The Quantum Determinant of the Elliptic Quantum Algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$

Daniel Issing

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This thesis is dedicated to my beloved grandma,

Linde,

to whom time and circumstances denied the opportunity to attend a university. Turning the fact that her more fortunate grandchildren could pursue this path into a source of pride, rather than bitterness, is only one of the beautiful life lessons I learned from her.

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Abstract

The central theme underlying this thesis is the quest for a better understanding of elliptic quantum algebras, a mathematical structure that rose to prominence in the late 1980s, following the (independent) discovery of quantum groups by Drinfel'd and Jimbo. We begin by familiarizing the reader with some indispensable background from Lie algebra theory before introducing the notion of (quantum) deformed algebra. The theoretical framework of Hopf algebras, which offers the necessary tools to grapple with these objects, will subsequently be outlined, and later on be extended to quasi-Hopf algebras. We will then see how these deformed quantum algebras can be generalized a so-called elliptic algebra, denoted by $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$. We will define a *quantum determinant*, and prove that it lies in the center of the algebra. Using these results, $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ will emerge as a quotient algebra from the general linear case. Some remarks how this research connects to the study of exactly solvable ('integrable') quantum systems round off the study, thus relating a mathematical theory to the physically observable.

Die zentrale Motivation dieser Masterarbeit ist das bessere Verständnis elliptischer Quantenalgebren, einer mathematischen Theorie, die zuerst in den späten 1980er Jahren entwickelt wurde. Sie sind ein Beispiel für sogenannte Quantengruppen, die vor mehr als 30 Jahren von Drinfel'd und Jimbo unabhängig voneinander entdeckt wurden. Nach einem detaillierten Überblick über zentrale Ergebnisse aus der Theorie von Lie-Algebren wird das Konzept der (Quanten-)Deformation von Algebren vorgestellt. Wir präsentieren ein theoretisches Rahmenmodell - Hopf-und Quasi-Hopf-Algebren - welches uns das nötige Rüstzeug an die Hand gibt, um diese Objekte adequat zu beschreiben. Anschließend wird erläutert, wie deformierte Algebren zu elliptischen Algebren verallgemeinert werden können. Für diese Algebra, die gewöhnlich mit $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ bezeichnet wird, definieren wir die sogenannte *Quantendeterminante*. Wir zeigen, dass ebenjene Determinante im Zentrum der Algebra liegt, und nutzen dies, um $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ als Quotientenalgebra zu definieren. Schließlich erläutere ich, welche Verbindung zwischen diesen Resultaten und der Theorie exakt lösbarer ('integrabler') Modelle besteht, um so eine Brücke zwischen mathematischen Objekten und physikalischen Observablen zu schlagen.

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Chapter 1

Introduction

It has by now become a cliché to attribute the extraordinary scientific advances during the last 200 years to the cross-fertilization between physical models and mathematical theorems. Time and again did we witness how techniques developed to deal with a conundrum arising from theoretical physics proved to be much richer from the mathematical point of view than anyone could reasonably have assumed when they were first introduced. Conversely, physicists generously serve themselves from the vast selection of mathematical tools in their day-to-day work, and it is by no means a rare occurrence to find them venturing into terrains that were though of as having no potential for real-world applications.
The present work falls squarely into this category. Quantum algebras were introduced and developed, sometimes by one and the same person, for both physical and mathematical reasons, and in many cases did a leap forward in one domain inspire new ideas in the other. So while the focus of my thesis is definitely an algebraic one, and whether or not the topics covered herein will prove 'useful' for understanding the empirical world around us is occasionally still a matter of speculation, physicists should not dismiss it prematurely. But let me provide you with a little more background first.

1.1 Quantum algebras for pragmatists

Maybe the most important reason why physicists became interested in quantum algebras can be found in the discovery of the quantum inverse scattering method by the so-called Leningrad School in the late 1970s [7, 8, 9, 27]. Details will have to be omitted here, but the general idea is as follows: In many physical models, symmetries can aptly be described by algebraic structures, the most famous examples certainly coming from Lie algebras. Among those, there is a special class known as *integrable* (aka exactly solvable) models that allow for the construction of an algebraic *transfer matrix*. This object can be split in such a way as to end up with N commuting operators, where N equals the degrees of freedom of the system, while ensuring that one of them contains the Hamiltonian of the model.

Let us look at an example, the Heisenberg spin-$\frac{1}{2}$ chain [2]. In this model, we examine a closed string of equidistantly distributed spins (one-dimensional) that are occupy either the 'up' or 'down' state, with only nearest-neighbor interactions assumed between them. The most general Hamiltonian for this model with N sites is given by

$$\mathcal{H} = \sum_{i=1}^{N} \left(j_x \sigma_i^x \sigma_{i+1}^x + j_y \sigma_i^y \sigma_{i+1}^y + j_z \sigma_i^z \sigma_{i+1}^z \right) , \tag{1.1}$$

where $\{j_x, j_y, j_z\}$ are coupling constants that might, for example, be the result of an external magnetic field, and $\{\sigma_i^x, \sigma_i^y, \sigma_i^z\}$ are the Pauli matrices acting in the i-th Hilbert space:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For ease of notation, we will also introduce

$$\sigma^+ := \frac{1}{2}(\sigma^x + i\sigma^y) \ , \ \sigma^- := \frac{1}{2}(\sigma^x - i\sigma^y)$$

In the easiest case, $j_x = j_y = j_z$ (isotropic or XXX model), and we find that $[\mathcal{H}, \sigma_i^\pm] = 0 = [\mathcal{H}, \sigma_i^z]$. Owing to the well-known fact that the Pauli matrices are representations of the $\mathfrak{sl}_2$ algebra, given by commutation relations

$$\begin{aligned} [J^+, J^-] &= 2J^z \\ [J^z, J^\pm] &= \pm J^\pm, \end{aligned} \tag{1.2}$$

the Hamiltonian is $\mathfrak{sl}_2$ symmetric. The key insight of the Leningrad school [8] was that these relations could alternative be encoded in what is known as the RLL relation:

$$R_{12} L_1 L_2 = L_2 L_1 R_{12} \tag{1.3}$$

where L is called a (in this case: 2×2) *Lax matrix* with generators (1.2) as entries, and R is a numerical matrix chosen such as to reproduce $\mathfrak{sl}_2$, and satisfying the 'star triangle' equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \tag{1.4}$$

where lower indices indicate the tensor space on which these objects act. From the Lax matrix, one constructs the transfer matrix, and then goes on to show that the Hamiltonian is integrable. Lastly, eigenstates and eigenenergies of the Hamiltonian are calculated analytically, using a suitable reference state [9]. This method is alternatively known as the *quantum inverse scattering method* or *algebraic Bethe ansatz*, paying homage to an earlier, exact solution of the Heisenberg spin chain found by Bethe [3].

This, to be sure, does not yet have anything to do with quantum algebras, but things change quickly as soon as we consider anisotropic spin chains. The symmetry algebra of the XXZ model ($j_x = j_y \neq j_z$), for example, is the quantum deformed $\mathcal{U}_q(\mathfrak{sl}_2)$ algebra, and the generic model (1.1) is related to the (elliptic) Sklyanin algebra [12, 13]. We will briefly encounter both cases later on, but for now, let us stress two possible ways to generalize this method: Either take a model with known symmetries and try to code the commutation relations into something of the form (1.3). This, for example, has been done for the reduced BCS model stemming from the study of superconductivity [4], whose $su(2)$-symmetry was the key in proving its integrability [32]. Alternatively, we may ask ourselves: Based on (1.4), what are the restrictions on the form of R, and furthermore, what algebras do they give rise to? Could these algebras be useful in modeling quantum interactions? It is the latter path that we will follow here, even though there is of course no guarantee that these structures will be useful for physical models.

There are many other areas where quantum algebras have been employed to better understand empirical phenomena, such as nuclear physics [39], but we shall not go into details at this point.

1.2 Quantum algebras for aesthetics

One may, of course, put aside the question of whether or not any physical relevance is to be found in such quantum algebras, and ask instead if their is anything intrinsic to them that would make their study a worthwhile occupation for the mathematician?

One possible way to answer this question is to point out their intricate connection to many different areas of mathematics, most importantly non-commutative geometry and knot theory [41]. Another answer invites us to consider the fundament on which those quantum algebras rest: Brick by brick, through loosened constraints or recombination, we move from the theory of Lie algebras (which, in itself, is already a beautiful framework with some surprising results) with finitely many generators to affine algebras with no such limitations. We go on to enlarge this algebra to a universal enveloping algebra, where generators appear not only in the form of monomes, but also in higher powers. Once this has been achieved, we begin to deform the Lie bracket (i.e. the defining relations of the algebra) by means of a generic, additional parameter. Although it is not at all self-evident, it turns out that there already exists a framework to describe these deformed, affine algebras, which goes by the name of Hopf algebras.

Among Hopf algebras, there is a subclass - the quasi-triangular ones - that have an extra, and rather particular, feature: They contain an invertible object, known as the universal $\mathcal{R}$-Matrix, satisfying an algebraic equivalent of eq. (1.4). Moreover, this $\mathcal{R}$-Matrix can be explicitly constructed, as we will show for the example of $\mathcal{U}_q(\hat{\mathfrak{g}})$. So rather than searching solutions for the 'star-triangle' equation (or generalizations thereof) by hand, we can now directly find numerical R-Matrices by simply choosing suitable representations for $\mathcal{R}$.

But it does not have to stop here: It is now known that one can deform said $\mathcal{R}$-Matrices in such a way as to yield yet another type of algebra, known as quantum elliptic algebras.

They contain a second deformation parameter, and in fact no longer remain within the boundaries laid out by Hopf algebras. What is most interesting about them is that these algebras originated from statistical mechanics [5, 6]. That there would be both a systematic way to construct them from 'simpler' quantum affine algebras **and** to include them in a unified framework[35] was beyond anyone's imagination. How non-intuitive this finding is will become more apparent when the notion of quasi-Hopf algebras is introduced later on. Regardless of precisely which motivation will ultimately compel the reader to continue beyond this point, then, we should provide her with a roadmap to avoid missing the bigger picture.

1.3 Structure of the thesis

The next chapter is dedicated the fix some notation and lay the groundwork for more advanced aspects. We therefore begin by presenting some familiar concepts from the study of Lie algebras, without attempting to give an exhaustive overview. These tools will be indispensable for later generalizations, such as affine and universally enveloping algebras. We will also encounter our first example of a quantum algebra, denoted $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$, in this chapter, and show it belongs to the much larger class of Hopf algebras.
The central equation in the study of quantum algebras is a matrix equality called the Yang-Baxter equation (YBE) with spectral parameter, which may be written down as

$$R_{12}(z/w)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(z/w) \tag{1.5}$$

with subscripts again indicating the tensor spaces in which these operators act. Perhaps the most fundamental insight, which can legitimately be viewed as having led to the birth of quantum groups, was the realization that any matrix solution $R(z)$ to (1.5) allows for the construction of an algebra that by virtue of the YBE is non-empty. This construction will be explained in detail, not only for $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$, but also for a second type of algebra we wish to consider in chapter 3. This other type, called the quantum elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, will be examined thoroughly, and its connection to $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ made explicit.
At the core of the thesis, however, is chapter 4, where the reader will first be confronted with quantum determinants. It is here were original contributions to the state of research can be found, when we will demonstrate that this determinant generates the center of the algebra. The last chapter summarizes our findings and provides an outlook into possible avenues for future research.

