \mathbb{N} \rightarrow \mathbb{N}$ of H in G is

$$\Delta_H^G(n) = \max\{d_H(1, h) \mid h \in H, d_G(1, h) \leq n\}.$$

where d_G and d_H denote the word metrics with respect to S and S' , respectively. Up to a natural equivalence, the function Δ_H^G does not depend on the choice of finite generating sets S and S' . H is said to be undistorted in G if there exists a $C \in \mathbb{R}_+$ such that $\Delta_H^G(n) \leq Cn$ for all $n \in \mathbb{N}$.

It is well known that for subgroups of hyperbolic groups, the notions of quasi-convexity, undistortedness and being quasi-isometrically embedded are all equivalent (see for instance [BH99, Corollary 3.6, Chapter III.Γ]).

Gersten showed that this can also be captured by the restriction homomorphism between the 1st ℓ^∞ -cohomology groups. More precisely, let G be a finitely generated group and let H be a finitely generated subgroup of G . Let S be a finite generating set for G and let $S' \subseteq S$ be a finite generating set for H . There are CW-complexes X of type $K(G, 1)$ and U of type $K(H, 1)$ so that U is a subcomplex of X with $\text{Cay}(G, S) = \widetilde{X}^{(1)}$ and $\text{Cay}(H, S') = \widetilde{U}^{(1)}$ where $\widetilde{X}, \widetilde{U}$ denote the universal covers of X and U , respectively. Let V be a normed $\mathbb{R}[G]$ -module and let $i: \widetilde{U} \hookrightarrow \widetilde{X}$ denote the inclusion map. We then get the induced restriction homomorphism as

$$\text{res} := i^*: H_{(\infty)}^1(G; V) = H_{(\infty)}^1(\widetilde{X}; V) \rightarrow H_{(\infty)}^1(\widetilde{U}; V) = H_{(\infty)}^1(H; V)$$

Theorem 4.5. [Ger98, Corollary 10.3] *Let G be a hyperbolic group and let H be a finitely generated subgroup. The following conditions are equivalent*

- (1) H is quasi-isometrically embedded in G .
- (2) H is a quasi-convex subgroup of G .
- (3) H is undistorted in G .
- (4) $\text{res}: H_{(\infty)}^1(G; V) \rightarrow H_{(\infty)}^1(H; V)$ is surjective.

As we have mentioned, equivalence of (1), (2), (3) is already well-established. We first show that (2) \implies (4). The result is actually even stronger, Gersten showed that the surjection splits:

Theorem 4.6. [Ger98, Theorem 7.3] *If H is a quasi-convex subgroup of an hyperbolic group G , then the restriction homomorphism $\text{res}: H_{(\infty)}^1(G; V) \rightarrow H_{(\infty)}^1(H; V)$ is surjective and furthermore, there exists a section of res that is a bounded linear map of normed $\mathbb{R}[G]$ -modules.*

Proof. Let d_G, d_H denote the corresponding word metrics on $\text{Cay}(G, S), \text{Cay}(H, S')$, respectively. Let $\rho_H: G \rightarrow H$ be a nearest point projection of G onto H , that is, for every $g \in G$, $d_G(g, \rho_H(g)) = d_G(g, H)$. We have that

Lemma 4.7. [Neu92, Lemma 1] *Let S be a convex subset of an δ -hyperbolic group G and let $\rho_S: G \rightarrow S$ denote the nearest point projection of G onto S . Then*

$$d_G(\rho_S(g), \rho_S(h)) \leq d_G(g, h) + 5\delta + 2$$

for any $g, h \in G$.

Replacing S with a quasi-convex subgroup H , we have that H is (λ, μ) -quasi-isometrically embedded in G for some $\lambda \in \mathbb{R}_{\geq 1}, \mu \in \mathbb{R}_{\geq 0}$. Therefore

$$\frac{1}{\lambda} d_H(\rho_H(g), \rho_H(h)) - \mu \leq d_G(\rho_H(g), \rho_H(h))$$

and

$$d_H(\rho_H(g), \rho_H(h)) \leq \lambda(d_G(g, h) + 5\delta + \mu + 2)$$

for all $g, h \in G$. If we pick $g, h \in G$ so that $d_G(g, h) = 1$, we obtain

$$d_H(\rho_H(g), \rho_H(h)) \leq \varepsilon$$

with $\varepsilon = \lambda(5\delta + \mu + 3) \in \mathbb{R}_+$.

