Comparing Different Mathematical Definitions of 2D CFT

Master's Thesis by

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May 10, 2016

I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment.

Acknowledgements

First of all, I would like to thank my advisor Prof. Schottenloher for the suggestion of the topic and writing such an eye-opening introductory book to the mathematical CFT without which this project would have not been possible, or at least would have been significantly harder.

I thank Dr. Dybalski for agreeing to be the second marker on such a short notice, and for teaching me AQFT and answering all my questions about the course.

My sincere gratitude goes to the Stack Exchange community, in particular, to Physics Stack Exchange, Mathematics Stack Exchange and MathOverflow. I think Stack Exchange is the best thing which happened to the Internet after Wikipedia. The users Marcel Bischoff and Anshuman deserve a separate mention because their answers came when I was very stuck and without their input this work would have been of way much poorer quality.

I would also like to thank Profs. Kac and Nikolov for the clarifications of their work which were very insightful.

For fruitful discussions and great googling skills I am grateful to Abhiram.

For feeling welcome in Germany I am very thankful to the DAAD, which has supported me during my first year, and to the Elite Netzwerk Bayern for funding such a great master's program.

I thank Robert for all the little things which have made my studies in Munich smoother and more fun.

For hospitality during my defense I am indebted to Andreas, Marin and Nina.

Last but not least, I would like to thank my family for its continuous support throughout the years.

Abstract

We give introductions into the representation theory of the Virasoro algebra, Wightman axioms and vertex algebras in the first part.

In the second part, we compare the above definitions. We give a proof of Lüscher and Mack [LM76] that a dilation invariant 2D QFT with an energy-momentum tensor gives rise to two commuting unitary representations of the Virasoro algebra.

We give a proof of Schottenloher [Sch08, p. 193] that associated to a Verma module $M(c, 0)$ of highest weight zero, there exists a vertex operator algebra of CFT type. This result was firstly proved by Frenkel and Zhu [FZ92]. We then recall another result of [FZ92] that related to $M(c, 0)$ there exists a Virasoro vertex operator algebra $L(c, 0)$. We follow [DL14] and show that $L(c, 0)$ is a unitary vertex operator algebra. The converse is a tautology—each conformal vertex algebra has at least one representation of the Virasoro algebra. Moreover, if we have a unitary vertex algebra, then this representation is unitary as well.

Finally we compare Wightman QFTs to vertex algebras. We present Kac's [Kac98, Sec. 1.2] proof that every Wightman Möbius CFT (a 2D Wightman QFT containing quasiprimary fields) gives rise to two commuting strongly-generated positive-energy Möbius conformal vertex algebras. If the number of generating fields of each conformal weight is finite, then these vertex algebras are unitary quasi-vertex operator algebras. As a corollary using Lüscher–Mack's Theorem we obtain that a Wightman CFT (a Wightman Möbius CFT with an energy-momentum tensor) gives rise to a conformal vertex algebra which furthermore becomes a unitary vertex operator algebra, if the number of generating fields of each conformal weight is finite. We reverse Kac's arguments and get a converse proof that two unitary (quasi)-vertex operator algebras can be combined to give a Wightman (Möbius) CFT.

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Introduction

One could argue that modern physics is the study of symmetries. Indeed, Noether's theorem states that symmetries correspond to conservation laws and this observation underlies most of the current physics. One such commonly arising symmetry is the conformal symmetry. Loosely speaking conformal symmetry means that our physical system under consideration is invariant under angle preserving maps. Such a symmetry may seem to be rather restrictive and indeed it is. However, there is an abundance of physical systems that can be treated as conformally invariant at least up to a very good approximation. More precisely, one of the most notable applications of a 2-dimensional conformal field theory (2D CFT), a field theory invariant under conformal transformations in 2 dimensions, is to statistical mechanics and string theory [DFMS99], [BLT13]. Among the newer developments one could mention AdS/CFT correspondence which was first formulated in high energy physics [Mal99] and now is also applied in condensed matter physics [Pir14].

2D CFTs are special among other CFTs because their Lie algebra contains the Virasoro algebra which is infinite dimensional. Thus, 2D CFTs are restricted even more than their higher-dimensional counterparts. This restrictiveness have led to many different mathematical axiomatizations of CFTs. We will present and explore the relationship between two of them: 2D CFT in Wightman framework and vertex algebras. Because of the importance of the Virasoro algebra to 2D CFTs, we also add its representation theory for completeness.

Wightman axioms [SW64] are the first attempt to define QFT rigorously. As such, they try to encompass the whole of QFT. Under some modifications they also describe 2D CFTs. The language of Wightman framework is functional analysis.

Vertex algebras [Bor86, FLM88] on the other hand are algebraic and describe only the chiral half of a 2D CFT. A 2D field is called chiral if it depends only on a single coordinate. Therefore, without a background in physics the fact that there should be a relationship between 2D Wightman CFT and vertex algebras is not obvious and even armed with such knowledge providing a detailed proof still requires some work. The master's thesis aims to fill in these gaps.

As far as we know, the first mathematically rigorous proof that from a 2D Wightman Möbius CFT one can construct two Möbius conformal vertex algebras was given by Kac in [Kac98]. In the same reference Kac also wrote that: "Under certain assumptions and with certain additional data one may reconstruct the whole QFT from these chiral algebras, but we shall not discuss this problem here". We were unable to find any references containing a proof of this plausible claim. The users of MathOverflow were not aware of any references either [Gyt], although the general idea was rather clear (see Marcel Bischoff's comment in [Gyt]). Since we found Kac's proof clear and natural, it was an obvious choice to base the thesis on it and give a converse proof, namely, that two vertex algebras can be combined into a 2D Wightman Möbius CFT by reversing the arguments of [Kac98]. Along the way we also managed to extend Kac's proof to conformal vertex algebras using Lüscher–Mack Theorem. For the converse proof we started with vertex operator algebras because there is a wealth of mathematical literature about them and the recent work [CKLW15] includes a lot of useful results. However, there should be a more general proof which would also include (Möbius) conformal vertex algebras which are not (quasi-)vertex operator algebras.

This work is divided into two parts. The first part gives the necessary background, whereas the second part explores the relationships. Experts in the field are encouraged to skip the first part altogether and use it just to refresh their memory for the well-known definitions, if needed.

We have chosen to present the material as follows:

- In Chapter 1 we show that the conformal group of the Minkowski plane $\mathbb{R}^{1,1}$ is $\mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})$ or $\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$ and its relation to $\mathrm{SO}^+(2,2)/\{\pm 1\}$ and $\mathrm{PSL}(2,\mathbb{R})$.

- Chapter 2 is concerned with the Virasoro algebra. We define the Virasoro algebra as the unique non-trivial universal central extension of the Witt algebra—a dense subalgebra of the vector fields on a circle. Moreover, we give an introduction to the representation theory of the Virasoro algebra.

- In Chapter 3 we start with the basics and carefully define vertex algebras and related notions of (Möbius) conformal vertex algebras and (quasi)-vertex operator algebras. No prior knowledge is assumed. We give full proofs of all the fundamentals and start relying on other sources for proofs only in the last section for which readily accessible sources are available, e.g. [CKLW15].

- In Chapter 4 we present the Wightman axioms for a scalar field. We prove the existence of Wightman distributions, which according to Wightman's Reconstruction Theorem 4.12 provide an equivalent description of the theory. We also define a Wightman (Möbius) CFT.

- Chapter 5 starts the second part. We prove the Lüscher–Mack Theorem which shows that 2D dilation invariant Wightman QFT gives rise to two commuting Virasoro algebras.

- Chapter 6 is rather trivial. We construct a vertex operator algebra from a Verma module of weight zero and note that the converse is a tautology.

- Chapter 7 is the highlight of this work. It contains Kac's Theorem that a Wightman (Möbius) CFT gives rise to two commuting (Möbius) conformal strongly-generated vertex algebras and a converse that two unitary (quasi)-vertex operator algebras give rise to a Wightman (Möbius) CFT.

Throughout the master's thesis we consider bosonic QFTs on the plane because we also wanted to make this work accessible and thus not cluttered with minor details. However, the generalization to superspaces including fermions is quite trivial and can be found in our main references: for Wightman axioms in [SW64, BLOT89] and for vertex algebras in [Kac98]. We are shy of examples because constructing them for general QFTs is rather hard and there is even a Millenium Prize for constructing a non-trivial QFT in $\mathbb{R}^4$ [JW00]. However, we provide full details for the transformations of scalar fields and the energy-momentum tensor from one framework to another in the proofs themselves.

It should be noted that Wightman axioms and vertex algebras are not the only mathematical definitions of 2D CFTs. Other mathematical definitions include Segal's axioms [Seg88] and conformal nets, see, e.g., [CKLW15]. Conformal nets describe chiral CFTs in the framework of algebraic QFT, whereas Segal's axioms describe full 2D CFTs on arbitrary genera, i.e. not only on $\mathbb{R}^2$ or the open disk, as considered in this work. Thus, Segal's axioms seem to be superior to other approaches. However, many different approaches to the same problem are often beneficial in providing more tools to tackle it and to gain familiarity with the problem in the simpler cases before embarking on the most general form of the problem.

Part I

Background

Chapter 1

Conformal Group

1.1 General Case

We start with some basic definitions as given in [Sch08]. Chapters 1 and 2 of [Sch08] are our main references for this chapter.

Definition 1.1. (Semi-)Riemannian manifold. A *semi-Riemannian manifold* is a smooth manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g. A *Riemannian manifold* is a semi-Riemannian manifold whose metric tensor is also positive-definite.

Definition 1.2. Conformal transformation. Let (M, g) and (M', g') be two semi-Riemannian manifolds of dimension n. Let $U \subset M$, $V \subset M'$ be open. A smooth mapping $f : U \to V$ of maximal rank is called a *conformal transformation* or *conformal map* if there exists a smooth function $\Omega : U \to \mathbb{R}_{>0}$ such that

$$f^* g' = \Omega^2 g,$$

where $f^* g'_p(X, Y) := g'_{f(p)}(D_p f(X), D_p f(Y))$ is the pullback of g' by f evaluated at a point $p \in U$ and $D_p f : T_p U \to T_p V$ is the derivative of f at the point $p \in U$. The function Ω is called the *conformal factor* of f.

Some authors also require a conformal transformation to be bijective and/or orientation preserving.

Locally in a chart (U, ϕ) of M we have

$$(f^* g')_{\mu\nu}(p) = g'_{ij}(f(p)) \partial_\mu f^i \partial_\nu f^j \quad \forall p \in U.$$

Hence f is conformal if and only if

$$\Omega^2 g_{\mu\nu} = (g'_{ij} \circ f) \partial_\mu f^i \partial_\nu f^j \tag{1.1}$$

in every coordinate patch.

Remark 1.3. Since we have required a conformal map to be of maximal rank, conformal maps are local diffeomorphisms.

Even though the definition of a conformal transformation is straightforward, it turns out that it is not trivial to sensibly define the conformal group. We state the general definition as given in [Sch08].

Definition 1.4. Conformal group. The *conformal group* $\mathrm{Conf}(\mathbb{R}^{p,q})$ is the connected component containing the identity in the group of conformal diffeomorphisms of the conformal compactification of $\mathbb{R}^{p,q}$.

In Section 1.3 we will see that this definition has to be modified for the Euclidean plane. Moreover, the Minkowski plane is also special, since it does not need a conformal compactification to make sense. We will show this in Section 1.2. Thus, the general definition of the conformal group boils down to cases $\mathbb{R}^{1,1}$, $\mathbb{R}^{2,0}$ and $\mathbb{R}^{p,q}$ with $p+q \geq 3$.

Theorem 1.5. Conformal group. *The conformal group* $\mathrm{Conf}(\mathbb{R}^{p,q})$ *of* $\mathbb{R}^{p,q}$ *is:*

1) $(p,\, q) \neq (1,1),\ p,q \geq 1$

$$\mathrm{Conf}(\mathbb{R}^{p,q}) = \begin{cases} \mathrm{SO}^+(p+1,q+1) & \text{if } -\mathrm{id} \notin \mathrm{SO}^+(p+1,q+1) \\ \mathrm{SO}^+(p+1,q+1)/\{\pm\,\mathrm{id}\} & \text{if } -\mathrm{id} \in \mathrm{SO}^+(p+1,q+1); \end{cases}$$

2) $(p,\, q) = (1,1)$

$$\mathrm{Conf}(\mathbb{R}^{1,1}) = \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1).$$

By the above, the groups $\mathrm{SO}^+(p,q)$ are the most important for CFT. We state here their generators before specializing to the 2-dimensional case. For a proof check [Sch08, Thms. 2.9 and 2.11].

Theorem 1.6. *The group* $\mathrm{SO}^+(p+1,q+1)$*, with* $p,q \geq 1,\ p+q = n$*, is isomorphic to the group generated by*

- *translations*
$$x \mapsto x + c,$$
- *special orthogonal transformations*
$$x \mapsto \Lambda x,$$
- *dilations*
$$x \mapsto e^{\lambda} x,$$
- *special conformal transformations*
$$x \mapsto \frac{x + |x|^2 b}{1 + 2\langle x, b\rangle + |x|^2 |b|^2}.$$

Here $x, b, c \in \mathbb{R}^n$, $\Lambda \in \mathrm{SO}^+(p,q)$, $\lambda \in \mathbb{R}$.

1.2 Conformal group of $\mathbb{R}^{1,1}$

In this section we will prove that $\mathrm{Conf}(\mathbb{R}^{1,1}) \cong \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$.

Proposition 1.7. *A smooth map $f = (u, v) : U \to \mathbb{R}^{1,1}$ is conformal if and only if*

$$u_x^2 > v_x^2 \quad \text{and} \quad u_x = v_y,\ u_y = v_x \quad \text{or} \quad u_x = -v_y,\ u_y = -v_x. \tag{1.2}$$

Here $U \subset \mathbb{R}^{1,1}$ is connected and open.

Proof. The condition of being conformal $f^*g = \Omega^2 g$ for $g = g^{1,1}$ with $\Omega^2 > 0$ is equivalent to the equations

$$u_x^2 - v_x^2 = \Omega^2, \quad u_x u_y - v_x v_y = 0, \quad u_y^2 - v_y^2 = -\Omega^2, \quad \Omega^2 > 0. \tag{1.3}$$

First assume that the map f is conformal. Then the equations (1.3) imply that $u_x^2 = \Omega^2 + v_x^2 > v_x^2$ and adding the first three of them we get

$$0 = u_x^2 - v_x^2 + u_y^2 - v_y^2 + 2u_x u_y - 2v_x v_y = (u_x + u_y)^2 - (v_x + v_y)^2.$$

Hence,

$$u_x + u_y = \pm(v_x + v_y). \tag{1.4}$$

Taking the positive root and using the second equation of (1.3) we get

$$\begin{aligned} 0 &= -u_x u_y + v_x v_y = u_x^2 - u_x^2 - u_x u_y + v_x v_y = u_x^2 - u_x(u_x + u_y) + v_x v_y \\ &= u_x^2 - u_x(v_x + v_y) + v_x v_y = (u_x - v_x)(u_x - v_y), \end{aligned}$$

i.e. $u_x = v_x$ or $u_x = v_y$. The solution $u_x = v_x$ contradicts $u_x^2 - v_x^2 = \Omega^2 > 0$. Thus, we have $u_x = v_y$ and by Equation (1.4) $u_y = v_x$ as required. Similarly taking the negative root in Equation (1.4) yields $u_x = -v_y$ and $u_y = -v_x$.

Now assume that the equations (1.2) are fulfilled. Setting $\Omega^2 := u_x^2 - v_x^2 > 0$ and substituting $u_y = \pm v_x$, $v_y = \pm u_x$ yields

$$u_y^2 - v_y^2 = v_x^2 - u_x^2 = -\Omega^2 \quad \text{and} \quad u_x u_y - v_x v_y = 0,$$

i.e. f is conformal. If $u_x = v_y$, $u_y = v_x$, then

$$\det Df = u_x v_y - u_y v_x = u_x^2 - v_x^2 > 0.$$

So f is orientation preserving in this case. Similarly, if $u_x = -v_y$ and $u_y = -v_x$, then f is orientation reversing. $\square$

The next lemma shows that in the case of $U = \mathbb{R}^{1,1}$, the global orientation-preserving conformal transformations can be conveniently described using light-cone coordinates $x^{\pm} = x \pm y$.

Lemma 1.8. *Given* $f \in C^\infty(\mathbb{R})$*, define* $f_\pm \in C^\infty(\mathbb{R}^2, \mathbb{R})$ *by* $f_\pm := f(x \pm y)$*. The map*

$$\begin{aligned} \phi : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) &\to C^\infty(\mathbb{R}^2, \mathbb{R}^2) \\ (f\,,\, g) \qquad &\mapsto \frac{1}{2}(f_+ + g_-\,,\, f_+ - g_-) \end{aligned}$$

has the following properties:

(a) $\operatorname{Im}\phi = \{(u,v) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2) \mid u_x = v_y,\ u_y = v_x\}$,

(b) $\phi(f,g)$ *is conformal* $\iff$ $f' > 0,\ g' > 0$ *or* $f' < 0,\ g' < 0$,

(c) $\phi(f,g)$ *is bijective* $\iff$ f *and* g *are bijective,*

(d) $\phi(f \circ h, g \circ k) = \phi(f,g) \circ \phi(h,k)$ *for* $f,g,h,k \in C^\infty(\mathbb{R})$*.*

Proof. (a) Let $(u,v) \in \operatorname{Im}\phi$, i.e. $(u,v) = \phi(f,g)$ for some $f,g \in C^\infty(\mathbb{R})$. From

$$\begin{aligned} u_x &= \frac{1}{2}(f'_+ + g'_-), \quad u_y = \frac{1}{2}(f'_+ - g'_-), \\ v_x &= \frac{1}{2}(f'_+ - g'_-), \quad v_y = \frac{1}{2}(f'_+ + g'_-), \end{aligned}$$

it follows that $u_x = v_y$ and $u_y = v_x$.

Conversely, let $(u,v) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ be such that $u_x = v_y$ and $u_y = v_x$. Then $u_{xx} = v_{yx} = v_{xy} = u_{yy}$. But this is just the one dimensional wave equation and it has solutions $u(x,y) = \frac{1}{2}(f_+(x,y) + g_-(x,y))$ with suitable $f, g \in C^\infty(\mathbb{R})$. Because of $v_x = u_y = \frac{1}{2}(f'_+ - g'_-)$ and $v_y = u_x = \frac{1}{2}(f'_+ + g'_-)$, we have $v = \frac{1}{2}(f_+ - g_-)$ where f and g might have to be translated by a constant.

(b) If $(u,v) = \phi(f,g)$, then $u_x^2 - v_x^2 = f'_+ g'_-$. Thus,

$$u_x^2 - v_x^2 > 0 \iff f'_+ g'_- > 0 \iff f'g' > 0.$$

(c) Let $\varphi = \phi(f,g)$. To prove the equivalence of injectivities note that

$$\varphi(x,y) = \varphi(x',y') \iff \begin{cases} f(x+y) + g(x-y) = f(x'+y') + g(x'-y') \\ f(x+y) - g(x-y) = f(x'+y') - g(x'-y') \end{cases}$$

$$\iff \begin{cases} f(x+y) = f(x'+y') \\ g(x-y) = g(x'-y') \end{cases} \Longleftarrow \begin{cases} x+y = x'+y' \\ x-y = x'-y' \end{cases} \iff \begin{cases} x = x' \\ y = y'. \end{cases}$$

Now if both f and g are injective, we also get the forward implication in step 3 and the equivalence diagram above shows that ϕ is also injective. On the other hand, if ϕ is injective, i.e. $\varphi(x,y)=\varphi(x',y') \iff x=x'$ and $y=y'$, then the diagram above shows that f and g are injective as well.

Let $(x',y')\in\mathbb{R}^2$ be arbitrary. If f and g are surjective, then $\exists\, s,t\in\mathbb{R}$ with $f(s)=x'+y'$, $g(t)=x'-y'$. Moreover, $\varphi(x,y)=(x',y')$ with $x:=\frac{1}{2}(s+t)$, $y:=\frac{1}{2}(s-t)$.

Conversely, fix $x'\in\mathbb{R}$ and assume that ϕ is surjective. Then $\exists(x,y)\in\mathbb{R}^2$ such that $\varphi(x,y)=(x',0)$. This implies that $f(x+y)=x'=g(x-y)$ and hence f and g are surjective.

(d) Set $\phi:=\phi(f,g)$ and $\psi:=\phi(h,k)$. We have

$$\phi\circ\psi=\frac{1}{2}\left(f_+\circ\psi+g_-\circ\psi\,,\,f_+\circ\psi-g_-\circ\psi\right),$$

where $f_+\circ\psi=f\left(1/2(h_++k_-)+1/2(h_+-k_-)\right)=f\circ h_+=(f\circ h)_+$ and other terms evaluate similarly. Thus,

$$\phi\circ\psi=\frac{1}{2}\left((f\circ h)_++(g\circ k)_-\,,\,(f\circ h)_+-(g\circ k)_-\right)=\phi(f\circ h\,,\,g\circ k)$$

as required. □

Proposition 1.9. *The group of orientation-preserving conformal diffeomorphisms*

$$\varphi:\mathbb{R}^{1,1}\to\mathbb{R}^{1,1}$$

is isomorphic to the group

$$(\mathrm{Diff}_+(\mathbb{R})\times\mathrm{Diff}_+(\mathbb{R}))\cup(\mathrm{Diff}_-(\mathbb{R})\times\mathrm{Diff}_-(\mathbb{R})).$$

Proof. The result follows from Lemma 1.8 and Proposition 1.7. □

From Proposition 1.9 we see that the conformal compactifications mentioned in the general Definition 1.4 are not necessary in Minkowski plane $\mathbb{R}^{1,1}$. Hence, it would make sense simply to define the conformal group $\mathrm{Conf}(\mathbb{R}^{1,1})$ as the identity component of the group of conformal transformations $\mathbb{R}^{1,1}\to\mathbb{R}^{1,1}$ which is isomorphic to $\mathrm{Diff}_+(\mathbb{R})\times\mathrm{Diff}_+(\mathbb{R})$ by Lemma 1.8. However, usually researchers want to work with a group of transformations on a compact manifold. Thus, $\mathbb{R}$ is replaced by the circle S:

$$\mathbb{R}^{1,1}\to S^{1,1}=S^1\times S^1\subset\mathbb{R}^{2,0}\times\mathbb{R}^{0,2}\cong\mathbb{R}^{2,2}.$$

From such reasoning it follows that a sensible definition for the conformal group $\mathrm{Conf}(\mathbb{R}^{1,1})$ is the identity component of the group of all conformal diffeomorphisms $S^{1,1}\to S^{1,1}$. Analogously to Theorem 1.9 the group of orientation-preserving conformal diffeomorphisms $S^{1,1}$ turns out to be isomorphic to

$$(\mathrm{Diff}_+(S) \times \mathrm{Diff}_+(S)) \cup (\mathrm{Diff}_-(S) \times \mathrm{Diff}_-(S)).$$

To prove this result, one simply has to consider 2π-periodic functions in the proof of Lemma 1.8. Therefore, we have:

Theorem 1.10. $\mathrm{Conf}(\mathbb{R}^{1,1}) \cong \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$.

By Theorem 1.10 we see that the conformal group of Minkowski plane $\mathrm{Conf}(\mathbb{R}^{1,1})$ is infinite dimensional. However, there is also a finite dimensional counterpart $\mathrm{SO}^+(2,2)/\{\pm 1\} \subset \mathrm{Conf}(\mathbb{R}^{1,1})$.

Definition 1.11. The *restricted conformal group* of the (compactified) Minkowski plane $\mathbb{R}^{1,1}$ is $\mathrm{SO}^+(2,2)/\{\pm 1\}$.

The group $\mathrm{SO}^+(2,2)/\{\pm 1\}$ consists of translations, Lorentz transformations, dilations and special conformal transformations [Sch08, Thm. 2.9]. If we introduce the light-cone coordinates

$$x_+ = x + y, \quad x_- = x - y,$$

then the restricted conformal group acts as

$$(A_+,\, A_-)(x_+, x_-) = \left(\frac{a_+ x_+ + b_+}{c_+ x_+ + d_+},\, \frac{a_- x_- + b_-}{c_- x_- + d_-} \right).$$

Thus,

$$\mathrm{SO}^+(2,2)/\{\pm 1\} \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}).$$

Because of this decoupling translations and special conformal transformations can be chosen as the generators of the group as the following proposition shows. It is a special case of Dickson's Theorem.

Proposition 1.12. *The group* $\mathrm{PSL}_2(F)$ *with* $F = \mathbb{C}$ *or* $F = \mathbb{R}$ *is generated by translations and special conformal transformations.*

Proof. We use the action of $\mathrm{PSL}_2(F)$ on one of the light-cone coordinates to identify translations with upper triangular matrices and special conformal transformations with lower triangular matrices. So we need to prove that

$$S = \left\langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \right\rangle$$

is the whole of $\mathrm{PSL}_2(F)$, i.e. that S contains dilations and Lorentz transformations.

First of all we note that a reflection (rotation by π) is equal to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and hence is in S. Furthermore, we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -e^{-x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & -e^{-x} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{x} & 0 \\ 0 & e^{-x} \end{pmatrix}$$

and hence dilations are in S. Moreover, every Lorentz transformation can be diagonalized

$$\begin{pmatrix} \cosh(\xi) & \sinh(\xi) \\ \sinh(\xi) & \cosh(\xi) \end{pmatrix} = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} e^{\xi} & 0 \\ 0 & e^{-\xi} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

□

1.3 Conformal group of $\mathbb{R}^{2,0}$

The next lemma shows why the general definition of conformal group 1.4 fails for the Euclidean plane.

Lemma 1.13. *Conformal transformations $f : U \to \mathbb{C}$ are the locally invertible holomorphic or antiholomorphic functions with conformal factor $|\det Df\,|$. Here U is an open and connected subset of $\mathbb{C}$.*

Proof. A smooth map $f : U \to \mathbb{C}$ on a connected open subset $U \subset \mathbb{C}$ is conformal according to Equation 1.1 with conformal factor $\Omega : U \to \mathbb{R}_{>0}$ if and only if for $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ we have

$$u_x^2 + v_x^2 = \Omega^2 = u_y^2 + v_y^2 > 0 \quad \text{and} \quad u_x u_y + v_x v_y = 0. \tag{1.5}$$

These equations are satisfied by holomorphic and antiholomorphic functions with the property $u_x^2 + v_x^2 > 0$, since by Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ for holomorphic functions and $u_x = -v_y$, $u_y = v_x$ for antiholomorphic. For holomorphic (antiholomorphic) functions $u_x^2 + v_x^2 > 0$ is equivalent to $\det Df > 0$ ($\det Df < 0$).

Conversely, given a conformal transformation $f = (u, v)$ the equations (1.5) imply that (u_x, v_x) and (u_y, v_y) are perpendicular vectors in $\mathbb{R}^{2,0}$ of equal length $\Omega > 0$. Hence, $(u_x, v_x) = (-v_y, u_y)$ or $(u_x, v_x) = (v_y, -u_y)$. These correspond to f being holomorphic or antiholomorphic with $\det Df > 0$ or $\det Df < 0$, respectively. $\square$

Corollary 1.14. The holomorphic maps $f : U \to \mathbb{C}$ with $f' \neq 0$ are in one-to-one correspondence with conformal orientation preserving maps $h : U \to \mathbb{C}$. Here $U \subset \mathbb{C}$ is open and connected.

Hence, the conformal compactification [Sch08, Rmk. 2.2 and Def. 2.7] does not exist: there are many noninjective conformal transformations. For example,

$$\mathbb{C}\backslash\{0\} \to \mathbb{C}, \quad z \mapsto z^k, \quad \text{with } k \in \mathbb{Z}\backslash\{-1, 0, 1\}.$$

Therefore, one is lead to a different definition for the Euclidean plane.

Definition 1.15. A *global* conformal transformation of $\mathbb{R}^{2,0}$ is an injective holomorphic function, defined on the whole of $\mathbb{C}$ with at most one exceptional point.

It follows that the group $\mathrm{Conf}(\mathbb{R}^{2,0})$ is isomorphic to the **Möbius group** which is the group of all holomorphic maps $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = \frac{az + b}{cz + d}, \quad cz + d \neq 0 \qquad \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Even though the matrices are in $\mathrm{SL}(2, \mathbb{C})$, the transformations are invariant under multiplication by -1. Hence,

$$\mathrm{Conf}(\mathbb{R}^{2,0}) \cong \mathrm{SL}(2, \mathbb{C})/\{\pm 1\} = \mathrm{PSL}(2, \mathbb{C}).$$

Moreover, there exist other well-known isomorphisms of $\mathrm{PSL}(2, \mathbb{C})$ and so we have

$$\mathrm{Conf}(\mathbb{R}^{2,0}) \cong \mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{SO}^+(3, 1) \cong \mathrm{Aut}(\widehat{\mathbf{C}}),$$

where $\widehat{\mathbf{C}}$ is the Riemann sphere.

Chapter 2

Virasoro Algebra

We present a short introduction to the Virasoro algebra which arises as a complexification of a restriction of the Lie algebra of $\mathrm{Conf}(\mathbb{R}^{1,1})$.

The rest of this chapter is arranged as follows:

- In Section 2.1 we present the general theory of central extensions of Lie algebras.
- The Witt algebra is defined and it is shown that the Virasoro algebra is the unique nontrivial universal central extension of it in Section 2.2.
- In Section 2.3 main tools of the representation theory of the Virasoro algebra are provided. The highlight of the section is the proof that the Virasoro algebra admits unitary representations for all $c > 1$, $h > 0$.

2.1 Central Extensions of Lie Algebras

Our main references for this section are [IK11] and [Sch08].

Throughout this section let F be a field of characteristic zero (usually $\mathbb{R}$ or $\mathbb{C}$).

Definition 2.1. Lie algebra. A *Lie algebra* is a vector space $\mathfrak{g}$ over some field F together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket*, which satisfies the following axioms $\forall a, b \in F$ and $\forall X, Y, Z \in \mathfrak{g}$:

- bilinearity
  $[aX + bY, Z] = a[X, Z] + b[Y, Z], \quad [Z, aX + bY] = a[Z, X] + b[Z, Y]$,
- alternating property
  $[X, X] = 0$,
- the Jacobi identity
  $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Definition 2.2. Abelian Lie algebra. A Lie algebra $\mathfrak{a}$ is called *abelian* if $[X, Y] = 0 \ \forall X, Y \in \mathfrak{a}$.

Definition 2.3. Central extension of Lie algebra. Let $\mathfrak{a}$ be an abelian Lie algebra over a field F and $\mathfrak{g}$ a Lie algebra over F. An *exact sequence* of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

is called a *central extension* of $\mathfrak{g}$ by $\mathfrak{a}$, if $[\iota(\mathfrak{a}), \mathfrak{h}] = 0$. Then $\mathfrak{a}$ is called the *kernel* of the central extension.

A sequence of maps is called *exact* if the kernel of each map is equal to the image of the previous map. In particular, here we have that ι is injective, π is surjective and $\ker \pi = \operatorname{Im} \iota \cong \mathfrak{a}$. Moreover, $[\iota(\mathfrak{a}), \mathfrak{h}] = 0$ implies that $\mathfrak{a}$ corresponds to an ideal in $\mathfrak{h}$ and hence $\mathfrak{g} \cong \mathfrak{h}/\mathfrak{a}$ via π.

Definition 2.4. Universal central extension of Lie algebra. A central extension

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

of $\mathfrak{g}$ is called a *universal central extension* if

- $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ i.e. $\mathfrak{h}$ is *perfect*,
- for all central extensions $\pi' : \mathfrak{h}' \to \mathfrak{g}$ there exists a Lie algebra homomorphism $\gamma : \mathfrak{h} \to \mathfrak{h}'$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\pi"] \arrow[d, "\gamma"] & \mathfrak{g} \arrow[d, "id"] \\ \mathfrak{h}' \arrow[r, "\pi'"] & \mathfrak{g} \end{tikzcd}$$

Lemma 2.5. *The Lie algebra homomorphism γ from Definition 2.4 is unique.*

Proof. Let $\gamma' \colon \mathfrak{h} \to \mathfrak{h}'$ be another homomorphism such that $\pi = \pi' \circ \gamma'$. Then $\forall X, Y \in \mathfrak{h}$ we have

$$\begin{aligned} (\gamma - \gamma')([X, Y]) &= [\gamma(X), \gamma(Y)] - [\gamma'(X), \gamma'(Y)] \\ &= [\gamma(X) - \gamma'(X), \gamma(Y)] + [\gamma'(X), \gamma(Y) - \gamma'(Y)] \end{aligned}$$

Now $\pi' \circ (\gamma(Z) - \gamma'(Z)) = \pi(Z) - \pi(Z) = 0$ i.e. $\gamma(Z) - \gamma'(Z) \in \ker \pi' = \iota'(\mathfrak{a}')$ $\forall Z \in \mathfrak{h}$. Since the extension $\pi' \colon \mathfrak{h}' \to \mathfrak{g}$ is central, $[\iota'(\mathfrak{a}'), \mathfrak{h}'] = 0$. Hence $(\gamma - \gamma')([X, Y]) = [\gamma(X) - \gamma'(X), \gamma(Y)] + [\gamma'(X), \gamma(Y) - \gamma'(Y)] = 0$ $\forall X, Y \in \mathfrak{h}$. Since $\mathfrak{h}$ is perfect, we get that $\gamma = \gamma'$. $\square$

Corollary 2.6. A universal central extension, if it exists, is unique up to Lie algebra isomorphism.