Chapter 2

A toolbox for quantum algebras

2.1 Basic notions: Lie algebras and their affine extensions, universal enveloping algebras

It is rather unusual that a theoretical physicist would stumble across deformed or, worse yet, dynamical algebras in her day-to-day work. By contrast, the simpler concept of Lie algebras, upon which these more general models are ultimately built, is arguably one of the most important pillars of modern physics. To keep reader dropout at a minimum, we will review some of their properties that are relevant in this context without trying to give an extensive presentation of vast subject that is Lie algebra theory. Subsequently, we will also introduce (affine) Kac-Moody and universal enveloping algebras; their features will be outlined only to the extent that they contribute to understanding the following sections. This section was inspired by [40] and [51].

2.1.1 Lie algebras, roots and basis

Definition 2.1.1 *A* ***Lie algebra*** $\mathfrak{g}$ *is a (finite-dimensional) vector space over a field* F *together with a Lie bracket* $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ *that is bilinear, antisymmetric and satisfies the Jacobi identity*

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad \forall x,y,z\in\mathfrak{g}$$

We will always assume $F=\mathbb{C}$ in the following and not explicitly use the qualifier 'complex'. Since it is a vector space, we can choose a basis $\mathcal{B}=\{t^a\}$ and simply define the Lie bracket on these *generators*. Bilinearity then ensures that it is determined for all elements of $\mathfrak{g}$. The defining relation involve structure constant f^{ab}_c that may equally be used to characterize the algebra:

$$[t^a,t^b]=f^{ab}_c t^c$$

The following subsets of Lie algebras will be useful for subsequent sections.

Definition 2.1.2 *(i) A (Lie)* ***subalgebra*** $\mathfrak{h}$ *of* $\mathfrak{g}$ *is a Lie algebra with* $\mathfrak{h}\subseteq\mathfrak{g}$ *that is closed under the application of the Lie bracket, i.e.* $[x,y]\in\mathfrak{h}$ *for* $x,y\in\mathfrak{h}$*.*

(ii) $\mathfrak{h} \subseteq \mathfrak{g}$ *is called an* ***ideal*** *iff* $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

(iii) The ***center*** *of an algebra is the set* $\{x\}$ *of all* $x \in \mathfrak{g}$ *such that* $[x, y] = 0\ \forall y \in \mathfrak{g}$.

Equipped with these definition, we introduce three particular classes of Lie algebras:

Definition 2.1.3 *(i) A Lie algebra that is commutative, i.e. satisfies* $[x, y] = 0\ \forall x, y \in \mathfrak{g}$*, is called* ***abelian****.*

(ii) If $\mathfrak{g}$ *is not abelian and contains only the two ideals* $\{0\}$ *and* $\mathfrak{g}$*, it is said to be* ***simple****.*

(iii) A Lie algebra is called ***semisimple*** *if* $\{0\}$ *is its only abelian ideal*

Remark. For Lie algebras, property (*iii*) can be shown to be equivalent to the statement that a Lie algebra is semisimple if it is the direct sum of simple Lie algebras; however this does not hold, for example, for so-called superalgebras [40].
From now on, we will restrict our attention to (complex) semisimple Lie algebras. For these, there is an easy way to define what is called a *Cartan subalgebra*, which we will need for later constructions.

Definition 2.1.4 *For a semisimple Lie algebra* $\mathfrak{g}$*, an element* $x \in \mathfrak{g}$ *is called* ***semisimple*** *if there exists a set of generators* $\{t^a\}$ *such that* $[x, t^a] \propto t^a$*. A* ***Cartan subalgebra*** $\mathfrak{h} \subset \mathfrak{g}$ *is a maximal abelian subalgebra such that all* $y \in \mathfrak{h}$ *are semisimple.*

'Maximal' simply means that there is no abelian subalgebra $\mathfrak{h}' \supsetneq \mathfrak{h}$ with the same properties. To avoid confusion later on, we remark that a simple Lie algebra is also semisimple, but not vice versa.

Cartan basis

The Cartan subalgebra is commonly characterized by a (maximal) set of linearly independent generators $H_i, i = 1, \ldots, r$ satisfying $[H_i, H_j] = 0$. As it turns out, all Cartan subalgebras are related by automorphisms, so that the number r of Cartan generators is in fact a property of the Lie algebra, and will be called the **rank**.
What is important for us here is that Cartan was able to show that for every simple Lie algebra, there exists a basis with r commuting generators H_i, and $(\dim \mathfrak{g} - r)$ generators E_α that are eigenvectors of the H_i under the action of the Lie bracket:

$$
\begin{aligned}
[H_i, H_j] &= 0 & \qquad [E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta} \\
[H_i, E_\alpha] &= \alpha^i E_\alpha & \qquad [E_\alpha, E_{-\alpha}] &= \sum_{i=1}^{r} \alpha^i H_i \, ,
\end{aligned}
$$

Here, for the sake of completeness, we also included the exchange relations among the E_α. $N_{\alpha\beta}$ will depend on the exact type of Lie algebra under investigation. Let $\alpha := (\alpha_1, \ldots, \alpha_r)$

be the vector in $\mathbb{R}^r$ formed from all eigenvalues to the Cartan generators H_i, then α is what is known as the *root* associated to the generator E_α. More on this below.

Example. A Cartan-Weyl basis spanning the special linear Lie algebra $\mathfrak{sl}_2$ is given by three generators H, E and F and the commutation relations $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$. The smallest non-trivial representation (more on this later) is given by setting

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

with the Lie bracket defined as the standard commutator of matrices, $[X, Y] = XY - YX$.

Root systems

Let us extend this already long list of definitions by yet a few more entries. Time and again, we will use the concept of *roots* to cast light on more intriguing algebras. Here is how to do it:

Definition 2.1.5 *Consider an element $\lambda \in \mathfrak{h}^*$ in the dual space of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, which is a linear functional on $\mathfrak{h}$. Define a subspace $\mathfrak{g}_\lambda \subset \mathfrak{g}$ by*

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} : [H, X] = \lambda(H)X \textit{ for all } H \in \mathfrak{h}\}. \tag{2.1}$$

*If $\lambda \neq 0$ and $\mathfrak{g}_\lambda$ is nonzero, we call λ a **root**, with $\mathfrak{g}_\lambda$ being its corresponding **root space**. The dimension of $\mathfrak{g}_\lambda$ is called the **multiplicity** of λ. Finally, the set of all λ is called a **root system** and will be referred to through Δ, where* $\dim(\Delta) = \dim(\mathfrak{h})$.

From definition 2.1.4, we can easily see that $\mathfrak{g}_0 = \mathfrak{h}$. Furthermore, it follows from the definition of the root space that $\mathfrak{g}$ can be written as a direct sum:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda$$

None of this might seem spectacular. But as it turns out, root systems are not just an interesting property of (semisimple) Lie algebras; they provide an elegant way to fully classify them. How so?
The key insight is that we can equip the root system with an Euclidean structure, that is, there exists a unique inner product $(\cdot|\cdot)$ (symmetric and bilinear) on the roots [40]. This inner product, in turn, is used to generate the *Cartan matrix*, which is the building block for *Dynkin diagrams* - a graphical representation of the root system that allows to judge if two Lie algebras are isomorphic. Given that we will only be interested in a very special class of Lie algebras (mostly of the general linear and special linear variety), we will refrain from explicitly constructing the geometric structure and simply state the major results.

Proposition 2.1.6 *Let $\{\lambda_1, \dots, \lambda_r\}$ be a basis of the root system Δ. Any $\lambda \in \Delta$ can be written as $\lambda = \sum^r a_i \lambda_i$, and we call this root*

(i) ***positive*** *if the first non-vanishing coefficient in the sum is positive; one writes $\lambda > 0$,*

(ii) ***negative*** *if the first non-vanishing coefficient is negative; we write $\lambda < 0$.*

(iii) We can define a ***lexicographical ordering*** *on Δ: For $\lambda, \gamma \in \Delta$, $\lambda > \gamma$ if $\lambda - \gamma$ is positive.*

The subsystem of positive (negative) roots will be labeled Δ^+ (Δ^-). Finally, there exists a subset $\Delta^0 \subset \Delta^+$ called a ***simple root system****, which consists of all positive roots that cannot be written as a linear combination of other positive roots. The number of simple roots corresponds to the rank of the algebra.*

Once we have an ordering and established the notion of a simple root system, we should take a closer look at its Euclidean structure. To this end, we first introduce the *Killing form* $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, which is characterized by the following properties (the object c appearing here is called a central extension, as it commutes with all generators):

- Symmetry: $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in \mathfrak{g}_c := \mathfrak{g}_L \oplus \mathbb{C}c$
- Bilinearity: $\kappa(x + \lambda y, z) = \kappa(x, z) + \lambda\kappa(y, z)$ for $x, y, z \in \mathfrak{g}_c$ and $\lambda \in \mathbb{C}$
- Invariance: $\kappa(x, [y, z]) = \kappa([x, y], z)$

For the case of a simple, finite Lie algebra, it is an established fact that

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}x \circ \mathrm{ad}y)$$

where $\mathrm{ad}x(\cdot) = [x, \cdot]$ is the adjoint map.
The Killing form is important to define a proper inner product, given that it is unique up to a multiplicative constant. We set $(\alpha|\beta) := \kappa(H_\alpha, H_\beta)$ for any two roots α and β and use this definition to introduce

Definition 2.1.7 *Given a simple root system $\Delta^0 = \{\lambda_1, \dots, \lambda_r\}$, the* ***Cartan matrix*** *is the $r \times r$ matrix A with entries*

$$A_{ij} = 2\frac{(\lambda_i|\lambda_j)}{(\lambda_j|\lambda_j)}.$$

It has the following properties:

(i) $A_{ii} = 2$ and $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$.

(ii) For $i \neq j$, A_{ij} are negative integers (unless $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$)

(iii) Any Cartan matrix can be symmetrized, that is, there exists a set of relatively coprime integers (satisfying $d_i A_{ij} = d_j A_{ji}$) that give rise to $A_{ij}^{sym} = d_i A_{ij}$. The symmetrized matrix thus obtained is positive-definite.

One can in fact show that the entries A_{ij} can only assume values in $\{2, 0, -1, -2, -3\}$ [61].