Let $f \in Z_{(\infty)}^1(\tilde{U}; V)$. Given an edge $e \in \tilde{X}^{(1)}$ between the vertices $g, h \in G$, we define $\sigma: Z_{(\infty)}^1(\tilde{U}; V) \rightarrow Z_{(\infty)}^1(\tilde{X}; V)$ as the linear extension of

$$\sigma(f)(e) = f(\langle [\rho_H(g), \rho_H(h)] \rangle)$$

to $C_1^{\text{cell}}(\tilde{X}; \mathbb{Z})$, where $[\rho_H(g), \rho_H(h)]$ denotes a geodesic path from $\rho_H(g)$ to $\rho_H(h)$ in $\tilde{U}^{(1)} = \text{Cay}(H, S')$. This is well-defined because for any other choice of a geodesic path $[\rho_H(g), \rho_H(h)]'$, we have

$$\langle [\rho_H(g), \rho_H(h)] \rangle - \langle [\rho_H(g), \rho_H(h)]' \rangle \in B_1^{\text{cell}}(\tilde{X}; \mathbb{Z})$$

and

$$f(\langle [\rho_H(g), \rho_H(h)] \rangle) = f(\langle [\rho_H(g), \rho_H(h)]' \rangle)$$

since f is a cocycle. By the same reason, $\sigma(f) \in Z_{\text{cell}}^1(\tilde{X}; \mathbb{Z})$. Moreover, as $\rho_H|_H = \text{id}_H$, it follows that $\sigma(f)|_H = f$. We now use the result we showed above to show that $\sigma(f)$ is bounded: $d_H(\rho_H(g), \rho_H(h)) = \ell([\rho_H(g), \rho_H(h)]) \leq \varepsilon$ for any $g, h \in G$ with $d_G(g, h) = 1$. Hence, we obtain

$$\|\sigma(f)(e)\|_1 \leq \|\langle [\rho_H(g), \rho_H(h)] \rangle\|_1 \cdot \|f\| \leq \varepsilon \|f\|$$

for any $e \in \tilde{X}^{(1)}$ and therefore $\|\sigma(f)\| \leq \varepsilon \|f\|$. This shows that $\sigma(f) \in Z_{(\infty)}^1(\tilde{X}; V)$ and also $\|\sigma\| \leq \varepsilon$. Linearity of σ is straightforward, so it follows that σ is a bounded linear map of $\mathbb{R}[G]$ -modules.

It only remains to show that σ descends to cohomology. We need to show that σ maps 1-coboundaries to 1-coboundaries. For any $f \in B_{(\infty)}^1(\tilde{U}; V)$, we have that $f = \delta k$ for some $k \in C_{(\infty)}^0(\tilde{U}; V) = \ell^\infty(H, V)$ and

$$\sigma(f)(e) = (\delta k)(\langle [\rho_H(g), \rho_H(h)] \rangle) = k(\partial \langle [\rho_H(g), \rho_H(h)] \rangle) = k(\rho_H(h)) - k(\rho_H(g)).$$

This shows that $\sigma(f) = \delta(k \circ \rho_H)$ where $k \circ \rho_H \in \ell^\infty(G, V) = C_{(\infty)}^0(\widetilde{X}; V)$. This finishes the proof. \square

For (4) \implies (1), we first explore some classes of bounded 1-cocycles on \widetilde{U} coming from unbounded metrics on $\text{Cay}(H, S')$. Given an unbounded metric d' on H and a base point $h_0 \in H$, consider the function $\varphi : H \rightarrow \mathbb{R}_{\geq 0}$ defined as

$$\varphi(h) = d'(h_0, h)$$

Then $\varphi \in C_{\text{cell}}^0(\widetilde{U}; \mathbb{R})$ but it is not bounded as d' is unbounded. Consider its coboundary (defined over edges of $\widetilde{U}^{(1)}$ for brevity):