Definition 2.7. Second cohomology group. By $H^2(\mathfrak{g},\mathfrak{a}) := Z^2(\mathfrak{g},\mathfrak{a})/B^2(\mathfrak{g},\mathfrak{a})$ the *second cohomology group* is defined where $\mathfrak{a}$ is regarded as a trivial $\mathfrak{g}$-module. Here $Z^2(\mathfrak{g},\mathfrak{a})$ (respectively $B^2(\mathfrak{g},\mathfrak{a})$) is called the *space of 2-cocycles* (respectively *2-coboundaries*) of $\mathfrak{g}$ with coefficients in $\mathfrak{a}$:

$$Z^2(\mathfrak{g},\mathfrak{a}) := \left\{ \Theta\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{a} \;\middle|\; \begin{array}{l} \forall X,Y,Z\in\mathfrak{g}: \\ 1.\ \Theta \text{ is bilinear,} \\ 2.\ \Theta(X,Y) = -\Theta(Y,X), \\ 3.\ \Theta(X,[Y,Z]) + \Theta(Y,[Z,X]) + \Theta(Z,[X,Y]) = 0 \end{array} \right\}$$

$$B^2(\mathfrak{g},\mathfrak{a}) := \{\Theta\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{a} \mid \exists\mu\colon \mathfrak{g}\to\mathfrak{a} \text{ linear, such that } \Theta(X,Y) = \mu([X,Y])\}\,.$$

Definition 2.8. Equivalent central extensions. Two central extensions of Lie algebra $\mathfrak{g}$ by $\mathfrak{a}$ are *equivalent* if there exists a Lie algebra isomorphism $\psi\colon \mathfrak{h}'\to\mathfrak{h}$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota'"] \arrow[d, "\mathrm{id}"] & \mathfrak{h}' \arrow[r, "\pi'"] \arrow[d, "\psi"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] & \mathfrak{h} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0 \end{tikzcd}$$

commutes.

Lemma 2.9. *There is a correspondence between 2-cocycles of $\mathfrak{g}$ with values in $\mathfrak{a}$ and central extensions of $\mathfrak{g}$ by $\mathfrak{a}$.*

Proof. Given $\Theta\in Z^2(\mathfrak{g},\mathfrak{a})$, define $\mathfrak{h} := \mathfrak{g}\oplus\mathfrak{a}$. Then define a bracket

$$[(X,V),(Y,W)]_{\mathfrak{h}} := ([X,Y]_{\mathfrak{g}}, \Theta(X,Y)) \quad \forall X,Y\in\mathfrak{g}, \forall V,W\in\mathfrak{a}.$$

It follows that this is a Lie bracket by definition of Θ. Thus $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$ is a Lie algebra. Therefore the exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\mathrm{pr}_1} \mathfrak{g} \longrightarrow 0$$

where ι is the inclusion and pr_1 is the projection onto the first variable, is a central extension of $\mathfrak{g}$.

Conversely, given a central extension

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

there is a linear map $\beta\colon \mathfrak{g}\to\mathfrak{h}$ with $\pi\circ\beta = \mathrm{id}_{\mathfrak{g}}$ (it is not a Lie algebra homomorphism in general). Let

$$\Theta_\beta(X,Y) := [\beta(X),\beta(Y)] - \beta([X,Y]) \quad \forall X,Y\in\mathfrak{g}. \tag{2.1}$$

Since π is a Lie algebra homomorphism,

$$\pi \circ \Theta_\beta(X,Y) = \pi([\beta(X),\beta(Y)]) - [X,Y] = 0 \quad \forall X,Y \in \mathfrak{g}$$

i.e. $\mathrm{Im}(\Theta_\beta) \subset \ker \pi = \mathrm{Im}(\iota) \cong \mathfrak{a}$. So we can interpret Θ_β as $\Theta_\beta \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$. Clearly Θ_β is bilinear and alternating. The Jacobi identity is proved by noticing that by linearity of β and the Jacobi identity on $\mathfrak{h}$ we have

$$\beta([X,[Y,Z]]) + \beta([Y,[Z,X]] + \beta([Z,[X,Y]]) = 0,$$

so that

$$\begin{aligned}
&\Theta_\beta(X,[Y,Z]) + \Theta_\beta(Y,[Z,X]) + \Theta_\beta(Z,[X,Y]) = \\
&\quad = [\beta(X), \beta([Y,Z])] + [\beta(Y), \beta([Z,X])] + [\beta(Z), \beta([X,Y])] = \\
&\quad = [\beta(X), ([\beta(Y),\beta(Z)] - \Theta_\beta(Y,Z))] + [\beta(Y), ([\beta(Z),\beta(X)] - \Theta_\beta(Z,X))] + \\
&\quad + [\beta(Z), ([\beta(X),\beta(Y)] - \Theta_\beta(X,Y))] = 0.
\end{aligned}$$

Here we have used again the Jacobi identity on $\mathfrak{h}$ and the fact that $\mathrm{Im}(\Theta_\beta) \in \mathfrak{a}$ with $[\mathfrak{a},\mathfrak{h}] = 0$. Thus, $\Theta_\beta \in Z^2(\mathfrak{g},\mathfrak{a})$. Moreover, $\mathfrak{h} \cong \mathfrak{g} \oplus \mathfrak{a}$ as vector spaces via the linear isomorphism

$$\psi \colon \mathfrak{g} \times \mathfrak{a} \to \mathfrak{h}, \quad (X,W) = X \oplus W \mapsto \beta(X) + W.$$

If we define the Lie bracket on $\mathfrak{g} \oplus \mathfrak{a}$ by

$$[X \oplus W, Y \oplus V]_{\mathfrak{g}\oplus\mathfrak{a}} := \beta([X,Y]_{\mathfrak{g}}) + \Theta_\beta(X,Y) \quad \forall X,Y \in \mathfrak{g}, \forall W,V \in \mathfrak{a}, \tag{2.2}$$

then the map ψ becomes a Lie algebra isomorphism. $\square$

Definition 2.10. Split exact sequence, trivial central extension. An exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

splits if there is a Lie algebra homomorphim $\beta \colon \mathfrak{g} \to \mathfrak{h}$ with $\pi \circ \beta = \mathrm{id}_{\mathfrak{g}}$. The homomorphism β is called a *splitting map*. A central extension which splits is called a *trivial extension*, since from the proof of Lemma 2.9 it is equivalent to the exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \oplus \mathfrak{a} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

where $\mathfrak{g} \oplus \mathfrak{a}$ has the Lie bracket $[X \oplus W, Y \oplus V]_{\mathfrak{g}\oplus\mathfrak{a}} = \beta([X,Y]_{\mathfrak{g}})$.

The following proposition "mods out" the trivial cases.

Proposition 2.11. *There exists a bijection between $H^2(\mathfrak{g}, \mathfrak{a})$ and the set of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$.*

Proof. Using Lemma 2.9 all that is left to prove is that two elements $\Theta, \Omega \in Z^2(\mathfrak{g}, \mathfrak{a})$ are such that $\Theta - \Omega \in B^2(\mathfrak{g}, \mathfrak{a})$ if and only if the central extensions defined by Θ and Ω are equivalent. Equivalently, we must show that every trivial extension is an extension defined by a coboundary and vice versa.

So let $\Theta \in B^2(\mathfrak{g}, \mathfrak{a})$, i.e. $\Theta(X, Y) = \mu([X, Y])$ for some $\mu \in \mathrm{Hom}_F(\mathfrak{g}, \mathfrak{a})$. Define a linear map $\beta\colon \mathfrak{g} \to \mathfrak{h}(\cong \mathfrak{g} \oplus \mathfrak{a})$ by $\beta(X) := X + \mu(X)$, $\forall X \in \mathfrak{g}$. Then

$$\begin{aligned} \beta([X, Y]) &= [X, Y]_{\mathfrak{g}} + \mu([X, Y]) = [X, Y]_{\mathfrak{g}} + \Theta(X, Y) \\ &= [X + \mu(X), Y + \mu(Y)]_{\mathfrak{h}} = [\beta(X), \beta(Y)]_{\mathfrak{h}}, \end{aligned}$$

i.e. β is a Lie algebra homomorphism. Hence, β is a splitting map.

Conversely, given a splitting map $\beta\colon \mathfrak{g} \to \mathfrak{h}(\cong \mathfrak{g} \oplus \mathfrak{a})$, then β has to be of the form $\beta(X) = X + \mu(X)$, $\forall X \in \mathfrak{g}$, for some suitable $\mu \in \mathrm{Hom}_F(\mathfrak{g}, \mathfrak{a})$ since $\pi \circ \beta = \mathrm{id}_{\mathfrak{g}}$. By definition of the bracket on $\mathfrak{h}$, $[\beta(X), \beta(Y)] = [X, Y] + \Theta(X, Y)$ for all $X, Y \in \mathfrak{g}$. Moreover, since β is a Lie algebra homomorphism we have $[\beta(X), \beta(Y)] = \beta([X, Y]) = [X, Y] + \mu([X, Y])$. Hence $\Theta(X, Y) = \mu([X, Y])$. □

Proposition 2.12. *A Lie algebra $\mathfrak{g}$ admits a universal central extension if and only if $\mathfrak{g}$ is perfect.*

Proof. First suppose that $\pi\colon \mathfrak{h} \to \mathfrak{g}$ is the universal central extension. By definition, $\mathfrak{h}$ is perfect. Hence,

$$\mathfrak{g} = \pi(\mathfrak{h}) = \pi([\mathfrak{h}, \mathfrak{h}]) = [\pi(\mathfrak{h}), \pi(\mathfrak{h})] = [\mathfrak{g}, \mathfrak{g}].$$

Now assume that $\mathfrak{g}$ is perfect. We set

$$\begin{aligned} W' \quad &:= \bigwedge\nolimits^2 \mathfrak{g} = (\mathfrak{g} \otimes \mathfrak{g})/\langle X \otimes Y + Y \otimes X \mid X, Y \in \mathfrak{g}\rangle_F\,, \\ B_2(\mathfrak{g}, F) &:= \{X \wedge [Y, Z] + Y \wedge [Z, X] + Z \wedge [X, Y] \mid X, Y, Z \in \mathfrak{g}\}\,, \end{aligned}$$

and $W := W'/B_2(\mathfrak{g}, F)$. Let $\Omega\colon W' \to W$ be the canonical projection. By definition, $\Omega \in Z^2(\mathfrak{g}, W)$. Let

$$0 \longrightarrow W \xrightarrow{\ \iota\ } \mathfrak{h}_\Omega \xrightarrow{\ \pi_\Omega\ } \mathfrak{g} \longrightarrow 0$$

be the central extension defined by Ω. Using this central extension, we construct the universal central extension of $\mathfrak{g}$.

Let $\mathfrak{a}$ be an arbitrary F-vector space (a Lie algebra with trivial bracket) and $\Theta \in Z^2(\mathfrak{g}, \mathfrak{a})$. Since $\Theta(X, Y) = -\Theta(Y, X)$, we have a F-linear map

$$\psi \colon W \to \mathfrak{a} \quad \text{such that} \quad \Omega(X, Y) \mapsto \Theta(X, Y).$$

We define $\phi' \colon \mathfrak{h}_\Omega \to \mathfrak{h}_\Theta$ by

$$\phi'((X, U)) := (X, \psi(U)).$$

Then, it is clear that the diagram

$$\begin{tikzcd} \mathfrak{h}_\Omega \arrow[r, "\pi_\Omega"] \arrow[d, "\phi'"] & \mathfrak{g} \arrow[d, "\mathrm{id}"] \\ \mathfrak{h}_\Theta \arrow[r, "\pi_\Theta"] & \mathfrak{g} \end{tikzcd}$$

commutes. Now set

$$\hat{\mathfrak{h}} := [\mathfrak{h}_\Omega, \mathfrak{h}_\Omega].$$

Since $\mathfrak{g}$ is perfect, it follows that $\hat{\mathfrak{h}} \oplus W = \mathfrak{h}_\Omega$. This implies that

$$\hat{\mathfrak{h}} = [\hat{\mathfrak{h}} \oplus W, \hat{\mathfrak{h}} \oplus W] = [\hat{\mathfrak{h}}, \hat{\mathfrak{h}}],$$

i.e. $\hat{\mathfrak{h}}$ is perfect. Moreover, if we set

$$\mathfrak{c} := W \cap \hat{\mathfrak{h}},$$

then we obtain a central extension

$$0 \longrightarrow \mathfrak{c} \longrightarrow \hat{\mathfrak{h}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

such that $\hat{\mathfrak{h}}$ is perfect. Defining $\phi := \phi'|_{\hat{\mathfrak{h}}}$ we get a commutative diagram

$$\begin{tikzcd} \hat{\mathfrak{h}} \arrow[r, "\pi_\Omega|_{\hat{\mathfrak{h}}}"] \arrow[d, "\phi"] & \mathfrak{g} \arrow[d, "\mathrm{id}"] \\ \mathfrak{h}_\Theta \arrow[r, "\pi_\Theta"] & \mathfrak{g}\,. \end{tikzcd}$$

Therefore, $0 \longrightarrow \mathfrak{c} \longrightarrow \hat{\mathfrak{h}} \longrightarrow \mathfrak{g} \longrightarrow 0$ is the universal central extension. $\square$

2.2 Witt Algebra

Our main references for this section are [Sch08] and [KR87].

The goal of this section is to prove that the Virasoro algebra is the unique universal nontrivial central extension of the Witt algebra.

Definition 2.13. Lie algebra of smooth vector fields. Let M be a smooth compact manifold. The space $\mathfrak{X}(M)$ is the *space of smooth vector fields* on M. Here we consider $X \in \mathfrak{X}(M)$ as a derivation $X \colon C^\infty(M) \to C^\infty(M)$, i.e. as an $\mathbb{R}$-linear map with

$$X(fg) = X(f)g + fX(g) \quad \forall f, g \in C^\infty(M).$$

The Lie bracket of $X, Y \in \mathfrak{X}(M)$ is the *commutator*

$$[X, Y] := X \circ Y - Y \circ X$$

which is also a derivation. Consequently, $(\mathfrak{X}(M), [\cdot, \cdot])$ is an infinite dimensional Lie algebra over $\mathbb{R}$.

We will be interested in the case when $M = S^1$. In this case, the space $C^\infty(S^1)$ can be described as the vector space $C^\infty_{2\pi}(\mathbb{R})$ of 2π-periodic functions $\mathbb{R} \to \mathbb{R}$. Then $\mathfrak{X}(S^1) = \{f\frac{d}{d\theta} | f \in C^\infty_{2\pi}(\mathbb{R})\}$ and $S^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$. For $X = f\frac{d}{d\theta}$ and $Y = g\frac{d}{d\theta}$ we get

$$[X, Y] = (fg' - f'g)\frac{d}{d\theta}.$$

Since f is smooth and periodic, it can be represented by a convergent Fourier series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\theta) + b_n \sin(n\theta)).$$

This leads to a natural (topological) generating system for $\mathfrak{X}(S^1)$:

$$\frac{d}{d\theta}, \quad \cos(n\theta)\frac{d}{d\theta}, \quad \sin(n\theta)\frac{d}{d\theta}.$$

Complexifying $\mathfrak{X}(S^1)$, i.e. by defining $\mathfrak{X}^{\mathbb{C}}(S^1) := \mathfrak{X}(S^1) \otimes \mathbb{C}$, we finally arrive at:

Definition 2.14. Witt algebra. The *Witt algebra* W is the linear span of L_n's over $\mathbb{C}$:

$$\mathsf{W} := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n,$$

where $L_n := z^{1-n}\frac{d}{dz} = -iz^{-n}\frac{d}{d\theta} = -ie^{-in\theta}\frac{d}{d\theta} \in \mathfrak{X}^{\mathbb{C}}(S^1)$ with $z = e^{i\theta}$ and $n \in \mathbb{Z}$.

We note that $L_n \colon C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C}),\ f \mapsto z^{1-n}f'$, so that to prove that W with the Lie bracket in $\mathfrak{X}^{\mathbb{C}}(S^1)$ is actually a Lie algebra over $\mathbb{C}$ we need to show that $[\mathsf{W}, \mathsf{W}] \subset \mathsf{W}$.

For $m, n \in \mathbb{Z}$ and $f \in C^\infty(S^1, \mathbb{C})$

$$L_m L_n f = z^{1-m}\frac{d}{dz}\left(z^{1-n}\frac{d}{dz}f\right) = (1-n)z^{1-m-n}\frac{d}{dz}f + z^{1-m}z^{1-n}\frac{d^2}{dz^2}f.$$

Therefore

$$\begin{aligned}[L_m, L_n]f &= L_m L_n f - L_n L_m f = ((1-n) - (1-m))z^{1-m-n}\frac{d}{dz}f \\ &= (m-n)L_{m+n}f\end{aligned}$$

as required. Note that this actually implies that $[\mathsf{W}, \mathsf{W}] = \mathsf{W}$, i.e. that W is perfect.

Theorem 2.15. $\dim H^2(\mathsf{W}, \mathbb{C}) = 1$.

Proof. Given an $\omega \in Z^2(\mathsf{W}, \mathbb{C})$, define $g_\omega : \mathsf{W} \to \mathbb{C}$ by

$$g_\omega(L_n) := \begin{cases} \omega(L_0, L_n)/n & \text{if } n \neq 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Then $\hat{\omega}(x, y) := \omega(x, y) + g_\omega([x, y])$ is such that

$$\hat{\omega} \in Z^2(\mathsf{W}, \mathbb{C}), \quad \omega - \hat{\omega} \in B^2(\mathsf{W}, \mathbb{C}) \quad \text{and} \quad \hat{\omega}(L_0, x) = 0 \quad \forall x \in \mathsf{W}.$$

Hence for any $\omega + B^2(\mathsf{W}, \mathbb{C}) \in H^2(\mathsf{W}, \mathbb{C})$ we can take its representative ω such that $\omega(L_0, x) = 0\ \forall x \in \mathsf{W}$. Moreover, since $\omega \in Z^2(\mathsf{W}, \mathbb{C})$, we have

$$\omega(L_m, [L_n, L_k]) + \omega(L_n, [L_k, L_m]) + \omega(L_k, [L_m, L_n]) = 0,$$

and therefore

$$(n-k)\omega(L_m, L_{n+k}) + (k-m)\omega(L_n, L_{k+m}) + (m-n)\omega(L_k, L_{m+n}) = 0. \quad (2.3)$$

First set $k = 0$ in (2.3) to get

$$(n+m)\omega(L_m, L_n) = 0,$$

since $\omega(L_0, x) = 0$ and $\omega(x, y) = -\omega(y, x)$. This implies that

$$\omega(L_m, L_n) = \delta_{m+n,0} f(m)$$

for some $f\colon \mathbb{Z} \to \mathbb{C}$ such that $-f(-m) = f(m)$ since $\omega(L_m, L_n) = -\omega(L_n, L_m)$. Plugging this into (2.3) with $m + n + k = 0$ gives

$$(2n + m)f(m) - (n + 2m)f(n) + (n - m)f(m + n) = 0. \tag{2.4}$$

Setting $m = 1$ in (2.4) gives us a linear recursion relation:

$$(n - 1)f(n + 1) = (n + 2)f(n) - (2n + 1)f(1). \tag{2.5}$$

Since $f(-n) = -f(n)$ we have $f(0) = 0$ and thus we have to solve (2.5) only for $n > 0$. The space of solutions of Equation (2.5) is at most 2-dimensional because if we know $f(1)$ and $f(2)$ we can calculate all $f(n)$'s using (2.5). Note that $f(n) = n$ and $f(n) = n^3$ are solutions. Hence the general solution is $f(n) = \alpha n + \beta n^3$, where $\alpha, \beta \in \mathbb{C}$. However $\omega \in B^2(\mathsf{W}, \mathbb{C})$ if and only if $f(n) = \alpha n$ (otherwise $f(n)$ is non-linear and thus $\omega \notin B^2(\mathsf{W}, \mathbb{C})$). Hence, to get a nontrivial central extension, we must set $\beta \neq 0$ and α can be arbitrary, so following the usual convention we set $\alpha := -\beta$. Hence, $f(n) = \beta(n^3 - n)$ and

$$\omega(L_m, L_n) = \delta_{m+n,0}\beta(n^3 - n). \tag{2.6}$$

Therefore, $\dim H^2(\mathsf{W}, \mathbb{C}) = 1$. $\square$

Combining the above theorem with the previous observation that the Witt algebra is perfect, we can define the Virasoro algebra as the unique universal nontrivial central extension of W by $\mathbb{C}$.

Definition 2.16. Virasoro algebra. The *Virasoro algebra*

$$\mathsf{Vir} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$$

is the Lie algebra which satisfies the following commutation relations:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}\,, \\ [\mathsf{Vir}, C] &= 0. \end{aligned}$$

Remark 2.17. The Virasoro algebra is defined by the nontrivial cocycle $\omega \in H^2(\mathsf{W}, \mathbb{C})$ from Theorem 2.15 by setting $\beta = C/12$ in Equation (2.6). Cf. Equation (2.1) and Equation (2.2) from the proof of Lemma 2.9.

2.3 Representation Theory of Virasoro Algebra

Our main references for this section are [Sch08] and [KR87].

Let V be a vector space over $\mathbb{C}$.

Definition 2.18. Hermitian form, inner product. A map

$$\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$$

is called a *Hermitian form* if it is complex antilinear in the first variable, complex linear in the second and satisfies

$$\langle v, w \rangle = \overline{\langle w, v \rangle} \quad \forall v, w \in V.$$

A Hermitian form is an *inner product* if moreover we have

$$\langle v, v \rangle > 0 \quad \forall v \in V \setminus \{0\}.$$

Definition 2.19. Unitary representation of Virasoro algebra. A map $\rho\colon \mathsf{Vir} \to \operatorname{End}_{\mathbb{C}} V$ is called a *representation* if it is a Lie algebra homomorphism. The representation ρ is called *unitary* if there is a positive semidefinite Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that $\forall v, w \in V$ and $\forall n \in \mathbb{Z}$ we have

$$\begin{aligned} \langle \rho(L_n)v, w \rangle &= \langle v, \rho(L_{-n})w \rangle, \\ \langle \rho(C)v, w \rangle &= \langle v, \rho(C)w \rangle. \end{aligned}$$

Definition 2.20. Cyclic vector. A vector $v \in V$ is called a *cyclic vector* for a representation $\rho\colon \mathsf{Vir} \to \operatorname{End} V$ if the set

$$\{\rho(X_1) \ldots \rho(X_m)v \mid X_j \in \mathsf{Vir} \text{ with } j \in \{1, \ldots, m\} \text{ and } m \in \mathbb{N}\}$$

spans the vector space V.

Definition 2.21. Highest weight representation, Virasoro module. A representation $\rho\colon \mathsf{Vir} \to \operatorname{End} V$ is called a *highest weight representation* if there are complex numbers $h, c \in \mathbb{C}$ and a cyclic vector $v_0 \in V$ such that

$$\begin{aligned} \rho(C)v_0 &= cv_0, \\ \rho(L_0)v_0 &= hv_0, \\ \rho(L_n)v_0 &= 0 \quad \forall n \in \mathbb{N}. \end{aligned}$$

The vector v_0 is then called the *highest weight vector* (or *vacuum vector*) and V is called a *Virasoro module* (via ρ) with *highest weight* (c, h) or simply a *Virasoro module for* (c, h).

Remark 2.22. The operator L_0 is often interpreted as the energy operator which is assumed to be diagonalizable with its spectrum bounded from below. With this assumption and the assumption that v_0 is an eigenvector of $\rho(L_0)$ with lowest eigenvalue $h \in \mathbb{R}$, any representation ρ preserving the energy spectrum property satisfies $\rho(L_n)v_0 = 0\ \forall n \in \mathbb{N}$. This follows by noting that for $w := \rho(L_n)v_0$ we have

$$\rho(L_0)w = \rho(L_n)\rho(L_0)v_0 - n\rho(L_n)v_0 = \rho(L_n)hv_0 - nw = (h-n)w.$$

Since we assumed h to be the lowest eigenvalue of $\rho(L_0)$, w has to vanish for $n > 0$.

Definition 2.23. Verma module. A *Verma module* for $c, h \in \mathbb{C}$ is a complex vector space $M(c,h)$ with a highest weight representation

$$\rho\colon \mathsf{Vir} \to \operatorname{End}_{\mathbb{C}} M(c,h)$$

and a highest weight vector $v_0 \in M(c,h)$ such that

$$\{\rho(L_{-n_1})\dots\rho(L_{-n_k})v_0 \mid n_1 \geq \dots \geq n_k \geq 1,\ k \in \mathbb{N}\} \cup \{v_0\} \tag{2.7}$$

is a vector space basis of $M(c,h)$.

Note that by the definition for fixed $c, h \in \mathbb{C}$ the Verma module $M(c,h)$ is unique up to isomorphism.

Lemma 2.24. *For all $c, h \in \mathbb{C}$ there exists a Verma module $M(c,h)$.*

Proof. Let

$$M(c,h) := \mathbb{C}v_0 \bigoplus \mathbb{C}\{v_{n_1\dots n_k} : n_1 \geq \dots \geq n_k \geq 1\}$$

be the complex vector space spanned by v_0 and $v_{n_1\dots n_k}$'s for $n_1 \geq \dots \geq n_k \geq 1$. We define a map

$$\rho\colon \mathsf{Vir} \to \operatorname{End}_{\mathbb{C}}(M(c,h))$$

by

$$\begin{aligned}
\rho(C) &:= c\,\mathrm{id}_{M(c,h)},\\
\rho(L_0)v_0 &:= hv_0,\\
\rho(L_0)v_{n_1\dots n_k} &:= \left(\sum_{j=1}^{k} n_j + h\right) v_{n_1\dots n_k},\\
\rho(L_n)v_0 &:= 0 \qquad \forall n \in \mathbb{N},\\
\rho(L_{-n})v_0 &:= v_n \qquad \forall n \in \mathbb{N},\\
\rho(L_{-n})v_{n_1\dots n_k} &:= v_{nn_1\dots n_k}\ \ \forall n \geq n_1.
\end{aligned}$$

For all other $v_{n_1 \dots n_k}$'s with $1 \leq n < n_1$, $\rho(L_{-n})v_{n_1 \dots n_k}$ can be obtained by permutation using the commutation relations $[L_m, L_n] = (m-n)L_{m+n}$ for $m \neq -n$. E.g. for $n_1 > n \geq n_2$:

$$\begin{aligned}\rho(L_{-n})v_{n_1\dots n_k} &= \rho(L_{-n})\rho(L_{-n_1})v_{n_2\dots n_k} \\ &= \big(\rho(L_{-n_1})\rho(L_{-n}) + (-n+n_1)\rho(L_{-(n+n_1)})\big)\, v_{n_2\dots n_k} \\ &= v_{n_1 n n_2 \dots n_k} + (n_1 - n)v_{(n_1+n)n_2\dots n_k}.\end{aligned}$$

So the above calculation guides us to define

$$\rho(L_{-n})v_{n_1\dots n_k} := v_{n_1 n n_2\dots n_k} + (n_1-n)v_{(n_1+n)n_2\dots n_k}.$$

Similarly we define $\rho(L_n)v_{n_1\dots n_k}$ $\forall n \in \mathbb{N}$ taking into account the commutation relations, e.g.

$$\rho(L_n)v_{n_1} := \begin{cases} 0 & n > n_1, \\ \left(2nh + \frac{n}{12}(n^2-1)c\right) v_0 & n = n_1, \\ (n+n_1)v_{n_1-n} & 0 < n < n_1. \end{cases}$$

Thus ρ is well-defined and $\mathbb{C}$-linear. It remains to show that ρ respects the commutation relations, so that it is actually a Lie algebra representation, i.e. that

$$[\rho(L_m), \rho(L_n)] = \rho([L_m, L_n]).$$

E.g. for $n \geq n_1$ we have

$$\begin{aligned}[\rho(L_0), \rho(L_{-n})]v_{n_1\dots n_k} &= \rho(L_0)v_{nn_1\dots n_k} - \rho(L_{-n})\left(\sum_{j=1}^{k} n_j + h\right) v_{n_1\dots n_k} \\ &= \left(\sum_{j=1}^{k} n_j + n + h\right) v_{nn_1\dots n_k} - \left(\sum_{j=1}^{k} n_j + h\right) v_{nn_1\dots n_k} \\ &= n v_{nn_1\dots n_k} = n\rho(L_{-n})v_{n_1\dots n_k} \\ &= \rho\left([L_0, L_{-n}]\right) v_{n_1\dots n_k}\end{aligned}$$

and for $n > m > n_1$

$$\begin{aligned}[\rho(L_{-m}), \rho(L_{-n})]v_{n_1\dots n_k} &= \rho(L_{-m})v_{nn_1\dots n_k} - v_{nmn_1\dots n_k} \\ &= v_{n\,m\,n_1\dots n_k} + (n-m)v_{(n+m)\,n_1\dots n_k} - v_{n\,m\,n_1\dots n_k} \\ &= (n-m)v_{(n+m)\,n_1\dots n_k} = (n-m)\rho(L_{-(m+n)})v_{n_1\dots n_k} \\ &= \rho([L_{-m}, L_{-n}])v_{n_1\dots n_k}.\end{aligned}$$

Other identities follow similarly. Hence ρ is a highest weight representation. Thus, by construction $M(c, h)$ is a Verma module. □

Corollary 2.25. Any Virasoro module V of highest weight (c, h) is isomorphic to a quotient of the corresponding Verma module $M(c, h)$. In particular, (c, h) determines $M(c, h)$ uniquely.

Proof. There exists a surjective homomorphism from $M(c, h)$ to V which maps the highest weight vector of $M(c, h)$ to the highest weight vector of V and commutes with the action of Vir since in $M(c, h)$ the set of vectors (2.7) is linearly independent. Hence quotienting out the kernel of this homomorphism we obtain the desired result. $\square$

Definition 2.26. Submodule of Virasoro module. A *submodule* U of a Virasoro module V is a $\mathbb{C}$-linear subspace of V with $\rho(D)U \subset U \;\; \forall D \in \mathsf{Vir}$, i.e. it is an invariant linear subspace of V.

Remark 2.27. Let V be a Virasoro module for $c, h \in \mathbb{C}$. Then there exists a direct sum decomposition $V = \bigoplus_{N \in \mathbb{N}_0} V_N$ where $V_0 := \mathbb{C}v_0$ and

$$V_N := \operatorname{span}\left(\left\{\rho(L_{-n_1})\dots\rho(L_{-n_k})v_0 \,\middle|\, n_1 \geq \dots \geq n_k \geq 1, \sum_{j=1}^{k} n_j = N, k \in \mathbb{N}\right\}\right).$$

The V_N's are eigenspaces of $\rho(L_0)$ with the eigenvalue $N + h$, i.e.

$$\rho(L_0)|_{V_N} = (N + h)\,\mathrm{id}_{V_N}.$$

This follows from the definition of a Virasoro module and from the commutation relations.

Lemma 2.28. *Let V be a Virasoro module for $c, h \in \mathbb{C}$ and U a submodule of V. Then*

$$U = \bigoplus_{N \in \mathbb{N}_0} (V_N \cap U).$$

Proof. Let $w = w_0 \oplus \dots \oplus w_s \in U$ with $w_j \in V_j$ for $j \in \{0, \dots, s\}$. Then

$$\begin{aligned} w &= w_0 + \dots + w_s, \\ \rho(L_0)w &= hw_0 + \dots + (s + h)w_s, \\ &\;\;\vdots \\ \rho(L_0)^s w &= h^s w_0 + \dots + (s + h)^s w_s. \end{aligned}$$

This is a system of linear equations for $w_0, \dots, w_s$ with a regular coefficient matrix. Hence, the $w_0, \dots, w_s$ are linear combinations of the $w, \dots, \rho(L_0)^s w \in U$. Thus $w_j \in V_j \cap U \;\; \forall j \in \{0, \dots, s\}$. $\square$

We will mostly need unitary representations of the Virasoro algebra. To define a suitable Hermitian form, we need the notion of an expectation value first.

Definition 2.29. Expectation value. Let $V = \bigoplus_{N\in\mathbb{N}_0} V_N$ be a Virasoro module and $w \in V$. Then according to Remark 2.27, w has a unique component $w_0 \in V_0$ with respect to the decomposition $\bigoplus_{N\in\mathbb{N}_0} V_N$. The *expectation value* of w, denoted $< w >$, is the coefficient of $w_0 \in V_0$ with respect to the basis v_0, i.e. $w_0 =< w > v_0$.

In what follows we will often abuse our notation and simply write L_n for $\rho(L_n)$.

Definition 2.30. Hermitian form on Verma module. Let $M = M(c, h)$ with $c, h \in \mathbb{R}$ be the Verma module with a highest weight representation $\rho\colon \mathsf{Vir} \to \mathrm{End}_{\mathbb{C}}(M(c, h))$ and let v_0 be the respective highest weight vector. A *Hermitian form on* M with respect to the basis $\{v_{n_1\dots n_k}\} \cup \{v_0\}$ is defined as

$$\langle v_{n_1\dots n_k}, v_{m_1\dots m_j}\rangle :=< L_{n_k} \dots L_{n_1} v_{m_1\dots m_j} >=< L_{n_k} \dots L_{n_1} L_{-m_1} \dots L_{-m_j} v_0 > .$$

Note that from the above definition it follows that

$$\langle v_0, v_0\rangle = 1 \quad \text{and} \quad \langle v_0, v_{n_1\dots n_k}\rangle = 0 = \langle v_{n_1\dots n_k}, v_0\rangle.$$

The condition $c, h \in \mathbb{R}$ implies that $\langle v, v'\rangle = \langle v', v\rangle$ for all basis vectors

$$v, v' \in B := \{v_{n_1\dots n_k} \mid n_1 \geq \cdots \geq n_k \geq 1\} \cup \{v_0\}.$$

The proof of the above consists of repeated use of the commutation relations of L_n's.