Example. For the special linear Lie algebra $\mathcal{A}_{N-1} = \mathfrak{sl}_N$ (which has dimension $N^2 - 1$ and rank $N - 1$), we can write down the root system as $\Delta = \{\epsilon_i - \epsilon_j\}$, where $1 \leq i \neq j \leq N$ and $(\epsilon_i|\epsilon_j) = \delta_{ij}$. The positive roots are those for which $j > i$, and the simple root system is usually written as $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{N-1} = \epsilon_{N-1} - \epsilon_N$. Starting with the latter, it is easy to determine the Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

As promised earlier one, we will now take the Cartan matrix to construct the Dynkin diagrams, which is an intuitively appealing way to classify simple Lie algebras (and hence semisimple Lie algebras, which are nothing but the direct sum of simple ones). The algorithm goes as follows:

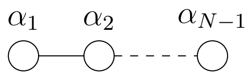
- For every simple root, draw a circle like this: $\bigcirc$
- Simple roots are connected by lines depending one the angle between them. More precisely, define $\cos^2 \Theta_{ij} : = \frac{A_{ij} \cdot A_{ji}}{4} = \frac{(\alpha_i|\alpha_j)^2}{(\alpha_i|\alpha_i)(\alpha_j|\alpha_j)}$, and draw $0, 1, 2$ or 3 lines between the roots according to the following scheme:

$\cos^2(\Theta_{ij})$	0	$1/4$	$1/2$	$3/4$
Θ_{ij}	$90°$	$60°$	$45°$	$30°$
Number of lines	0	1	2	3

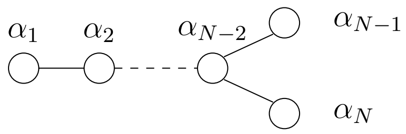
- If there is more than one line between two simple roots, we draw an arrow pointing from the longer to the shorter one, where the length of a root is obviously determined by the scalar product $(\cdot|\cdot)$.

Even though we will not need the full classification scheme to proceed, it is worth pointing out that Lie algebras fall in five different categories: Special unitary ($\mathfrak{sl}(N)$), two types of special orthogonal ($\mathfrak{so}(2N)$ and $\mathfrak{so}(2N+1)$), symplectic ($\mathfrak{sp}(2N)$) and a number of isolated examples know as G_2, F_4, E_6, E_7 and E_8.

Example. The Dynkin diagram for $\mathcal{A}_N$ looks deceivingly simple:



A bit more involved is the Dynkin diagram for the special orthogonal Lie algebra $\mathcal{D}_N \cong \mathfrak{so}(2N)$:



This would be a remarkable discovery in its own right, but the usefulness of these diagrams runs even deeper. Spoiler alert: Modifying them just a little bit (talk about *extended Dynkin diagrams*) will be enough to capture the properties of *affine Lie algebras*, an 'infinite' version of ordinary Lie algebras that will be introduced below.

Representation theory: Some remarks

Up to now, all that was said applied to abstracts elements of an algebra. By contrast, in physics, we are often interested in the linear action of a group on elements of a vector space. Hence, **representations** of Lie algebras are of vital interest for any physicist. There is no need to examine the nuances of representation theory[1], but a few remarks are in place.

Definition 2.1.8 *Let $\mathfrak{g}$ be a Lie algebra and V a (finite) vector space. A Lie algebra homomorphism $\pi\colon \mathfrak{g} \to gl(V)$ is called a (finite-dimensional)* ***representation*** *of $\mathfrak{g}$. If π is injective, the representation is called* ***faithful****.*

For anyone who came to know Lie algebras mainly as the underlying structure of certain classes of matrices (traceless, symplectic etc), this might seem tantamount to breaking a butterfly on a wheel: Are we trying to represent matrices through matrices? There are several ways to answer this question; for our purposes, one is especially relevant: We will usually define an algebra through abstract commutation relations without any reference to matrices, and it is here where the grain of truth contained in the old (and probably apocryphal) adage 'No calculation without representation' becomes apparent.

Definition 2.1.9 *Let $\mathfrak{g}$, V and π be as before. A subspace $W \subset V$ is called* ***invariant*** *if $\pi(x)w \in W$ for all $w \in W, x \in \mathfrak{g}$. If a representation has no non-trivial subspaces (invariant subspaces other than $\{0\}$ and V), it is called* ***irreducible***

Example. Given an element X of a Lie algebra $\mathfrak{g}$, one defines $\mathrm{ad}\colon \mathfrak{g} \to gl(\mathfrak{g})$ by setting $\mathrm{ad}(X)Y = [X, Y]$. This is called the **adjoint representation**.

[1] The interested reader can find those, for example, in [60]

Definition 2.1.10 *For a (complex) representation $\pi : \mathfrak{g} \to gl(V)$, the subspace V_λ given by*

$$V_\lambda := \{v \in V | \forall\, H \in \mathfrak{h}\,, \pi(H)v = \lambda(H)v\} \tag{2.2}$$

where $\mathfrak{h}$ is the Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$, and λ is a linear functional on $\mathfrak{h}$, is called the ***weight space*** *of V associated to the* ***weight*** *λ. If the weight space is non-zero, an element $v \in V_\lambda$ is called a* ***weight vector****. Because we are looking at the Cartan subalgebra, v is a simultaneous eigenvector for the action of elements from $\mathfrak{h}$ with eigenvalues given by the elements of λ.*

The attentive reader might have already noticed a striking similarity with the definition of root spaces and roots (cf. eq. 2.1). In fact, if we look at the special case of the adjoint representation, non-zero weights *are* the roots previously introduced. This connection allows us to define what is known as a *fundamental weight*, which we need later to understand the structure of elliptic algebras.

Definition 2.1.11 *Let Δ^0 be the set of simple roots on $\mathfrak{g}$, and define the* ***coroot*** *associated to*[2] *$\alpha \in \Delta^0$ by*

$$H_\alpha := \frac{2}{(\alpha|\alpha)}\alpha.$$

The ***fundamental weights*** *$\omega_1, \omega_2, \ldots$ are a basis of a subspace $\mathfrak{h}_0 \subset \mathfrak{h}$ dual to the set of coroots obtained from the simple roots of $\mathfrak{g}$, i.e. they satisfy*

$$2\frac{(\omega_i|\alpha_j)}{\alpha_j|\alpha_j)} = \delta_{ij}.$$

Here, $\alpha_j \in \Delta^0$, and $(\cdot|\cdot)$ is the inner product we already encountered multiple times before.

For the rest of the thesis, we will often use curvilinear letters (such as $\mathcal{R}$) to refer to abstract algebraic objects, and regular font (or an expression involving the representation map, as in $\pi(\mathcal{R})$) for their concrete representations. We hope this will help to avoid unnecessary confusion.

Serre-Chevalley basis

With all this machinery, we should now revisit the notion of a basis again. As we saw before, the Cartan-Weyl basis is intuitively appealing because of its explicitness, but it is not the only type of basis used in actual calculations. Nor is it necessarily the most effective one - as we shall see later on, when dealing with an infinite number of generators, there are more elegant ways to go about one's business. The leading contender bears the name *Serre-Chevalley basis*, and takes the Cartan matrix as its point of origin.

[2]In fact, this definition is valid not just for simple, but for any kind of root.

Definition 2.1.12 *Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{h} = \{h_1, \ldots, h_r\}$ its Cartan subalgebra. Denote the Cartan matrix by (A_{ij}), and let $\Delta^0 = \{\alpha_1, \ldots, \alpha_r\}$ be a simple root system with corresponding generators $\{e_i\}, \{f_i\}$. The **Serre-Chevalley basis** is defined by the following relations between the $3r$ generators:*

$$\begin{aligned} [h_i, h_j] &= 0 & \qquad [h_i, f_j] &= -A_{ij} f_j \\ [h_i, e_j] &= A_{ij} e_j & \qquad [e_i, f_j] &= \delta_{ij} h_i \end{aligned}$$

*In addition, we have the **Serre relations***

$$(ad\ e_i)^{1-A_{ij}}\, e_j = 0 \qquad (ad\ f_i)^{1-A_{ij}}\, f_j = 0, \tag{2.3}$$

where $(ad\ e_i)(e_j) = [e_i, e_j]$. They should be understood as a way to generate the additional elements necessary to form a basis.

Remark. For the simple generators, the difference between Cartan-Weyl and Serre-Chevalley is just a different normalization:

$$h_i = \frac{2}{(\alpha_i|\alpha_i)} H_{\alpha_i}\,, \qquad [e_i, f_i] = h_i$$

Note how we use caps with the root as a subscript for Cartan-Weyl, but lower case letters and numerical indices for Serre-Chevalley.

Example. For $\mathfrak{sl}_2$, we see no new structures (compared to Cartan-Weyl). Scaling it up by one dimension to $\mathfrak{sl}_3$, however, we get a first set of non-trivial Serre relations:

$$[e_1, [e_1, e_2]] = [e_2, [e_2, e_1]] = [f_1, [f_1, f_2]] = [f_2, [f_2, f_1]] = 0$$

There is visibly some redundancy in here, only two of them provide new information: In particular, they tell us that $[e_1, e_2]$ and $[f_1, f_2]$ are nonzero, so that we get $3 \times 2 + 2 = 8$ generators in total (which is also the dimension of $\mathfrak{sl}_3$, as it should be).
The advantage of the Serre-Chevalley presentation will become obvious in the next section(s).

2.1.2 (Affine) Kac-Moody and universal enveloping algebras

Look back at the definition of the Cartan matrix in (2.1.7). Viewed in isolation, the requirements for it seem to come out of the blue, and one might well wonder what happens if these conditions are relaxed a bit. What, for example, happens if we drop the part about the symmetrized matrix having to be positive-definite? It turns out that this gives us access to a much larger class of algebras, called *Kac-Moody algebras* after their discoverers. Kac-Moody algebras are labeled finite, affine and indefinite, depending on the form of the Cartan matrix, and we will henceforth only consider the affine case.

Definition 2.1.13 *Let (A_{ij}) be a complex $(r+1) \times (r+1)$ matrix of rank r subject to the following conditions:*

- *All diagonal entries $A_{ii} = 2$.*
- *The off-diagonal-entries are negative integers, with the exception of $A_{ij} = 0 \leftrightarrow A_{ji} = 0$*
- *The matrix is symmetrizable, and the symmetrized matrix is positive-semidefinite.*

The algebra built from this matrix (cf. the Serre-Chevalley presentation) is called an ***affine Kac-Moody algebra****.*

This might not look like a revolutionary leap, but in fact, the implications are profound. Most importantly, algebras obtained through this procedure are no longer finite (they possess infinitely many generators), as we shall see below. What's more, having studied finite Lie algebras and their classification through Dynkin diagrams, we can carry over a lot of the insights obtained in those simpler cases.
In practice, we will be interested in affine Kac-Moody algebras obtained from a Lie algebra $\mathfrak{g}$, which we will denote by $\widehat{\mathfrak{g}}$. Starting from the Cartan matrix of a simple Lie algebra, the construction of *extended Cartan matrix* is by no means arbitrary, but instead follows a clear trajectory.
Recall that every positive root α of a simple Lie algebra can be written as a linear combination of simple roots, i.e. $\alpha = \sum a^i \alpha_i$ for $\alpha_i \in \Delta^0$. The corresponding sum $h_\alpha : = \sum a^i$ is called the *height* of the root, and there exists a unique root $-\alpha_0 = \sum b^i \alpha_i$ such that α_0 has maximal height [40]. The extended Cartan matrix is built from the root system $\widehat{\Delta}^0 = \Delta^0 \cup \{\alpha_0\}$. It satisfies the requirements listed in (2.1.13), and is thus an affine Kac-Moody algebra.
Unsurprisingly, another concept also carries over to the affine case: Just as there is an extended Cartan matrix, so do we have extended Dynkin diagrams. They work very much in the same way, the only difference being that we can now have up to four lines connecting two dots (i.e. when $A_{ij}A_{ji} = 4$).