$$f_{d', h_0}(e) := (\delta\varphi)(e) = d'(h_0, h) - d'(h_0, g)$$

where e is an edge in $\widetilde{U}^{(1)}$ from g to h . We then have that $\|f_{d', h_0}(e)\| \leq d'(h, g) = 1$ and $f_{d', h_0} \in Z_{(\infty)}^1(\widetilde{U}; \mathbb{R})$. Then for any closed path p in $\widetilde{U}^{(1)}$ from g to h , we have that

$$f_{d', h_0}(\langle p \rangle) = d'(h_0, h) - d'(h_0, g).$$

We note further that for a different choice of a basepoint $h_1 \in H$, $[f_{d', h_0}] = [f_{d', h_1}]$ as $f_{d', h_0} - f_{d', h_1} = \delta b$ with $b(h) = d'(h_0, h) - d'(h_1, h) \leq d'(h_0, h_1)$. So we say $[f_{d'}]$ to refer to the cohomology class of f_{d', h_0} for any $h_0 \in H$. We can now state the converse direction:

Theorem 4.8. [Ger98, Theorem 10.1] *A finitely generated subgroup H of a finitely generated group G is quasi-isometrically embedded if and only if there is a left-invariant metric d' on H quasi-isometric to the word metric d_H of H such that $[f_{d'}] \in H_{(\infty)}^1(H; \mathbb{R})$ is in $\text{im}(\text{res}: H_{(\infty)}^1(G; \mathbb{R}) \rightarrow H_{(\infty)}^1(H; \mathbb{R}))$.*

Proof. Let d' be a left-invariant metric on H quasi-isometric to the word metric d_H of H such that $[f_{d'}]$ is in the image of res . Then there exists a 1-cocycle $F \in Z_{(\infty)}^1(\widetilde{X}; \mathbb{R})$ with $\text{res}([F]) = [F|_H] = [f_{d'}]$. Then for any $h \in H$ there exists a $k \in \ell^\infty(H, \mathbb{R})$, such that $F|_H - f_{d', h} = \delta k$.

Fix a basepoint $h_0 \in H$ and let $h \in H$. Choose geodesics $[h_0, h]_G$ in $\text{Cay}(G, S)$ and $[h_0, h]_H$ in $\text{Cay}(H, S')$. We then have that

$$\langle [h_0, h]_G \rangle - i_* \langle [h_0, h]_H \rangle \in Z_1^{\text{cell}}(\widetilde{X}; \mathbb{Z}) = B_1^{\text{cell}}(\widetilde{X}; \mathbb{Z}).$$

Therefore

$$\begin{aligned} F(\langle [h_0, h]_G \rangle) &= F(i_* \langle [h_0, h]_H \rangle) \\ &= (i^* F)(\langle [h_0, h]_H \rangle) \\ &= (f_{d', h_0} + \delta k)(\langle [h_0, h]_H \rangle) \\ &= d'(h_0, h) + k(h) - k(h_0) \end{aligned}$$

and it follows that $d'(h_0, h) \leq F(\langle [h_0, h]_G \rangle) + 2\|k\| \leq \|F\| d_G(h_0, h) + 2\|k\|$ as $\|\langle [h_0, h]_G \rangle\|_1 = d_G(h_0, h)$. Then $d'(g, h) \leq \|F\| d_G(g, h) + 2\|k\|$ for any $g, h \in H$ by the left-invariance of d' . Finally, since d' is quasi-isometric to d_H , we obtain $d_H(g, h) \leq \lambda d_G(g, h) + \mu$ for some $\lambda \in \mathbb{R}_{\geq 1}$ and $\mu \in \mathbb{R}_{\geq 0}$. Since $d_H(g, h) \geq d_G(g, h)$ is always true, it follows that H is quasi-isometrically embedded in G .

Conversely, if H is quasi-isometrically embedded in G , let $F = f_{d_G, h_0} \in Z_{(\infty)}^1(\widetilde{X}; \mathbb{R})$, $d' = d_G|_H$. Then d' is quasi-isometric to d_H and $F|_H = f_{d', h}$ for any $h \in H$ and . It follows that $\text{res}([F]) = [f_{d'}]$. \square

Corollary 4.9. *Let H be a finitely generated subgroup of a finitely generated group G . If the restriction homomorphism $\text{res}: H_{(\infty)}^1(G; \mathbb{R}) \rightarrow H_{(\infty)}^1(H; \mathbb{R})$ is surjective, then $H < G$ is quasi-isometrically embedded.*

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