The map $\langle\cdot,\cdot\rangle\colon B \times B \to \mathbb{R}$ has an $\mathbb{R}$-bilinear continuation to $M \times M$ which is $\mathbb{C}$-antilinear in the first and $\mathbb{C}$-linear in the second variable: for $w, w' \in M$ with unique representations $w = \sum \lambda_j w_j$, $w' = \sum \mu_k w'_k$ with respect to basis vectors $w_j, w'_k \in B$, one defines

$$\langle w, w'\rangle := \sum\sum \overline{\lambda}_j \mu_k \langle w_j, w'_k\rangle.$$

By the above discussion, the map $\langle\cdot,\cdot\rangle\colon M \times M \to \mathbb{C}$ is a Hermitian form. However, it is not positive definite or positive semidefinite in general. To check this, the Kac determinant is used. Before defining it, we need some more results about the Hermitian form.

Theorem 2.31. *Let $c, h \in \mathbb{R}$ and $M = M(c, h)$. Then*

(a) $\langle\cdot,\cdot\rangle\colon M\times M\to\mathbb{C}$ *is the unique Hermitian form satisfying* $\langle v_0,v_0\rangle=1$, $\langle L_n v,w\rangle=\langle v,L_{-n}w\rangle$ *and* $\langle Cv,w\rangle=\langle v,Cw\rangle\quad\forall v,w\in M,\ \forall n\in\mathbb{Z}$.

(b) *The eigenspaces of* L_0 *are pairwise orthogonal, i.e. if* $M\neq N$, *then* $\langle v,w\rangle=0\quad\forall v\in V_M,\forall w\in V_N$.

(c) *The maximum proper submodule of* M *is* $\ker\langle\cdot,\cdot\rangle$.

Proof. (a) That the identity

$$\langle L_n v,w\rangle=\langle v,L_{-n}w\rangle$$

holds can be seen using commutation relations. The uniqueness of such a form follows from

$$\langle v_{n_1\ldots n_k},v_{m_1\ldots m_j}\rangle=\langle v_0,L_{n_k}\ldots L_{n_1}v_{m_1\ldots m_j}\rangle.$$

(b) Assume that $N>M$. Then any $\langle v,w\rangle$ with $v\in V_N$ and $w\in V_M$ can be written as a sum of elements of the form $< L_{n_k}\ldots L_{n_1}L_{-m_1}\ldots L_{-m_j}v_0 >$ with $n_1+\ldots n_k=N$ and $m_1+\ldots m_j=M$. However using the commutation relations we can move L_n's to front and get a sum of expectation values where $L_s, s\in\mathbb{N}$, acts directly on v_0. Thus, $< L_{n_k}\ldots L_{n_1}L_{-m_1}\ldots L_{-m_j}v_0 >= 0$ and hence $\langle v,w\rangle=0$.

(c) If $v\in\ker\langle\cdot,\cdot\rangle:=\{u\in M\mid\langle w,u\rangle=0\ \forall w\in M\}$, then $L_n v\in\ker\langle\cdot,\cdot\rangle\ \forall n\in\mathbb{Z}$ because $\langle w,L_n v\rangle=\langle L_{-n}w,v\rangle=0$. Moreover, $v_0\notin M$ since $\langle v_0,v_0\rangle=1$. Hence, $\ker\langle\cdot,\cdot\rangle$ is a proper submodule of M.

To prove maximality, let $U\subset M$ be an arbitrary proper submodule and let $u\in U$. For $n_1\geq\cdots\geq n_k\geq 1$ one has $\langle v_{n_1\ldots n_k},u\rangle=\langle v_0,L_{n_k}\ldots L_{n_1}u\rangle$. If $\langle v_{n_1\ldots n_k},u\rangle\neq 0$, then $< L_{n_k}\ldots L_{n_1}u >\neq 0$. By Lemma 2.28 and part (b) of the current theorem we see that in this case $v_0\in U$ because $L_{n_k}\ldots L_{n_1}u\in U$, and that $v_{n_1\ldots n_k}\in U$. Since $v_{n_1\ldots n_k}$ is an arbitrary basis vector of M, this implies that $M=U$ contradicting properness of $U\subset M$. Thus, $\langle v_{n_1\ldots n_k},u\rangle=0$. Similarly, $\langle v_0,u\rangle=0$, so $u\in\ker\langle\cdot,\cdot\rangle$. □

Remark 2.32. $M(c,h)/\ker\langle\cdot,\cdot\rangle$ is a Virasoro module with a nondegenerate Hermitian form $\langle\cdot,\cdot\rangle$. However, in general $\langle\cdot,\cdot\rangle$ is not definite.

Corollary 2.33. If $\langle\cdot,\cdot\rangle$ is positive semidefinite, then $c\geq 0$ and $h\geq 0$.

Proof. We have

$$\langle v_n,v_n\rangle=\langle v_0,L_nL_{-n}v_0\rangle=\langle v_0,[L_n,L_{-n}]v_0\rangle=2nh+\frac{c}{12}(n^3-n)\quad\forall n\in\mathbb{N}.$$

Now $\langle v_1,v_1\rangle\geq 0\iff h\geq 0$. Moreover, $\langle v_n,v_n\rangle\geq 0\iff 2nh+\frac{c}{12}(n^3-n)\geq 0$. Therefore, $\langle v_n,v_n\rangle\geq 0$ is valid for all $n\in\mathbb{N}$ if and only if $c\geq 0$ and $h\geq 0$. □

We need some general results before continuing with unitarity.

Definition 2.34. (In)decomposable representation. A representation M is *indecomposable* if there are no invariant proper subspaces U, V of M such that $M = U \oplus V$. Otherwise M is *decomposable*.

Definition 2.35. (Ir)reducible representation. A representation M is called *irreducible* if there is no invariant proper subspace V of M. Otherwise M is called *reducible*.

Theorem 2.36. *For each (c, h) we have*

(a) *The Verma module $M(c, h)$ is indecomposable.*

(b) *There exists a unique maximal subrepresentation $J(c, h)$ of $M(c, h)$ and*

$$L(c, h) := M(c, h)/J(c, h) \tag{2.8}$$

is the unique irreducible highest weight representation with highest weight (c, h).

Proof. (a) Let V, W be invariant subspaces of $M = M(c, h)$ and $M = V \oplus W$. By Lemma 2.28 there exist direct sum decompositions

$$V = \bigoplus (M_j \cap V) \quad \text{and} \quad W = \bigoplus (M_j \cap W).$$

Since $\dim M_0 = 1$, this implies that $M_0 \cap V = 0$ or $M_0 \cap W = 0$. So the highest weight vector v_0 is contained in V or in W. But if v_0 belongs to a subrepresentation, then this subrepresentation must coincide with M.

(b) By Lemma 2.28 all proper subrepresentations are graded. Thus, their sum is graded too. The sum is also a proper subrepresentation since it does not contain the vacuum vector v_0. The maximal subrepresentation $J(c, h)$ is thus the sum of all proper subrepresentations. Hence the proof.

□

Remark 2.37. Combining Theorem 2.36 (b) with Theorem 2.31 (c) we see that $J(c, h) = \ker\langle\cdot, \cdot\rangle$ and hence $L(c, h)$ is the unique unitary positive definite highest weight representation of Vir, provided that $M(c, h)$ is unitary and positive semidefinite. Indeed, if $\rho : \mathsf{Vir} \to End_{\mathbb{C}}(V)$ is a positive definite unitary highest weight representation with vacuum vector $v_0' \in V$ and Hermitian form $\langle\cdot, \cdot\rangle'$ we can define a surjective linear homomorphism $\varphi : M(c, h) \to V$

$$v_0 \mapsto v_0', \quad v_{n_1 \dots n_k} \mapsto \rho(L_{-n_1 \cdots -n_k}) v_0',$$

which also respects the Hermitian forms:

$$\langle \varphi(v), \varphi(w) \rangle' = \langle v, w \rangle.$$

Therefore, $\langle\cdot,\cdot\rangle$ is positive semidefinite and φ factorizes over $L(c,h)$ leading to an isomorphism $\bar{\varphi} : L(c,h) \to V$.

Definition 2.38. Let $P(N) := dim_{\mathbb{C}} V_N$ and $\{b_1, \dots, b_{P(N)}\}$ be a basis of V_N. We define matrices A^N by $A^N_{ij} := \langle b_i, b_j\rangle$ for $i,j \in \{1,\dots,P(N)\}$.

Clearly, $\langle\cdot,\cdot\rangle$ is positive semidefinite if all matrices A^N are positive semidefinite. For $N = 0$ and $N = 1$ we get $A^0 = (1)$ and $A^1 = (2h)$ with respect to the bases $\{v_0\}$ and $\{v_1\}$. For example, to get A^2 we calculate

$$\begin{aligned}\langle v_2, v_2\rangle &=< L_2 L_{-2} v_0 >=< 4L_0 v_0 + \frac{c}{2} v_0 >= 4h + \frac{c}{2},\\ \langle v_{1,1}, v_{1,1}\rangle &= 8h^2 + 4h,\\ \langle v_2, v_{1,1}\rangle &= 6h.\end{aligned}$$

Thus, relative to the basis $\{v_2, v_{1,1}\}$

$$A^2 = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 8h^2 + 4h \end{pmatrix}.$$

Therefore, A^2 is (for $c \geq 0$ and $h \geq 0$) positive semidefinite if and only if

$$\det A^2 = 2h(16h^2 - 10h + 2hc + c) \geq 0.$$

Theorem 2.39. Kac determinant formula. *The Kac determinant* $\det A^N$ *depends on* (c,h) *as follows*

$$\det A^N(c,h) = K_N \prod_{\substack{p,q\in\mathbb{N}\\ pq\leq N}} (h - h_{p,q}(c))^{P(N-pq)},$$

where $K_N \geq 0$ *is a constant,* $P(N)$ *is as in Definition 2.38 and*

$$h_{p,q}(c) := \frac{1}{48}((13-c)(p^2+q^2) + \sqrt{(c-1)(c-25)}(p^2-q^2) - 24pq - 2 + 2c).$$

For a proof check [KR87, Chap. 8] or [IK11, Chap. 4].

Theorem 2.40. *Let* $c,h \in \mathbb{R}$.

(i) $M(c,h)$ *is unitary positive definite for* $c > 1, h > 0$ *and positive semidefinite for* $c \geq 1, h \geq 0$.

(ii) $M(c,h)$ *is unitary in the region* $0 \leq c < 1$, $h > 0$ *if and only if* $(c,h) = (c(m), h_{p,q}(m))$ *where*

$$c(m) = 1 - \frac{6}{(m+2)(m+3)}, \quad m \in \mathbb{N}_0, \tag{2.9}$$

$$h_{p,q}(m) = \frac{((m+3)p-(m+2)q)^2-1}{4(m+2)(m+3)}, \quad p,q \in \mathbb{N} \text{ and } 1 \leq p \leq q \leq m+1.$$

For a proof of (ii) see [FQS86] where the authors have shown that the Hermitian form $\langle\cdot,\cdot\rangle$ can only be unitary in the region $0 \leq c < 1$ for $(c(m), h_{p,q}(m))$ and [GKO86] where the authors have proved that $M(c,h)$ actually gives a unitary representation in all these cases.

To prove part (i) we first need an example of a Virasoro algebra representation.

2.3.1 Fock Space Representation of Virasoro Algebra

Definition 2.41. Heisenberg algebra. Let H be the *Heisenberg algebra*, the complex Lie algebra with a basis $\{a_n, \hbar \mid n \in \mathbb{Z}\}$ subject to the commutation relations

$$[a_m, a_n] = m\delta_{m+n,0}\hbar, \qquad [\hbar, a_n] = 0 \qquad \forall m,n \in \mathbb{Z}. \tag{2.10}$$

Define the Fock space $\mathsf{S} := \mathbb{C}[x_1, x_2, \dots]$; this is the space of polynomials in infinitely many variables $x_1, x_2, \dots$.

Given $\mu, \hbar \in \mathbb{R}$, define the following representation ρ of H on S $\forall n \in \mathbb{N}$:

$$\begin{aligned} \rho(a_n) &:= \frac{\partial}{\partial x_n}, \\ \rho(a_{-n}) &:= n x_n, \\ \rho(a_0) &:= \mu \operatorname{id}_{\mathsf{S}}, \\ \rho(\hbar) &:= \hbar \operatorname{id}_{\mathsf{S}} . \end{aligned} \tag{2.11}$$

Clearly the commutation relations (2.10) hold in Fock representation (2.11). Moreover, the Fock representation is irreducible and unitary.

Lemma 2.42. *If $\hbar \neq 0$, then the representation* (2.11) *is irreducible.*

Proof. Any polynomial in S can be reduced to a multiple of 1 by successive application of a_n's with $n > 0$. Then the successive application of a_{-n} with $n > 0$ can give any other polynomial in S provided that $\hbar \neq 0$. $\square$

Lemma 2.43. *For each $\mu \in \mathbb{R}$ there exists a unique positive definite Hermitian form $\langle\cdot,\cdot\rangle$ on S such that*

$$\langle 1, 1\rangle = 1 \quad \text{and} \quad \langle \rho(a_n)f, g\rangle = \langle f, \rho(a_{-n})g\rangle \quad \forall f,g \in \mathsf{S},\ \forall n \in \mathbb{Z}.$$

Here and in what follows 1 is the vacuum vector.

Proof. First, we need to prove that the Hermitian form of two distinct monomials is zero. So let $f, g \in \mathsf{S}$ be two distinct monomials. Then there exists an index $n \in \mathbb{N}$ and exponents $k \neq l,\ k, l \geq 0$, such that $f = x_n^k f_1$ and $g = x_n^l g_1$ for suitable monomials f_1, g_1 independent of x_n. Without loss of generality assume that $k < l$. We now calculate $\langle f, g\rangle n^{k+1}$ in two different ways:

$$\left\langle (\rho(a_n))^{k+1} f, x_n^{l-k-1} g_1 \right\rangle = \left\langle \left(\frac{\partial}{\partial x_n}\right)^{k+1} x_n^k f_1, x_n^{l-k-1} g_1 \right\rangle = \left\langle 0, x_n^{l-k-1} g_1 \right\rangle = 0$$

and

$$\langle (\rho(a_n))^{k+1} f, x_n^{l-k-1} g_1\rangle = \langle f, (\rho(a_{-n}))^{k+1} x_n^{l-k-1} g_1\rangle = \langle f, n^{k+1} x_n^l g_1\rangle = \langle f, g\rangle n^{k+1}.$$

Thus, $\langle f, g\rangle = 0$. Moreover,

$$\langle f, f\rangle = \langle f, n^{-k}(\rho(a_{-n}))^k f_1\rangle = n^{-k}\langle \rho(a_n)^k x_n^k f_1, f_1\rangle = \frac{k!}{n^k}\langle f_1, f_1\rangle.$$

By definition $\langle 1, 1\rangle = 1$. Thus it follows that for monomials $f = x_{n_1}^{k_1} x_{n_2}^{k_2} \dots x_{n_r}^{k_r}$ with $n_1 < n_2 < \cdots < n_r$

$$\langle f, f\rangle = \frac{k_1!\, k_2!\, \dots k_r!}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}}. \tag{2.12}$$

Since the monomials constitute a (Hamel) basis of S, $\langle \cdot, \cdot \rangle$ is uniquely determined as a positive definite Hermitian form by (2.12) and the orthogonality condition. Reversing the arguments, by using (2.12) and the orthogonality condition $\langle f, g\rangle = 0$ for distinct monomials $f, g \in \mathsf{S}$ as a definition of $\langle \cdot, \cdot \rangle$, we obtain a Hermitian form on S with the desired properties. □

Note that $\rho(a_n)^* = \rho(a_{-n})$ and for each $n > 0$ the operator $\rho(a_n)$ is an annihilation operator whereas $\rho(a_n)^*$ is a creation operator. This justifies another common name of the Heisenberg algebra—the *oscillator algebra*.

Set $\hbar = 1$ and let

$$\rho(L_n) := \frac{1}{2}\sum_{k\in\mathbb{Z}} {:}\rho(a_{n-k})\rho(a_k){:} \quad n \in \mathbb{Z},$$

where the colons indicate *normal ordering* defined by

$${:}\rho(a_i)\rho(a_j){:}= \begin{cases} \rho(a_i)\rho(a_j) & \text{if } i \leq j \\ \rho(a_j)\rho(a_i) & \text{if } i > j. \end{cases}$$

Due to normal ordering, when an operator $\rho(L_n)$ is applied to any vector of S only a finite number of terms in the sum are non-zero. Hence, $\rho(L_n) : \mathsf{S} \to \mathsf{S}$ is a well-defined map. From now on, we abuse our notation and write L_n for $\rho(L_n)$ and similarly for $\rho(a_n)$.

Proposition 2.44. *The L_n's satisfy the commutation relations*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.13}$$

Thus the map $\rho : \mathsf{H} \to \mathrm{End}_{\mathbb{C}}\,\mathsf{S}$ is a representation of the Virasoro algebra in the Fock space S for $c = 1$.

Proof. We define a cutoff function ψ on $\mathbb{R}$ by:

$$\psi(x) = \begin{cases} 1 & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x| > 1. \end{cases}$$

Let

$$L_n(\varepsilon) = \frac{1}{2}\sum_{j\in\mathbb{Z}} {:}a_{n-j}a_j{:}\, \psi(\varepsilon j)\,.$$

Notice that $L_n(\varepsilon)$ is a finite sum if $\varepsilon \neq 0$ and that $L_n(\varepsilon) \to L_n$ as $\varepsilon \to 0$. In particular, the latter statement means that given $v \in \mathsf{S}$, $L_n(\varepsilon)(v) = L_n(v)$ for ε sufficiently small. Furthermore, note that $L_n(\varepsilon)$ differs from $1/2\sum_{j\in\mathbb{Z}} a_{n-j}a_j\,\psi(\varepsilon j)$ by a finite sum of scalars. These terms drop out of the commutator $[a_k, L_n(\varepsilon)]$. Hence

$$\begin{aligned} [a_k, L_n(\varepsilon)] =& \frac{1}{2}\sum_j [a_k, a_{n-j}a_j]\psi(\varepsilon j) \\ =& \frac{1}{2}\sum_j [a_k, a_{n-j}]a_j\psi(\varepsilon j) + \frac{1}{2}\sum_j a_{n-j}[a_k, a_j]\psi(\varepsilon j) \\ =& \frac{1}{2}ka_{k+n}\psi(\varepsilon(k+n)) + \frac{1}{2}ka_{n+k}\psi(-\varepsilon k) \end{aligned}$$

where for the last equality we have used the Heisenberg commutation relations (2.10). Letting $\varepsilon \to 0$ gives us

$$[a_k, L_n] = ka_{k+n} \quad \forall k, n \in \mathbb{Z}.$$

Using this result we calculate

$$\begin{aligned} [L_m(\varepsilon), L_n] &= \frac{1}{2}\sum_j [a_{m-j}a_j, L_n]\psi(\varepsilon j) \\ &= \frac{1}{2}\sum_j ja_{m-j}a_{j+n}\psi(\varepsilon j) + \frac{1}{2}\sum_j (m-j)a_{m-j+n}a_j\psi(\varepsilon j). \end{aligned} \tag{2.14}$$

We note that

$$\begin{aligned}\sum_j a_{m-j}a_{j+n}\psi(\varepsilon j) &= \sum_{\frac{m-n}{2}\leq j} {:}a_{m-j}a_{j+n}{:}\ \psi(\varepsilon j) + \sum_{j<\frac{m-n}{2}} a_{m-j}a_{j+n}\psi(\varepsilon j)\\ &= \sum_j {:}a_{m-j}a_{j+n}{:}\ \psi(\varepsilon j) + \delta_{m+n,0}\sum_{j<\frac{m-n}{2}} (m-j)\psi(\varepsilon j)\\ &= \sum_j {:}a_{m-j}a_{j+n}{:}\ \psi(\varepsilon j) + \delta_{m+n,0}\sum_{j<m}(m-j)\psi(\varepsilon j).\end{aligned} \tag{2.15}$$

Similarly

$$\sum_j a_{m-j+n}a_j\psi(\varepsilon j) = \sum_j {:}a_{m-j+n}a_j{:}\ \psi(\varepsilon j) - \delta_{m+n,0}\sum_{j<0} j\,\psi(\varepsilon j). \tag{2.16}$$

Plugging Equations (2.15) and (2.16) into Equation (2.14) we get

$$\begin{aligned}&[L_m(\varepsilon),L_n] =\\ &\quad \frac{1}{2}\sum_j j\ {:}a_{m-j}a_{j+n}{:}\ \psi(\varepsilon j) + \frac{1}{2}\sum_j (m-j)\ {:}a_{m-j+n}a_j{:}\\ &\quad +\delta_{m+n,0}\left(\frac{1}{2}\sum_{j=0}^{m-1}(m-j)j\,\chi_{[1,\infty)}(m) - \frac{1}{2}\sum_{j=m}^{-1}(m-j)j\,\chi_{(-\infty,-1]}(m)\right)\psi(\varepsilon j).\end{aligned}$$

Here $\chi_A(x)$ is the characteristic function. Both of the sums under the bracket sum up to $1/12(m^3-m)$. Making a variable transformation $j \mapsto j-n$ in the first sum and taking the limit $\varepsilon \to 0$ we get the desired result (2.13). □

Remark 2.45. One can also prove Proposition 2.44 without using a cutoff function. However, this method requires more calculations to treat all the different cases separately. See, e.g., [Sch08, Chap. 7].

Corollary 2.46. The representation of Proposition 2.44 yields a positive definite unitary highest weight representation of the Virasoro algebra with the higest weight $c = 1,\ h = 1/2\,\mu^2$, where $\mu \in \mathbb{R}$ is such that $\rho(a_0) := \mu\,\mathrm{id}_{\mathsf{S}}$.

Proof. For the highest weight vector $v_0 := 1$ let

$$V := \mathrm{span}_{\mathbb{C}}\{L_n v_0 \mid n \in \mathbb{Z}\}.$$

The restrictions of $\rho(L_n)$ to the subspace $V \subset \mathsf{S}$ of the Fock space S define a highest weight representation of Vir with the highest weight $(1, 1/2\,\mu^2)$ and Virasoro module V. □

Remark 2.47. In most cases $\mathsf{S} = V$. But it does not hold, e.g., if $\mu = 0$.

More unitary highest weight representations can be constructed by taking tensor products:

$$(\rho \otimes \rho)(L_n)(f_1 \otimes f_2) := (\rho(L_n)f_1) \otimes f_2 + f_1 \otimes (\rho(L_n)f_2) \quad \forall\, (f_1 \otimes f_2) \in V \otimes V.$$

The Hermitian form on $V \otimes V$ is defined by

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \langle f_1, g_1 \rangle \langle f_2, g_2 \rangle.$$

These observations lead to the following.

Proposition 2.48. *The representation $\rho \otimes \rho : \mathsf{Vir} \to \mathrm{End}_{\mathbb{C}}(V \otimes V)$ is unitary positive definite with highest weight $c = 2,\ h = \mu^2 \neq 0$. Iterating we obtain positive semidefinite unitary highest weight representations $\forall (c, h) \in \mathbb{N} \times \mathbb{R}$ which are positive definite if $c \geq 2$ and $h > 0$.*

Now we can finally prove Theorem 2.40 (i).

Proof of Theorem 2.40 (i). Let

$$\varphi_{p,q} = \begin{cases} h - h_{q,q}(c) & \text{if} \quad p = q, \\ (h - h_{p,q}(c))(h - h_{q,p}(c)) & \text{if} \quad p \neq q. \end{cases}$$

Then by Theorem 2.39

$$\det A^N(c, h) = K_N \prod_{\substack{p,q \in \mathbb{N} \\ pq \leq N, q \leq p}} \varphi_{p,q}^{P(N-pq)}.$$

For $1 \leq p, q \leq N$ and $c > 1,\ h > 0$ we have

$$\begin{aligned} \varphi_{q,q} &= h + \frac{1}{24}(c-1)(q^2-1) > 0, \\ \varphi_{p,q} &= \left(h - \left(\frac{p-q}{2} \right)^2 \right)^2 + \frac{h}{24}(p^2 + q^2 - 2)(c-1) + \\ &\quad + \frac{1}{576}(p^2-1)(q^2-1)(c-1)^2 + \frac{1}{48}(c-1)(p-q)^2(pq+1) > 0. \end{aligned}$$

Hence, $\det A^N(c, h) > 0$ for all $c > 1,\ h > 0$. This implies that the Hermitian form $\langle \cdot, \cdot \rangle$ is positive definite in the entire region $c > 1,\ h > 0$ if there is just one example $M(c, h)$ with $c > 1,\ h > 0$ such that $\langle \cdot, \cdot \rangle$ is positive definite. Proposition 2.48 shows that we have positive semidefinite representations for $c \in \mathbb{N}$, $h \geq 0$, and positive definite for $c = 2, 3, \ldots,\ h > 0$ thereby proving Theorem 2.40 (i). $\square$

Chapter 3

Vertex Algebras

Borcherds introduced vertex algebras in [Bor86] to understand Frenkel's work on the Lie algebra whose Dynkin diagram is the Leech lattice. Then Frenkel, Lepowsky and Meurman modified the definition and added some natural assumptions to vertex algebras which led to vertex operator algebras. This allowed them to construct the moonshine module [FLM88]—a vertex operator algebra with the monster group, the largest sporadic finite simple group, being its symmetry group. Finally in [Bor92] Borcherds proved the Conway–Norton monstrous moonshine conjecture [CN79] for the moonshine module. The conjecture relates the monster group and modular functions, so it was rather unexpected. For this and related work, Borcherds was awarded a Fields Medal in 1998. Thus, vertex algebras are definitely of interest to mathematicians.

The interest of physicists stems from the fact that Frenkel, Lepowsky and Meurman were using ideas from conformal field theory and string theory in their work. Thus, it is no surprise that vertex (operator) algebras can be viewed as a mathematical axiomatization of chiral conformal field theory and indeed we will see that, for example, the operator product expansion, a crucial assumption made in 2D CFT, can be rigorously proved in vertex algebras (Theorem 3.30). A nice, but a little bit outdated, overview of these connections can be found in the introduction of [FLM88].

To understand the current work, no prior knowledge of vertex algebras is assumed. We present full proofs up to Section 3.6. Our particular choice of material is tailored so that we are able to give full details of the proof of Kac's Theorem 7.1 up to the level found in [Kac98]. In the last section, however, some proofs are skipped, but freely available references are given. As elsewhere in this work, we only consider bosonic theories, but the generalization to vertex superalgebras which also include fermions is rather trivial, see, e.g., [Kac98].

Our main references for sections 3.1–3.5 is [Kac98] and [Sch08]. For Section 3.6 we have mostly used [CKLW15].

3.1 Formal Distributions

Throughout this chapter let $Z = \{z_1, \dots, z_n\}$ be a set of variables and U be a vector space over $\mathbb{C}$. A **formal distribution** is a series

$$a(z_1, \dots, z_n) = \sum_{j \in \mathbb{Z}^n} a_j z^j = \sum_{j \in \mathbb{Z}^n} a_{j_1,\dots,j_n} z_1^{j_1} \dots z_n^{j_n},$$

with coefficients $a_j \in U$. The **vector space of formal distributions** over $\mathbb{C}$ will be denoted by $U[[z_1^{\pm}, \dots, z_n^{\pm}]] = U[[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}]]$. It contains the subspace of **Laurent polynomials**

$$U[z_1^{\pm}, \dots, z_n^{\pm}] = \left\{a \in U[[z_1^{\pm}, \dots, z_n^{\pm}]] \mid \exists k, l : a_j = 0 \text{ except for } k \le j \le l\right\},$$

with the partial order on $\mathbb{Z}^n$ defined by $i \le j \iff i_\mu \le j_\mu \, \forall \mu \in \{1, 2, \dots, n\}$. The space of **formal Laurent series** is

$$U((z)) = \left\{a \in V[[z^{\pm}]] \mid \exists k \in \mathbb{Z} \, \forall j \in \mathbb{Z} : j < k \implies a_j = 0\right\}.$$

A formal distribution can be always multiplied by a Laurent polynomial (provided that the product of coefficients is defined), but two formal distributions cannot be multiplied in general. For each product of two formal distributions, we need to check that it converges in the algebraic sense, i.e. the coefficient of each monomial $z_1^{j_1} \dots z_n^{j_n}$ must be a finite or at least a convergent sum. Here and further by multiplication of formal distributions we mean the usual Cauchy product: for $a(z) = \sum_n a_n z^n$ and $b(z) = \sum_n b_n z^n$, the Cauchy product is

$$a(z)b(z) = \sum_n \left(\sum_{i+j=n} a_i b_j \right) z^n.$$

Given a formal distribution $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$, the **residue** is defined as

$$\operatorname{Res}_z a(z) = a_{-1}.$$

Defining the **derivative** of a formal distribution $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ by

$$\partial a(z) := \sum_{n \in \mathbb{Z}} n a_n z^{n-1},$$

we note that $\operatorname{Res}_z \partial d(z) = 0$ for any distribution $d(z)$. Hence, the integration by parts formula holds, provided that $a(z)b(z)$ is defined:

$$\operatorname{Res}_z \partial a(z) b(z) = -\operatorname{Res}_z a(z) \partial b(z). \tag{3.1}$$

We will also need the **formal delta function** $\delta(z-w)$ which is the formal distribution in z and w with values in $\mathbb{C}$

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{n-1}w^{-n}. \tag{3.2}$$

Given a rational function $f(z,w)$ with poles only at $z=0$, $w=0$ or $|z|=|w|$, we denote by $\iota_{z,w}f$ ($\iota_{w,z}f$) the **power series expansion** of f in the domain $|z|>|w|$ ($|w|>|z|$).

E.g. for $j\in\mathbb{N}_0$ we have

$$\iota_{z,w}\frac{1}{(z-w)^{j+1}} = \sum_{m=0}^{\infty}\binom{m}{j} z^{-m-1}w^{m-j}, \tag{3.3a}$$

$$\iota_{w,z}\frac{1}{(z-w)^{j+1}} = -\sum_{m=-1}^{-\infty}\binom{m}{j} z^{-m-1}w^{m-j} \tag{3.3b}$$

and it follows that

$$D_w^j\delta(z-w) = \iota_{z,w}\frac{1}{(z-w)^{j+1}} - \iota_{w,z}\frac{1}{(z-w)^{j+1}} \tag{3.4a}$$

$$= \sum_{m\in\mathbb{Z}}\binom{m}{j} z^{-m-1}w^{m-j}, \tag{3.4b}$$

with

$$D_w^j := \frac{\partial_w^j}{j!}.$$

The next proposition justifies the name formal delta function.

Proposition 3.1. *We have for all formal distributions $f(z)\in U[[z,z^{-1}]]$:*

(a) *$f(z)\delta(z-w)$ is well-defined,*

(b) $f(z)\delta(z-w) = f(w)\delta(z-w)$,

(c) $\operatorname{Res}_z f(z)\delta(z-w) = f(w)$,

(d) $\delta(z-w) = \delta(w-z)$,

(e) $\partial_z\delta(z-w) = -\partial_w\delta(z-w)$,

(f) $(z-w)D_w^{j+1}\delta(z-w) = D_w^j\delta(z-w)$ *with* $j\in\mathbb{N}_0$,

(g) $(z-w)^{j+1}D_w^j\delta(z-w) = 0$ *if* $j\in\mathbb{N}_0$.

Here $D_w^0 = 1$ is understood.

Proof. Note that

$$\delta(z-w) = \sum_{k+n+1=0} z^k w^n = \delta(w-z)$$

and

$$\delta(z-w) = \sum_{n,k\in\mathbb{Z}} \delta_{k,-n-1} z^k w^n \in U[[z^{\pm}, w^{\pm}]].$$

Thus, the product $f(z)\delta(z-w)$ is well-defined. Moreover, for $f(z) = \sum_{k\in\mathbb{Z}} f_k z^k$ we have

$$f(z)\delta(z-w) = \sum_{n,k\in\mathbb{Z}} f_k z^{k-n-1} w^n = \sum_{k\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} f_{k+n+1} w^n \right) z^k \implies$$

$$\implies \operatorname{Res}_z f(z)\delta(z-w) = f(w).$$

Furthermore,

$$f(w)\delta(z-w) = \sum_{n,k\in\mathbb{Z}} f_k w^k z^{-n-1} w^n = \sum_{n,k\in\mathbb{Z}} f_k z^{k-n-1} w^n = f(z)\delta(w-z)$$

by the above. This proves parts (a)-(d). Part (e) follows from the definition of the formal delta function (3.2) by direct calculation. To prove (f), we use Equation (3.4a)

$$(z-w)D_w^{j+1}\delta(z-w) = (z-w)\sum_{m\in\mathbb{Z}} \binom{m}{j+1} z^{-m-1} w^{m-j-1} =$$

$$\sum_{m\in\mathbb{Z}} \binom{m}{j+1} z^{-m} w^{m-j-1} - \sum_{m\in\mathbb{Z}} \binom{m}{j+1} z^{-m-1} w^{m-j} =$$

$$\sum_{m'\in\mathbb{Z}} \binom{m'+1}{j+1} z^{-m'-1} w^{m'-j} - \sum_{m\in\mathbb{Z}} \binom{m}{j+1} z^{-m-1} w^{m-j} =$$

$$\sum_{m\in\mathbb{Z}} \left(\binom{m+1}{j+1} - \binom{m}{j+1} \right) z^{-m-1} w^{m-j} =$$

$$\sum_{m\in\mathbb{Z}} \binom{m}{j} z^{-m-1} w^{m-j} = D_w^j \delta(z-w).$$

Part (g) follows by repeated application of (f) and by symmetry property (b):

$$(z-w)^{j+1} D_w^j \delta(z-w) = (z-w)\delta(z-w) = z\delta(z-w) - w\delta(z-w) = 0.$$

$\square$

The next proposition will be useful for OPEs.