Example. The Kac-Moody algebras $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sl}}_3$ are encapsulated in the following matrices (the last row and column correspond to the extension by the highest root):

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{2.4}$$

Instead of the hat notation, you will also find them referred to as $A_1^{(1)}$ and $A_2^{(1)}$, respectively[3]. They have the following Dynkin diagrams:

[3] This notation is to be preferred when it comes to classifying affine Lie algebras. The letter and subscripts stem from the classification of simple Lie algebras, and the superscript indicates that there are other matrices of the same rank satisfying the condition (2.1.13). Our construction leads to *untwisted* affine algebras, while we will not touch upon the so-called *twisted* affine algebras.

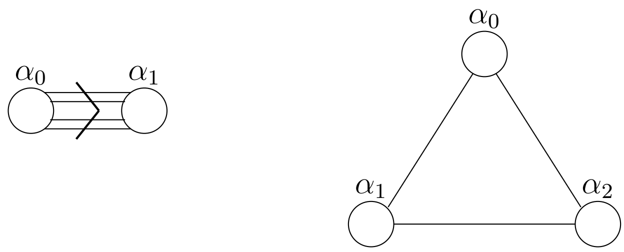


Figure 2.1: The extended Dynkin diagrams of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sl}}_3$, respectively.

Let us now show how a Cartan matrix such (2.4) can lead to an infinite number of generators by studying the Serre relations (2.3), turning the implicit Serre-Chevalley formulation into the explicit Cartan-Weyl presentation. In order to be able to graphically display the result, we will do so for the affine algebra $\widehat{\mathfrak{sl}}_2$.

Example. The 2×3 generators associated to the two simple roots α_0 and α_1 will be called $e_i \sim E_{\alpha_i}, f_i \sim F_{\alpha_i}$ and h_i, where $i = 0, 1$. From the Serre relations (2.3), we find that (focusing only on the positive part here)

$$[e_0, [e_0, [e_0, e_1]]] = [e_1, [e_1, [e_1, e_0]]] = 0.$$

The two simple roots will henceforth be denoted by $\alpha_0 = \delta - \alpha$ and $\alpha_1 = \alpha$, where δ is sometimes called the imaginary direction, stemming from the labeling of roots of the form $n\delta$ as *imaginary*[4]. We remind the reader that whenever $[E_\alpha, E_\beta] \neq 0$, then $\alpha + \beta$ is a root. Since the commutator associated to α_0 and α_1 satisfies $[E_{\delta-\alpha}, E_\alpha] \neq 0$, we know that $\alpha_0 + \alpha_1 = \delta \sim E_\delta$ is a root. We continue:

- $E_{2\delta-\alpha} := [E_{\delta-\alpha}, [E_{\delta-\alpha}, E_\alpha]] \neq 0 \Rightarrow 2\alpha_0 + \alpha_1 = 2\delta - \alpha$ is a root
- $E_{\delta+\alpha} := [E_\alpha, [E_{\delta-\alpha}, E_\alpha]] \neq 0 \Rightarrow \delta + \alpha$ is root

However, not all commutators are nonzero:

- $[E_{\delta-\alpha}, E_{2\delta-\alpha}] = [E_{\delta-\alpha}, [E_{\delta-\alpha}, [E_{\delta-\alpha}, E_\alpha]]] = 0 \Rightarrow 3\delta - 2\alpha$ is not a root
- $[E_\alpha, E_{\delta+\alpha}] = [E_\alpha, [E_\alpha, [E_\alpha, E_{\delta-\alpha}]]] = 0 \Rightarrow \delta + 2\alpha$ is not a root

By iterating this algorithm, one finds that the root system of $\widehat{\mathfrak{sl}}_2$ has the form

$$\Delta = \{m\delta \pm \alpha | m \in \mathbb{Z}\} \cup \{m\delta | m \in \mathbb{Z} \setminus \{0\}\},$$

which can also be depicted graphically as seen in fig. 2.2.

[4]In fact, a root α is simply called imaginary if the scalar product $(\alpha|\alpha) = 0$, an *real* if said product is strictly positive, i.e. non-degenerate.

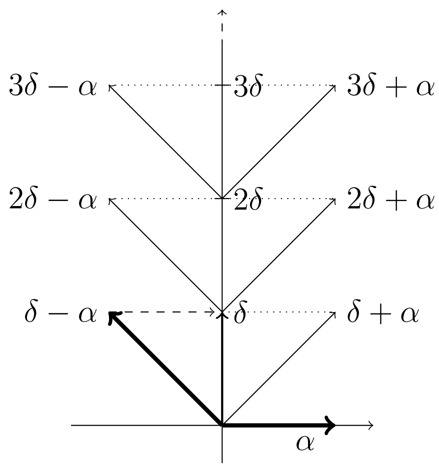


Figure 2.2: Construction of the (infinite) set of roots for $\widehat{\mathfrak{sl}}_2$. The algebra is said to be of *finite growth*, as for each level $n \sim n\delta(\pm\alpha)$, the number of generators varies as a power of of $|n|$.

Digression: A formal look at affine Kac-Moody algebras

Constructing affine algebras from the Serre-Chevalley basis is a very practical approach for the 'quantization' procedures we will introduce shortly hereafter. Nevertheless, it will also be helpful, indeed necessary, to formalize the concept a bit. Later on, when we roll out the Drinfel'd twistors, the purpose of this detour will become clearer. The approach relies on *extensions* of *loop algebras*, and provides an equivalent way to set up the algebra.

Definition 2.1.14 *Let $\mathfrak{g}$ be a (complex, semisimple) Lie algebra with generators t^a, $a = 1, \ldots, \dim \mathfrak{g}$ with the usual exchange relations*

$$[t^a, t^b] = f^{ab}_c t^c\,,$$

where we choose a basis in which the structure constants f^{abc} are totally antisymmetric (this is always possible). Let $z \in S^1$ be a complex variable on the unit circle, and consider the ring of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ with elements $\sum_{\mathbb{Z}} c_k z^k$. The ***loop algebra*** *$\mathfrak{g}_L$ of $\mathfrak{g}$ is the tensor product*

$$\mathfrak{g}_L = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$$

with natural basis $T^a_m := z^m \otimes T^a, m \in \mathbb{Z}$ and exchange relations

$$[T^a_m, T^b_n] = f^{ab}_c T^c_{m+n}.$$

Moreover, it satisfies the Jacobi identity

$$[T^a_k, [T^b_m, T^c_n]] + [T^b_m, [T^c_n, T^a_k]] + [T^c_n, [T^a_k, T^b_m]] = 0.$$

Turning $\mathfrak{g}$ into a loop algebra, however, will not be enough. In fact, going back to our affine Cartan matrix (def. 2.1.13), we can define what one calls *dual Coxeter labels* $\alpha_i^\vee$, which are the entries of a (right) eigenvector of the Cartan matrix with eigenvalue 0, i.e. $0 = \sum_{j=1}^r A^{ij}\alpha_i^\vee$. Let furthermore $\{h^i\}$ be a basis of the Cartan subalgebra of $\mathfrak{g}$ and set $c := \sum_j a_j^\vee h_j$. Evidently, $[c, h_i] = 0$ for all generators of the Cartan subalgebra, and we also find

$$[c, e_i] = \sum_j a_j^\vee [h_j, e_i] = \sum_j a_j^\vee A_{ji} e_i = 0 \qquad [c, f_i] = -\sum_j a_j^\vee A_{ji} f_i = 0$$

Since every other generator is defined by iteration from the Serre-Chevalley triplet $\{e_i, f_i, h_i\}$, we can conclude $[c, \widehat{\mathfrak{g}}] = 0$, which means that c is in the center of the affine algebra $\widehat{\mathfrak{g}}$. It is therefore called the *central element*.

But such an element is not yet present in the loop algebra $\mathfrak{g}_L$, so we will have to introduce it by hand. Generally speaking, one can ask: Are there any non-trivial possibilities to deform the Lie bracket such as to obtain

$$[T_m^a, T_n^b] = f_c^{ab}\, T_{m+n}^c + \omega(T_m^a, T_n^b)\, c$$

by introducing an additional generator c into the exchange relations? The answer is yes, and if we want c to lie in the center of the algebra (which makes it a *central extension*), the Jacobi identity imposes tight constraints on the bilinear form $\omega(\cdot,\cdot)$. Up to a multiplicative constant, the solution is unique, and has been shown [61] to be of the form

$$[T_m^a, T_n^b] = f_c^{ab} T_{m+n}^c + m\, \delta_{m+n,0}\, \kappa(t^a, t^b) c, \tag{2.5}$$

with $\kappa(\cdot,\cdot)$ the Killing form, and obviously $[c, T_m^a] = 0$ by definition.

However, this creates a problem: As a direct consequence of the commutation relation (2.5), any element in the centrally extended loop algebra $\mathfrak{g}_c := \mathfrak{g} \bigoplus \mathbb{C}c$ can be written as the Lie bracket of two elements $x, y \in \mathfrak{g}_c$; we have indeed $[\mathfrak{g}_c, \mathfrak{g}_c] = \mathfrak{g}_c$. But then, the invariance condition tells us that

$$\kappa([x,y], c) = \kappa(x, \underbrace{[y,c]}_{=0}) = 0.$$

In other words, there exists a non-trivial element - the central element c - whose Killing form with all other elements in $\mathfrak{g}_c$ evaluates to zero: The Killing form is degenerate! Luckily, there is a remedy to it, which consists in adding a second extension. This generator is called the *derivation*[5] d, which, as per requirement, commutes with c, and also satisfies $[d, T_m^a] = mT_m^a$. If you recall the definition of T_m^a, you will see that d, in a sense, measures

[5]For the interested reader, we note that d can be represented by $z\frac{d}{dz}$.

the power of the complex variable z in the same way an ordinary derivative acts on a monomial. The complete, affine Kac-Moody algebra is thus given by

$$\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

and, because the derivation never appears on the right-hand side of any Lie bracket, we have $[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] = \mathfrak{g}_c$. Consequentially, the Killing form becomes non-degenerate, and we say that the *derived* algebra is smaller than $\widehat{\mathfrak{g}}$ itself.
We are just one concept short of moving away from the 'classical' case and considering quantum algebras. This concept is the *universal enveloping algebra*.