Proposition 3.2. *If $a(z,w) \in U[[z^{\pm}, w^{\pm}]]$ is such that $(z-w)^N a(z,w) = 0$ for some $N \in \mathbb{N}$, then it can be uniquely written as*

$$a(z,w) = \sum_{j=0}^{N-1} c^j(w) D_w^j \delta(z-w), \tag{3.5}$$

with

$$c^j(w) = \mathrm{Res}_z (z-w)^j a(z,w). \tag{3.6}$$

Proof. We have

$$(z-w)^N \sum_{j=0}^{N-1} c^j(w) D_w^j \delta(z-w) = 0$$

by Proposition 3.1 (g).

We prove the converse by induction. For $N = 1$ we have

$$0 = (z-w)a(z,w) = \sum_{m,n\in\mathbb{Z}} a_{m,n} z^{m+1} w^n - \sum_{m,n\in\mathbb{Z}} a_{m,n} z^m w^{n+1} =$$
$$= \sum_{m,n\in\mathbb{Z}} (a_{m,n+1} - a_{m+1,n}) z^{m+1} w^{n+1}.$$

Thus, $a_{m,n+1} = a_{m+1,n}$ $\forall n, m \in \mathbb{Z}$. Hence, $a_{0,n+1} = a_{1,n} = a_{k,n-k+1}$ $\forall m, k \in \mathbb{Z}$. This implies

$$a(z,w) = \sum_{n,k\in\mathbb{Z}} a_{k,n-k+1} z^k w^{n-k+1} = \sum_{n\in\mathbb{Z}} a_{-1,n+2} w^{n+2} \sum_{k\in\mathbb{Z}} z^k w^{-k-1} =$$
$$= c^0(w)\delta(z-w)$$

with $c^0(w) = \sum_{n\in\mathbb{Z}} a_{-1,n} w^n$ as required.

Now let $a(z,w)$ be such that

$$0 = (z-w)^{N+1} a(z,w) = (z-w)^N \left((z-w)a(z,w)\right).$$

By induction hypothesis

$$(z-w)a(z,w) = \sum_{j=0}^{N-1} d^j(w) D_w^j \delta(z-w)$$

thus applying ∂_z gives

$$\begin{aligned} a(z,w) + (z-w)\partial_z a(z,w) &= \sum_{j=0}^{N-1} d^j(w) D_w^j \partial_z \delta(z-w) = \\ &= -\sum_{j=0}^{N-1} d^j(w)(j+1) D_w^{j+1} \delta(z-w). \qquad (3.7) \end{aligned}$$

Here we have used $\partial_z \delta(z-w) = -\partial_w \delta(z-w)$ from Proposition 3.1. Application of the induction hypothesis to

$$0 = \partial_z \left((z-w)^{N+1} a(z,w)\right) = (z-w)^N ((N+1)a(z,w) + (z-w)\partial_z a(z,w))$$

yields

$$(N+1)a(z,w) + (z-w)\partial_z a(z,w) = \sum_{j=0}^{N-1} e^j(w) D_w^j \delta(z-w). \qquad (3.8)$$

Subtracting Equation (3.7) from Equation (3.8) gives

$$Na(z,w) = \sum_{j=0}^{N-1} e^j(w) D_w^j \delta(z-w) + \sum_{j=1}^{N} j\, d^{j-1}(w) D_w^j \delta(z-w)$$

which implies that

$$a(z,w) = \sum_{j=0}^{N} c^j(w) D_w^j \delta(z-w)$$

for suitable $c^j(w) \in U[[w^{\pm}]]$ as required.

We now prove the formula for $c^j(w)$ using Proposition 3.1. From part (g) we see that

$$\mathrm{Res}_z((z-w)^n c^j(w) D_w^j \delta(z-w)) = 0$$

if $j < n$. If $j = n$, then by (f), (b) and (c)

$$\mathrm{Res}_z((z-w)^n c^j(w) D_w^j \delta(z-w)) = c^n(w).$$

Finally, if $j > n$, then (e) and integration by parts (3.1) gives

$$\mathrm{Res}_z((z-w)^n c^j(w) D_w^j \delta(z-w)) = \mathrm{Res}_z \left((z-w)^n c^j(w)(-1)^j D_z^j \delta(z-w)\right) = 0.$$

Thus, the coefficient equation (3.6) holds and therefore the expansion (3.5) is unique. $\square$

Remark 3.3. Note that (3.5) is equivalent to

$$a_{(m,n)} = \sum_{j=0}^{N-1} \binom{m}{j} c^j_{(m+n-j)}\,, \tag{3.9}$$

as follows from (3.4b) by comparing coefficients.

3.2 Locality and Normal Ordering

Let the vector space U over $\mathbb{C}$ be also associative. On U one naturally has the **commutator** $[a,b] = ab - ba$. The most important example for us of U is $\operatorname{End} V$ of a vector space V.

Definition 3.4. Locality. Two formal distributions $a(z), b(z) \in U[[z^{\pm}]]$ are *(mutually) local* if

$$(z-w)^N [a(z), b(w)] = 0 \quad \text{for} \quad N \gg 0.$$

Here $N \gg 0$ means that there exits $n \in \mathbb{N}_0$ such that $\forall N \geq n$ the statement holds.

Remark 3.5. Differentiating $(z-w)^N[a(z), b(w)] = 0$ and multiplying by $(z-w)$ gives $(z-w)^{N+1}[\partial a(z), b(w)] = 0$. Hence, if a and b are mutually local, ∂a and b are mutually local as well.

Our next goal is to formulate some equivalent definitions of locality. However, we need some notation first. Instead of $a(z) = \sum_{m\in\mathbb{Z}} a_m z^m$ we will often write $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$. This makes it easy to calculate the coefficients:

$$a_{(n)} = a_{-n-1} = \operatorname{Res}_z \left(a(z) z^n\right).$$

We break $a(z)$ into

$$a(z)_- := \sum_{n\geq 0} a_{(n)} z^{-n-1}, \quad a(z)_+ := \sum_{n<0} a_{(n)} z^{-n-1}.$$

Note that the above decomposition is the only way to break $a(z)$ into a sum of “positive” and “negative” parts such that

$$\left(\partial a(z)\right)_{\pm} = \partial \left(a(z)_{\pm}\right). \tag{3.10}$$

Definition 3.6. The *normally ordered product* of two formal distributions $a(z), b(z) \in U[[z^{\pm}]]$ is the distribution

$$\mathbf{:}a(z)b(w)\mathbf{:}= a(z)_+ b(w) + b(w) a(z)_-.$$

Note that the definition implies

$$a(z)b(w) = \quad [a(z)_-, b(w)] + :a(z)b(w): , \tag{3.11a}$$
$$b(w)a(z) = -[a(z)_+, b(w)] + :a(z)b(w): . \tag{3.11b}$$

Theorem 3.7. Equivalent definitions of locality. *Let $a(z), b(z) \in U[[z^\pm]]$ and $N \in \mathbb{N}$. The following are equivalent:*

(a) *$a(z)$ and $b(z)$ are mutually local with $(z-w)^N[a(z), b(w)] = 0$,*

(b) $[a(z), b(w)] = \sum_{j=0}^{N-1} c^j(w) D_w^j \delta(z-w)$, *where $c^j(w) \in U[[w^\pm]]$,*

(c) $$[a(z)_-, b(w)] = \sum_{j=0}^{N-1} \left(\iota_{z,w}\frac{1}{(z-w)^{j+1}}\right) c^j(w),$$
$$-[a(z)_+, b(w)] = \sum_{j=0}^{N-1} \left(\iota_{w,z}\frac{1}{(z-w)^{j+1}}\right) c^j(w),$$

(d) $$a(z)b(w) = \sum_{j=0}^{N-1} \left(\iota_{z,w}\frac{1}{(z-w)^{j+1}}\right) c^j(w) + :a(z)b(w): ,$$
$$b(w)a(z) = \sum_{j=0}^{N-1} \left(\iota_{w,z}\frac{1}{(z-w)^{j+1}}\right) c^j(w) + :a(z)b(w): ,$$
where $c^j(w) \in U[[w^\pm]]$,

(e) $$[a_{(m)}, b_{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} c^j_{(m+n-j)}, \quad m, n \in \mathbb{Z},$$

(f) $$[a_{(m)}, b(w)] = \sum_{j=0}^{N-1} \binom{m}{j} c^j(w) w^{m-j}, \quad m \in \mathbb{Z}.$$

Proof. We have

$$\text{(a)} \iff \text{(b)} \iff \text{(c)} \iff \text{(d)}$$

by Proposition 3.2, taking all terms in (b) with negative (resp. non-negative) powers of z and using (3.4a), and equations (3.11) respectively. Finally, (e) and (f) are equivalent to (b) by Remark 3.3. □

Remark 3.8. Abusing our notation of Theorem 3.7 (d) gives:

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}} + :a(z)b(w): . \tag{3.12a}$$

Often we will simplify even more and write just the singular part

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}. \tag{3.12b}$$

Such notation is very common in physics. The condition $|z| > |w|$ is implicit.

Equations (3.12a) and (3.12b) are called the **operator product expansion (OPE)**. By Theorem 3.7 we can calculate all brackets between all coefficients of mutually local formal distributions $a(z)$ and $b(z)$ using only the singular part of the OPE. Hence, the importance of OPE. Moreover, defining the **n-th product** ($n \in \mathbb{N}_0$) on the space of formal distributions to be

$$a(w)_{(n)}b(w) = \operatorname{Res}_z \left([a(z), b(w)](z-w)^n\right) \tag{3.13}$$

and combining this with Proposition 3.2 and Theorem 3.7 (d) for two mutually local distributions gives

$$[a(z), b(w)] = \sum_{j=0}^{N-1} \left(a(w)_{(j)}b(w)\right) D_w^j \delta(z-w) \tag{3.14a}$$

which by Theorem 3.7 and abuse of notation is equivalent to

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{a(w)_{(j)}b(w)}{(z-w)^{j+1}} + \mathbf{:}a(z)b(w)\mathbf{:}\,. \tag{3.14b}$$

So we have equivalent formulations of OPE.

We now consider other notions inspired by physics.

Definition 3.9. Hamiltonian, conformal weight. A diagonalizable derivation of the associative algebra U will be called *Hamiltonian* and denoted H. Its action on the space of formal distributions with values in U will be given coefficient-wise.

We say that a formal distribution $a = a(z, w, \ldots)$ with values in U is an *eigendistribution for* H of *conformal weight* $h \in \mathbb{C}$ if

$$(H - h - z\partial_z - w\partial_w - \ldots)\, a = 0.$$

The following proposition can be proved by straightforward computations.

Proposition 3.10. *Given to eigendistributions* a *and* b *with conformal weights* h *and* h' *respectively, we have*

- $\partial_z a$ *is an eigendistribution of conformal weight* $h+1$,

- $:a(z)b(w):$ *is an eigendistribution of conformal weight* $h + h'$,
- *the* n*-th OPE coefficient of* $[a(z), b(w)]$ *is an eigendistribution of conformal weight* $h + h' - n - 1$ *with* $n \in \mathbb{N}$,
- *if* f *is a homogeneous function of degree* j, *then* fa *is an eigendistribution of conformal weight* $h - j$.

Corollary 3.11. The summands of an OPE

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}},$$

where $a(z)$ and $b(z)$ are two mutually local eigendistributions of conformal weights h and h', have the same conformal weight $h + h'$.

It is convenient to write

$$a(z) = \sum_{n \in -h+\mathbb{Z}} a_n z^{-n-h}$$

for eigendistributions of conformal weight h. In this case, the condition for $a(z)$ to be an eigendistribution of conformal weight h is equivalent to

$$[H, a_n] = -na_n.$$

Example 3.12. Virasoro formal distribution with central charge C. Let V be a vector space and consider a representation of Virasoro algebra Vir on it, such that $L_n \in \operatorname{End} V$ and $C = c\,\mathrm{id}_V$ with $c \in \mathbb{C}$. Then

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

is a formal distribution with coefficients in $\operatorname{End} V$. We compute

$$\begin{aligned}[L(z), L(w)] &= \sum_{m,n\in\mathbb{Z}} [L_m, L_n] z^{-m-2} w^{-n-2} \\ &= \sum_{m,n\in\mathbb{Z}} (m-n) L_{m+n} z^{-m-2} w^{-n-2} + \sum_{m\in\mathbb{Z}} \frac{m}{12}(m^2-1) z^{-m-2} w^{m-2} C.\end{aligned}$$

Substituting $k = m + n$ and then $j = m + 1$ gives

$$\begin{aligned}
&\sum_{m,n}(m-n)L_{m+n}z^{-m-2}w^{-n-2} = \sum_{k,m}(2m-k)L_k z^{-m-2}w^{-k+m-2} = \\
&\quad = \sum_{k,j}(2j-k-2)L_k z^{-j-1}w^{-k+j-3} = \\
&\quad = 2\sum_{k,j} L_k w^{-k-2} j z^{-j-1}w^{j-1} + \sum_{k,j}(-k-2)L_k w^{-k-3}z^{-j-1}w^{j} = \\
&\quad = 2L(w)\partial_w\delta(z-w) + \partial_w L(w)\delta(z-w).
\end{aligned}$$

For the remaining term we get by substituting $m = n - 1$

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} \frac{m}{12}(m^2-1)z^{-m-2}w^{m-2}C = \\
&\qquad\qquad = \frac{C}{12}\sum_{m\in\mathbb{Z}} n(n-1)(n-2)z^{-n-1}w^{n-3} = \frac{C}{12}\partial_w^3\delta(z-w).
\end{aligned}$$

Thus,

$$[L(z), L(w)] = \frac{C}{2}D_w^3\delta(z-w) + 2L(w)D_w\delta(z-w) + \partial_w L(w)\delta(z-w) \quad (3.15)$$

or equivalently using Theorem 3.7 and Remark 3.8

$$L(z)L(w) \sim \frac{C/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}. \quad (3.16)$$

Note that $L(z)$ is basically the formal distribution version of the energy-momentum tensor which is usually written $T(z)$ in CFT. But T denotes the infinitesimal translation operator in vertex algebras, so that's why we write $L(z)$ instead.

3.3 Fields and Dong's Lemma

Throughout this section, let V be a vector space.

Definition 3.13. Field in formal distributions. A formal distribution $a(z) = \sum a_{(n)}z^{-n-1} \in \operatorname{End} V[[z^{\pm}]]$ is called a *field* if $\forall v \in V$

$$a_{(n)}(v) = 0 \quad \text{for} \quad n \gg 0.$$

The collection of fields on a vector space V will be denoted $\mathscr{F}(V)$.

The definition means that $a(z)v$ is a formal Laurent series in z (i.e. $a(z)v \in V[[z]][z^{-1}]$).

For fields normal ordering can be extended to coinciding points.

Definition 3.14. Normally ordered product. Given two fields $a(z)$ and $b(z)$ we define

$$\boldsymbol{:}a(z)b(z)\boldsymbol{:}= a(z)_+b(z) + b(z)a(z)_-. \tag{3.17}$$

From

$$\boldsymbol{:}a(z)b(z)\boldsymbol{:}_{(n)}= \sum_{j=-1}^{-\infty} a_{(j)}b_{(n-j-1)} + \sum_{j=0}^{\infty} b_{(n-j-1)}a_{(j)}.$$

it follows that upon application on $v \in V$ each of the two sums gives only a finite number of non-zero summands. Thus, $\boldsymbol{:}a(z)b(z)\boldsymbol{:}$ is a well-defined formal distribution. Note that the assumption that both $a(z)$ and $b(z)$ are fields was necessary. That is why we were only able to define normally ordered product of general formal distributions in two variables in Definition 3.6. Furthermore, from (3.17) it is clear that $\boldsymbol{:}a(z)b(z)\boldsymbol{:}$ is a field, since for all $v \in V$ $b(z)v$ is a formal Laurent series in z, hence $a(z)_+b(z)v$ is a formal Laurent series in z. Similarly for the other summand. Therefore, the space of fields forms an algebra with respect to the normally ordered product (which in general is not associative).

Another useful property is that the derivative $\partial a(z)$ of a field $a(z)$ is a field and due to (3.10) ∂ is a derivation of the normally ordered product

$$\partial \boldsymbol{:}a(z)b(z)\boldsymbol{:}=\boldsymbol{:}\partial a(z)b(z)\boldsymbol{:} + \boldsymbol{:}a(z)\partial b(z)\boldsymbol{:}\ .$$

The existence of normally ordered product allows us to define the n-th product between the fields $\forall n \in \mathbb{Z}$.

Definition 3.15. n-th product of fields. We define the *n-th product* for $n \in \mathbb{Z}$ as

$$a(w)_{(n)}b(w) = \begin{cases} \mathrm{Res}_z\left([a(z), b(w)](z-w)^n\right) & \text{if} \quad n \geq 0 \\ \boldsymbol{:}D^{(-n-1)}a(w)b(w)\boldsymbol{:} & \text{if} \quad n < 0. \end{cases}$$

The n-th product of fields can be written in a single formula.

Lemma 3.16. *For all n-th products of fields we have*

$$a(w)_{(n)}b(w) = \mathrm{Res}_z\left(a(z)\, b(w)\, \iota_{z,w}(z-w)^n - b(w)\, a(z)\, \iota_{w,z}(z-w)^n\right), \tag{3.18}$$

where $n \in \mathbb{Z}$.

Proof. For $n \geq 0$, Equation (3.18) obviously coincides with (3.13). For $n < 0$, the lemma follows from the general Cauchy formulas for any formal distribution $a(z)$ and $k \in \mathbb{N}_0$

$$\operatorname{Res}_z a(z)\iota_{z,w}\frac{1}{(z-w)^{k+1}} = +D^k a(w)_+, \tag{3.19a}$$

$$\operatorname{Res}_z a(z)\iota_{w,z}\frac{1}{(z-w)^{k+1}} = -D^k a(w)_-. \tag{3.19b}$$

A straightforward use of definitions proves the $k = 0$ case. Differentiating the $k = 0$ case k times by w gives the required result. □

Proposition 3.17. *For all fields $a(w)$, $b(w)$ and $\forall n \in \mathbb{Z}$ holds*

$$\partial a(w)_{(n)}b(w) = -na(w)_{(n-1)}b(w), \tag{3.20a}$$

$$a(w)_{(n)}\partial b(w) = +na(w)_{(n-1)}b(w) + \partial\left(a(w)_{(n)}b(w)\right). \tag{3.20b}$$

Hence, ∂ is a derivation on all n-th products.

Proof. We will only prove the $n < 0$ case of the formula (3.20a). The other proofs are similar.

Given $n < 0$, first of all set $n = -j-1$. Then $j \in \mathbb{N}_0$, and using equations (3.18) and (3.3) together with the standard properties of binomial coefficients we have

$$\partial a(w)_{(n)}b(w) =$$

$$\operatorname{Res}_z\left(\sum_{k\in\mathbb{Z}} ka_k z^{k-1} b(w)\sum_{m=0}^{\infty}\binom{m}{j}z^{-m-1}w^{m-j} + b(w)\sum_{k\in\mathbb{Z}} ka_k z^{k-1}\sum_{m=-1}^{-\infty}\binom{m}{j}\right) =$$

$$\sum_{m=0}^{\infty}\binom{m}{j}(m+1)a_{m+1}b(w)w^{m-j} + b(w)\sum_{m=-2}^{-\infty}\binom{m}{j}(m+1)a_{m+1}w^{m-j} =$$

$$\sum_{m=1}^{\infty}\binom{m-1}{j}ma_m b(w)w^{m-j-1} + b(w)\sum_{m=-1}^{-\infty}\binom{m-1}{j}ma_m w^{m-j-1}.$$

On the other hand,

$$
\begin{aligned}
&- na(w)_{(n-1)}b(w) = \\
&\quad (j+1)\operatorname{Res}_z \left(\sum_{k\in\mathbb{Z}} a_k z^k b(w) \sum_{m=0}^{\infty} \binom{m}{j+1} z^{-m-1} w^{m-j-1} + \right. \\
&\quad \left. b(w) \sum_{k\in\mathbb{Z}} a_k z^k \sum_{m=-1}^{-\infty} \binom{m}{j+1} z^{-m-1} w^{m-j-1} \right) = \\
&\quad (j+1)\left(\sum_{m=1}^{\infty} \binom{m}{j+1} a_m b(w) w^{m-j-1} + b(w) \sum_{m=-1}^{-\infty} \binom{m}{j+1} a_m w^{m-j-1} \right).
\end{aligned}
$$

Thus, $\partial a(w)_{(n)}b(w) = -na(w)_{(n-1)}b(w)$ as required. □

We now prove two technical lemmas which will be used in the next section.

Lemma 3.18. *Let $a(z) = \sum_n a_{(n)}z^{-n-1}$ and $b(z) = \sum_n b_{(n)}z^{-n-1}$ be fields with values in $\operatorname{End} V$ and let $|0\rangle \in V$ be a vector such that*

$$a_{(n)}|0\rangle = 0 \quad \textit{and} \quad b_{(n)}|0\rangle = 0, \quad \forall n \in \mathbb{N}_0.$$

Then $\big(a(z)_{(n)}b(z)\big)\,|0\rangle$ is a V-valued formal distribution $\forall n \in \mathbb{Z}$ which does not include any negative powers of z and has a constant term $a_{(n)}b_{(-1)}|0\rangle$.

Proof. Let $k \in \mathbb{N}_0$. We consider two cases. Firstly,

$$
\begin{aligned}
\big(a(z)_{(-k-1)}b(z)\big)\,|0\rangle &= {:}D^k a(z) b(z){:}\,|0\rangle = D^k(a(z))_+\, b(z)|0\rangle \\
&= (D^k a(z))_+\, b(z)_+|0\rangle.
\end{aligned}
$$

Here we have used (3.10). Secondly,

$$
\begin{aligned}
\big(a(z)_{(k)}b(z)\big)\,|0\rangle &= \sum_{j=0}^{k} \binom{k}{j} (-z)^{k-j} [a_{(j)}, b(z)]\,|0\rangle \\
&= \sum_{j=0}^{k} \binom{k}{j} (-z)^{k-j} a_{(j)} b(z)_+|0\rangle.
\end{aligned}
$$

This proves the lemma. □

Lemma 3.19. Dong's Lemma. *Given pairwise mutually local fields (resp. formal distributions) $a(z)$, $b(z)$ and $c(z)$, we have that $a(z)_{(n)}b(z)$ and $c(z)$ are mutually local fields (resp. formal distributions) for all $n \in \mathbb{Z}$ (resp. $n \in \mathbb{N}$). In particular, ${:}a(z)b(z){:}$ and $c(z)$ are mutually local fields if the conditions of the lemma are fulfilled.*

Proof. We will show that for $M \gg 0$

$$(z_2 - z_3)^M A = (z_2 - z_3)^M B, \tag{3.21}$$

where

$$A = \iota_{z_1,z_2}(z_1 - z_2)^n a(z_1)b(z_2)c(z_3) - \iota_{z_2,z_1}(z_1 - z_2)^n b(z_2)a(z_1)c(z_3), \tag{3.22a}$$
$$B = \iota_{z_1,z_2}(z_1 - z_2)^n c(z_3)a(z_1)b(z_2) - \iota_{z_2,z_1}(z_1 - z_2)^n c(z_3)b(z_2)a(z_1). \tag{3.22b}$$

This suffices since applying Res_{z_1} to both sides of Equation (3.21) and setting $z_2 = z$, $z_3 = w$ proves the lemma due to Equation (3.18).

Since $a(z)$, $b(z)$ and $c(z)$ are pairwise mutually local, we get for $r \gg 0$

$$(z_1 - z_2)^r a(z_1)b(z_2) = (z_1 - z_2)^r b(z_2)a(z_1), \tag{3.23a}$$
$$(z_2 - z_3)^r b(z_2)c(z_3) = (z_2 - z_3)^r c(z_3)b(z_2), \tag{3.23b}$$
$$(z_1 - z_3)^r a(z_1)c(z_3) = (z_1 - z_3)^r c(z_3)a(z_1). \tag{3.23c}$$

If we take r sufficiently large, then $n \geq -r$. Pick such an $r \in \mathbb{N}$. Furthermore, take $M = 4r$ and use

$$(z_2 - z_3)^{3r} = \sum_{s=0}^{3r} \binom{3r}{s} (z_2 - z_1)^{3r-s}(z_1 - z_3)^s$$

to write down the left-hand side of Equation (3.21) as

$$\sum_{s=0}^{3r} \binom{3r}{s} (z_2 - z_1)^{3r-s}(z_1 - z_3)^s(z_2 - z_3)^r A. \tag{3.24}$$

If $3r - s + n \geq r$, then $(z_1 - z_2)^{3r-s}\iota_{z_1,z_2}(z_1 - z_2)^n = (z_1 - z_2)^{r'}$ where $r' \geq r$. Thus, using (3.23a) we see that the s-th summand in Equation (3.24) is 0 for $0 \leq s \leq r$. Hence, the left-hand of (3.21) becomes

$$\sum_{s=r+1}^{3r} \binom{3r}{s} (z_2 - z_1)^{3r-s}(z_1 - z_3)^s(z_2 - z_3)^r A. \tag{3.25}$$

Analogously, the right-hand side of (3.21) equals

$$\sum_{s=r+1}^{3r} \binom{3r}{s} (z_2 - z_1)^{3r-s}(z_1 - z_3)^s(z_2 - z_3)^r B. \tag{3.26}$$

From the locality assumptions (3.23b) and (3.23c), it follows that the equations (3.25) and (3.26) are equal thereby proving the lemma. □

3.4 Vertex Algebras

We are now ready to define one of the central definitions of this work.

Definition 3.20. Vertex Algebra. A *vertex algebra* is the following data:

- a vector space V (the space of states),
- a vector $|0\rangle \in V$ (the vacuum vector),
- a map $T \in \operatorname{End} V$ (infinitesimal translation operator),
- a linear map $Y(\cdot, z) : V \to \mathscr{F}(V)$ (the state-field correspondence)

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \operatorname{End} V.$$

This data is subject to the following axioms $\forall\, a, b \in V$:

V1. Translation covariance.

$$[T, Y(a,z)] = \partial Y(a,z)$$

V2. Locality.

$$(z-w)^N [Y(a,z), Y(b,w)] = 0,$$

for some $N \in \mathbb{N}$ depending on a and b.

V3. Vacuum.

$$T|0\rangle = 0, \quad Y(|0\rangle, z) = \mathrm{id}_V, \quad Y(a,z)|0\rangle|_{z=0} = a.$$

Now we want to prove the Existence Theorem 3.28 for vertex algebras which states when a vector space is a vertex algebra. Before that, we need some preliminary results first.

Proposition 3.21. *For all fields $a(w)$, $b(w)$ and $\forall n \in \mathbb{Z}$ holds*

$$\operatorname{ad} T\left(a(w)_{(n)} b(w)\right) = \left(\operatorname{ad} T a(w)\right)_{(n)} b(w) + a(w)_{(n)} \operatorname{ad} T(b(w)),$$

i.e. $\operatorname{ad} T$ *is a derivation on all n-th products.*

Proof. We have using Equation (3.18) and linearity of $\operatorname{ad} T$

$$\begin{aligned}
&\operatorname{ad} T\left(a(w)_{(n)} b(w)\right) = \\
&\quad = \operatorname{ad} T \left(\operatorname{Res}_z \big(a(z)b(w)\iota_{z,w}(z-w)^n - b(w)a(z)\iota_{w,z}(z-w)^n\big)\right) = \\
&\quad = \operatorname{Res}_z \left(\operatorname{ad} T \left(a(z)b(w)\iota_{z,w}(z-w)^n - b(w)a(z)\iota_{w,z}(z-w)^n\right)\right) = \\
&\quad = \operatorname{Res}_z \left([T, a(z)b(w)]\iota_{z,w}(z-w)^n - [T, b(w)a(z)]\iota_{w,z}(z-w)^n\right). \qquad (3.27a)
\end{aligned}$$

Moreover,

$$(\operatorname{ad} Ta(w))_{(n)}b(w) = \\ \operatorname{Res}_z\left([T,a(z)]b(w)\iota_{z,w}(z-w)^n - b(w)[T,a(z)]\iota_{w,z}(z-w)^n\right) \tag{3.27b}$$

and

$$a(w)_{(n)} \operatorname{ad} T(b(w)) = \\ \operatorname{Res}_z(a(z)[T,b(w)]\iota_{z,w}(z-w)^n - [T,b(w)]a(z)\iota_{w,z}(z-w)^n). \tag{3.27c}$$

Thus, adding equations (3.27b) and (3.27c) we obtain Equation (3.27a) as required. □

An analogue of Cauchy problem can be stated and solved for formal series.

Lemma 3.22. *Let U be a vector space and $S \in \operatorname{End} U$. The initial value problem*

$$\frac{d}{dz}f(z) = Sf(z), \quad f(0) = f_0,$$

with $f(z) \in U[[z]]$, has a unique solution of the form

$$f(z) = \sum_{n\in\mathbb{N}_0} f_n z^n, \quad f_n \in U.$$

In fact, $f(z) = e^{zS}f_0 = \sum 1/n!\, Sf_n z^n$.

Proof. The differential equation means $\sum(n+1)f_{n+1}z^n = \sum Sf_n z^n$ which implies that $(n+1)f_{n+1} = Sf_n\ \forall n \in \mathbb{N}_0$. This is equivalent to $f_n = 1/n!\, S^n f_0$. □

The following proposition is the first step in the proof that a Möbius conformal vertex algebra has an action of $\mathrm{PSL}(2,\mathbb{C})$ (Proposition 3.35).

Proposition 3.23. (a) *Given a vertex algebra V we have $\forall a \in V$*

$$Y(a,z)|0\rangle = e^{zT}(a) \tag{3.28}$$

$$e^{wT}Y(a,z)e^{-wT} = Y(a,z+w), \tag{3.29}$$

$$e^{wT}Y(a,z)_\pm e^{-wT} = Y(a,z+w)_\pm. \tag{3.30}$$

The last 2 equalities are in $\operatorname{End} V[[z^\pm]][[w]]$ which means that $(z+w)^n$ is replaced by its expansion $\iota_{z,w}(z+w)^n = \sum_{k\geq 0}\binom{n}{k}z^{n-k}w^k \in \mathbb{C}[[z^\pm]][[w]]$.
(b) *It holds $\forall a,b \in V$ and $\forall n \in \mathbb{Z}$ that*

$$Y(a_{(n)}b,z)|0\rangle = \left(Y(a,z)_{(n)}Y(b,z)\right)|0\rangle. \tag{3.31}$$

Proof. Let $f(z) = Y(a,z)|0\rangle$ which is in $V[[z]]$ because of the vacuum axiom V3. Using the translation covariance V1 and $T|0\rangle = 0$ from V3, we obtain the differential equation $\partial f(z) = Tf(z)$. Applying Lemma 3.22 to $U = V$ and $S = T$ gives us $f(z) = e^{zT}a$ proving the first equality.

To prove the second equation, we will apply Lemma 3.22 to $U = \operatorname{End} V[[z^{\pm}]]$ and $S = \operatorname{ad} T$. First, observe that $\partial_w(e^{wT}Y(a,z)e^{-wT}) = [T, e^{wT}Y(a,z)e^{-wT}] = \operatorname{ad} T(e^{wT}Y(a,z)e^{-wT})$. Furthermore, $\partial_w Y(a, z+w) = [T, Y(a, z+w)]$ by translation covariance V1. Both of these differential equations are of the form $\partial_w f = (\operatorname{ad} T)(f)$ and have the same initial value $f_0 = Y(a,z) \in \operatorname{End} V[[z^{\pm}]]$. Therefore, their solutions are the same by Lemma 3.22. This proves the second equation.

Equation (3.30) follows from the splitting $[T, Y(a,z)_{\pm}] = \partial Y(a,z)_{\pm}$.

To prove (3.31), first of all note that both ∂_z and $\operatorname{ad} T$ are derivations of all n-th products by propositions 3.17 and 3.21. Moreover, by vacuum (V3) and translation covariance (V1) axioms, both sides of (3.31) satisfy the differential equation of Lemma 3.22. The initial conditions also coincide by the vacuum axiom V3 and Lemma 3.18. $\square$

Theorem 3.24. Uniqueness [God89]. *Let V be a vertex algebra and let $B(z) \in \operatorname{End} V[[z^{\pm}]]$ be a field which is mutually local with all the fields $Y(a,z)$, $a \in V$. We have that*

$$\text{if} \quad B(z)|0\rangle = e^{zT}b \text{ for some } b \in V, \quad \text{then} \quad B(z) = Y(b,z).$$

Proof. Locality of $B(z)$ means that

$$(z-w)^N B(z)Y(a,w)|0\rangle = (z-w)^N Y(a,w)B(z)|0\rangle.$$

Applying to the left-hand side formula (3.28) and using the second assumption of the theorem for the right-hand side we obtain

$$(z-w)^N B(z)e^{wT}a = (z-w)^N Y(a,w)e^{zT}b. \tag{3.32}$$

Using formula (3.28) once more for $e^{zT}b$ we can write the right-hand side as

$$(z-w)^N Y(a,w)Y(b,z)|0\rangle = (z-w)^N Y(b,z)Y(a,w)|0\rangle$$

where last equality holds for sufficiently large N by locality. Applying formula (3.28) yet again to the last equation and equating it with the left-hand side of (3.32) we get

$$(z-w)^N B(z)e^{wT}a = (z-w)^N Y(b,z)e^{wT}a.$$

Setting $w = 0$ and dividing by z^N we find that $B(z)a = Y(b,z)a$ $\forall a \in V$. Hence, $B(z) = Y(b,z)$. $\square$

Remark 3.25. The assumption $B(z)|0\rangle = e^{zT}b$ of Theorem 3.24 holds if

$$B(z)|0\rangle|_{z=0} = b \quad \text{and} \quad \partial B(z)|0\rangle = TB(z)|0\rangle \tag{3.33}$$

do. This follows from Lemma 3.22. Note that only the first condition is not sufficient as the example $B(z) = (1+z)Y(b,z)$ shows.