Definition 2.1.15 *Let $\mathfrak{g}$ be a Lie (or Kac-Moody) algebra of dimension N. Denote by $\mathfrak{g}^{\otimes} := \mathbb{C} \bigoplus \mathfrak{g} \bigoplus (\mathfrak{g} \otimes \mathfrak{g}) \bigoplus \ldots$ the tensor algebra over $\mathfrak{g}$, and consider an ideal $\mathcal{I}$ of $\mathfrak{g}^{\otimes}$ that is generated by $[x, y] - (x \otimes y - y \otimes x)$, where $x, y \in \mathfrak{g}$. The* ***universal enveloping algebra*** *$\mathcal{U}(\mathfrak{g})$ is defined as the quotient $\mathfrak{g}^{\otimes}/\mathcal{I}$, and its basis is given by the Poincaré-Birkhoff-Witt theorem as*

$$b_1^{i_1} \ldots b_N^{i_N} \quad \text{with} \quad i_1, \ldots, i_N \geq 0,$$

where $\{b_j\}$ is a basis of $\mathfrak{g}$.

Less formally speaking, the universal enveloping algebra is the largest algebra you get when you try to embed $\mathfrak{g}$ into an associative algebra $\mathcal{A}$ such that there is a one-to-one correspondence between the abstract Lie bracket $[\cdot, \cdot]$ and the commutator in $\mathcal{A}$. In the next section, we will see that we can naturally define what is called a co-product on $\mathcal{U}(\mathfrak{g})$, and thus equip it with the structure of a *Hopf algebra*.

2.2 Quantum deformations

Having learned about these algebraic notion, we can go one step further and investigate a first example of a deformed quantum algebra. But first, a word of warning: Although these structures are canonically referred to as *quantum groups*[6], this does not mean they play the same role in quantum field theories as do classical groups in classical field theories. It is only in a formal sense that the deformation of an algebra by a parameter q can be understood as a 'quantization' procedure, but we will not go into detail here [59].
What we wish to construct now is a quantum universal enveloping affine Kac-Moody algebra, which we will denote by $\mathcal{U}_q(\widehat{\mathfrak{g}})$. Because that is an awfully long name, we will just call it *quantum affine algebra* from now on.

[6]The term 'quantum group' is bound to create confusion. Not only is the 'group' part of it misleading, but even the literature has not settled for a generally accepted and non-ambiguous meaning. What is usually meant by quantum groups are quasi-triangular Hopf algebras (as introduced below), and we will stick to this convention.

Definition 2.2.1 *Let $\mathbb{C}[q]$ be the ring of rational functions in the indeterminate q. The* ***quantum affine algebra*** *$\mathcal{U}_q(\widehat{\mathfrak{sl}}_N)$ is defined as the unital algebra over $\mathbb{C}[q]$ with generators $k_i^{\pm}$, e_i and f_i, where $0 \leq i \leq r$ (corresponding to the number of simple roots), subject to the exchange relations*

$$\begin{array}{lll} \left[k_i^{\pm}, k_j^{\pm}\right] = 0 & k_i^+ k_i^- = k_i^- k_i^+ = 1 & \\ k_i^{\pm} e_j = q^{\pm A_{ij}^{sym}} e_j k_i^{\pm} & k_i^{\pm} f_j = q^{\mp A_{ij}^{sym}} f_j k_i^{\pm} & [e_i, f_j] = \delta_{ij} \dfrac{k_i^+ - k_i^-}{q - q^{-1}}. \end{array}$$

For $i \neq j$, we have two more relations (called ***q-Serre relations****), namely*

$$(ad_q e_i)^{1-A_{ij}}(e_j) = 0 \qquad (ad_q f_i)^{1-A_{ij}}(f_j) = 0. \tag{2.6}$$

Again, the Cartan matrix and its symmetrized cousin, (A_{ij}) and (A_{ij}^{sym}), appear, and we introduced the q-adjoint $(ad_q e_i)(e_j) = e_i e_j - q^{A_{ij}^{sym}} e_j e_i$. These relations can be understood as the q-deformed equivalent of the Serre relations[7] (2.3).

To touch base with earlier results, we should remark here that by writing $k_i^{\pm} \to q^{\pm h_i}$ and taking the 'classical' limit $q \to 1$, we recover the algebra $\widehat{\mathfrak{g}}$ in the Serre-Chevalley presentation (2.1.12). But we should be cautious here; even though we use the same expressions for the generators of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ and $\widehat{\mathfrak{g}}$, they are manifestly not the same.

2.2.1 Hopf structure for $\mathcal{U}_q(\widehat{\mathfrak{g}})$

The algebraic structure of the just-defined object is hardly in plain sight. As it turns out, the proper way to classify it is through so-called *Hopf algebras*. What is hidden behind this name?

Definition 2.2.2 *Given a unital associative algebra $\mathfrak{A}$ over $\mathbb{C}$, we equip $\mathfrak{A}$ with a co-algebra structure to generate a* ***Hopf (bi-)algebra****. More precisely, we have*

- *two algebra homomorphisms $\Delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ and $\epsilon\colon \mathfrak{A} \to \mathbb{C}$, called the co-product and co-unit, respectively,*
- *two co-algebra homomorphisms*[8] *$m\colon \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ and $\iota\colon \mathbb{C} \to \mathfrak{A}$, called the product and unit,*
- *as well as an anti-homomorphism $S\colon \mathfrak{A} \to \mathfrak{A}$ called the antipode,*

[7] The reader might rightfully wonder why we cannot simply reabsorb the deformation back into the definition of the generators and recover the non-deformed case. Showing that the deformation is not trivial, and classifying non-equivalent deformations, requires arguments from the theory of Chevalley cohomologies, which we do not wish to present here. See [59] and the appendix of [45] for a detailed presentation.

[8] Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras as stipulated in the definition, then $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is a co-algebra homomorphism iff $\Delta_{\mathfrak{B}} \circ \Phi = (\Phi \otimes \Phi) \circ \Delta_{\mathfrak{A}}$ and $\epsilon_{\mathfrak{B}} \circ \Phi = \epsilon_{\mathfrak{A}}$.

which satisfy the following set of equations:

$$\begin{aligned}
&(id \otimes \Delta)(\Delta(x)) = (\Delta \otimes id)(\Delta(x)) \qquad \forall x \in \mathfrak{A} \\
&(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id \\
&m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \iota \circ \epsilon
\end{aligned}$$

The first of these equations is sometimes referred to as co-associativity.

There exists a subclass of Hopf algebras that is dubbed co-commutative: If σ is a map that flips the contributors to a tensor product as in $\sigma(x \otimes y) = y \otimes x$ for any pair $x, y \in \mathfrak{A}$, co-commutativity means that the opposite co-product $\Delta^{\text{op}} :\, = \sigma \circ \Delta = \Delta$. Even more important is yet another subclass:

Definition 2.2.3 *A Hopf algebra is called **quasi-triangular** if there exists an invertible element $\mathcal{R} \in \mathfrak{A} \otimes \mathfrak{A}$ that satisfies*

$$\begin{aligned}
&\Delta^{op}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \qquad \forall x \in \mathfrak{A} \\
&(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23} \\
&(id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}
\end{aligned} \tag{2.7}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes \mathbb{I}$ and so on. We will refer to this object as the universal $\mathcal{R}$-Matrix. It is not hard to show that it satisfies the famous Yang-Baxter equation (YBE) in $\mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A}$:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{2.8}$$

Furthermore, the R-Matrix satisfies

$$\begin{aligned}
&(\epsilon \otimes id)(\mathcal{R}) = (id \otimes \epsilon)(\mathcal{R}) = 1 \\
&(S \otimes id)(\mathcal{R}) = (id \otimes S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}.
\end{aligned}$$

It will not come as a surprise now that our quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ is indeed a quasi-triangular Hopf algebra (QTHA); which was not necessarily obvious when the algebra was first introduced in [25]. In the interest of full disclosure, here is the Hopf structure for $\mathcal{U}_q(\widehat{\mathfrak{g}})$, conveniently defined on the generators of the algebra:

$$\begin{aligned}
&\Delta(k_i^{\pm}) = k_i^{\pm} \otimes k_i^{\pm} && \Delta(e_i) = e_i \otimes 1 + k_i^{+} \otimes e_i && \Delta(f_i) = f_i \otimes k_i^{-} + 1 \otimes f_i \\
&S(k_i^{\pm}) = k_i^{\mp} && S(e_i) = -k_i^{-} e_i && S(f_i) = f_i k_i^{+} \\
&\epsilon(k_i^{\pm}) = \epsilon(e_i) = \epsilon(f_i) = 0 && \epsilon(1) = 1 &&
\end{aligned}$$

To convince yourself of that this set-up is consistent with the defining features of a Hopf algebra (c.f. (2.2.2)) is a matter of straightforward verification, and it is indeed always possible to equip a universal enveloping algebra with a Hopf structure. A trickier question to ask is this: How do we know $\mathcal{U}_q(\widehat{\mathfrak{g}})$ is also quasi-triangular? How do we find the universal $\mathcal{R}$-Matrix? The next section will therefore explain how the construction works explicitly, and also offer guidance for finding a concrete realization of $\mathcal{R}$.

2.2.2 Construction of the R-Matrix for $\mathcal{U}_q(\widehat{\mathfrak{g}})$

By now, it should have become clear why the Serre-Chevalley basis was introduced for affine algebras - the presentation is a lot more economical, since we only need the exchange relations between the finitely many generators associated to the simple roots plus the ($q-$)Serre relations (2.6). But if that is the case, you may ask, why bother introducing a second type of basis? The short answer to this question is that they are needed to construct the universal R-Matrix, the object that guarantees quasi-triangularity. Let us see how this can be done.

First, we will have to define a normal ordering among the positive roots: For $\alpha, \beta \in \Delta^+$, we will write $\alpha \prec \alpha + \beta \prec \beta$ iff $\alpha + \beta \in \Delta^+$ and there are no other positive roots α', β' whose sum gives[9] $\alpha + \beta$. Once the ordering is fixed, we can construct all other Cartan-Weyl generators by induction from the generators that belong to the simple roots $\alpha_0, \dots, \alpha_r$, as pioneered by [26]. Let $F_\alpha := E_{-\alpha}$, then we have

$$E_{\alpha+\beta} = [E_\alpha, E_\beta]_q := E_\alpha E_\beta - q^{(\alpha|\beta)} E_\beta E_\alpha \tag{2.9}$$

$$F_{\alpha+\beta} = [F_\beta, F_\alpha]_{1/q} := F_\beta F_\alpha - q^{-(\alpha|\beta)} F_\alpha F_\beta \tag{2.10}$$

where $\alpha \prec \alpha + \beta \prec \beta$, $(\cdot|\cdot)$ is the standard scalar product and $[\cdot, \cdot]_q$ is called the q-deformed commutator.

Next, we determine some commutation relations that flow into the definition of the R-Matrix. In particular, for a real root $\gamma = \sum_{i=0}^r n_i \alpha_i \in \Delta^+$,

$$[E_\gamma, F_\gamma] = \eta_\gamma \frac{k_\gamma^+ - k_\gamma^-}{q - q^{-1}}, \tag{2.11}$$

where we defined $k_\gamma^\pm = \prod_{i=0}^r k_i^{\pm n_i}$, and η_γ is a proportionality factor to be determined. We would like to extend this to generators belonging to imaginary roots $n\delta$, but this does not work with the generators obtained from (2.9). Instead, one has to introduce a new set of generators $\breve{E}_{n\delta}^{(i)}$, which, contrary to the 'real' generators, can have a multiplicity i greater than 1. If $E_{n\delta}$ are the generators calculated from the iteration (2.9), then the $\breve{E}_{n\delta}^{(i)}$ are derived from those using Schur polynomials (dropping the multiplicity index here):

$$\breve{E}_{n\delta} = \sum_{p_1 + 2p_2 + \dots + np_n = n} \frac{(q^{-1} - q)^{\sum p_i - 1} (\sum_{i=1}^n p_i - 1)!}{p_1! \dots p_n!} E_\delta^{p_1} \dots E_{n\delta}^{p_n}$$

The new generators satisfy

$$[\breve{E}_{n\delta}^{(i)}, \breve{F}_{m\delta}^{(j)}] = a_{ij}(n) \frac{(k_\delta^+)^n - (k_\delta^-)^n}{q - q^{-1}} \delta_{m+n,0},$$

[9]The attentive reader will note that this ordering is not unique, one could just as well inverse the order and write $\beta \prec \alpha + \beta \prec \alpha$. Discriminating in favor of one of them will not lead to a loss of generality.

where

$$a_{ij}(n) = \frac{q^{nA_{ij}^{\text{sym}}} - q^{-nA_{ij}^{\text{sym}}}}{n(q-q^{-1})}.$$

All the remaining exchange relations can be found in [26], but we will not need them to proceed.
We are now in a position to explicitly write down the form of the universal $\mathcal{R}$-Matrix. Let us define the inverse matrices $(c_{ij}(n)) = (a_{ij}(n))^{-1}$ and $(d_{ij}) = (A_{ij}^{\text{sym}})^{-1}$, and introduce the q-exponential

$$\exp_q(x) = \sum_{n\geq 0} \frac{x^n}{(n)_q!}, \qquad \text{where} \qquad (n)_q! := (1)_q(2)_q \dots (n)_q \qquad \text{and} \qquad (a)_q = \left(\frac{1-q^a}{1-q}\right).$$

Then the R-Matrix of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ can be expressed as

$$\mathcal{R}[\mathcal{U}_q(\widehat{\mathfrak{g}})] = \left(\prod_{\gamma\in\Delta^+}^{\longrightarrow} \widehat{\mathcal{R}}_\gamma\right)\mathcal{K}, \tag{2.12}$$

where the (infinite) product is carried out with respect to the chosen normal ordering. Apart from a multiplicative constant, it is the unique solution that satisfies the requirements of an R-Matrix as laid down in (2.7) [26]. The factors appearing in (2.12) have the following form:

$$\mathcal{K} = q^{\sum_{i,j} d_{ij} h_{\alpha_i} \otimes h_{\alpha_j}} \tag{2.13}$$

$$\widehat{\mathcal{R}}_\gamma = \exp_{q^{-(\gamma|\gamma)}}\left((q-q^{-1})\eta_\gamma^{-1} E_\gamma \otimes F_\gamma\right) \tag{2.14}$$

or, in case of imaginary roots,

$$\widehat{\mathcal{R}}_{n\delta} = \exp\left((q-q^{-1})\sum_{i,j} c_{ij}(n)\breve{E}_{n\delta}^{(i)} \otimes \breve{F}_{n\delta}^{(j)}\right).$$

The relationship between H_{α_i} and k_i is given by $k_i^{\pm} = q^{\pm H_{\alpha_i}}$.

The R-Matrix of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$

Formula (2.12) is a remarkable finding, but it does not come very handy. In fact, in its current form, it is not particularly useful for the journey ahead of us, especially not for the FRT formalism to be introduced below. But we have learned above that we can make such objects more calculation-friendly by choosing a suitable representation for the generators. For the special linear Kac-Moody algebra, the R-Matrix is known in full generality [18], but we will confine our demonstration to the case $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$.
There are two different two-dimensional (evaluation) representations π_z that are commonly used, the homogeneous and the principal gradation. Focusing on the latter, π_z acts on the

generators as (e_{ij} are the elementary matrices)

$$\begin{aligned}
\pi_z(e_1) &= \pi_z(E_\alpha) = ze_{12} & \pi_z(f_1) &= \pi_z(F_\alpha) = z^{-1}e_{21} \\
\pi_z(e_0) &= \pi_z(E_{\delta-\alpha}) = ze_{21} & \pi_z(f_0) &= \pi_z(F_{\delta-\alpha}) = z^{-1}e_{12} \\
\pi_z(h_1) &= \pi_z(H_\alpha) = e_{11} - e_{22} & \pi_z(h_0) &= \pi_z(H_{\delta-\alpha}) = e_{22} - e_{11}
\end{aligned} \tag{2.15}$$

The fundamental representation of the $\mathcal{R}$-Matrix is calculated as $R(z_1/z_2) = (\pi_{z_1} \otimes \pi_{z_2})\mathcal{R}$, and we will often not write down explicitly that we are working in a certain representation. For the generators associated to non-simple roots, we obtain the following expressions, using (2.9):

$$\begin{aligned}
\pi_z(E_{n\delta+\alpha}) &= (-1)^n \frac{z^{2n+1}}{q^n} e_{12} & \pi_z(F_{n\delta+\alpha}) &= (-1)^n \frac{z^{-(2n+1)}}{q^n} e_{12} \\
\pi_z(E_{(n+1)\delta-\alpha}) &= (-1)^n \frac{z^{2n+1}}{q^n} e_{21} & \pi_z(F_{(n+1)\delta-\alpha}) &= (-1)^n \frac{z^{-(2n+1)}}{q^n} e_{21} \\
\pi_z(E_{(n+1)\delta}) &= (-1)^n \frac{z^{2n+2}}{q^n} (e_{11} - q^{-2}e_{22}) & \pi_z(E_{(n+1)\delta}) &= (-1)^n \frac{z^{-(2n+2)}}{q^n} (e_{11} - q^{-2}e_{22})
\end{aligned}$$

The multiplicity of the imaginary roots is one, so we dropped the subscript here. This is not quite the end of the story, as we noticed before; we will still have to modify the 'imaginary' generators by means of the Schur polynomials. After the dust has settled, one finds

$$\pi_z(\breve{E}_{n\delta}) = (-1)^{n+1} \frac{z^{2n}}{n} [n]_q (e_{11} - q^{-2n}e_{22}) \qquad \pi_z(\breve{F}_{n\delta}) = (-1)^{n+1} \frac{z^{-2n}}{n} [n]_q (e_{11} - q^{-2n}e_{22})$$

with

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \, .$$

The matrix $\mathcal{K}$ in (2.13) is easy to calculate and evaluates to

$$\mathcal{K} = \operatorname{diag}\left(q^{1/2}, q^{-1/2}, q^{-1/2}, q^{1/2}\right) .$$

A bit more involved are the calculations for $\widehat{\mathcal{R}}_{n\delta\pm\alpha}$, for which we need to know the factor η_γ from the commutation relations (2.11), and the scalar product of two opposite roots. It turns out that $\eta_{n\delta\pm\alpha} = \frac{z^{2n\pm1}}{q^n}$, and $(n\delta \pm \alpha| - n\delta \mp \alpha) = 2$. We got lucky: The matrix inside the q-exponential is nilpotent, so that an otherwise infinite series breaks of after just two terms. For example,

$$\begin{aligned}
(\pi_{z_1} \otimes \pi_{z_2})(\widehat{\mathcal{R}}_{n\delta+\alpha}) &= \exp_{q^{-2}} \left((q - q^{-1}) \left(\frac{z_1}{z_2}\right)^{2n+1} e_{12} \otimes e_{21} \right) \\
&= \mathbb{I} + (q - q^{-1}) z^{2n+1} e_{23},
\end{aligned}$$

which is now a 4×4 matrix where we set $z = z_1/z_2$. We furthermore find

$$\begin{aligned}
(\pi_{z_1} \otimes \pi_{z_2})(\widehat{\mathcal{R}}_{n\delta-\alpha}) &= \mathbb{I} + (q - q^{-1}) z^{2n-1} e_{32} \\
(\pi_{z_1} \otimes \pi_{z_2})(\widehat{\mathcal{R}}_{n\delta-\alpha}) &= \exp\left(\operatorname{diag}(a_n, -q^{2n}a_n, -q^{-2n}a_n, a_n)\right) ,
\end{aligned}$$

where

$$a_n \equiv a_n(z) = \binom{z^{2n}}{n} \frac{1-q^{-2n}}{1+q^{-2n}}$$

To carry out the product (2.12), we need to fix the normal ordering. Starting with the simple roots α and $\delta - \alpha$, we set $\alpha \prec \delta \prec \delta - \alpha$, and all other roots are placed into this scheme in an iterative manner, such as to get

$$\alpha \prec \delta + \alpha \prec \cdots \prec \infty\delta + \alpha \prec \delta \prec 2\delta \prec \cdots \prec \infty\delta \prec \infty\delta - \alpha \prec \cdots \prec 2\delta - \alpha \prec \delta - \alpha.$$

The rest, then, is just a straightforward multiplication of (infinitely many) matrices that adds very little to our understanding. So we will confine ourselves to simply quoting the final result, which reads

$$R[\mathcal{U}_q(\widehat{\mathfrak{sl}_2})] = \rho_2(z^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta(z) & \gamma(z) & 0 \\ 0 & \gamma(z) & \beta(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.16}$$

where the entries are given as

$$\beta(z) = \frac{q(1-z^2)}{1-q^2z^2} \qquad \gamma(z) = \frac{z(1-q^2)}{1-q^2z^2} \tag{2.17}$$

and the normalization factor $\rho_2(z^2)$ is expressed through the general formula

$$\rho_N(z^2) = q^{\frac{1}{N}-1} \frac{(q^2z^2;q^{2N})_\infty (q^{2N-2}z^2;q^{2N})_\infty}{(z^2;q^{2N})_\infty (q^{2N}z^2;q^{2N})_\infty}. \tag{2.18}$$

The infinite products $(z,a)_\infty$ are defined in appendix A.
This result has been generalized by M. Jimbo first in the fundamental representation with homogeneous grading [18]. For the principal gradation and generic N, the resulting matrix reads

$$\begin{aligned} R^{(p)}(z) = \rho_N(z^2)\Big[&\sum_i e_{ii} \otimes e_{ii} + \frac{q(1-z^2)}{1-q^2z^2} e_{ii} \otimes e_{jj} \\ &+ \frac{z(1-q^2)}{1-q^2z^2} \left(\sum_{i<j} z^{(2j-2i-N)/N} + \sum_{i>j} z^{(2j-2i+N)/N} \right) e_{ij} \otimes e_{ji} \Big]. \end{aligned} \tag{2.19}$$

In the next chapter, we will see how to manipulate this to obtain the elliptic quantum algebra, but let us first understand the algebraic properties this matrix gives rise to.