Note that Goddard's Uniqueness Theorem is a vertex algebra analogue of the corollary of Reeh–Schlieder's Theorem 4.8.

Proposition 3.26. *Let V be a vertex algebra. One has*

$$Y(a_{(n)}b, z) = Y(a,z)_{(n)}Y(b,z) \tag{3.34}$$

$\forall a, b \in V$, and $\forall n \in \mathbb{Z}$.

Proof. Let $B(z) = Y(a,z)_{(n)}Y(b,z)$. By (3.31) and (3.28) we have

$$B(z)|0\rangle = Y(a_{(n)}b, z)|0\rangle = e^{zT}(a_{(n)}b).$$

Moreover, by Dong's Lemma 3.19, $B(z)$ is local with respect to all vertex operators $Y(c,z)$. Thus, Theorem 3.24 gives the required result. $\square$

Corollary 3.27. (a) For arbitrary collections of vectors $a^1, \ldots, a^n$ of vertex algebra V and arbitrary collections $k_1, \ldots, k_n$ of positive integers it holds

$$:D^{k_1-1}Y(a^1,z)\ldots D^{k_n-1}Y(a^n,z):= Y\left(a^1_{(-k_1)}\ldots a^n_{(-k_n)}|0\rangle, z\right).$$

(b) We have $\forall a, b \in V$ and $\forall n \in \mathbb{N}$:

$$:D^nY(a,z)Y(b,z):= Y(a_{(-n-1)}b, z).$$

(c) It holds $\forall a \in V$

$$Y(Ta, z) = \partial Y(a,z). \tag{3.35}$$

Proof. Parts (a) and (b) follow from Proposition 3.26 by Definition 3.15. The case (c) follows from (a) by setting $n = 1$ and $k_1 = 2$ and noting that $Ta = a_{(-2)}|0\rangle$. $\square$

We are finally ready to prove the existence theorem.

Theorem 3.28. Existence. *Let V be a vector space with an endomorphism $T \in \operatorname{End} V$ and a vector $|0\rangle \in V$. Let $(a^\alpha(z))_{\alpha\in I}$ (I an index set) be a collection of fields such that the following conditions are satisfied $\forall \alpha, \beta \in I$:*

(1) $[T, a^\alpha(z)] = \partial a^\alpha(z)$,

(2) $T|0\rangle = 0$ *and* $a^\alpha(z)|0\rangle|_{z=0} = a^\alpha$,

(3) *the linear map* $\sum_\alpha \mathbb{C}a^\alpha(z) \to \sum_\alpha \mathbb{C}a^\alpha$ *defined by* $a^\alpha(z) \mapsto a^\alpha$ *is injective,*

(4) $a^\alpha(z)$ *and* $a^\beta(z)$ *are mutually local,*

(5) *the vectors* $a^{\alpha_1}_{(j_1)} \ldots a^{\alpha_n}_{(j_n)}|0\rangle$ *with* $j_s \in \mathbb{Z}$, $\alpha_s \in I$ *span* V.

Then the definition

$$Y\left(a^{\alpha_1}_{(j_1)} \ldots a^{\alpha_n}_{(j_n)}|0\rangle, z\right) = a^{\alpha_1}(z)_{(j_1)}(a^{\alpha_2}(z)_{(j_2)}(\ldots(a^{\alpha_n}(z)_{(j_n)}\,\mathrm{id}_V)) \tag{3.36}$$

yields a unique structure of a vertex algebra on V *with the vacuum vector* $|0\rangle$, *the translation operator* T *and*

$$Y(a^\alpha, z) = a^\alpha(z) \quad \forall \alpha \in I. \tag{3.37}$$

Proof. Choose a basis for V using the vectors of the form (5) and define $Y(a, z)$ by formula (3.36).

The operators $\operatorname{ad} T$ and ∂ are derivations of n-th products by propositions 3.17 and 3.21. Furthermore, by assumption (1), $\operatorname{ad} T = \partial$ on the fields $a^\alpha(z)$. Thus, $\operatorname{ad} T = \partial$ on all the Y's proving the translation covariance axiom V1.

The locality axiom V2 holds due to (4), Remark 3.5 and Dong's Lemma 3.19.

The first two equations of the vacuum axiom V3 are trivially satisfied due to our assumption (2) and the defining equation (3.36). To prove that

$$Y(a^\alpha, z)|0\rangle|_{z=0} = a^\alpha,$$

we first note that by (2) we have $a^\alpha = a^\alpha(z)|0\rangle|_{z=0} = a^\alpha_{(-1)}|0\rangle$. This implies that $Y(a^\alpha, z)|0\rangle|_{z=0} = Y(a^\alpha_{(-1)}|0\rangle, z)|0\rangle|_{z=0} = a^\alpha(z)_{(-1)}|0\rangle|_{z=0} = a^\alpha$ using Equation (3.36). All the equalities are well-defined because of the injectivity assumption (3).

To prove that our vertex algebra is basis-independent and hence well-defined, note that if we chose another basis out of the monomials (5), we would get a structure of another vertex algebra on V which may differ from our original one. But all the fields of the new structure would be mutually local with respect to those of the old structure and would satisfy Equation (3.33). Thus, by Remark 3.25 and the Uniqueness Theorem 3.24, it follows that these vertex algebra structures would coincide. Therefore, Equation (3.36) is well-defined and Equation (3.37) holds.

□

Definition 3.29. Generating set of fields. A *generating set of fields* is a collection of fields of a vertex algebra V satisfying condition (5) of Theorem 3.28. If condition (5) holds restricted to $j_s < 0$, then such a collection is called a *strongly generating set of fields.*

The operator product expansion is usually assumed in 2D CFT in physics. The following theorem shows that it can be deduced from the axioms of a vertex algebra.

Theorem 3.30. OPE for vertex algebras. *Let V be a vertex algebra and $a, b \in V$. In the domain $|z| > |w|$ one has*

$$Y(a,z)Y(b,w) = \sum_{n=0}^{\infty} \frac{Y(a_{(n)}b, w)}{(z-w)^{n+1}} + {:}Y(a,z)Y(b,w){:}\,. \tag{3.38a}$$

Equivalently

$$[Y(a,z), Y(b,w)] = \sum_{n=0}^{\infty} Y(a_{(n)}b, w) D_w^n \delta(z-w). \tag{3.38b}$$

Proof. Fix $a, b \in V$. Then by the axiom of locality V2 we have

$$(z-w)^{N(a,b)}[Y(a,z), Y(b,w)] = 0$$

for some $N(a,b) \in \mathbb{N}$ depending on a and b. Thus, $Y(a,z)$ and $Y(b,z)$ are mutually local formal distributions (Definition 3.4) and satisfy Equation (3.14b)

$$Y(a,z)Y(b,w) = \sum_{j=0}^{N(a,b)-1} \frac{Y(a,w)_{(j)}Y(b,w)}{(z-w)^{j+1}} + {:}Y(a,z)Y(b,w){:}\,.$$

Here, as usual, the domain $|z| > |w|$ is implicit. By Proposition 3.26 we have $Y(a,w)_{(j)}Y(b,w) = Y(a_{(j)}b, w)$. Hence, allowing the sum to go to infinity we obtain

$$Y(a,z)Y(b,w) = \sum_{j=0}^{\infty} \frac{Y(a,w)_{(j)}Y(b,w)}{(z-w)^{j+1}} + {:}Y(a,z)Y(b,w){:} \qquad \forall a, b \in V.$$

Now, (3.38b) is equivalent to (3.38a) by the same reasoning which gave us the equivalence between equations (3.14a) and (3.14b) in Section 3.2. $\square$

We also obtain a useful corollary.

Corollary 3.31. Borcherds commutator formulas. The vertex algebra commutator OPE (3.38b) is equivalent to each of the following formulas

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} \left(a_{(j)}b\right)_{(m+n-j)} \tag{3.39a}$$

$$[a_{(m)}, Y(b,w)] = \sum_{j\geq 0} \binom{m}{j} Y(a_{(j)}b, w)w^{m-j}. \tag{3.39b}$$

Proof. To prove

$$(3.39a) \implies (3.39b) \implies (3.38b)$$

one has to multiply by the indeterminate with respective power and sum over.

The converse,

$$(3.38b) \implies (3.39b) \implies (3.39a)$$

is proved by multiplying with z^m and taking the residue Res_z, and then multiplying by w^n and taking Res_w. $\square$

3.5 Möbius Conformal and Conformal Vertex Algebras

Now we add some more structure to vertex algebras which will allow us to define quasiprimary fields.

Definition 3.32. A vertex algebra V is called *graded* if there is a diagonalizable operator H on V such that

$$[H, Y(a,z)] = z\partial Y(a,z) + Y(Ha, z). \tag{3.40}$$

Proposition 3.33. *A field $Y(a,z)$ of a graded vertex algebra V with diagonalizable operator H has conformal weight $h \in \mathbb{C}$ with respect to the Hamiltonian* $\operatorname{ad} H$ *if and only if $Ha = ha$.*

Proof. Use Definition 3.9 together with linearity of $Y(a,z)$ in the first argument. $\square$

Due to the above proposition, we will abuse our terminology and call H a **Hamiltonian** of a vertex algebra V if Equation (3.40) holds. Moreover, a graded vertex algebra whose Hamiltonian is bounded below by zero will be called a **positive-energy** vertex algebra.

Definition 3.34. Möbius conformal vertex algebra. A vertex algebra V graded by H is called *Möbius conformal* if there exists an operator T^* on V such that T^* decreases the conformal weight by 1 and

$$[T^*, Y(a,z)] = z^2 \partial Y(a,z) + 2zY(Ha,z) + Y(T^*a,z) \tag{3.41}$$

for all $a \in V$. We will also call a field $Y(a,z)$ of a Möbius conformal vertex algebra with weight h *quasiprimary* if

$$[T^*, Y(a,z)] = (z^2\partial + 2hz)Y(a,z). \tag{3.42}$$

Note that $Y(a,z)$ is a quasiprimary field of conformal weight h if and only if

$$Ha = ha, \qquad T^*a = 0. \tag{3.43}$$

Thus, we will call vectors satisfying (3.43) **quasiprimary** (of weight h).

The following proposition will be very important in Section 7.2, where we construct a Wightman CFT from vertex algebras.

Proposition 3.35. *We have*

$$H|0\rangle = T^*|0\rangle = 0, \tag{3.44}$$

i.e. the vacuum vector is quasiprimary of weight 0*. Moreover,*

$$[H,T] = T, \quad [H,T^*] = -T^*, \quad [T^*,T] = 2H, \tag{3.45}$$

i.e. H, T *and* T^* *form a representation of* $\mathfrak{sl}(2,\mathbb{C})$*. It also holds that*

(a) $e^{\lambda T}\, Y(a,z)\, e^{-\lambda T} = Y(a, z+\lambda), \qquad |\lambda| < |z|,$

(b) $\lambda^H\, Y(a,z)\, \lambda^{-H} = Y(\lambda^H a, \lambda z),$

(c) $e^{\lambda T^*} Y(a,z) e^{-\lambda T^*} = Y\left(e^{\lambda(1-\lambda z)T^*}(1-\lambda z)^{-2H}a, \dfrac{z}{1-\lambda z}\right), \quad |\lambda z| < 1.$

Proof. Write for a field of conformal weight h

$$Y(a,z) = \sum_{n \in -h+\mathbb{Z}} a_n z^{-n-h},$$

i.e. shift the coefficients so that

$$a_{(n)} = a_{n-h+1}$$

holds. Then (3.40) is equivalent to

$$[H, a_n] = -na_n. \tag{3.46}$$

Similarly, $[T, Y(a, z)] = \partial Y(a, z)$ is equivalent to

$$[T, a_n] = (-n - h + 1)a_{n-1}.$$

Equation (3.40) implies that $H|0\rangle = 0$ and hence

$$[H, T] = T$$

since both sides give the same result applied on a_n's and annihilate $|0\rangle$.

The other commutation relations follow similarly by noting that (3.41) with $a = |0\rangle$ gives $T^*|0\rangle = 0$, and that Equation (3.41) in component form for $a \in V$ of conformal weight h is

$$[T^*, a_n] = -(n - h + 1)a_{n+1} + (T^*a)_{n+1}.$$

Part (a) has been already proved in Proposition 3.23 and is restated here for convenience.

Integrating (3.46) we get

$$\lambda^H a_n \lambda^{-H} = \lambda^{-n} a_n \tag{3.47}$$

which is equivalent to (b).

Now we prove (c). Write

$$e^{\lambda T^*} Y(a, z) e^{-\lambda T^*} = Y\left(A(\lambda)a, \frac{z}{1 - \lambda z}\right),$$

with $A(\lambda)$ a formal power series in λ with coefficients in $\mathrm{Hom}\,(V, \mathrm{End}\, V[[z, z^{-1}]])$. Differentiate both sides by λ and use (3.41) to get:

$$\frac{dA(\lambda)}{d\lambda} = z^2 \partial_z A(\lambda) + 2zA(\lambda)H + A(\lambda)T^*.$$

By Lemma 3.22, this equation has a unique solution. To check that $A(\lambda) = e^{\lambda(1-\lambda z)T^*}(1 - \lambda z)^{-2H}$ solves this equation, use (3.45) and that $\mathrm{ad}\, T^*$ is a derivation. □

Remark 3.36. By Proposition 3.35, we can identify T, T^*, H with the corresponding $\mathfrak{sl}(2, \mathbb{C})$ generators by

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^* = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

It is a well-know fact that the exponentiation of $\mathfrak{sl}(2,\mathbb{C})$ is not surjective with the usual argument being that there are no elements in $\mathfrak{sl}(2,\mathbb{C})$ which under the exponentiation are mapped to the elements of $\mathrm{SL}_2(\mathbb{C})$ whose trace is less than or equal to -2. However, in $\mathrm{PSL}(2,\mathbb{C})$ we can choose a representative of positive trace for each equivalence class. Thus,

$$\exp : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$$

is onto. From Proposition 3.35 it follows that $\mathrm{PSL}_2(\mathbb{C})$ acts on the variable z by

$$z \mapsto \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

and that

$$e^{\lambda T} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad e^{\lambda T^*} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}, \quad e^{\lambda H} = \begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix}.$$

Adding an action of the Virasoro algebra gives a conformal vertex algebra in which primary fields can be defined.

Definition 3.37. Conformal vertex algebra. A *Virasoro field with central charge* c is a field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \in \operatorname{End} U[[z^{\pm}]]$, U some vector space, with the OPE

$$L(z)L(w) \sim \frac{C/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)} \tag{3.48}$$

such that $C = c\,\mathrm{id}_U$ with $c \in \mathbb{C}$.

A *conformal vector* of a vertex algebra V is a vector ν such that the corresponding field $Y(\nu,z) = \sum_{n\in\mathbb{Z}} \nu_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L^{\nu}_n z^{-n-2}$ is a Virasoro field with central charge c satisfying

(a) $L^{\nu}_{-1} = T$,

(b) L^{ν}_0 is diagonalizable on V.

The number c is called the *central charge* of ν.

A *conformal vertex algebra* (of rank c) is a vertex algebra having a conformal vector ν (with central charge c). Then the field $Y(\nu,z)$ is called an *energy-momentum field* of the vertex algebra V.

Note that each conformal vertex algebra is Möbius conformal with

$$T = L_{-1}, \quad H = L_0, \quad T^* = L_1.$$

Indeed by Equation (3.38a) we have $\forall a \in V$

$$Y(\nu, z)Y(a, w) \sim \sum_{n \geq -1} \frac{Y(L_n a, w)}{(z-w)^{n+2}} \tag{3.49}$$

which by Corollary 3.31 is equivalent to

$$[L_m, Y(a, z)] = \sum_{j \geq -1} \binom{m+1}{j+1} Y(L_j a, z) z^{m-j}.$$

Setting $m = 0$ gives Equation (3.40) and setting $m = 1$ gives Equation (3.41). The calculation

$$L_0(L_1 a) = [L_0, L_1]a + L_1 L_0 a = -L_1 a + L_1 ha = (h-1)L_1 a$$

shows that L_1 decreases the conformal weight by 1, as required.

If $L_0 = ha$, then Equation (3.49) becomes

$$Y(\nu, z)Y(a, w) \sim \frac{\partial Y(a, w)}{z-w} + \frac{hY(a, w)}{(z-w)^2} + \dots$$

where we have used $Y(Ta, z) = \partial Y(a, z)$ (Equation (3.35)).

Definition 3.38. Primary field. A field $Y(a, z)$ of a conformal vertex algebra V is *primary* of conformal weight h if

$$Y(\nu, z)Y(a, w) \sim \frac{\partial Y(a, w)}{z-w} + \frac{hY(a, w)}{(z-w)^2}.$$

All equivalent definitions of primary fields used by physicists also hold in conformal vertex algebras.

Proposition 3.39. *The following are equivalent:*

(a) $Y(a, z)$ *is primary of conformal weight* h*,*

(b) $L_n a = \delta_{n,0}\, ha \quad \forall n \in \mathbb{N}$*,*

(c) $[L_m, Y(a, z)] = z^m(z\partial + h(m+1))Y(a, z) \quad \forall m \in \mathbb{Z}$*,*

(d) $[L_m, a_n] = ((h-1)m - n)a_{m+n} \quad \forall m, n \in \mathbb{Z}$*.*

Proof. Equation (3.49) together with the definition, gives equivalence of (a) and (b).

By Theorem 3.30, the OPE of a primary field is equivalent to

$$\begin{aligned}
[Y(\nu,z),Y(a,w)] &= \partial Y(a,w)\delta(z-w) + hY(a,w)\partial_w\delta(z-w) \\
&= \sum_{m\in\mathbb{Z}}(-m-1)a_{(m)}w^{-m-2}\sum_{n\in\mathbb{Z}}z^{-n-1}w^n + \\
&\quad + h\sum_{m\in\mathbb{Z}}a_{(m)}w^{-m-1}\sum_{n\in\mathbb{Z}}nz^{-n-1}w^{n-1} \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(-m-1+h(n+1))a_{(m)}w^{n-m-1}z^{-n-2}.
\end{aligned}$$

But we also have

$$[Y(\nu,z),Y(a,w)] = \sum[L_n,Y(a,w)]z^{-n-2}$$

and so

$$\begin{aligned}
[L_m,Y(a,z)] &= \sum_{n\in\mathbb{Z}}(-n-1+h(m+1))a_{(n)}z^{m-n-1} \qquad (3.50)\\
&= z^{m+1}\sum_{n\in\mathbb{Z}}(-n-1)a_{(n)}z^{-n-2} + z^m h(m+1)\sum_{n\in\mathbb{Z}}a_{(n)}z^{-n-1} \\
&= z^{m+1}\partial Y(a,z) + z^m h(m+1)Y(a,z).
\end{aligned}$$

Thus, a primary field $Y(a,z)$ satisfies (c) and the converse is also true since the reasoning above can be reversed.

Now (d) is equivalent to (c) since by definition and (3.50)

$$\begin{aligned}
[L_m,Y(a,z)] &= \sum_{n\in\mathbb{Z}}[L_m,a_{(n)}]z^{-n-1} \\
&= \sum_{n\in\mathbb{Z}}(-m-n-1+h(m+1))a_{(m+n)}z^{-n-1}.
\end{aligned}$$

Thus, remembering that $a_{(n)} = a_{n-h+1}$ and comparing the coefficients gives the required result. □

Due to Proposition 3.39, a **primary vector** is defined as a vector satisfying $L_0a = ha$ and $L_na = 0$ for $n \geq 1$.

Example 3.40. The vacuum vector $|0\rangle$ is primary, since by V3 we have $Y(\nu,z)|0\rangle|_{z=0} = \nu$. This also shows that $\nu = L_{-2}|0\rangle$ and hence ν is quasiprimary of conformal weight 2, but not primary unless $c = 0$ by Equation (3.48).

3.6 Vertex Operator Algebras and Unitarity

Even more assumptions on vertex algebras are usually natural in physics. The following assumptions will allow us to obtain a transparent construction of Wightman 2D CFT in Section 7.2.

Definition 3.41. Vertex operator algebra. A *vertex operator algebra* (VOA) is a conformal vertex algebra such that

(i) $V = \bigoplus_{n\in\mathbb{Z}} V_n$, where $V_n := \ker(L_0 - n\,\mathrm{id}_V)$,

(ii) $V_n = \{0\}$ for n sufficiently small,

(iii) $\dim V_n < \infty$.

The subspaces V_n providing the grading are called *homogeneous subspaces*. If $V_n = 0$ for $n < 0$ and $V_0 = \mathbb{C}|0\rangle$, then the vertex operator algebra is of *CFT type*.

Remark 3.42. If we were to replace the conformal vertex algebra with a Möbius conformal vertex algebra in Definition 3.41, then we would get a **quasi-vertex operator algebra** (q-VOA) [FHL93, Sec. 2.8].

Throughout this section, fix V to be a (q-)VOA, and let $F = \mathbb{Z}$ for a VOA and $F = \{-1, 0, 1\}$ for a q-VOA which is not a VOA.

The following proposition is due to Roitman [Roi04].

Proposition 3.43. *Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a graded module over the Lie algebra $\mathfrak{sl}_2 = \Bbbk T + \Bbbk H + \Bbbk T^*$, where $\deg T = 1$, $\deg T^* = -1$, T^* is locally nilpotent, $H|_{M_d} = d$ and $\Bbbk$ is a field of characteristic 0. Furthermore, assume that the $\mathfrak{sl}_2$ commutation relations (3.45) hold. Then $M_d = (T^*)^{1-d} M_1$ for all $d < 0$.*

Corollary 3.44. By Proposition 3.43 we have that for $n < 0$, $V_n = L_1^{1-n} V_1 \subset L_1^{-n} V_0$. Hence, if $V_0 = \mathbb{C}|0\rangle$, then condition (ii) is equivalent to the condition $V_n = \{0\}$ for all $n < 0$. Thus, if $V_0 = \mathbb{C}|0\rangle$, then V is of CFT type.

We say that a map $\phi : V \to V$ is an **antilinear automorphism** of a q-VOA V, if it is an antilinear isomorphism such that $\phi(u_n v) = \phi(u)_n \phi(v)$ $\forall u, v \in V$, $\forall n \in \mathbb{Z}$, and $\phi(|0\rangle) = |0\rangle$. If V is a VOA, then we require ϕ to also satisfy $\phi(\nu) = \nu$.

Let $(\cdot,\cdot)$ be a bilinear form on V. If $(\cdot,\cdot)$ satisfies

$$(Y(a,z)b, c) = (b, Y(e^{zL_1}(-z^{-2})^{L_0} a, z^{-1})c) \quad \forall a, b, c \in V,$$

then it will be called an **invariant bilinear form**. By [Li94, Prop. 2.6], any invariant bilinear form on a (q-)VOA is in fact symmetric.

Remark 3.45. By direct calculation it follows that $(\cdot,\cdot)$ is invariant if and only if

$$(a_n b, c) = (-1)^{h_a} \sum_{k\in\mathbb{N}_0} \frac{1}{k!}(b, (L_1^k a)_{-n} c) \tag{3.51}$$

for all $b, c \in V$ and for all homogeneous $a \in V$. If V is a VOA, then for L_n's this boils down to

$$(L_n a, b) = (a, L_{-n} b) \quad a, b \in V,\ n \in \mathbb{Z} \tag{3.52}$$

and thus the case $n = 0$ shows that $(V_i, V_j) = 0$ if $i \neq j$.

Note that for general q-VOAs the implication from (3.51) to (3.52) does not work because there is no conformal vector ν. So in the case of a q-VOA which is not a VOA, $(L_1 a, b) = (a, L_{-1} b)$ has to be assumed along with (3.51) for $(V_i, V_j) = 0$ if $i \neq j$ to hold [Roi04].

Now let $(\cdot|\cdot)$ be an inner product on V, linear in the second variable. We also want it to be **normalized**, i.e. $(\Omega|\Omega) = 1$, and $(\cdot|\cdot)$ **invariant**, i.e. there exists a VOA antilinear automorphism θ of V such that $(\theta \cdot |\cdot)$ is an invariant bilinear form on V. We call θ a **PCT** operator associated with $(\cdot|\cdot)$. By definition $\theta(\nu) = \nu$, so θ commutes with all L_n's. Equation (3.51) implies that

$$(a_n b|c) = \left(\theta\left((\theta^{-1}a)_n \theta^{-1} b\right) \Big| c\right) = (b|(\theta^{-1} e^{L_1} (-1)^{L_0} a)_{-n} c) \tag{3.53}$$

$\forall a, b, c \in V$ and $\forall n \in \mathbb{Z}$. If a is quasiprimary, then

$$(a_n b|c) = (-1)^{h_a} (b|(\theta^{-1} a)_{-n} c),$$

$\forall b, c \in V$ and $\forall n \in \mathbb{Z}$. In particular,

$$(L_n a|b) = (a|L_{-n} b) \tag{3.54}$$

$\forall a, b \in V$ and $\forall n \in \mathbb{Z}$. Thus, the corresponding representations of the Virasoro algebra and its $\mathfrak{sl}(2,\mathbb{C})$ subalgebra $\mathbb{C}\{L_{-1}, L_0, L_1\}$ are unitary and therefore completely reducible. In particular, we have $V_n = 0$ for $n < 0$ by Proposition 3.43. Similarly, for a q-VOA.

Proposition 3.46. *Let V be a q-VOA with a normalized invariant inner product $(\cdot|\cdot)$. Then there exists a unique PCT operator θ associated with $(\cdot|\cdot)$. Furthermore, θ is an antiunitary involution.*

Proof. Let $\tilde\theta$ be another PCT operator associated with $(\cdot|\cdot)$. Equation (3.53) shows that $(\theta^{-1} e^{L_1} (-1)^{L_0} a)_n = (\tilde\theta^{-1} e^{L_1} (-1)^{L_0} a)_n$ and hence $\theta^{-1} e^{L_1} (-1)^{L_0} a =$

$\tilde{\theta}^{-1}e^{L_1}(-1)^{L_0}a$ for all $a \in V$. Now the surjectivity of $e^{L_1}(-1)^{L_0}$ implies that $\theta = \tilde{\theta}$.

From Equation (3.53) and symmetry of the bilinear form, it follows that $a = (e^{L_1}(-1)^{L_0})^2\theta^{-2}a$ for all $a \in V$. Equation (3.47) shows that

$$(-1)^{L_0}e^{L_1}(-1)^{L_0} = e^{-L_1}$$

and hence $(e^{L_1}(-1)^{L_0})^2 = 1$. Thus, $\theta^2 = 1$, i.e. θ is an involution. Moreover, we see that $(\theta a|\theta b) = (\theta^2 b|a) = (b|a)\ \forall a, b \in V$ by the symmetry of the invariant bilinear form $(\theta \cdot |\cdot)$. Therefore, θ is antiunitary. □

This leads to the following definition.

Definition 3.47. A *unitary (quasi-)vertex operator algebra* is a pair $(V, (\cdot|\cdot))$ where V is a (quasi-)vertex operator algebra and $(\cdot|\cdot)$ is a normalized invariant inner product on V.

Remark 3.48. Note that the requirement that $\dim V_n < \infty$ was not used until now. Thus, if we have a Möbius conformal vertex algebra with integer grading, whose Hamiltonian is bounded below, we can also define a **unitary Möbius conformal vertex algebra** paralleling Definition 3.47.

Many of the well-known VOAs have unitary examples: Virasoro (see Section 6.1), affine, Heisenberg and lattice VOAs. Moreover, the moonshine VOA $V^\natural$ is also unitary. For proofs see [DL14].

The definition of unitarity does not seem to have much in common with the notion of unitarity used in QFT. However, [CKLW15, Sec. 5.2] shows that these two notions are equivalent for VOAs of CFT type, but the proof works for q-VOAs with $V_0 = \mathbb{C}|0\rangle$ as well. In particular, we have [CKLW15, Thm. 5.16]:

Theorem 3.49. *Let V be a (q-)VOA with a normalized inner product $(\cdot|\cdot)$ and $V_0 = \mathbb{C}|0\rangle$. Then the following are equivalent:*

(a) $(V, (\cdot|\cdot))$ *is a unitary (q-)VOA,*

(b) $(V, (\cdot|\cdot))$ *has a unitary Möbius symmetry and every vertex operator has a local adjoint.*

We now give the definitions and some results of the notions used in the above theorem. More details and the proof can be found in [CKLW15, Sec. 5.2].

For a (q-)VOA with a normalized inner product $(\cdot|\cdot)$ to have **unitary Möbius symmetry** means that $\forall a, b \in V$

$$(L_n a|b) = (a|L_{-n} b), \quad n = -1, 0, 1.$$

For an operator $A \in \operatorname{End} V$ to have an adjoint on V (with respect to $(\cdot|\cdot)$) means that $\exists A^+ \in \operatorname{End} V$ such that

$$(a|Ab) = (A^+ a|b), \quad \forall a, b \in V.$$

If A^+ exists, then it is unique and called the **adjoint** of A on V.

Remark 3.50. Let $\mathcal{H}$ be the Hilbert space completion of $(V, (\cdot|\cdot))$. Then an operator $A \in \operatorname{End} V$ can be considered as a densely defined operator on $\mathcal{H}$. Thus, A^+ exists if and only if the domain of the Hilbert space adjoint A^* of A contains V and if this is the case we have $A^+ \subset A^*$, i.e. $A^+ = A^*|_V$.

Lemma 3.51. *Let $(V, (\cdot|\cdot))$ have unitary Möbius symmetry. Then the adjoint a_n^+ of a_n on V exists $\forall a \in V$ and $\forall n \in \mathbb{Z}$. Moreover, we have $a_{-n}^+ b = 0$ for $n \gg 0$.*

Proof. The finite-dimensional subspaces $V_n = \ker(L_0 - n \operatorname{id}_V)$ of V are pairwise orthogonal by unitary Möbius symmetry. Since $a_n(V_k) \subset V_{k-n}$, the operator $a_n|_{V_k}$ can be regarded as an operator between two finite-dimensional inner product spaces and thus it has an adjoint $(a_n|_{V_k})^* \in \operatorname{Hom}(V_{k-n}, V_k)$ which is well-defined. It follows that

$$a_n^+ := \bigoplus_{k\in\mathbb{Z}} (a_n|_{V_k})^*$$

is the adjoint of a_n. This shows that $a_{-n}^+(V_k) \subset V_{k-n}$ and hence $a_{-n}^+ b = 0$ for $n \gg 0$. $\square$

The lemma implies that for $a \in V$ the formal series

$$Y(a,z)^+ := \sum_{n\in\mathbb{Z}} a_{(n)}^+ z^{n+1} = \sum_{n\in\mathbb{Z}} a_{(-n-2)}^+ z^{-n-1}$$

is well-defined and is a field on V. So we say that a vertex operator $Y(a,z)$, $a \in V$, has a **local adjoint** if $\forall b \in V$ the fields $Y(a,z)^+$ and $Y(b,z)$ are mutually local, i.e.

$$(z-w)^N [Y(a,z)^+, Y(b,w)] = 0, \quad N \gg 0.$$

Chapter 4

Wightman QFT

Formulated in 1950s, Wightman's axioms of QFT are the first attempt at putting QFT on a rigorous mathematical footing. Even though the axioms are very natural, it turned out to be very difficult to construct examples. To date there are no non-trivial examples of Wightman QFTs in 4D. Nevertheless, the CPT and Spin-Statistics theorems can be proved in Wightman framework. Moreover, the statement that "knowing all the fields is the same as knowing all the correlation functions" is made explicit by Wightman Reconstruction Theorem 4.12 and its converse 4.10. Last but not least, the current work shows that Wightman axioms are sufficiently general to incorporate many aspects of 2D genus 0 CFTs.

The main reference for this chapter is the book by Wightman and Streater [SW64], but we have also used [Sch08], [Kac98] and [BLOT89]. We only consider bosonic theories, but a generalization to include fermions is easily obtained. See, e.g., [SW64] or [BLOT89].

4.1 Preliminaries

Before stating the axioms we need some definitions.

Definition 4.1. Schwartz space, tempered distribution. Let

$$\mathscr{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta\}$$

be the *Schwartz space* of rapidly decreasing smooth functions. Here α, β are multi-indices and

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|$$

are seminorms. The elements of $\mathscr{S}(\mathbb{R}^n)$ are called *test functions* and the dual space consists of *(tempered) distributions* which are linear functionals $\mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$, continuous with respect to all seminorms.