2.2.3 FRT formalism

One of the pathbreaking discoveries in the study of quantum groups was made by what is known as the Leningrad school. Faddeev, Reshetikhin and Takhtajan (FRT) had already

done pioneering work on quantum integrable models when they noticed that the full potential of their work had not been realized yet [21]. The R-Matrix, an example of which we presented in (2.19), was the starting point for many exact solution of quantum models, but only many years later did they see that it could also be used to define an algebra.
The point of departure is called the Yang-Baxter equation, which we have already encountered in connection with the Hopf algebras (cf. def. 2.2.2). Starting from the YBE at the universal level, cf. eq. (2.8), we apply the evaluation map to $\mathcal{R}$ as in $\pi(z_1) \otimes \pi(z_2)\mathcal{R}_{12} = R_{12}(\frac{z_1}{z_2})$. For the universal YBE, the evaluation map is $\pi(z_1) \otimes \pi(z_2) \otimes \pi(z_3)$, and, discovering that only the ratios of the spectral parameters matter, we set $z_1/z_3 =: z$, $z_2/z_3 =: w$ and finally arrive at

$$R_{12}(z/w)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(z/w). \tag{2.20}$$

The key insight of FRT was that every matrix $R(z)$ that is a solution of (2.20) can be used as the building block of an algebra. Since it was proven long ago [18] that the R-Matrix (2.19) indeed satisfies the YBE, we are ready to try out this second way. To this end, we introduce generators $L^{\pm}_{ij}(z) = \sum_{k\geq 0} L^{\pm}_{ij}(\mp k)z^{\pm k}$ that are conveniently encapsulated in an $N \times N$ matrix:

$$L^{\pm}(z) = \begin{pmatrix} L^{\pm}_{11}(z) & \dots & L^{\pm}_{1N}(z) \\ \vdots & \ddots & \vdots \\ L^{\pm}_{N1}(z) & \dots & L^{\pm}_{NN}(z) \end{pmatrix} \tag{2.21}$$

By making these generators subject to the constraints (often referred to as *RLL*)

$$R_{12}(\frac{z}{w})L^{\pm}_1(z)L^{\pm}_2(w) = L^{\pm}_2(w)L^{\pm}_1(z)R_{12}(\frac{z}{w}) \tag{2.22}$$

$$R_{12}(q^c\frac{z}{w})L^{+}_1(z)L^{-}_2(w) = L^{+}_2(w)L^{-}_1(z)R_{12}(q^{-c}\frac{z}{w}), \tag{2.23}$$

it can be shown [23] that one thus recovers[10] $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$. Note that the *central charge* c appears here.
You may wonder: Is that all there is to it? No caveats? But to ask this question is to underestimate the power of the YBE: For one thing, since it depends on a spectral parameter, there is no general way to find all the solutions to the YBE, as it requires solving functional equations. And while it is certainly possible to find isolated solutions, extending the classification scheme pioneered by Skylanin and Kulish [11] based on the universal $\mathcal{R}$-Matrix proved to be an insurmountable obstacle. But more importantly, without the YBE, there is no guarantee that the relations (2.22) refer to an actual algebra, rather than just being formal definitions. However, by requiring $R(z)$ to satisfy (2.20), we know that at least one representation of $L^{\pm}(z)$ exists, and thus the algebra is non-empty, and we are ensured that it is well-defined.

[10] The FRT formalism defines algebras based on $\widehat{\mathfrak{gl}}_N$, rather than $\widehat{\mathfrak{sl}}_N$, so we need to divide out certain parts. In analogy to the classical case, this will be the job of the *quantum determinant*, which we will discuss in great detail later on.

Finally, the Hopf structure is given by [23] see article

$$\Delta L_{ij}^{\pm}(z) = \sum_{k=1}^{N} L_{kj}^{\pm}(zq^{\mp(1\otimes c/2)}) \otimes L_{ik}^{\pm}(zq^{\mp(c/2\otimes 1)}) \tag{2.24}$$

for the co-product, and the respective expressions for antipode an co-unit are $S(L^{\pm}(z)) = (L^{\pm}(z))^{-1}$ and $\epsilon(L^{\pm}(z)) = 1$.

Before moving on, let us again stress how the two approaches - the FRT formalism and what we may call the Serre-Chevalley method - differ. Recall that in the case of the latter, we start with the exchange relations among finitely many generators e_i, f_i and k_i associated to roots of the underlying Lie algebra ((2.2.1)). We furthermore specify the Serre-Chevalley relations for e_i and f_i through which all the other (infinitely many) generators are produced. Finally, we define the Hopf structure of the algebra, and explicitly construct the universal $\mathcal{R}$-Matrix based on the Serre-Chevalley generators.

By contrast, the FRT formalism turns things upside down. We start with a particular representation of the $\mathcal{R}$-Matrix and use it to define exchange relations between an infinite ensemble of generators, denoted as $L_{ij}^{\pm}(\mp k)$. As we have seen above, *any* solution to the YBE can be used to formally set up an algebra, and one might be tempted to think that this is the preferred way to do it - after all, it is a lot more straightforward and allows to define a much larger class of algebras (even though not all of them are necessarily of physical interest). However, the non-trivial part, even given a concrete solution to the YBE, is to figure out how the Hopf structure looks like for these new generators. And in fact, the next chapter shows that trying to do so may result in contradictions that just cannot be rectified.

Chapter 3

Plot twist: Making matters elliptic

If chapter 2 created the feeling that the affine quantum algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ is rather well understood, your intuition has done its job. The same is not necessarily true for a second example from this class, the elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, which is the subject of this chapter. Ideally, we would proceed by extending the already familiar FRT formalism using a different R-Matrix. So why do we not just figure out what this matrix should be, and then equip the generators with a Hopf structure in a straightforward way?

We shall in fact try to do exactly this. The relevant R-Matrix might seem to come a bit out of nowhere, as it stems from an altogether different context (Baxter's eight-vertex model from statistical mechanics [5]). Setting up the FRT formalism for it, we will understand that the concepts we introduced so far are not sufficient to do justice to the true nature of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, which, most importantly, is no longer a simple Hopf algebra. As we shall see, we arrive at the $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ algebra by twisting the universal $\mathcal{R}$-Matrix inherited from $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$. Only after we understand this construction can we begin attempting to represent $\mathcal{R}$, and we will do this for the special case $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$, to date the only example for which the explicit calculation succeeded. With our toolkit thus refined, we will explain why the R-Matrix introduced *ad hoc* in section 3.1 is almost certainly the right candidate to work with.

3.1 FRT formalism for $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$

As we saw in the preceding chapter, the starting point for the FRT formalism is the Yang-Baxter equation with spectral parameter (2.20): Any matrix that satisfies it provides the structure algebra, complete with (at least) one representation.What other types of algebras does the YBE offers?

To this date, the known solutions to the YBE with spectral parameter are either of the rational, trigonometric (such as $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$) or elliptic variety. The latter class is the one that will keep us busy for the rest of the thesis. The classification initiated by Kulish and Sklyanin [11] in 1982 has, in fact, not seen the discovery of any new types of solutions since their pioneering work.

Oddly enough, the source of inspiration for the study of quantum elliptic algebra comes from a complete unrelated domain, that of statistical mechanics. Back in the early 1970s, Baxter [6, 57] managed to solve what is known as the *eight-vertex model*: A square lattice model in which each state is represented by a configuration of arrows at a vertex, and periodic boundary conditions are imposed[1]. In the absence of external fields, his demonstration involves a symmetric matrix which is a solution to the Yang-Baxter equation. As we will shortly see, this matrix is similar[2] to the R-Matrix of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$ derived much later by Jimbo et al. in a purely algebraic context [35]. The model was later generalized to a $\mathbb{Z}_N$-symmetric matrix that still satisfies the YBE [10], and important properties of this matrix were established in [17, 19].
Let there be no doubt: There is no proof yet that the R-Matrices we will now present correspond to a representation of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ as defined in section 3.3 for $N > 2$. This should not produce too much irritation, though: the fact that for $N = 2$, the results from the quasi-Hopf formulation coincide with $R[\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)]$ obtained directly from the eight-vertex model, plus the fact that solutions to the YBE constitute an extremely restricted class, renders it rather unlikely that these R-Matrices will have an altogether different form. And finally, given that these matrices satisfy YBE, the FRT formalism introduced in the previous chapter allows us to define an algebra regardless, using RLL relations. Whether this is in fact a quasi-triangular quasi-Hopf algebra (QTQHA, see below) remains an open question, as this can only be said with certainty about Jimbo's construction [35] to be discussed later on.

In Belavin-Baxter parametrization, the matrix we inherit from statistical mechanics has the form

$$\mathcal{Z}(z,p,q) = z^{2/N-2}\frac{1}{\kappa_N(z^2)}\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\zeta,\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\xi+\zeta,\tau)}\sum_{(\alpha_1,\alpha_2)\in\mathbb{Z}_N\times\mathbb{Z}_N} W_{(\alpha_1,\alpha_2)}(\xi,\zeta,\tau)I_{(\alpha_1,\alpha_2)}\otimes I^{-1}_{(\alpha_1,\alpha_2)} \tag{3.1}$$

with normalization factor

$$\frac{1}{\kappa_N(z^2)} = \frac{(q^{2N}z^{-2};p,q^{2N})_\infty(q^2z^2;p,q^{2N})_\infty(pz^{-2};p,q^{2N})_\infty(pq^{2N-2}z^2;p,q^{2N})_\infty}{(q^{2N}z^2;p,q^{2N})_\infty(q^2z^{-2};p,q^{2N})_\infty(pz^2;p,q^{2N})_\infty(pq^{2N-2}z^{-2};p,q^{2N})_\infty}, \tag{3.2}$$

and the relations between the variables z, q, p and ξ, ζ, τ are merely a matter of exponentiation:

$$z = e^{i\pi\xi}, \quad q = e^{i\pi\zeta}, \quad p = e^{2i\pi\tau}$$

[1]There are eight allowed configurations, hence the name.

[2]Two R-Matrices R and R' are said to be *similar* if there is a non-degenerate operator $\mathcal{O}$, acting on the representation space V, such that $R' = (\mathcal{O}\otimes\mathcal{O})R(\mathcal{O}\otimes\mathcal{O})^{-1}$.