Definition 4.2. Wightman field. Let $\mathscr{O}(\mathcal{H})$ be the set of all densely defined operators on a Hilbert space $\mathcal{H}$. Denote by $\langle \cdot, \cdot \rangle$ the inner product of $\mathcal{H}$. A *(Wightman) field* ϕ on a manifold M is a tempered operator-valued distribution, i.e. a map $\phi : \mathscr{S}(M) \to \mathscr{O}(\mathcal{H})$, such that there exists a dense subspace $\mathcal{D} \subset \mathcal{H}$ satisfying

- $\mathcal{D} \subset \mathcal{D}_{\phi(f)} \quad \forall f \in \mathscr{S}(M)$,
- the induced map $\mathscr{S} \to \operatorname{End} \mathcal{D}$, $f \mapsto \phi(f)|_{\mathcal{D}}$ is linear,
- $\forall v \in \mathcal{D}, \forall w \in \mathcal{H}$ the assignment $f \mapsto \langle w, \phi(f)(v)\rangle$ is a tempered distribution.

Minkowski space, Lorentz group, Poincaré group.

Let $M = (\mathbb{R}^{1,d-1}, g)$ be a d-dimensional **Minkowski space**, i.e. the vector space $\mathbb{R}^d$ with metric

$$|x-y|^2 = (x^0-y^0)^2 - \sum_{i=1}^{d-1}(x^i-y^i)^2,$$

so that $g = \operatorname{diag}(1, \underbrace{-1, -1, \ldots, -1}_{d-1})$.

Given $A, B \subset M$, we say that A and B are **spacelike separated** if $\forall a \in A$ and $\forall b \in B$ we have $|a-b|^2 < 0$. Let the **forward (light)cone** $\bar{V}_+$ be the set $\{x \in M \mid |x|^2 \geq 0,\, x^0 \geq 0\}$. Define **causal order on** M by $x \geq y \iff x - y \in \bar{V}_+$. We will also often write

$$a^\mu b_\mu = a \cdot b.$$

Definition 4.3. Lorentz group, Poincaré group.
In d dimensions:

- $\mathcal{L} := \mathrm{O}(1, d-1) = \{\Lambda \in \mathrm{GL}(d) \mid \Lambda\, g\, \Lambda^T = g\}$—*full Lorentz group*, preserves the metric;
- $\mathcal{L}_+ := \mathrm{SO}(1, d-1) = \{\Lambda \in \mathrm{O}(1,d) \mid \det \Lambda = 1\}$—*proper Lorentz group*, preserves orientation;
- $\mathcal{L}^\uparrow := \{\Lambda \in \mathrm{O}(1,d) \mid e\,\Lambda\, e^T \geq 0\}$—*orthochronous Lorentz group*, preserves the direction of time, here $e = (1,0,0,\ldots,0) \in M$;
- $\mathcal{L}_+^\uparrow := \mathcal{L}^\uparrow \cap \mathcal{L}_+ = \mathrm{SO}^+(1, d-1)$—*(proper orthochronous) Lorentz group.*

The d-dimensional **(proper orthochronous) Poincaré group** is defined as

$$\mathcal{P}_+^\uparrow := \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow.$$

It is a set of pairs $(q, \Lambda) \in (\mathbb{R}^d, \mathcal{L}_+^\uparrow)$ with multiplication:

$$(q_1, \Lambda_1) \cdot (q_2, \Lambda_2) := (q_1 + \Lambda_1 q_2, \Lambda_1 \Lambda_2),$$

and $\mathcal{P}_+^\uparrow$ acts continuously on the test functions $\mathscr{S}(\mathbb{R}^d)$ from the left by

$$(q, \Lambda) f(x) := f(\Lambda^{-1}(x-q)).$$

Note that equivalently one can define the Poincaré group $\mathcal{P}_+^\uparrow$ as the **identity component** (maximal connected subset containing the identity) of the group of all transformations of M preserving the metric. Similarly, the Lorentz group $\mathcal{L}_+^\uparrow$ can be defined as the group of all unimodular linear transformations of M preserving the lightcone $\bar{V}_+$. Therefore, the Poincaré group preserves the causal order and thus the spacelike separation.

4.2 Wightman Axioms

Now we define a bosonic Wightman QFT for an at most countable collection of scalar fields. See, e.g., [BLOT89, Sec. 8.2] for generalizations to arbitrary bosonic and fermionic fields.

Definition 4.4. Wightman QFT. A *Wightman quantum field theory* in d dimensions is:

- the projective space $\mathrm{P}(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ (the space of states),
- the vector $\Omega \in \mathcal{H}$ such that $\langle \Omega, \Omega \rangle = 1$ (the vacuum vector),
- a continuous unitary representation $(q, \Lambda) \mapsto U(q, \Lambda)$ of the Poincaré group $\mathcal{P}_+^\uparrow$,
- a collection of fields ϕ_a and their adjoints ϕ_a^*, $a \in I$ with I an at most countable index set,
$$\phi_a : \mathscr{S}(\mathbb{R}^d) \to \mathscr{O}(\mathcal{H}).$$

One requires this data to satisfy the following axioms:

W1. Covariance. It holds

$$U(q, \Lambda)\phi_a(f)U(q, \Lambda)^{-1} = \phi_a((q, \Lambda)f), \tag{4.1}$$

$\forall f \in \mathscr{S}(\mathbb{R}^d)$, $\forall (q, \Lambda) \in \mathcal{P}_+^\uparrow$.

Note that by Stone's theorem $U(q, 1) = \exp\left(i \sum_{k=0}^{d-1} q^k P_k\right)$ with P_k self-adjoint and commuting operators on $\mathcal{H}$.

W2. Stable vacuum and spectrum condition. We have

$$U(q, \Lambda)\Omega = \Omega,$$

$\forall (q, \Lambda) \in \mathcal{P}_+^\uparrow$. The simultaneous spectrum of all the operators $P_0, \ldots, P_{d-1}$ is contained in the forward light cone $\bar{V}_+$.

W3. Cyclicity of the vacuum. The vacuum Ω is in the domain of any polynomial in the $\phi_a(f)$'s and their adjoints. Let $\mathcal{D}_0 \subset \mathcal{H}$ be the subspace spanned by such polynomials

$$\phi_{a_1}(f_1)\phi_{a_2}(f_2)\ldots\phi_{a_m}(f_m)\Omega$$

and their adjoints. We assume that $\mathcal{D}_0$ is dense in $\mathcal{H}$. Clearly, $\Omega \in \mathcal{D}_0$.

Sometimes a weaker version of W3 is used.

W3$^{\text{weak}}$. Dense domain. There exists a linear set $\mathcal{D}$ dense in $\mathcal{H}$ such that the domain of each smeared operator $\phi_a(f)$ contains $\mathcal{D}$. Same holds for adjoints $\phi_a(f)^*$. Moreover,

$$\Omega \in \mathcal{D}, \quad U(q,\Lambda)\mathcal{D} \subset \mathcal{D}, \quad \phi_a(f)\mathcal{D} \subset \mathcal{D}, \quad \phi_a(f)^*\mathcal{D} \subset \mathcal{D}.$$

W4. Locality. If the supports of $f, g \in \mathscr{S}(\mathbb{R}^d)$ are spacelike separated, then on the common dense domain

$$[\phi_a(f), \phi_b(g)] = 0.$$

Similarly,

$$[\phi_a(f), \phi_b(g)^*] = 0.$$

Remark 4.5. By abuse of notation, we will often write $\phi(x)$ instead of $\phi(f)$. With this notational simplification, the equivariance condition (4.1) becomes

$$U(q,\Lambda)\phi_a(x)U(q,\Lambda)^{-1} = \phi_a(\Lambda x + q) \tag{4.2}$$

and the adjoint is simply $\phi^*(x)$ which acts by $\phi^*(f) = \phi\left(\overline{f}\right)^*$.

Remark 4.6. Note that by definition, we have

$$\phi_a(f)\mathcal{D}_0 \subset \mathcal{D}_0$$

for all the fields. Moreover, we have by W1 and W2

$$\begin{aligned}
&U(q,\Lambda)\phi_{a_1}(f_1)\phi_{a_2}(f_2)\ldots\phi_{a_m}(f_m)|0\rangle = \\
&\quad U(q,\Lambda)\phi_{a_1}(f_1)U(q,\Lambda)^{-1}U(q,\Lambda)\phi_{a_2}(f_2)\ldots\phi_{a_m}(f_m)U(q,\Lambda)^{-1}|0\rangle = \\
&\quad \phi_{a_1}((q,\Lambda)f_1)\phi_{a_2}((q,\Lambda)f_2)\ldots\phi_{a_m}((q,\Lambda)f_m)|0\rangle,
\end{aligned}$$

i.e.

$$U(q,\Lambda)\mathcal{D}_0 \subset \mathcal{D}_0.$$

Moreover, locally the translation covariance (Equation (4.2) with $\Lambda = \mathrm{id}$) is

$$i[P_k, \phi_a(x)] = \partial_{x_k}\phi_a(x). \tag{4.3}$$

Thus, using that $P_k\Omega = 0$ by W2, we conclude that

$$P_k\mathcal{D}_0 \subset \mathcal{D}_0. \tag{4.4}$$

Remark 4.7. For more general fields the covariance axiom W1 is replaced by

$$U(q,\Lambda)\phi_j(f)U(q,\Lambda)^{-1} = \sum_{k=1}^{m} R_{jk}(\Lambda^{-1})\phi_k((q,\Lambda)\cdot f), \tag{4.5}$$

where $R : G \to \mathrm{GL}(V)$ is a finite-dimensional representation of the corresponding Lorentz group $\mathcal{L}_+^{\uparrow}$ (or its double cover) on $\mathbb{R}^m$ or $\mathbb{C}^m$. If the representation is non-trivial, then a field is a collection of ϕ_i's transforming into each other under (4.5). See [BLOT89, Sect. 8.2] or [SW64] for more details.

The following corollary of the well-known Reeh and Schlieder's Theorem will be useful later. The proof can be found in [BLOT89, Sec. 8.2 D].

Corollary 4.8. Reeh–Schlieder. Let $X = \phi_{a_1}(x_1)\dots\phi_{a_n}(x_n)$ be some product of Wightman fields. If $X\Omega = 0$, then $X = 0$.

Later on we will need a technical lemma which can be found in [LM75, Sec. 2] which itself is based on [Gla74].

Lemma 4.9. *In an n-dimensional Wightman QFT satisfying W1–W4, the vector*

$$\Psi(x_1,\dots,x_n) := \phi_{a_1}(x_1)\dots\phi_{a_n}(x_n)\Omega,$$

where ϕ_{a_i} are some Wightman fields, extends to a vector-valued analytic function

$$\Psi(z_1,\dots,z_n), \quad z_k := x_k + iy_k,$$

on a connected domain which includes the Euclidean points with $z_k = (iy_k^0, \vec{x}_k)$ such that $y_k^0 > 0$ for all k and $z_i^0 \neq z_j^0$ if $i \neq j$. Here $\vec{x}_k := (x_k^1,\dots,x_k^{n-1})$.

Proof. We will prove the lemma for the simplified case of a self-adjoint scalar field ϕ. The general case follows similarly.

Let

$$\Psi(x_1,\dots,x_n) := \phi(x_1)\dots\phi(x_n)\Omega.$$

Note that $\Psi \in \mathcal{H}$ by W3.

By Poincare covariance (4.1) from W1, we have translation covariance

$$U(q,1)\phi(x)U(q,1)^{-1} = \phi(x+q). \tag{4.6}$$

Applying both sides of this equation to the vacuum and using the invariance of the vacuum vector $U(q,\Lambda)\Omega = \Omega$ from W2 as well as $U(q,1) = \exp i(q^\mu P_\mu)$ we obtain

$$\phi_a(x+q)\Omega = e^{iq^\mu P_\mu}\phi_a(x)\Omega.$$

Thus,

$$\begin{aligned}\Psi(x_1,\ldots,x_n) &= e^{ix_1^\mu P_\mu}\phi(0)e^{-ix_1^\mu P_\mu}e^{ix_2^\mu P_\mu}\ldots\phi(0)\Omega = \\ &= e^{ix_1^\mu P_\mu}\phi(0)e^{i(x_2^\mu - x_1^\mu)P_\mu}\ldots\phi(0)\Omega = \\ &= \int \mathrm{d}^n p\, \mathrm{d}^n q_1 \ldots \mathrm{d}^n q_{n-1}\tilde{\Psi}(p,q_1,\ldots,q_{n-1})e^{i\left(p_\mu x_1^\mu + \sum (q_\mu)_j (x_{j+1}^\mu - x_j^\mu)\right)}.\end{aligned}$$

By spectrum assumption from W2, $\tilde{\Psi}$ is non-zero only if $p^0 \geq 0$ and all $q_i^0 \geq 0$. Thus, Ψ can be analytically continued to a vector-valued analytic function $\Psi \in \mathcal{H}$, i.e. we have

$$\Psi(z_1,\ldots,z_n) \text{ of } z_k = x_k + iy_k \quad \text{defined and holomorphic for } y_1 \in V_+ \text{ and } y_j - y_i \in V_+ \text{ if } j > i. \tag{4.7}$$

Fix π to be any permutation of $(1,\ldots,n)$ and let

$$\Psi^\pi(z_1,\ldots,z_n) := \Psi(z_{\pi(1)},\ldots,z_{\pi(n)}), \quad z_k = x_k + iy_k.$$

By the above, Ψ^π is well-defined and holomorphic in a domain containing the Euclidean points with $0 < y^0_{\pi(1)} < \ldots < y^0_{\pi(n)}$. Furthermore, by locality W4

$$\Psi^\pi(x_1,\ldots,x_n) = \Psi(x_1,\ldots,x_n) \text{ for real } x_k \text{ such that } (x_i - x_j)^2 < 0 \ \ \forall i \neq j,$$

i.e. all Ψ^π's are equal on a real neighborhood. Now the Edge of the Wedge Theorem (see, e.g., [SW64]) shows that they are analytic continuations of one and the same analytic function. Moreover, the domain of analyticity of this function must contain the domains of analyticity of each Ψ^π. □

4.3 Wightman Distributions and Reconstruction

In this section we will show that there exist tempered distributions which provide an equivalent description of Wightman QFT. Here we will only consider scalar fields for simplicity.

Let $\phi_1, \ldots, \phi_n$ be scalar fields of a d-dimensional Wightman QFT. The function

$$W_n(f_1, \ldots, f_n) := \langle \Omega, \phi_1(f_1) \ldots \phi_n(f_n) \Omega \rangle$$

is well-defined by W3$^{\text{weak}}$ for $f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}^d)$ and is a separately continuous multilinear functional. By the Schwartz Nuclear Theorem [SW64, Thm. 2-1] this functional can be uniquely extended to a tempered distribution in $\mathscr{S}'((\mathbb{R}^d)^n) = \mathscr{S}'(\mathbb{R}^{d \cdot n})$. This distribution will be also denoted W_n. Such distribution is called a **Wightman distribution**, a **vacuum expectation value** or a **correlation function**.

Theorem 4.10. *Given a d-dimensional Wightman QFT satisfying W1, W2, W3^{weak} and W4, the Wightman distributions $W_n \in \mathscr{S}'(\mathbb{R}^{d \cdot n})$, $n \in \mathbb{N}$, associated to it have the following properties:*

WD1. Covariance. We have

$$W_n(f) = W_n((q, \Lambda) f) \quad \forall (q, \Lambda) \in \mathcal{P}_+^\uparrow.$$

WD2. Spectrum condition. There exists a distribution $W_n' \in \mathscr{S}'(\mathbb{R}^{d(n-1)})$ supported in the product $\bar{V}_+^{n-1} \subset \mathbb{R}^{d(n-1)}$ of forward cones such that

$$W_n(x_1, \ldots, x_n) = \int_{\mathbb{R}^{d(n-1)}} W_n'(p) e^{i \sum p_j \cdot (x_{j+1} - x_j)} \mathrm{d}p,$$

where $p = (p_1, \ldots, p_{n-1}) \in \mathbb{R}^{d(n-1)}$ and $\mathrm{d}p = \mathrm{d}p_1 \ldots \mathrm{d}p_{n-1}$.

WD3. Hermiticity. We have

$$\langle \Omega, \phi_1(x_1) \ldots \phi_n(x_n) \Omega \rangle = \overline{\langle \Omega, \phi_n^*(x_n) \ldots \phi_1^*(x_1) \Omega \rangle}.$$

WD4. Locality. For all $n \in \mathbb{N}$ and $1 \leq j \leq n-1$

$$W_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = W_n(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n)$$

if $(x_j - x_{j+1})^2 < 0$.

WD5. Positive definiteness. For any sequence $\{f_j\}$ of test functions, $f_j \in \mathscr{S}(\mathbb{R}^{d \cdot j})$, with $f_j = 0$ except for a finite number of j's, it holds that

$$\sum_{j,k=0}^{\infty} \int \overline{f}_j(x_1, \ldots, x_j) W_{jk}(x_j, \ldots x_1, y_1, \ldots, y_k) \times \\ \times f_k(y_1, \ldots, y_k) \, \mathrm{d}x_1 \ldots \mathrm{d}x_j \, \mathrm{d}y_1 \ldots \mathrm{d}y_k \geq 0. \tag{4.8}$$

Here by W_{jk} we mean

$$\langle \Omega, \phi_{jj}^*(x_j) \ldots \phi_{j1}^*(x_1) \phi_{k1}(y_1) \ldots \phi_{kk}(y_k) \Omega \rangle$$

and ϕ_{jk} can be any field of our theory. Furthermore, if (4.8) is zero for some $\{f_j\}$, then (4.8) is zero for any sequence $\{g_j\}$

$$g_0 = 0, \quad g_1 = g(x_1)f_0, \quad g_2 = g(x_1)f_1(x_2), \quad g_3 = g(x_1)f_2(x_2, x_3), \ldots \tag{4.9}$$

with $g \in \mathscr{S}(\mathbb{R}^d)$ arbitrary.

Proof. WD1 follows from W1 and WD4 from W4.

Hermiticity WD3 follows from

$$\langle \Omega, \phi_1(f_1) \ldots \phi_n(f_n)\Omega = \overline{\langle \Omega, (\phi_n(f_n))^* \ldots (\phi_1(f_1))^*\Omega \rangle}$$

and the fact that this relation extends from test functions of the form $f_1(x_1) \ldots f_n(x_n)$ to the whole of $\mathscr{S}(\mathbb{R}^{d \cdot m})$ by the Schwartz Nuclear Theorem.

The inequalities (4.8) of WD5 are equivalent to the fact that the norm of the state

$$\Psi = f_0\Omega + \phi_{11}(f_1)\Omega + \int \phi_{21}(x_1)\phi_{22}(x_2)f_2(x_1, x_2)\,\mathrm{d}x_1\mathrm{d}x_2\Omega + \ldots$$

is non-negative. If the norm is zero, then $\Psi = 0$ and hence $\mathrm{pr}_j(g)\Psi = 0$ for any component j of the test function g. Thus, (4.9) holds.

WD2 will be proved in Proposition 4.11. □

By the covariance of the fields,

$$W_n(x_1, \ldots, x_n) = W_n(\Lambda x_1 + q, \ldots, \Lambda x_n + q) \quad \forall (q, \Lambda) \in \mathcal{P}_+^{\uparrow}.$$

Here and further we abuse our notation for the correlation functions as we often do for the fields (cf. Remark 4.5). It follows that Wightman distributions are translation invariant

$$W_n(x_1, \ldots, x_n) = W_n(x_1 + q, \ldots, x_n + q).$$

Thus, the distributions depend only on the differences

$$\xi_i := x_i - x_{i-1}$$

and we define

$$w_n(\xi_1, \ldots, \xi_{n-1}) := W_n(x_1, \ldots, x_n).$$

Proposition 4.11. *The Fourier transform $\widehat{w}_n$ has its support in the product $(\bar{V}_+)^{n-1}$ of the forward cones $\bar{V}_+ \subset \mathbb{R}^d$. Thus,*

$$W_n(x) = (2\pi)^{-d(n-1)} \int_{\mathbb{R}^{d(n-1)}} \widehat{w}_n(p) e^{-i\sum p_j \cdot (x_j - x_{j+1})} dp.$$

Proof. Since $U(x,1)^{-1} = U(-x,1) = e^{-ix^\mu P_\mu}$ for $x \in \mathbb{R}^d$, the spectrum condition W2 implies

$$\int_{\mathbb{R}^d} e^{ix^\mu p_\mu} U(x,1)^{-1}\, v\, dx = 0 \quad \forall v \in \mathcal{H} \quad \text{if } p \notin \bar{V}_+. \tag{4.10}$$

Note that

$$w_n(\xi_1, \dots, \xi_j + x, \xi_{j+1}, \dots, \xi_{n-1}) = W_n(x_1, \dots, x_j, x_{j+1} - x, \dots, x_n - x).$$

Thus, the Fourier transform of w_n with respect to x gives

$$\begin{aligned}\int_{\mathbb{R}^d} & w_n(\xi_1, \dots, \xi_j + x, \xi_{j+1}, \dots, \xi_{n-1}) e^{ip_j \cdot x} dx = \\ &= \left\langle \Omega, \phi_1(x_1) \dots \phi_j(x_j) \int_{\mathbb{R}^d} \phi_{j+1}(x_{j+1} - x) \dots \phi_n(x_n - x) e^{ip_j \cdot x} \Omega dx \right\rangle = \\ &= \left\langle \Omega, \phi_1(x_1) \dots \phi_j(x_j) \int_{\mathbb{R}^d} e^{ip_j \cdot x} U(x,1)^{-1} \phi_{j+1}(x_{j+1}) \dots \phi_n(x_n) \Omega dx \right\rangle = 0,\end{aligned}$$

if $p_j \notin \bar{V}_+$ by (4.10) with $v = \phi_{j+1}(x_{j+1}) \dots \phi_n(x_n)\Omega$. Therefore,

$$\widehat{w}_n(p_1, \dots, p_{n-1}) = 0$$

if $p_j \notin \bar{V}_+$ for at least one index j. □

We state the cluster decomposition property for completeness, but do not give a proof since we will not use it. This property ensures that the Wightman QFT obtained via the Wightman Reconstruction Theorem from the Wightman distributions has a unique vacuum. For a proof with a mass gap see [SW64] and references therein.

WD6. Cluster Decomposition Property. For a space-like vector q

$$\begin{aligned}W_n(x_1, \dots, x_j, x_{j+1} + \lambda q, x_{j+2} + \lambda q, \dots, x_n + \lambda q) \to \\ W_j(x_1, \dots, x_j) W_{n-j}(x_{j+1}, \dots, x_n)\end{aligned}$$

as $\lambda \to \infty$ with convergence in $\mathscr{S}'$.

The following proof is based on [Sch08]. For a more explicit proof, which also uses WD6 and hence proves the uniqueness up to a unitary transformation of the resulting Wightman QFT, see [SW64]. For simplicity, we provide a proof only for a single self-adjoint scalar field.

Theorem 4.12. Wightman Reconstruction Theorem. *For a sequence of tempered distributions (W_n), $W_n \in \mathscr{S}'(\mathbb{R}^{d \cdot n})$, satisfying WD1–WD5, there exists a Wightman QFT satisfying W1, W2, W3^{weak} and W4.*

Proof. Let

$$\underline{\mathscr{S}} := \bigoplus_{n=0}^{\infty} \mathscr{S}(\mathbb{R}^{d\cdot n})$$

be the vector space of finite sequences $\underline{f} = (f_0, f_1, f_2, \ldots)$, i.e. $f_0 \in \mathbb{C}$, $f_n \in \mathscr{S}(\mathbb{R}^{d\cdot n})$ and all but finitely many of test functions f_n are zero. We define multiplication on $\underline{\mathscr{S}}$ by

$$\begin{aligned} \underline{f} \times \underline{g} &:= (h_n), \\ h_n &:= \sum_{i=0}^{n} f_i(x_1, \ldots, x_i) g_{n-i}(x_{i+1}, \ldots, x_n). \end{aligned}$$

Note that $\underline{\mathscr{S}}$ forms an associative algebra with unit $\underline{1} = (1, 0, 0, \ldots)$. We put the direct limit topology on $\underline{\mathscr{S}}$ to make it into a complete separable locally convex space. Each continuous linear functional $\mu : \underline{\mathscr{S}} \to \mathbb{C}$ can be represented by sequences (μ_n) of tempered distributions $\mu_n \in \mathscr{S}'_n : \mu((f_n)) = \sum \mu_n(f_n)$. For each functional λ of this form which is also positive semi-definite, i.e. $\lambda(\overline{\underline{f}} \times \underline{f}) \geq 0$ for all $\underline{f} \in \underline{\mathscr{S}}$, the subspace

$$J = \left\{ \underline{f} \in \underline{\mathscr{S}} : \lambda\left(\overline{\underline{f}} \times \underline{f}\right) = 0 \right\}$$

is an ideal of the algebra $\underline{\mathscr{S}}$. Then on the quotient $\underline{\mathscr{S}}/J$ the positive semi-definite functional λ gives rise to a positive definite Hermitian scalar product by setting $\omega(\underline{f}, \underline{g}) := \lambda\left(\overline{\underline{f}} \times \underline{g}\right)$. Thus, completing $\underline{\mathscr{S}}/J$ with respect to this scalar product produces a Hilbert space $\mathcal{H}$.

Now set $\lambda := (W_n)$. By WD5, the continuous functional λ is positive semi-definite and hence provides the Hilbert space $\mathcal{H}$ constructed above. For the vacuum vector we set $\Omega := \iota(\underline{1})$ where $\iota(\underline{f})$ denotes an equivalence class from the dense domain $\mathcal{D} := \underline{\mathscr{S}}/J$. We define the field operator ϕ on $\mathcal{D}$ by

$$\phi(f)\iota\left(\underline{g}\right) := \iota\left(\underline{g} \times f\right)$$

for all $f \in \mathscr{S}$. Here f denotes the sequence $(0, f, 0, \ldots)$. For $\underline{g}, \underline{h} \in \underline{\mathscr{S}}$ the mapping

$$f \mapsto \left\langle \iota(\underline{h}), \phi(f)\iota(\underline{g}) \right\rangle = \lambda\left(\underline{h} \times (\underline{g} \times f)\right)$$

is a tempered distribution by continuity of λ. Thus, ϕ is indeed a field operator (Definition 4.2). Furthermore, $\phi(f)\mathcal{D} \subset \mathcal{D}$ and $\Omega \in \mathcal{D}$.

Now we draw our attention to covariance. First of all, we need to define a unitary representation of the Poincaré group $\mathcal{P}_+^{\uparrow}$ on $\mathcal{H}$. We start by considering the natural action $\underline{f} \mapsto (q, \Lambda)\underline{f}$ of $\mathcal{P}_+^{\uparrow}$ on $\underline{\mathscr{S}}$ given term-wise by

$$(q, \Lambda) f_k(x_1, \ldots, x_k) := f_k(\Lambda^{-1}(x_1 - q), \ldots, \Lambda^{-1}(x_k - q)),$$

where $(q,\Lambda) \in \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow \cong \mathcal{P}_+^\uparrow$. This leads to a homomorphism $\mathcal{P}_+^\uparrow \to \mathrm{GL}(\underline{\mathscr{L}})$. By the covariance WD1, we have that if $\underline{f} \in J$ and $(q,\Lambda) \in \mathcal{P}_+^\uparrow$, then $(q,\Lambda)\underline{f} \in J$. Thus,

$$U(q,\Lambda)\iota(\underline{f}) := \iota\left((q,\Lambda)\underline{f}\right)$$

is well-defined on the dense domain $\mathcal{D} \subset \mathcal{H}$ and satisfies

$$\left\langle U(q,\Lambda)\iota(\underline{f}), U(q,\Lambda)\iota(\underline{f})\right\rangle = \langle \iota(\underline{f}), \iota(\underline{f})\rangle.$$

Therefore, we get a unitary representation of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$ such that $U(q,\Lambda)\Omega = \Omega$ and $U(q,\Lambda)\phi(f)U(q,\Lambda)^{-1} = \phi((q,\Lambda)f)$ because $U(q,\Lambda)$ respects the multiplication $\times$ of $\underline{\mathscr{L}}$. This proves W1 and W3$^{\text{weak}}$.

Now the spectrum axiom W2 follows from WD2 by noting that

$$\left\{(f_n) \,\middle|\, f_0 = 0,\ \widehat{f}(p_1,\ldots,p_n) = 0 \text{ in a neighborhood of } (\bar{V}_+)_n\right\} \subset J,$$

where $(\bar{V}_+)_n = \{p \mid p_1 + \ldots + p_n \in \bar{V}_+,\ j = 1,\ldots,N\}$. Similarly the locality axiom W4 holds by noting that J contains the ideal generated by linear combinations of the form

$$f_n(x_1,\ldots,x_n) = g(x_1,\ldots,x_j,x_{j+1},\ldots,x_n) - g(x_1,\ldots,x_{j+1},x_j,\ldots,x_n)$$

with $g(x_1,\ldots,x_n) = 0$ if $(x_{j+1}-x_j)^2 \geq 0$. □

4.4 Wightman CFT

To get a conformal Wightman QFT, we extend the symmetry group of our system from the Poincaré group $\mathcal{P}_+^\uparrow$ to the (restricted) conformal group.

W1$^{\text{conf}}$. Conformal covariance. The continuous unitary representation of the Poincaré group extends to a continuous unitary representation of the (restricted) conformal group $(q,\Lambda,b) \mapsto U(q,\Lambda,b)$ such that

$$U(q,\Lambda,b)\Omega = \Omega \tag{4.11}$$

$\forall (q,\Lambda,b) \in \mathrm{Conf}(\mathbb{R}^{1,d-1})$ and conformal covariance holds for some collection of fields of the QFT which we call *quasiprimary.* The other fields are just Poincaré covariant.

We assume that in 2D a quasiprimary field ϕ_a of *scaling dimension* Δ_a and *spin* s_a transforms as

$$U(q,\Lambda,b)\phi_a(f)U(q,\Lambda,b)^{-1} = \varphi_a(b,x)\ \phi_a((q,\Lambda,b)\cdot f), \tag{4.12}$$

with

$$\varphi_a(b,x) = (1 + (b^0 + b^1)(x^0 - x^1))^{-\Delta_a - s_a} (1 + (b^0 - b^1)(x^0 + x^1))^{-\Delta_a + s_a}.$$

We also assume that $s, \Delta \in \mathbb{R}$ for all fields.

Note that (4.12) is just the transformation law of a scalar field, so we should set $s_a = 0$, but we keep s_a for making the upcoming discussion clearer (cf. Remark 4.7). The most general transformation laws can be found in [MS69].

Clearly, stronger covariance of the fields leads to stronger covariance of Wightman distributions and so we call such distributions **conformally covariant**.

Remark 4.13. Note that by (4.12) for a special conformal transformation in 2D it holds

$$U(0,1,b)\phi_a(x)U(0,1,b)^{-1} = \varphi_a(b,x)\,\phi_a(x^b) \tag{4.13}$$

and that from Stone's Theorem it follows that $U(0,1,b) = \exp i \sum_{n=0}^{1} b^n K_n$, where K_n are self-adjoint and commuting operators on $\mathcal{H}$. Here we let x^b to denote a special conformal transformation with parameter b

$$x \mapsto \frac{x + |x|^2 b}{1 + 2\langle x, b\rangle + |x|^2|b|^2}.$$

Hence, locally we have

$$i\,[K_0, \phi_a(x)] = \left(|x|^2\partial_0 - 2x^0 E - 2\Delta_a x^0 + 2s_a x^1\right)\phi_a(x) \tag{4.14a}$$

$$i\,[K_1, \phi_a(x)] = \left(|x|^2\partial_1 + 2x^1 E + 2\Delta_a x^1 - 2s_a x^0\right)\phi_a(x) \tag{4.14b}$$

with $E = x^0\partial_0 + x^1\partial_1$.

Sometimes the axiom W1$^{\text{conf}}$ is too strong. To prove the Lüscher–Mack Theorem, only dilation covariance will suffice. Thus, we state the axiom of dilation covariance here separately.

W1$^{\text{dil}}$. Dilation covariance. There exists a unitary representation U' of the dilation group such that for $\lambda > 0$ we have

$$U'(\lambda)\Omega = \Omega$$

and

$$U'(\lambda)\phi(x)U'(\lambda)^{-1} = \lambda^{\Delta}\phi(\lambda x)$$

for some fields which we call dilation covariant. Other fields are just Poincaré covariant. Here Δ is the scaling dimension of ϕ.

Another very important axiom usually made in 2D CFT is:

W5. Existence of energy-momentum tensor. In the operator algebra generated by the fields $\{\phi_a\}_{a\in I}$ there is a dilation covariant local field $T_{\mu\nu}(x),\ \mu,\nu \in \{0,1\}$, with the following properties:

$$T_{\mu\nu} = T_{\nu\mu}, \quad T^*_{\mu\nu} = T_{\mu\nu}, \tag{4.15a}$$

$$\partial^\mu T_{\mu\nu} = 0, \tag{4.15b}$$

$$\Delta(T_{\mu\nu}) = 2, \tag{4.15c}$$

where Δ is the scaling dimension. Moreover, we assume that the generators P_μ can be expressed in terms of $T_{\mu\nu}$:

$$\int \mathrm{d}x^1 [T_{0\mu}(x^0, x^1), \phi(y)] = [P_\mu, \phi(y)] = -i\partial_\mu\phi(y). \tag{4.16}$$

We are now ready to give one of the central definitions of this work.

Definition 4.14. Wightman (Möbius) CFT. A 2D Wightman QFT satisfying W1$^{\mathrm{conf}}$–W4 is called *Wightman Möbius CFT*. If Wightman Möbius CFT contains an energy-momentum tensor, i.e. it also satisfies W5, then it is a *Wightman CFT*.

Part II

Comparisons

Chapter 5

Wightman Axioms and Virasoro Algebra

The goal of this chapter is to prove that a 2D dilation invariant Wightman QFT with an energy-momentum tensor gives rise to two commuting unitary Virasoro algebras as was first proved by Lüscher and Mack in [LM76].

This chapter is based on the original source [LM76], the talk [Lüs88] and [FST89].

5.1 Lüscher–Mack Theorem

We will use light-cone coordinates in this section:

$$\begin{aligned} t &= x^0 - x^1, & \partial_t &= \frac{1}{2}(\partial_0 - \partial_1), \\ \bar{t} &= x^0 + x^1, & \partial_{\bar{t}} &= \frac{1}{2}(\partial_0 + \partial_1), \end{aligned}$$

so that

$$\Theta := T_{tt} = \frac{1}{4}(T_{00} - 2T_{01} + T_{11}), \qquad \bar{\Theta} := T_{\bar{t}\bar{t}} = \frac{1}{4}(T_{00} + 2T_{01} + T_{11}), \quad (5.1)$$

$$T_{t\bar{t}} = T_{\bar{t}t} = \frac{1}{4}(T_{00} - T_{11}),$$

where $T_{\mu\nu}$ are components of the energy-momentum tensor defined in W5.

Lemma 5.1. *In 2D dilation invariant Wightman QFT with an energy-momentum tensor, i.e. a 2D Wightman QFT satisfying W1^{dil}–W5, it holds:*

- $\mathrm{tr}(T_{\alpha\beta}) = T^{\mu}{}_{\mu} = 0$,
- $\partial_{\bar{t}}\Theta = 0, \quad \partial_t\bar{\Theta} = 0 \quad$ *and* $\quad [\Theta(t), \bar{\Theta}(\bar{t})] = 0 \quad \forall t, \bar{t} \in \mathbb{R}.$

Proof. Direct calculation implies that

$$\partial_t\bar{\Theta} + \partial_{\bar{t}}T_{t\bar{t}} = 0 \quad \text{and} \quad \partial_{\bar{t}}\Theta + \partial_t T_{\bar{t}t} = 0. \qquad (5.2)$$

In 2D, Lorentz boosts are just squeeze mappings

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}$$

and the tensor field $T_{\mu\nu}$ transforms under Lorentz transformations as

$$U(\Lambda)\, T_{\mu\nu}(\vec{x})\, U(\Lambda)^{-1} = (\Lambda^{-1})_{\mu}{}^{\alpha}(\Lambda^{-1})_{\nu}{}^{\beta}\, T_{\alpha\beta}(\Lambda\vec{x}),$$

where $(\Lambda^{-1})_{\mu}{}^{\nu} = \Lambda^{\mu}{}_{\nu}$. Thus,

$$U(\Lambda)\, \bar{\Theta}(\vec{x})\, U(\Lambda)^{-1} = e^{2\xi}\, \bar{\Theta}\left(e^{\xi}\bar{t}, e^{-\xi}t\right).$$

Moreover, under dilations

$$U'(\lambda)\, \bar{\Theta}(\vec{x})\, U'(\lambda)^{-1} = \lambda^{2}\, \bar{\Theta}(\lambda\vec{x}).$$

Combining these transformations with $\lambda = e^{-\xi}$ we obtain

$$U(\Lambda)U'(\lambda)\, \bar{\Theta}(\vec{x})\, U'(\lambda)^{-1}U(\Lambda)^{-1} = \bar{\Theta}(\bar{t}, \lambda^{2}t).$$

From Theorem 4.10 it follows that

$$\left\langle \Omega, \bar{\Theta}(\bar{t}_1, t_1)\bar{\Theta}(\bar{t}_2, t_2)\Omega \right\rangle = \frac{A}{(\bar{t}_1 - \bar{t}_2 - i\varepsilon)^4}, \quad \bar{t}_1 \neq \bar{t}_2, \quad A \in \mathbb{C}. \tag{5.3}$$

Here ε means that we take the limit $\varepsilon \to 0$, i.e. our $(x \pm i\varepsilon)^n = (x \pm i0)^n$ with

$$(x \pm i0)^n = \lim_{y \to 0^+} (x \pm iy)^n.$$

See [GS64] for more details.

We apply $\partial/\partial t_1$ and ∂/t_2 to get

$$\langle \Omega,\, \partial_{t_1}\bar{\Theta}(\bar{t}_1, t_1)\, \partial_{t_2}\bar{\Theta}(\bar{t}_2, t_2)\Omega \rangle = 0.$$

Thus, by analytic continuation this distribution is identically zero throughout. Therefore, we have

$$\partial_t \bar{\Theta}\Omega = 0$$

and the Corollary of Reeh–Schlieder Theorem 4.8 implies that

$$\partial_t \bar{\Theta} = 0,$$

i.e. $\bar{\Theta}$ depends only on $\bar{t}$. Similarly, $\partial_{\bar{t}}\Theta = 0$. Hence, from (5.2) it follows that

$$\partial_{\bar{t}} T_{t\bar{t}} = \partial_t T_{t\bar{t}} = 0,$$

i.e. that $T_{t\bar{t}}$ is constant. But

$$U'(\lambda)\, T_{t\bar{t}}(\bar{t}, t)\, U'(\lambda)^{-1} = \lambda^2\, T_{t\bar{t}}(\lambda\bar{t}, \lambda t),$$

so $T_{t\bar{t}} = 0$. Therefore,

$$\operatorname{tr}(T_{\alpha\beta}) = T^{\mu}{}_{\mu} = g^{\mu\nu}T_{\mu\nu} = T_{00} - T_{11} = 4T_{t\bar{t}} = 0,$$

as required. Now

$$[\Theta(t), \bar{\Theta}(\bar{t})] = 0 \qquad \forall t, \bar{t} \in \mathbb{R},$$

by locality and the fact that Θ depends only on t and $\bar{\Theta}$ depends only on $\bar{t}$. □

Proposition 5.2. *We have*

$$[\Theta(t_1), \Theta(t_2)] = \frac{c}{24\pi} i^3\delta'''(t_1 - t_2) + 2i\delta'(t_1 - t_2)\Theta(t_2) - i\delta(t_1 - t_2)\partial\Theta(t_2),$$
$$[\bar{\Theta}(\bar{t}_1), \bar{\Theta}(\bar{t}_2)] = \frac{\bar{c}}{24\pi} i^3\delta'''(\bar{t}_1 - \bar{t}_2) + 2i\delta'(\bar{t}_1 - \bar{t}_2)\bar{\Theta}(\bar{t}_2) - i\delta(\bar{t}_1 - \bar{t}_2)\bar{\partial}\bar{\Theta}(\bar{t}_2)$$

with $c, \bar{c} \geq 0$. If parity is conserved, then $c = \bar{c}$.

Proof. By locality, $[\Theta(t_1), \Theta(t_2)] = 0$ if $t_1 \neq t_2$ with $t_1, t_2 \in \mathbb{R}$. Let

$$O_k(t_1) = \frac{i}{k!} \int t_2^k [\Theta(t_1 + t_2), \Theta(t_1)]\, \mathrm{d}t_2, \qquad k \in \mathbb{N}_0.$$

The O_k's are local self-adjoint fields. Therefore, using that $\Delta(\Theta) = 2$ by W5 and the definition of O_k's we get

$$U(\lambda) O_k(t) U(\lambda)^{-1} = \lambda^{3-k} O_k(\lambda t).$$

Moreover, O_k's are covariant under translations. Hence

$$\begin{aligned}\langle \Omega, O_k(t_1) O_k(t_2)\Omega\rangle &= A_k(t_1 - t_2 - i\varepsilon)^{2k-6} \\ &\overset{k\geq 3}{=\!=} (-1)^{k-3} A_k \int \frac{dp}{2\pi} e^{-ip(t_1 - t_2)} \delta^{(2k-6)}(p), \quad A_k \in \mathbb{C}.\end{aligned}$$

So by Bochner–Schwartz theorem (see [RS75]), these distributions are not positive for $k \geq 4$. Hence, $O_k = 0$ for $k \geq 4$. Moreover, $O_3(t)$ is independent of t. By locality, O_3 commutes with all the fields. It is therefore proportional to the unit operator so set

$$O_3 = -\frac{c}{24\pi}, \quad c \in \mathbb{C}.$$

Recall that by assumption (4.16) from W5, $\Theta(t)$ generates translations:

$$\int \mathrm{d}t_1 \,[\Theta(t_1), \Theta(t_2)] = -i\partial\Theta(t_2).$$

Thus,

$$O_0(t) = \partial_t\Theta(t). \tag{5.4}$$

Let $|\psi\rangle \in \mathcal{H}$ be arbitrary. Then by regularity theorem for tempered distributions [RS80, Thm. V.10], we can write

$$\langle\psi, [\Theta(t_1 + t_2), \Theta(t_1)]\Omega\rangle = \sum_{k=0}^{K} \delta^{(k)}(t_2)\psi_k(t_1),$$

where $K \in \mathbb{N}_0$ and $\psi_k(t_1)$ are some distributions. It follows from [RS80, p. 177] that

$$\psi_k(t) = -i(-1)^k\langle\psi, O_k(t)\Omega\rangle.$$

In particular, $\psi_k = 0$ for $k \geq 4$ and

$$[\Theta(t_1 + t_2), \Theta(t_1)] = -i\sum_{k=0}^{3}(-1)^k\delta^{(k)}(t_2)O_k(t_1) \tag{5.5}$$

holds on the vacuum and thus as an operator equality by the Reeh–Schlieder Theorem (Corollary 4.8).

To determine $O_1(t)$ and $O_2(t)$ we use $[\Theta(t_1), \Theta(t_2)] = -[\Theta(t_2), \Theta(t_1)]$ and (5.5) to obtain

$$-\sum_{k=0}^{3}(-1)^k\delta^{(k)}(t_2)O_k(t_1) = \sum_{k=0}^{3}(-1)^k\delta^{(k)}(-t_2)O_k(t_1 + t_2). \tag{5.6}$$

Note that

$$\delta^{(k)}(-t_2)O_k(t_1 + t_2) = \sum_{l=0}^{k}(-1)^l\binom{k}{l}\delta^{(l)}(t_2)\frac{\partial^{k-l}}{\partial t_1^{k-l}}O_k(t_1).$$

Plugging this equation into (5.6) and equating the coefficients of $\delta^{(2)}(t_2)$'s we get

$$O_2(t) = \sum_{k=2}^{3}(-1)^{k+1}\frac{k(k-1)}{2}\frac{\partial^{k-2}}{\partial t^{k-2}}O_k(t) = -O_2(t) \implies O_2 = 0,$$

where we have used that O_3 is a constant. Moreover, the terms with $\delta(t_2)$ give

$$O_0(t) = \sum_{k=0}^{3} (-1)^{k+1} \frac{\partial^k}{\partial t^k} O_k(t) = -O_0(t) + \frac{\partial}{\partial t} O_1(t) \implies$$
$$\implies \frac{\partial}{\partial t} O_1(t) = 2O_0(t) \overset{(5.4)}{=\!=} 2\frac{\partial}{\partial t}\Theta(t) \implies O_1(t) = 2\Theta(t),$$

where the last implication is by locality and dilation invariance. Plugging the expressions of $O_0(t)$, $O_1(t)$ and O_3 into (5.5) we prove the commutation relation for Θ.

The proof for $\bar{\Theta}$ is analogous.

To prove that $c \geq 0$, we first of all note that

$$\langle \Omega, [\Theta(t_1), \Theta(t_2)]\, \Omega \rangle = -i\frac{c}{24\pi}\delta'''(t_1 - t_2) \tag{5.7}$$

since $\langle \Omega, \Theta(t)\, \Omega \rangle = 0$ by translation and dilation invariance. Moreover, the "unbarred" version of (5.3) gives

$$\langle \Omega, \Theta(t_1)\Theta(t_2)\Omega \rangle = \frac{A}{(t_1 - t_2 - i\varepsilon)^4}, \quad t_1 \neq t_2, \quad A \in \mathbb{C}.$$

Hence, using

$$\delta'''(t) = -\frac{6}{2\pi i}\left((t - i\varepsilon)^{-4} - (t + i\varepsilon)^{-4}\right)$$

we see that $A = c/8\pi^2$. Then the Fourier transform

$$(2\pi)^2 \langle \Omega, \Theta(t_1)\Theta(t_2)\Omega \rangle = \frac{c}{2(t_1 - t_2 - i\varepsilon)^4} = -\frac{\mathrm{d}^3}{\mathrm{d}t_{12}^3}\frac{c}{12(t_{12} - i\varepsilon)}$$
$$= -i\frac{c}{12}\frac{\mathrm{d}^3}{\mathrm{d}t_{12}^3}\int_0^\infty e^{-ipt_{12}}\mathrm{d}p = \frac{c}{12}\int_0^\infty p^3 e^{-ipt_{12}}\mathrm{d}p$$

implies that we must have $c \geq 0$ to ensure the positivity of the correlation function.

To show that $c = \bar{c}$ if parity is conserved, we use parity invariance ($t = \bar{t}$) to get that $\Theta = \bar{\Theta}$ and so by (5.7)

$$-i\frac{c}{24\pi}\delta'''(t_1 - t_2) = -i\frac{\bar{c}}{24\pi}\delta'''(t_1 - t_2) \implies c = \bar{c}.$$

□

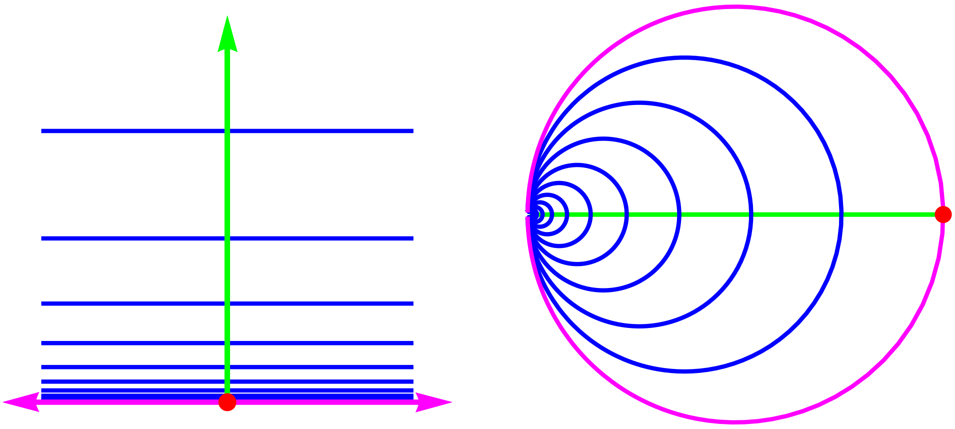

Figure 5.1: Cayley transform is a biholomorphic map from the open upper half-plane to the open unit disk

Remark 5.3. Note that as an operator equation we have

$$\begin{aligned}[\Theta(t_1), \Theta(t_2)] &= \frac{c}{24\pi} i^3 \delta'''(t_1 - t_2) + 2i\delta'(t_1 - t_2)\Theta(t_2) - i\delta(t_1 - t_2)\partial\Theta(t_2) \\ &= \frac{c}{24\pi} i^3 \delta'''(t_1 - t_2) + i\delta'(t_1 - t_2)\left\{\Theta(t_1) + \Theta(t_2)\right\} \end{aligned} \tag{5.8}$$

and similarly for $\bar{\Theta}$.

By Lemma 4.9 each vector $\Psi(x_1, \dots, x_n) := \phi_{a_1}(x_1) \dots \phi_{a_n}(x_n)|0\rangle$ of $\mathcal{H}$ extends analytically to the domain containing

$$\{\operatorname{Im} t_1, \operatorname{Im} \bar{t}_1 > 0\} \times \cdots \times \{\operatorname{Im} t_n, \operatorname{Im} \bar{t}_n > 0\} \;\text{ such that }\; t_i + \bar{t}_i \neq t_j + \bar{t}_j \text{ if } i \neq j.$$

Since vectors of the form Ψ are dense in the Hilbert space $\mathcal{H}$ by W3, each field's ϕ_a domain of definition extends to Schwartz functions on $\mathbb{C}^2$ supported in $\operatorname{Im} t, \operatorname{Im} \bar{t} \geq 0$. This allows us to compactify the Minkowski space

$$z = \frac{1 + it/2}{1 - it/2}, \quad \bar{z} = \frac{1 + i\bar{t}/2}{1 - i\bar{t}/2}.$$

Under these transformations, the domain $\operatorname{Im} t > 0$, $\operatorname{Im} \bar{t} > 0$ is mapped to the domain $|z| < 1$, $|\bar{z}| < 1$. Define the holomorphic energy-momentum tensor in the domain $|z| < 1$ by

$$T(z) := 2\pi \left(\frac{2}{1+z}\right)^4 \Theta(t), \tag{5.9}$$

with $t = 2i(1-z)(1+z)^{-1}$ and similarly for the antiholomorphic tensor $\overline{T}(\bar{z})$.

Theorem 5.4. *Every 2D dilation invariant Wightman QFT with an energy-momentum tensor, i.e. a 2D Wightman QFT satisfying W1^{dil}–W5, gives rise to two commuting unitary representations of the Virasoro algebra with central charges c and $\bar{c}$ with generators defined by*

$$L_n := \oint_{S^1} z^{n+1} T(z) \frac{\mathrm{d}z}{2\pi i}, \qquad \bar{L}_n := \oint_{S^1} \bar{z}^{n+1} \overline{T}(\bar{z}) \frac{\mathrm{d}\bar{z}}{2\pi i}.$$

Proof. We define the circular delta function by

$$\oint_{S^1} \delta_c(z - z_0) f(z) \frac{\mathrm{d}z}{2\pi i} = f(z_0),$$

so that $\delta_c(z - z_0)\,\mathrm{d}z = 1/(z - z_0)\,\mathrm{d}z$ by Cauchy's integral formula. Using the definition of $T(z)$, Proposition 5.2 and Remark 5.3 we get

$$[T(z), T(w)] = \frac{c}{12} \delta_c'''(z - w) + \delta_c'(z - w)\,(T(z) + T(w)).$$

Defining

$$L_n := \oint_{S^1} z^{n+1} T(z) \frac{\mathrm{d}z}{2\pi i},$$

we get by direct calculation

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{5.10}$$

as required.

To prove unitarity, i.e. that $L_n^* = L_{-n}$, we note that

$$L_0 = \oint_{S^1} z\,T(z) \frac{\mathrm{d}z}{2\pi i} = \int_{-\pi}^{\pi} \Theta\left(2 \tan \frac{\alpha}{2}\right) \frac{\mathrm{d}\alpha}{\cos^4(\alpha/2)} = \int_{-\infty}^{\infty} \Theta(t) \left(1 + \frac{t^2}{4}\right) \mathrm{d}t$$

is a self-adjoint operator. From (5.10) it follows that

$$[L_n, L_0] = n L_n \implies [L_0, L_n^*] = n L_n^* \implies L_n^* = L_{-n}.$$

□

Chapter 6

Virasoro Algebra and Vertex Algebras

In this chapter we study the relationship between vertex algebras and the Virasoro algebra. As an attentive reader could have already guessed, the relationship is rather trivial.

6.1 From Virasoro Algebra to Virasoro Vertex Algebra

By Poincaré–Birkhoff–Witt Theorem, an equivalent definition of the Verma module $M(c,h)$ is obtained by setting

$$M(c,h) = U(\mathsf{Vir}) \otimes_{U(\mathfrak{b})} \mathbb{C},$$

where $\mathfrak{b} = (\oplus_{n\geq 1}\mathbb{C}L_n) \oplus (\mathbb{C}L_0 \oplus \mathbb{C}C)$ is a subalgebra of Vir, $U(\mathsf{Vir})$ is the universal enveloping algebra of Vir and $\mathbb{C}$ denotes a 1-dimensional $\mathfrak{b}$-module

$$\begin{aligned} L_n|0\rangle &= 0, \quad n \geq 1, \\ L_0|0\rangle &= h|0\rangle, \\ C|0\rangle &= c|0\rangle. \end{aligned}$$

This should be compared with our explicit construction of Lemma 2.24.

Frenkel and Zhu have shown in [FZ92] that

$$\overline{M(c,0)} = M(c,0)/\left(U(\mathsf{Vir})L_{-1}|0\rangle \otimes |0\rangle\right)$$

has a vertex operator algebra structure with the conformal vector $\nu = L_{-2}|0\rangle$. We present here an explicit construction as given in [Sch08, p. 193] in subsection "Virasoro Vertex Algebra".

Proposition 6.1. *The quotient $\overline{M(c,0)}$ gives rise to a vertex operator algebra of CFT type.*

Proof. We give a construction similar to that of a Verma module $M(c,0)$ in Lemma 2.24.

Let $\overline{M(c,0)}$ be a vector space with a basis

$$\{v_{n_1\ldots n_k} \mid n_1 \geq \ldots \geq n_k \geq 2,\ n_j \in \mathbb{N},\ k \in \mathbb{N}\} \cup \{|0\rangle\}$$

together with the following action of Vir on $\overline{M(c,0)}$ for all $n, n_j \in \mathbb{Z}$ such that $n_1 \geq \ldots \geq n_k \geq 2$, $k \in \mathbb{N}$:

$$\begin{aligned} C &:= c\,\mathrm{id}, \\ L_n|0\rangle &:= 0, && n \geq -1, \\ L_0 v_{n_1\ldots n_k} &:= \left(\sum_{j=1}^{k} n_j\right) v_{n_1\ldots n_k}, \\ L_{-n}|0\rangle &:= v_n, && n \geq 2, \\ L_{-n} v_{n_1\ldots n_k} &:= v_{nn_1\ldots n_k}, && n \geq n_1. \end{aligned}$$

Other actions of L_n's on general $v \in \overline{M(c,0)}$ follow from the commutation relations of the Virasoro algebra. Defining $L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ we see that $L(z)$ is a field, as follows by generalizing

$$L_m v_n = L_m L_{-n}|0\rangle = L_{-n} L_m|0\rangle + (m+n) L_{m-n}|0\rangle = 0 \quad m \gg 0,$$

to arbitrary $v \in \overline{M(c,0)}$. Moreover, it is a Virasoro field as follows from Definition 3.37 and Example 3.12. Hence, $L(z)$ is also local with respect to itself. For the asymptotic state using $L_n|0\rangle = 0$ $\forall n \geq -1$ and $L_{-n}|0\rangle = v_n$ $\forall n \geq 2$ we get

$$L(z)|0\rangle|_{z=0} = \sum_{n\leq -2} L_n z^{-n-2}|0\rangle|_{z=0} = L_{-2}|0\rangle = v_2.$$

Moreover, note that

$$\begin{aligned} [L_{-1}, L(z)] &= \sum_{n\in\mathbb{Z}} [L_{-1}, L_n] z^{-n-2} = \sum_{n\in\mathbb{Z}} (-1-n) L_{-1+n} z^{-n-2} = \\ &= \sum_{n\in\mathbb{Z}} (-n-2) L_n z^{-n-3} = \partial L(z). \end{aligned}$$

Thus, setting $T := L_{-1}$ for clarity, we apply Theorem 3.28 to get a vertex algebra with a single strongly generating field $L(z)$ (Definition 3.29). Clearly, it is a vertex operator algebra of CFT type with conformal vector v_2 (Definition 3.41). $\square$

In the same paper [FZ92], Frenkel and Zhu have also shown that $L(c,0) = M(c,0)/J(c,0)$, where $J(c,0)$ is the maximal invariant subspace such that $L(c,0)$ is an irreducible highest weight representation of the Virasoro algebra (Theorem 2.36), is the unique irreducible quotient VOA of $\overline{M(c,0)}$. The VOA $L(c,0)$ is called the **Virasoro VOA** with central charge c.

Now we prove that $L(c,0)$ is a unitary VOA for $c \in \mathbb{R}$. The proof is due to Dong and Lin [DL14].

For $c \in \mathbb{R}$ define an antilinear map $\overline{\varphi} : \overline{M(c,0)} \to \overline{M(c,0)}$ by

$$L_{-n_1} \dots L_{-n_k}|0\rangle \mapsto L_{-n_1} \dots L_{-n_k}|0\rangle, \quad n_1 \geq \dots \geq n_k \geq 2.$$

Lemma 6.2. *The map $\overline{\varphi}$ is an antilinear involution of the VOA $\overline{M(c,0)}$ $\forall c \in \mathbb{R}$. Moreover, φ induces an antilinear involution φ of $L(c,0)$.*

Proof. Since $\overline{\varphi}^2 = \mathrm{id}$, it suffices to prove that $\overline{\varphi}$ is an antilinear automorphism. Let U be a subspace of $\overline{M(c,0)}$ defined by

$$U = \{u \in \overline{M(c,0)} \mid \varphi(u_n v) = \varphi(u)_n \varphi(v) \quad \forall v \in \overline{M(c,0)}\,,\ \forall n \in \mathbb{Z}\}.$$

By associativity of $\operatorname{End}\overline{M(c,0)}$, it follows that if $a, b \in U$, then $a_m b \in U$ $\forall m \in \mathbb{Z}$. Moreover, $|0\rangle \in U$ and $\nu = L_{-2}|0\rangle \in U$. Hence, $U = \overline{M(c,0)}$ since $\overline{M(c,0)}$ is generated by ν, as required.

Let $\overline{J(c,0)}$ be the maximal proper L-submodule of $\overline{M(c,0)}$. Then $\overline{\varphi}(\overline{J(c,0)})$ is a proper L-submodule of $\overline{M(c,0)}$. Thus, $\overline{\varphi}(\overline{J(c,0)}) \subset \overline{J(c,0)}$ and so $\overline{\varphi}$ induces an antilinear involution φ of $L(c,0)$. $\square$

All in all, rather unsurprisingly, we get a result equivalent to the representation theory of the Virasoro algebra (cf. Theorem 2.40).

Theorem 6.3. *Let $c \in \mathbb{R}$ and φ be the antilinear involution of $L(c,0)$ defined above. Then $(L(c,0), \varphi)$ is a unitary VOA if and only if $c \geq 1$ or $c = c(m)$ for $m \in \mathbb{N}_0$, where $c(m)$ is defined in Equation (2.9).*

Proof. If $c \geq 1$ or $c = c(m)$ for some $m \in \mathbb{N}_0$, then there exists a Hermitian form $(\cdot,\cdot)$ on $L(c,h)$ (Definition 2.30) and if it also satisfies

$$(L_n v, w) = (v, L_{-n} w), \quad (Cv, w) = (v, Cw), \quad (|0\rangle, |0\rangle) = 1,$$

then it is positive definite on $L(c,h)$ by Remark 2.37 and Theorem 2.40. So we just need to prove the invariance property. By [FZ92], the vertex operators $L(n)$ of $L(c,h)$ and Virasoro generators L_n coincide on $L(c,0)$, i.e. we have $L(n)u = L_n u$ for all $u \in L(c,0)$. We continue writing L_n for vertex operators as elsewhere in this work. This implies that $(L_n u, v) = (u, L_{-n} v)$ for all $u, v \in L(c,0)$. Hence,

$$\begin{aligned}(u, Y\big(e^{zL_1}(-z^{-2})^{L_0}\nu, z^{-1}\big)v) &= z^{-4}(u, Y(\nu, z^{-1})v) = \\ &= \sum_{n\in\mathbb{Z}} (u, \nu_{(n+1)}v) z^{n-2} = \sum_{n\in\mathbb{Z}} (u, L_n v) z^{n-2} = \sum_{n\in\mathbb{Z}} (L_{-n}u, v) z^{n-2} \\ &= \sum_{n\in\mathbb{Z}} (\nu_{(-n+1)}u, v) z^{n-2} = (Y(\nu, z)u, v) = (Y(\varphi(\nu), z)u, v).\end{aligned}$$

Since $L(c,0)$ is generated by ν, $(L(c,0),\varphi)$ is a unitary VOA by [DL14, Prop. 2.11] which states that a VOA is unitary if the invariant property holds on its generators.

Conversely, if $(L(c,0),\varphi)$ is a unitary VOA, then $L(c,0)$ is a unitary module of the Virasoro algebra by [DL14, Lem. 2.5]. Therefore, $c \geq 1$ or $c = c(m)$, as required. □

6.2 From Vertex Algebra to Virasoro Algebra

Now the converse is a tautology—every conformal vertex algebra has at least one representation of the Virasoro algebra encoded in itself. Moreover, a unitary vertex algebra contains a unitary representation of the Virasoro algebra.

Chapter 7

Wightman QFT and Vertex Algebras

In this chapter we will show that a Wightman (Möbius) CFT gives rise to two commuting (Möbius) conformal vertex algebras and conversely that two unitary (quasi)-vertex operator algebras can be combined to give a Wightman (Möbius) CFT. The only other reference providing the converse proof of which we are aware of is [Nik04]. However, in [Nik04], Nikolov studies higher dimensional vertex algebras and gets a one-to-one correspondence between them and Wightman QFTs with global conformal invariance [NT01]. The relationship between these higher dimensional vertex algebras and the vertex algebras used elsewhere in our work is not explicitly discussed in [Nik04]. Moreover, the proof itself is different from ours. Therefore, we hope that our proof will still be useful.

7.1 From Wightman CFT to Vertex Algebras

The following theorem is due to Kac [Kac98, Sec. 1.2]. We have also used [FST89] for clarifications and minor changes in normalization.

Theorem 7.1. [Kac98]. *Every Wightman Möbius CFT gives rise to two commuting strongly-generated positive-energy Möbius conformal vertex algebras. Moreover, if conformal weights are integers and the number of the generating fields of each conformal weight is finite, then these algebras are also unitary quasi-vertex operator algebras.*

Proof. Introduce the light cone coordinates $t := x^0 - x^1$ and $\bar{t} := x^0 + x^1$ so that $|x|^2 = t\bar{t}$. Define

$$P := \frac{1}{2}(P_0 - P_1) \quad \text{and} \quad \bar{P} := \frac{1}{2}(P_0 + P_1).$$

Furthermore, let

$$|0\rangle := \Omega.$$

In the light-cone coordinates special conformal transformations decouple

$$t^b = \frac{t}{1 + b^+ t}, \qquad \bar{t}^b = \frac{\bar{t}}{1 + b^- \bar{t}},$$

where $b^{\pm} = b^0 \pm b^1$. Since translations and special conformal transformations generate the whole of PSL$(2,\mathbb{R})$ by Proposition 1.12, the restricted conformal group acts as

$$\gamma(t,\bar{t}\,) = \left(\frac{at+b}{ct+d}, \frac{\bar{a}\bar{t}\,+\bar{b}}{\bar{c}\bar{t}\,+\bar{d}}\right),$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in SL_2(\mathbb{R}).$$

The transformation law for quasiprimary fields (4.12) becomes

$$U(\gamma)\phi_a(t,\bar{t}\,)U(\gamma)^{-1} = (ct+d)^{-2h_a}\left(\bar{c}\bar{t}\,+\bar{d}\right)^{-2\bar{h}_a}\phi_a(\gamma(t,\bar{t}\,)) \tag{7.1}$$

with $h = (\Delta + s)/2$ and $\bar{h} = (\Delta - s)/2$.

Define

$$K := -\frac{1}{2}(K_0 + K_1) \quad \text{and} \quad \bar{K} := \frac{1}{2}(K_1 - K_0).$$

We now focus on the t coordinate, but the same holds for $\bar{t}$.

Let D be the generator of dilations in t, i.e.

$$e^{i\lambda D}\phi(t,\bar{t})e^{-i\lambda D} = e^{\lambda h}\phi(e^{\lambda}t,\bar{t}), \quad \lambda > 0. \tag{7.2}$$

Then from (4.3), (4.14) and (7.2) it follows that in light-cone coordinates

$$\begin{aligned} i[P,\phi_a(t,\bar{t}\,)] &= \partial_t\phi_a(t,\bar{t}\,), && (7.3a)\\ i[D,\phi(t,\bar{t}\,)] &= (t\partial_t + h_a)\phi(t,\bar{t}\,) && (7.3b)\\ i[K,\phi_a(t,\bar{t}\,)] &= (t^2\partial_t + 2h_a t)\phi_a(t,\bar{t}\,), && (7.3c) \end{aligned}$$

with $\partial_t := 1/2(\partial_0 - \partial_1)$. Note that to prove (7.3c) it might be easier to start from (7.1) with γ being special conformal transformation, see [Ansb]. We also have

$$[P,K]\phi_a(t,\bar{t})|0\rangle = [[P,K],\phi_a(t,\bar{t})]\,|0\rangle$$

and similarly for the others, so that equations (7.3) imply

$$[D,P] = -iP, \quad [D,K] = iK, \quad [P,K] = 2iD$$

on $\mathcal{D}_0$, i.e. D,P,K form a representation of $\mathfrak{sl}(2,\mathbb{C})$. In particular, if we set $P = -iA$, $D = -iB$ and $K = -iC$ with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

then

$$wAw^{-1} = C \quad \text{with} \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}). \tag{7.4}$$

By Lemma 4.9 each vector $\Psi(x_1, \ldots, x_n) := \phi_{a_1}(x_1) \ldots \phi_{a_n}(x_n)|0\rangle$ of $\mathcal{H}$ extends analytically to the domain containing

$$\{\operatorname{Im} t_1, \operatorname{Im} \bar{t}_1 > 0\} \cdots \times \{\operatorname{Im} t_n, \operatorname{Im} \bar{t}_n > 0\} \quad \text{such that} \quad t_i + \bar{t}_i \neq t_j + \bar{t}_j \text{ if } i \neq j.$$

Since vectors of the form Ψ are dense in the Hilbert space $\mathcal{H}$ by W3, each field's ϕ_a domain of definition extends to Schwartz functions on $\mathbb{C}^2$ supported in $\operatorname{Im} t, \operatorname{Im} \bar{t} \geq 0$. This allows us to make conformal transformations defined everywhere by compactifying the Minkowski space using the Cayley transform (Figure 5.1)

$$z = \frac{1 + it/2}{1 - it/2}, \quad \bar{z} = \frac{1 + i\bar{t}/2}{1 - i\bar{t}/2}.$$

Under these transformations, the domain $\operatorname{Im} t > 0$, $\operatorname{Im} \bar{t} > 0$ is mapped to the domain $|z| < 1$, $|\bar{z}| < 1$. Define the new fields in the domain $|z| < 1$, $|\bar{z}| < 1$ by

$$Y(a, z, \bar{z}) := 2\pi \left(\frac{2}{1+z} \right)^{2h_a} \left(\frac{2}{1+\bar{z}} \right)^{2\bar{h}_a} \phi_a(t, \bar{t}),$$

with $t = 2i(1-z)(1+z)^{-1}$ and $\bar{t} = 2i(1-\bar{z})(1+\bar{z})^{-1}$ (cf. (5.9)). By the analytic extension,

$$a := Y(a, z, \bar{z})|0\rangle|_{z, \bar{z}=0} \tag{7.5}$$

is a well-defined vector in $\mathcal{D}_0$. Furthermore, $Y(a, z, \bar{z}) \mapsto a$ is a linear injective map.

Define

$$T := P - \frac{1}{4}K - iD, \qquad T^* := P - \frac{1}{4}K + iD,$$
$$H := P + \frac{1}{4}K.$$

By direct calculation from (7.3) it follows that

$$[T, Y(a, z, \bar{z})] = \partial_z Y(a, z, \bar{z}), \tag{7.6a}$$
$$[H, Y(a, z, \bar{z})] = (z\partial_z + h_a) Y(a, z, \bar{z}), \tag{7.6b}$$
$$[T^*, Y(a, z, \bar{z})] = (z^2 \partial_z + 2h_a z) Y(a, z, \bar{z}). \tag{7.6c}$$

Moreover, operators T, H and T^* annihilate the vacuum since P and K do. Thus, we have

$$[H, T] = T, \quad [H, T^*] = -T^*, \quad [T^*, T] = 2H$$

on $\mathcal{D}_0$ (cf. [Ansa]).

Applying (7.6b) to the vacuum and letting $z = \bar{z} = 0$ we obtain

$$Ha = h_a a. \tag{7.7}$$

The operator P is self-adjoint and semi-definite on $\mathcal{H}$ by W2. Same holds for K due to (7.4). Hence, by definition, H is also self-adjoint semi-definite. Therefore, conformal weights are non-negative real numbers.

The locality axiom W4 in light-cone coordinates is

$$\phi_a(t,\bar{t})\phi_b(t',\bar{t}') = \phi_b(t',\bar{t}')\phi_a(t,\bar{t}) \quad \text{if} \quad (t-t')(\bar{t}-\bar{t}') < 0. \tag{7.8}$$

Let us now consider the right chiral Wightman fields—fields satisfying $\partial_{\bar{t}}\phi_a = 0$. Then the locality condition becomes

$$\phi_a(t)\phi_b(t') = \phi_b(t')\phi_a(t) \quad \text{if} \quad t \neq t'$$

and since Wightman fields are operator-valued distributions we have

$$[\phi_a(t), \phi_b(t')] = \sum_{j\geq 0} \delta^{(j)}(t-t')\psi_j(t')$$

for some fields $\psi_j(t')$. For fields $\psi_j(t')$ the general Wightman axioms W1–W4 hold, but they are not necessarily quasiprimary as defined in W1$^{\text{conf}}$. So let us add such fields to our algebra to obtain:

$$[Y(a,z), Y(b,z')] = \sum_{j\geq 0} \delta^{(j)}(z-z')Y(c_j,z').$$

The map $Y(c_j,z')|0\rangle|_{z=0} = c_j$ is also well-defined, since we used only the general Wightman axioms W1–W4 to extend the fields in Lemma 4.9.

Now the Wightman field $Y(c_j,z')$ has conformal weight $h_a + h_b - j - 1$ as can be seen by applying $[H, \cdot\,]$ to both sides of this equality and using (7.6b) with Proposition 3.33. The positivity of conformal weights implies that the sum on the right-hand side is finite. Thus,

$$(z-z')^N[Y(a,z), Y(b,z')] = 0 \quad \text{for} \quad N \gg 0,$$

by the properties of the delta distribution.

Now we want to write the Wightman fields in a Fourier series

$$Y(a,z) = \sum_n a_{(n)} z^{-n-1}, \tag{7.9}$$

with $a_{(n)} \in \operatorname{End} \mathcal{D}_0$. However, it is not obvious that such an expansion is well-defined. Since it is an operator equality, it suffices to prove the equality on $\mathcal{D}_0$, i.e. we have to prove that

$$Y(a,z)Y(b_1,w_1)\dots Y(b_n,w_n)|0\rangle = \sum_{k,k_1,\dots,k_n} a_{(k)}b_{(k_1)}\dots b_{(k_n)}z^{-k-1}w_1^{-k_1-1}\dots w_n^{-k_n-1}|0\rangle.$$

Note that in $|z|<1$ with $h \geq 0$ the function

$$\frac{1}{(1+z)^{2h}}$$

is holomorphic and hence analytic. Therefore, $Y(a,z)Y(b_1,w_1)\dots Y(b_n,w_n)|0\rangle$ is analytic, since $\phi_a(t)\phi_{a_1}(t_1)\dots\phi_{a_n}(t_n)|0\rangle$ is analytic as proven above.

Let V be the subspace of $\mathcal{D}_0$ spanned by all polynomials in the $a_{(n)}$ applied to the vacuum vector $|0\rangle$. Clearly V is invariant with respect to all $a_{(n)}$'s and with respect to T since by (7.6a) we have

$$\sum_n [T,a_{(n)}]z^{-n-1} = \sum_n (-n-1)a_{(n)}z^{-n-2} = \sum_n -na_{(n-1)}z^{-n-1}.$$

Thus,

$$[T,a_{(n)}] = -na_{(n-1)}$$

and because $T|0\rangle = 0$ by $\mathrm{W1}^{\mathrm{conf}}$,

$$Ta_{(n)}|0\rangle = -na_{(n-1)}|0\rangle. \tag{7.10}$$

Now we prove that $Y(a,z)$'s are fields in vertex algebra sense (Definition 3.13). Similarly like for T in (7.10), Equation (7.6b) gives

$$[H,a_{(n)}] = (h_a-n-1)a_{(n)}.$$

Given $v = b_{(j)}|0\rangle \in V$ we get

$$\begin{aligned}(h_a-n-1)a_{(n)}v = [H,a_{(n)}]v = Ha_{(n)}v - a_{(n)}Hb_{(j)}|0\rangle &= \\ = Ha_{(n)}v - a_{(n)}[H,b_{(j)}]|0\rangle &= Ha_{(n)}v - a_{(n)}(h_b-j-1)v.\end{aligned}$$

Hence,

$$Ha_{(n)}v = (h_a+h_b-j-n-2)a_{(n)}v.$$

Thus, the Wightman field $Y(a_{(n)}v,z)$ has conformal weight $h_a+h_b-j-n-2$, since

$$Ha = h_a a \iff [H,Y(a,z,\bar{z})] = (z\partial_z + h_a)Y(a,z,\bar{z}).$$

But conformal weights are non-negative real numbers. Hence, $a_{(n)}v = 0$ for $n \gg 0$. The above reasoning clearly holds $\forall v \in V$. Therefore, the Wightman fields $Y(a,z)$ for $a \in V$ are also vertex algebra fields and we can use the Existence Theorem 3.28 to obtain a vertex algebra.

Combining the expansion (7.9) with the definition of a (7.5) we obtain

$$a_{(n)}|0\rangle = 0 \quad \forall n \geq 0. \tag{7.11}$$

Moreover, note that given two generators $a_{(m)}$, $m \geq 0$, and $b_{(j)}$, $j < 0$, their commutator is $[a_{(m)}, b_{(j)}] = \sum_{k<m} c_{(k)}$ for some generators $c_{(k)}$ with $k < m$ by Borcherds commutator formula (3.39a). Thus, by generalizing the simple calculation

$$a_{(m)}b_{(j)}|0\rangle = b_{(j)}a_{(m)}|0\rangle + [a_{(m)}, b_{(j)}]|0\rangle = 0 + \sum_{k<0} c_{(k)}|0\rangle$$

it follows that V is strongly generated by the fields $Y(a,z)$ (Definition 3.29).

Now if we take the left chiral fields, i.e. fields satisfying $\partial_t \phi_i = 0$, and apply the same reasoning as above, we obtain the left vertex algebra $\bar{V}$ with the same vacuum vector $|0\rangle$, the infinitesimal translation operator $\overline{T}$ and fields $Y(\bar{a},\bar{z})$ with $\bar{a} \in \bar{V}$. From (7.8) we see that locality in the mixed chiral case boils down to $\phi_a(t)\phi_{\bar{a}}(\bar{t}) = \phi_{\bar{a}}(\bar{t})\phi_a(t)$ for all t and $\bar{t}$ hence

$$[Y(a,z), Y(\bar{a},\bar{z})] = 0 \qquad \forall a \in V,\ \forall \bar{a} \in \bar{V}.$$

This finishes the first part of the theorem.

To prove unitarity, we first of all have to show that if $h_a = \bar{h}_a = 0$, then $a = \lambda|0\rangle$ with $\lambda \in \mathbb{C}$. If $h_a = 0$, then $T^*a = 0$ since $h \geq 0$. Using Wightman inner product and unitarity of the representation of the restricted conformal group together with $[T^*, T] = 2H$, we get

$$\|Ta\|^2 = 2h_a\|a\|^2 = 0.$$

Thus, a is annihilated by all of the $\mathfrak{sl}(2,\mathbb{C})$ generators T, T^* and H. Hence, it is invariant under $\mathrm{PSL}(2,\mathbb{C})$ and in particular under the Poincaré group. By the uniqueness of the vacuum vector, $a = \lambda|0\rangle$ as required. Therefore, if the extra assumptions of the theorem hold, then unitarity follows by Remark 3.50 and Theorem 3.49. $\square$

If we also assume the existence of the energy-momentum tensor, we get two conformal vertex algebras.

Corollary 7.2. A Wightman CFT gives rise to two commuting strongly-generated unitary positive-energy conformal vertex algebras. Moreover, if conformal weights are integers and the number of the generating fields of each conformal weight is finite, then these algebras are also unitary VOAs of CFT type.

Proof. We use the Lüscher–Mack Theorem 5.4 to get an energy-momentum field $T(z)$ in vertex algebra sense. It gives rise to conformal vector $\nu = T(z)|0\rangle|_{z=0}$. Similarly, for the antichiral part. The rest follows by Theorem 7.1. □

7.2 From Vertex Algebras to Wightman CFT

We will show in this section that two unitary vertex operator algebras can be combined to give distributions satisfying all axioms of conformal Wightman distributions. Thus, we can use the Wightman Reconstruction Theorem 4.12 to get a Wightman CFT. The uniqueness of the vacuum vector follows if we assume that our VOAs have a single vacuum vector. The idea of the proof is to reverse the arguments of Kac's Theorem 7.1.

We summarize this discussion in a theorem.

Theorem 7.3. *Given two unitary vertex operator algebras V and $\bar{V}$, one can construct a Wightman CFT.*

We will see in the proof that the energy-momentum tensor of a VOA gives the existence of Wightman energy-momentum tensor W5 and it does not imply anything else. Thus, we get a corollary.

Corollary 7.4. Given two quasi-vertex operator algebras, one can construct a Wightman Möbius CFT.

Throughout this section, set the notation in accordance with Wightman framework

$$L_{-1} := T, \quad L_1 := T^*, \quad L_0 := H, \quad \Omega := |0\rangle.$$

Now we introduce vertex algebra correlation functions which are well-known and can be found in [FLM88] or [FBZ04]. We have also used [CKLW15] for the discussion of the contragradient module.

Let V be a vertex algebra and V^* be the dual of V, i.e. the space of linear functions $\varphi : V \to \mathbb{C}$. Let $\langle \cdot, \cdot \rangle$ be the natural pairing between V^* and

V. Then for $a_1, \ldots, a_n, v \in V$ and $\varphi \in V^*$

$$\langle \varphi, Y(a_1, z_1) \ldots Y(a_n, z_n) v \rangle$$

is a formal power series in $\mathbb{C}[[z_1^{\pm}, \ldots, z_m^{\pm}]]$. Such series are called **correlation functions** (in the sense of vertex algebra). Note that $v = Y(v, z)\Omega|_{z=0}$ by the vacuum axiom V3. Thus, it suffices to consider only the case $v = \Omega$.

Proposition 7.5. *Let V be a vertex algebra, $\varphi \in V^*$ and let $a_1, \ldots, a_n \in V$. Then there exists a series*

$$M^{\varphi}_{a_1 \ldots a_n}(z_1, \ldots, z_n) \in \mathbb{C}[[z_1, \ldots, z_n]][(z_i - z_j)^{-1}]_{i \neq j}$$

with the following property:

For arbitrary permutation σ of $\{1, \ldots, n\}$, the correlation function

$$\langle \varphi, Y(a_{\sigma(1)}, z_{\sigma(1)}) \ldots Y(a_{\sigma(n)}, z_{\sigma(n)}) \Omega \rangle$$

is the expansion in $\mathbb{C}((z_{\sigma(1)})) \ldots ((z_{\sigma(n)}))$ of $M^{\varphi}_{a_1, \ldots, a_n}(z_1, \ldots, z_n)$.

Proof. By the definition of vertex algebra 3.20, $Y(a, z)$ is a field and hence $\langle \varphi, Y(a, z) v \rangle \in \mathbb{C}((z))$ for all $a, v \in V$. Thus, by induction

$$\langle \varphi, Y(a_{\sigma(1)}, z_{\sigma(1)}) \ldots Y(a_{\sigma(n)}, z_{\sigma(n)}) \Omega \rangle \in \mathbb{C}((z_{\sigma(1)})) \ldots ((z_{\sigma(n)})).$$

By locality V2, there exist positive even integers $N_{ij} \in 2\mathbb{N}$ such that

$$(z_i - z_j)^{N_{ij}} [Y(a_i, z_i), Y(a_j, z_j)] = 0.$$

Thus, the series

$$\prod_{i<j} (z_i - z_j)^{N_{ij}} \langle \varphi, Y(a_{\sigma(1)}, z_{\sigma(1)}) \ldots Y(a_{\sigma(n)}, z_{\sigma(n)}) \Omega \rangle$$

is independent of the permutation σ. Furthermore, by V3, $Y(a, z)\Omega \in V[[z]]$ and combining this with permutation invariance we get that the series contains only non-negative powers of z_i, $1 \leq i \leq n$. Therefore,

$$\prod_{i<j} (z_i - z_j)^{N_{ij}} \langle \varphi, Y(a_{\sigma(1)}, z_{\sigma(1)}) \ldots Y(a_{\sigma(n)}, z_{\sigma(n)} \Omega \rangle$$

is the same as

$$\prod_{i<j} (z_i - z_j)^{N_{ij}} \langle \varphi, Y(a_1, z_1) \ldots Y(a_n, z_n) \Omega \rangle$$

in $\mathbb{C}[[z_1, \ldots, z_n]]$. Dividing the last series by $\prod_{i<j} (z_i - z_j)^{N_{ij}}$, we obtain the required series $M^{\varphi}_{a_1 \ldots a_n}(z_1, \ldots, z_n) \in \mathbb{C}[[z_1, \ldots, z_n]][(z_i - z_j)^{-1}]_{i \neq j}$.

$\square$

Remark 7.6. For general $v \in V$, $M^{\varphi,v}_{a_1 \dots a_n}(z_1, \dots, z_n)$ would belong to

$$\mathbb{C}[[z_1, \dots, z_n]][z_1^{-1}, \dots, z_n^{-1}, (z_i - z_j)^{-1}]_{i \neq j}.$$

Now consider a VOA. The grading allows us to define the **restricted dual** [FHL93] of a vertex operator algebra V as

$$V' := \bigoplus_{n \in \mathbb{Z}} V_n^*,$$

i.e. as the space of linear functionals on V vanishing on all but finitely many V_n. Note that for $a, v \in V$ and $v' \in V'$

$$\langle v', Y(a, z)v \rangle \in \mathbb{C}[z, z^{-1}]$$

or equivalently

$$\langle v', Y(a, z)\Omega \rangle \in \mathbb{C}[z]$$

because $Y(a, z)$ is a field and v' belongs to the restricted dual V'. Hence, application of Proposition 7.5 to a vertex operator algebra gives

$$M^{v'}_{a_1 \dots a_n}(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n][(z_i - z_j)^{-1}]_{i \neq j}.$$

Moreover, if we specialize from the case of arbitrary formal variables to the case $z_i \in \mathbb{C}$, we obtain the following version of Proposition 7.5.

Corollary 7.7. Let V be a vertex operator algebra, $a_1, \dots, a_n, v \in V$ and $v' \in V'$. For arbitrary permutations σ of $\{1, \dots, n\}$, the correlation functions

$$\langle v', Y(a_{\sigma(1)}, z_{\sigma(1)}) \dots Y(a_{\sigma(n)}, z_{\sigma(n)})v \rangle$$

with $z_i \in \mathbb{C}$, $1 \leq i \leq n$, are absolutely convergent to a common rational function $M^{v',v}_{a_1 \dots a_n}(z_1, \dots, z_n)$ in the domains

$$\left|z_{\sigma(1)}\right| > \dots > \left|z_{\sigma(n)}\right| > 0.$$

In light of Corollary 7.7, we will call the rational functions

$$M^{v',v}_{a_1 \dots a_n}(z_1, \dots, z_n)$$

analytic extensions of VOA **correlation functions**.

The restricted dual V' becomes a V-module by setting

$$\langle Y'(a, z)b', c \rangle = \langle b', Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})c \rangle \quad \forall a, c \in V,\ \forall b' \in V'.$$

This formula determines the field $Y'(a, z)$ on V' and implies that the map $a \mapsto Y'(a, z)$ is a V-module. See [FHL93, Sections 4.1 and 5.2] for the definition and a proof. Note that the V-module structure on V' depends not only on the vertex algebra structure of V, but also on L_1. We will call the module V' the **contragradient module** and the fields $Y'(a,z)$ **adjoint vertex operators**. However, the endomorphisms $a'_{(n)} \in \operatorname{End} V$ of the formal series $Y'(a,z) = \sum_{n\in\mathbb{Z}} a'_{(n)} z^{-n-1}$ are not the adjoint endomorphisms of $a_{(n)}$. In particular, we have

$$\langle L'_n a', c\rangle = \langle a', L_{-n} b\rangle \qquad a' \in V',\ b \in V,\ n \in \mathbb{Z},$$

with $L'_n = \nu'_{(n+1)}$. This implies that $L'_0 = na'$ for $a' \in V^*_n$, i.e. V' is a $\mathbb{Z}$-graded V module.

If we let $(\cdot,\cdot)$ to be an invariant bilinear form on V, then by Remark 3.45 $(V_i, V_j) = 0$ if $i \neq j$. Hence,

$$(a,\cdot) \in V' \quad \forall a \in V$$

and the map $a \mapsto (a,\cdot)$ is a module homomorphism from V to V'. On the other hand, given a module homomorphism $\varphi : V \to V'$, the bilinear form

$$(a,b) := \langle \varphi(a), b\rangle \tag{7.12}$$

is invariant. By finite-dimensionality of the homogeneous subspaces and grading-preserving property, each V-module homomorphism from V to V' is injective if and only if it is surjective. We have proved a well-known result:

Proposition 7.8. *The restricted dual V' is isomorphic to V as a V-module if and only if there exists a non-degenerate invariant bilinear form on V.*

Now let $(V, (\cdot|\cdot))$ be a unitary VOA with PCT operator θ. By definition, we have that $(\cdot,\cdot) := (\theta\cdot|\cdot)$ is an invariant bilinear form on V. Fix $\varphi : V \to V'$ to be an isomorphism between V and V' and set

$$\Omega' := \varphi(\Omega).$$

Let

$$M_{a_1\dots a_n}(z_1,\dots,z_n) := M^{\Omega',\Omega}_{a_1\dots a_n}(z_1,\dots,z_n)$$

be VOA **vacuum expectation values (VEVs)**. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}),$$

define

$$g_A(z) = \frac{az+b}{cz+d}$$

to be a Möbius transformation. In particular, set

$$g_1^\lambda(z) = \frac{z}{1-\lambda z}, \quad g_0^\lambda(z) = e^\lambda z, \quad g_{-1}^\lambda(z) = z + \lambda.$$

Proposition 7.9. *Let V be a unitary VOA. Then the VOA vacuum expectation values of quasiprimary fields are Möbius covariant.*

Proof. Using Proposition 3.35, V3, Equations (3.52) and (7.12) we have

$$\begin{aligned}
&\prod_{i=1}^{n}\left(\frac{d}{dz_i}g_m^\lambda(z_i)\right)^{h_i}\left\langle \Omega', Y(a_1, g_m^\lambda(z))\dots Y(a_n, g_m^\lambda(z_n))\Omega\right\rangle = \\
&\quad = \left\langle \Omega', e^{\lambda L_m}Y(a_1,z_1)\dots Y(a_n,z_n)e^{-\lambda L_m}\Omega\right\rangle \\
&\quad = \left(\Omega, e^{\lambda L_m}Y(a_1,z_1)\dots Y(a_n,z_n)\Omega\right) \\
&\quad = \left(e^{\lambda L_{-m}}\Omega, Y(a_1,z_1)\dots Y(a_n,z_n)\Omega\right) = (\Omega, Y(a_1,z_1)\dots Y(a_n,z_n)\Omega) \\
&\quad = \langle \Omega', Y(a_1,z_1)\dots Y(a_n,z_n)\Omega\rangle\,.
\end{aligned} \tag{7.13}$$

Since the transformations of the form $g_1^\lambda(z)$ and $g_{-1}^\lambda(z)$ generate $\mathrm{PSL}(2,\mathbb{C})$ by Proposition 1.12, it follows that

$$M_{a_1\dots a_n}(z_1,\dots,z_n) = \prod_{i=1}^{n}\left(\frac{d}{dz_i}g_A(z_i)\right)^{h_i} M_{a_1\dots a_n}\left(g_A(z_1),\dots,g_A(z_n)\right) \tag{7.14}$$

$\forall A \in \mathrm{SL}(2,\mathbb{C})$, as required. □

Now we restrict the variables z_i to the open unit disk in $\mathbb{C}^n$. We use the inverse Cayley transform

$$t = 2i\frac{1-z}{1+z}$$

to map the open unit disk to the open upper half-plane $\operatorname{Im} t > 0$ (Figure 5.1). We define the transformed fields for quasiprimary vectors $a \in V_h$ on the upper-half plane as

$$\phi_a(t) := \frac{1}{2\pi}\left(\frac{2i}{2i+t}\right)^{2h} Y(a,z) \tag{7.15}$$

with $z = (1+it/2)(1-it/2)^{-1}$. We also define the corresponding correlation functions as

$$W_{a_1\dots a_n}(t_1,\dots,t_n) := \frac{1}{(2\pi)^n}\prod_{j=1}^{n}\left(\frac{2i}{2i+t_j}\right)^{2h_j} M_{a_1\dots a_n}(z_1,\dots,z_n)$$

and call their limit as $\operatorname{Im} t_i \to 0$ **lightcone VEVs**.

Proposition 7.10. *The lightcone VEVs are Möbius covariant tempered distributions.*

Proof. We prove temperedness first. The correlation functions W are rational by rationality of M's and the fact that the inverse Cayley transform is rational. We let $\operatorname{Im} t_i \to 0$ and use

$$\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$$

with f being the numerator of W and g one over the denominator. The limit of the numerator simply returns a polynomial, whereas one over the denominator gives a tempered distribution containing factors of the form

$$\frac{1}{(t_i - t_j \pm i\varepsilon)^k}, \quad k \in \mathbb{N}_0.$$

Since the product of a function of at most polynomial growth with a tempered distribution is a tempered distribution, we get that the lightcone VEVs are tempered distributions.

The inverse Cayley transform can be viewed as a change of basis matrix

$$\begin{pmatrix} -2i & 2i \\ 1 & 1 \end{pmatrix}.$$

Thus, defining

$$P = \frac{1}{4}(2L_0 + L_{-1} + L_1), \quad K = 2L_0 - L_{-1} - L_1, \quad D = \frac{1}{2i}(L_1 - L_{-1}) \tag{7.16}$$

and using Proposition 3.35 we see that

$$e^{i\lambda P} \quad \text{maps} \quad t \mapsto t + \lambda, \tag{7.17}$$
$$e^{i\lambda D} \qquad\qquad t \mapsto e^{\lambda} t, \tag{7.18}$$
$$e^{i\lambda K} \qquad\qquad t \mapsto \frac{t}{1 - \lambda t}. \tag{7.19}$$

Therefore, the operators P, D and K are infinitesimal generators of translations, dilations and special conformal transformations, respectively. In a unitary VOA, L_0 is self-adjoint and L_1 is the adjoint of L_{-1} and vice versa. Hence, P, D and K are self-adjoint. Therefore, by Stone's Theorem $U_q(A) := e^{iqA}$ are strongly continuous one-parameter unitary groups, where $A = P, D$ or K, and $q \in \mathbb{R}$. By V3 and Proposition 3.35 we have $L_{-1}\Omega = L_0\Omega = L_1\Omega = 0$. Hence,

$$e^{iq_1 P}\Omega = e^{iq_2 D}\Omega = e^{iq_3 K}\Omega = \Omega \quad \forall q_1, q_2, q_3 \in \mathbb{R}, \tag{7.20}$$

i.e. the vacuum is fixed under global conformal transformations. Hence, by Proposition 7.9 lightcone vacuum expectation values of quasiprimary fields $W_{a_1 \ldots a_n}$ are Möbius covariant. $\square$

We now take a second vertex operator algebra $(\bar{V}, \bar{Y}, \bar{\Omega}, \bar{\nu})$ and mimic the construction of full field algebras [HK07]. Let

$$V_f := V \otimes \bar{V}$$

be the full vector space and let the full vertex operators be

$$\mathcal{Y}_{a,\bar{a}}(z, \bar{z}) := Y(a, z) \otimes \bar{Y}(\bar{a}, \bar{z}).$$

Here we identify $\bar{z}$ with the complex conjugate of z. Then the full vertex operators act as

$$\mathcal{Y}_{a,\bar{a}}(z, \bar{z})(v \otimes \bar{v}) = Y(a, z)v \otimes Y(\bar{a}, \bar{z})\bar{v},$$

$\forall a, v \in V$, $\forall \bar{a}, \bar{v} \in \bar{V}$. We also define an inner product on V_f by

$$(a \otimes \bar{a}|b \otimes \bar{b})_f = (a|b)(\bar{a}|\bar{b}).$$

By [DL14, Prop. 2.9] a tensor product of unitary VOAs is a unitary VOA with conformal vector $\boldsymbol{\nu} = \nu \otimes \bar{\Omega} + \Omega \otimes \bar{\nu}$ and the vacuum vector $\mathbf{1} := \Omega \otimes \bar{\Omega}$.

The action of the full vertex operators implies that the full VOA correlation functions are

$$\mathcal{M}_{a_1,\bar{a}_1 \ldots a_n,\bar{a}_n}(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) = M_{a_1 \ldots a_n}(z_1, \ldots, z_n)\bar{M}_{a_1 \ldots a_n}(\bar{z}_1, \ldots, \bar{z}_n).$$

In particular, since we have proved in Corollary 7.7 that M's are symmetric if $z_i \neq z_j$, the full VOA correlation functions are also symmetric if $z_i \neq z_j$, i.e.

$$\mathcal{M}_{a_1,\bar{a}_1 \ldots a_i,\bar{a}_i\, a_{i+1},\bar{a}_{i+1} \ldots a_n,\bar{a}_n}(\vec{z}_1, \ldots, \vec{z}_i, \vec{z}_{i+1}, \ldots, \vec{z}_n) = \mathcal{M}_{a_1,\bar{a}_1 \ldots a_{i+1},\bar{a}_{i+1}\, a_i,\bar{a}_i \ldots a_n,\bar{a}_n}(\vec{z}_1, \ldots, \vec{z}_{i+1}, \vec{z}_i, \ldots, \vec{z}_n)$$

with $z_i \neq z_{i+1}$ and $\vec{z} = (z, \bar{z})$.

Applying the inverse Cayley transform to the fields (7.15) we get full lightcone fields

$$\Phi_{a,\bar{a}}\left(x^0, x^1\right) := \phi_a(t) \otimes \bar{\phi}_{\bar{a}}(\bar{t}),$$

where $t = x^0 - x^1$, $\bar{t} = x^0 + x^1$, with the corresponding **full Wightman VEVs**

$$\mathcal{W}_{a_1,\bar{a}_1 \ldots a_n,\bar{a}_n}(x_1, \ldots, x_n) = W_{a_1 \ldots a_n}(t_1, \ldots t_n)\bar{W}_{\bar{a}_1 \ldots \bar{a}_n}(\bar{t}_1, \ldots, \bar{t}_n), \tag{7.21}$$

where

$$\operatorname{Im} t_i, \operatorname{Im} \bar{t}_i \to 0$$

is implicit. Moreover, we define the operators on the Minkowski plane as

$$P_0 = P \otimes \mathrm{id} + \mathrm{id} \otimes \bar{P}, \qquad P_1 = -P \otimes \mathrm{id} + \mathrm{id} \otimes \bar{P}, \tag{7.22a}$$
$$K_0 = -K \otimes \mathrm{id} - \mathrm{id} \otimes \bar{K}, \quad K_1 = -K \otimes \mathrm{id} + \mathrm{id} \otimes \bar{K}. \tag{7.22b}$$

Now we are ready to prove the main theorem of this section.

Theorem 7.3. *Given two unitary vertex operator algebras V and $\bar{V}$, one can construct a Wightman CFT.*

Proof. Conformal covariance and temperedness obviously hold for full Wightman vacuum expectation values by Proposition 7.10 and Equation (7.22).

Conformal covariance also includes translation invariance and hence we can define for $n \geq 2$

$$w(\zeta_1, \ldots, \zeta_{n-1}) = W_{a_1 \ldots a_n}(t_1, \ldots, t_n), \quad \zeta_i = t_i - t_{i+1}.$$

We have

$$w(\zeta_1, \ldots, \zeta_n) = \int \hat{w}(p_1, \ldots, p_n) e^{i \sum p_j t_j} \mathrm{d}p, \quad \forall \operatorname{Im} t_j > 0.$$

Thus, $\hat{w}(p_1, \ldots, p_n) = 0$ if at least one of $p_j < 0$. Combining this with the definition of full Wightman distributions (7.21), we see that this is precisely the spectrum property WD2.

Now write

$$\mathcal{W}(x_1, \ldots, x_n) = \mathcal{W}_{a_1, \bar{a}_1 \ldots a_n, \bar{a}_n}(x_1, \ldots, x_n).$$

We want to show that

$$\mathcal{W}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \mathcal{W}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \tag{7.23}$$

if $(x_i - x_{i+1})^2 < 0$. It holds that $(x_i - x_{i+1})^2 = (t_i - t_{i+1})(\bar{t}_i - \bar{t}_{i+1})$ and so it suffices to prove that (7.23) holds whenever $(t_i - t_{i+1}) < 0$ and $(\bar{t}_i - \bar{t}_{i+1}) > 0$. But this follows from the symmetry of VOA correlation functions, since we have identified $\bar{z}$ with the complex conjugate of z.

Wightman positivity WD5 was used to prove the existence of positive-definite scalar product on the Hilbert space constructed in the Wightman Reconstruction Theorem 4.12. However, a unitary VOA already has an inner product which can be used for Wightman reconstruction so WD5 is unnecessary.

Combining all of the above observations, we get a Wightman Möbius CFT.

To get the energy-momentum tensor, let

$$\Theta(t) = \frac{1}{2\pi}\left(\frac{2i}{2i+t}\right)^4 Y(\nu, z), \quad \bar{\Theta}(\bar{t}) = \frac{1}{2\pi}\left(\frac{2i}{2i+\bar{t}}\right)^4 \bar{Y}(\bar{\nu}, \bar{z})$$

be fields acting on V and $\bar{V}$, respectively, where ν is the conformal vector of V and $\bar{\nu}$ of $\bar{V}$. Set

$$\begin{aligned} T_{00}(x^0, x^1) &= T_{11}(x^0, x^1) = \Theta(t) \otimes \mathrm{id} + \mathrm{id} \otimes \bar{\Theta}(\bar{t}), \\ T_{01}(x^0, x^1) &= T_{10}(x^0, x^1) = \mathrm{id} \otimes \bar{\Theta}(\bar{t}) - \Theta(t) \otimes \mathrm{id}. \end{aligned}$$

Then the full Wightman VEVs containing these fields give rise to Wightman fields satisfying all requirements of W5. In particular, self-adjointness follows from the fact that after the reconstruction we have a unitary representation of Möbius group. Thus, $L_0 = L_0^*$ and

$$L_0 = \int_{-\infty}^{+\infty} \left(1 + \frac{t^4}{4}\right) \Theta_W(t)\, \mathrm{d}t$$

give the required result, where $\Theta_W(t)$ denotes the chiral part of the Wightman energy-momentum tensor in lightcone coordinates. Similarly, for $\bar{\Theta}_W(\bar{t})$.

□

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