The functions $\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$ with $(\gamma_1, \gamma_2) \in \frac{1}{N}\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$ are known as Jacobi Theta functions with rational characteristics, and they are defined in appendix A. We introduce the $N \times N$ matrix g defined by

$$g_{ij} = \omega^i \delta_{ij}\,, \qquad 1 \leq i, j \leq N \quad \text{with} \quad \omega = e^{2i\pi/N} \tag{3.3}$$

and the $N \times N$ matrix h such that

$$h_{ij} = \delta_{i+1,j}\,, \qquad 1 \leq i, j \leq N. \tag{3.4}$$

Here, as in many other cases, the addition of indices should be understood modulo N. With these definitions, $I_{(\alpha_1,\alpha_2)}$ can be conveniently expressed as $I_{(\alpha_1,\alpha_2)} = g^{\alpha_2} h^{\alpha_1}$, while the functions $W_{(\alpha_1,\alpha_2)}$ are given by

$$W_{(\alpha_1,\alpha_2)}(\xi, \zeta, \tau) = \frac{\vartheta \begin{bmatrix} \frac{1}{2} + \alpha_1/N \\ \frac{1}{2} + \alpha_2/N \end{bmatrix} (\xi + \zeta/N, \tau)}{N\vartheta \begin{bmatrix} \frac{1}{2} + \alpha_1/N \\ \frac{1}{2} + \alpha_2/N \end{bmatrix} (\zeta/N, \tau)}$$

We will not be using the matrix (3.1), but one that is similar to it. In more precise language, set

$$R(z, p, q) = (g^{\frac{1}{2}} \otimes g^{\frac{1}{2}}) \mathcal{Z}(z, p, q) (g^{-\frac{1}{2}} \otimes g^{-\frac{1}{2}}). \tag{3.5}$$

For later purposes, we will also define yet another matrix

$$\tilde{R}_{12}(z) = \tau_N(q^{\frac{1}{2}} z^{-1}) R_{12}(z) \tag{3.6}$$

with a conversion factor

$$\tau_N(z) = z^{\frac{2}{N}-2} \frac{\Theta_{q^{2N}}(qz^2)}{\Theta_{q^{2N}}(qz^{-2})}$$

that satisfies $(\tau_N(q^N z) = \tau_N(z))$ and $(\tau_N(z)\tau_N(z^{-1}))$.

Now, the presentation in the Belavin-Baxter basis (3.1) is not always the handiest way to go about it. We may instead rewrite it by explicitly spelling out the exact form of the matrix entries [36]. To do so, we will take $R(z) = \sum R^{b,d}_{a,c}(z)\,(e_{a,b} \otimes e_{c,d})$ and specify

$$R^{b,d}_{a,c}(z) = \eta(z)\, S^b_{a,c}(z) \omega^{(a+c-b-d)/2} \delta_{a+c,b+d} \qquad \forall\, a, b, c = 1, \ldots, N \tag{3.7}$$

with indices understood modulo N, where

$$S^b_{a,c}(z) = z^{\frac{2(b-a)}{N}} q^{\frac{2(c-b)}{N}} p^{\frac{(b-a)(c-b)}{N}} \frac{\Theta_{p^N}(p^{N+c-a} q^2 z^2)}{\Theta_{p^N}(p^{N+c-b} z^2)\Theta_{p^N}(p^{N+b-a} q^2)} \tag{3.8}$$

which is manifestly $\mathbb{Z}_N$-symmetric, and defined for any index $c \in \mathbb{Z}$, since $S^b_{a,c+N}(z) = S^b_{a,c}(z)$ (and likewise for a and b). Recall that $\omega = e^{2\pi i/N}$, so that the factor $\omega^{(a+c-b-d)/2}$ necessarily evaluates to ± 1. Finally, the normalization factor is given by

$$\eta(z) = \frac{z^{\frac{2}{N}}}{\kappa_N(z^2)} \frac{(p^N, p^N)^3_\infty}{(p, p)^3_\infty} \frac{\Theta_p(q^2)\Theta_p(pz^2)}{\Theta_p(q^2 z^2)}. \tag{3.9}$$

Since the mid-1980s, it is known that the matrices (3.5) and (3.7) satisfy a number of remarkable properties [17, 19] the proof of which we will spare the reader at this point. Here is a comprehensive list, where starred bullet points are equally valid for $\tilde{R}(z)$ (and vice versa):

- Yang-Baxter-equation*:

$$R_{12}(z)R_{13}(w)R_{23}(w/z) = R_{23}(w/z)R_{13}(w)R_{12}(z)\,, \tag{3.10}$$

- Unitarity:

$$R_{12}(z)\,R_{21}(z^{-1}) = 1\,, \tag{3.11}$$

- Regularity (P_{12} is the permutation matrix):

$$R_{12}(1) = P_{12}\,, \tag{3.12}$$

- Crossing-symmetry:

$$R_{12}(z)^{t_2}\,R_{21}(z^{-1}q^{-N})^{t_2} = 1\,, \tag{3.13}$$

- Antisymmetry

$$R_{12}(-z) = \omega\,(g^{-1}\otimes\mathbb{I})\,R_{12}(z)\,(g\otimes\mathbb{I})\,, \tag{3.14}$$

- Quasi-periodicity

$$\tilde{R}_{12}(-zp^{\frac{1}{2}}) = (g^{\frac{1}{2}}hg^{\frac{1}{2}}\otimes\mathbb{I})^{-1}\,\tilde{R}_{21}(z^{-1})^{-1}\,(g^{\frac{1}{2}}hg^{\frac{1}{2}}\otimes\mathbb{I})\,, \tag{3.15}$$

- Invariance*:

$$(h\otimes h)\,R_{12}(z) = R_{12}(z)\,(h\otimes h)\,, \tag{3.16}$$

- Quasi-unitarity* (really just a consequence of (3.11) and (3.13))

$$\Big(R_{12}(x)^{t_2}\Big)^{-1} = \Big(R_{12}(q^N x)^{-1}\Big)^{t_2}\,. \tag{3.17}$$

The unitarity property for $\tilde{R}_{12}$ now reads

$$\tilde{R}_{12}(z)\,\tilde{R}_{21}(z^{-1}) = \tau_N(q^{\frac{1}{2}}z)\,\tau_N(q^{\frac{1}{2}}z^{-1}) \equiv \mathcal{U}(z), \tag{3.18}$$

where the function $\mathcal{U}(z)$ is defined as

$$\mathcal{U}(z) = q^{\frac{2}{N}-2}\,\frac{\Theta_{q^{2N}}(q^2z^2)\,\Theta_{q^{2N}}(q^2z^{-2})}{\Theta_{q^{2N}}(z^2)\Theta_{q^{2N}}(z^{-2})}\,. \tag{3.19}$$

Again, let us emphasize that $\tilde{R}_{12}(z)$ is the matrix that defines $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, while $R_{12}(z)$ was introduced purely for convenience.

Finally, we also need to write down the RLL relations. Similarly to what we did for $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$, we consider a free, associative algebra generated by operators $L_{ij}[n]$, compactly represented by the formal series

$$L_{ij}(z) = \sum_{n\in\mathbb{Z}} L_{ij}[n]z^n \qquad i,j \in \mathbb{Z}_N, n \in \mathbb{Z} \tag{3.20}$$

and encapsulated into the so-called Lax matrix:

$$L(z) = \sum_{i,j=1}^{N} L_{ij}(z)e_{ij},$$

The RLL relations now read

$$\tilde{R}_{12}(\frac{z}{w})L_1(z)L_2(w) = L_2(w)L_1(z)\tilde{R}^{\star}_{12}(\frac{z}{w}). \tag{3.21}$$

The indices refer to the spaces in which the Lax operators and the matrix $R(z)$ operate, with $L_1(z) = L(z) \otimes \mathbb{I}$ and $L_2(z) = \mathbb{I} \otimes L(z)$, just like before. The second R-matrix appearing in (3.21) is related to the original R-matrix through $\tilde{R}^{\star}_{12}(z,q,p) = \tilde{R}_{12}(z,q,p^{\star} = pq^{-2c})$, where c is the central charge [25].
That this cannot be a Hopf algebra is not immediately obvious; in fact, we did not elaborate on the co-algebra structure at all so far. But ultimately, the problem stems from the fact that the R-Matrices appearing on the L.H.S. and R.H.S. of (3.21) have different elliptic nomes p and $p^{\star}$. Together with the fact that (3.20) involves both positive and negative powers of z (unlikely the power series found in (2.21)), this prevents us from applying the usual co-product formula $\Delta(L) = L \dot{\otimes} L$. We have no choice but to revisit our algebraic toolbox and rummage for an even more general framework.

3.2 Quasi-Hopf algebras and Drinfel'd twists

Historically, the study of elliptic quantum algebras began with the discovery of the first elliptic solutions to the YBE by Baxter [6] in the early 1970s, a result generalized by Belavin a decade later [10]. Algebraic structures related to those solutions took again roughly a decade to see the light of day [25, 24], and a few years later Frønsdal suggested that quasi-Hopf algebras would be the right framework to deal with these exotic structures ([33, 34]. The same year, Jimbo et al ([35]) succeeded in making explicit the twist that would take one from the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}_2)$ to an elliptic algebra. The following two sections were inspired by [35].

3.2.1 Beyond Hopf

(Quasi-triangular) Hopf algebras were introduced in def. 2.2.2; and we discussed them in some detail there. While especially the quasi-triangularity is an extremely powerful feature, it is not enough to capture the essence of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$. The extended notion that does the job for us goes by the name of *quasi-Hopf algebra*, which we will henceforth define.

Definition 3.2.1 *Let $\mathfrak{A}$ be a unital, associative algebra over $\mathbb{C}$ with a co-algebra structure familiar from Hopf algebras: two algebra homomorphisms $\Delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ and $\epsilon\colon \mathfrak{A} \to \mathbb{C}$ (co-product and co-unit), two co-algebra homomorphisms $m\colon \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ and $\iota\colon \mathbb{C} \to \mathfrak{A}$ (product and unit) and an anti-homomorphism $S\colon \mathfrak{A} \to \mathfrak{A}$. In addition, we also need two elements $\alpha, \beta \in \mathcal{A}$ and an invertible quantity $\Phi \in \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A}$. We call $\mathfrak{A}$ a **quasi-Hopf algebra** if the following constraints are satisfied:*

$$
\begin{aligned}
&(id \otimes \Delta)(\Delta(x)) = \Phi(\Delta \otimes id)(\Delta(x))\Phi^{-1} \quad \forall x \in \mathfrak{A}\\
&(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id\\
&(id \otimes id \otimes \Delta)(\Phi) \cdot (\Delta \otimes id \otimes id)(\Phi) = (1 \otimes \Phi) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (\Phi \otimes 1)\\
&(id \otimes \epsilon \otimes id)(\Phi) = 1
\end{aligned}
$$

Additionally, by setting $\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$ for any $x \in \mathfrak{A}$, and fixing the notation

$$\Phi = \sum_i \phi_i^{(1)} \otimes \phi_i^{(2)} \otimes \phi_i^{(3)}, \quad \Phi^{-1} = \sum_i \psi_i^{(1)} \otimes \psi_i^{(2)} \otimes \psi_i^{(3)},$$

we also need to impose four more constraints on the antipode, valid for all $x \in \mathfrak{A}$:

$$
\begin{aligned}
&\sum_i S(x_i^{(1)})\alpha\, x_i^{(2)} = \epsilon(x)\alpha \qquad &&\sum_i S(\psi_i^{(1)})\alpha\, \psi_i^{(2)} \beta\, S(\psi_i^{(3)}) = 1\\
&\sum_i x_i^{(1)} \beta\, S(x_i^{(2)}) = \epsilon(x)\beta \qquad &&\sum_i \phi_i^{(1)} \beta\, S(\phi_i^{(2)})\alpha\, \phi_i^{(3)} = 1
\end{aligned}
$$

Unfortunately, these modifications also mean that we will have to adapt our definition of quasi-triangularity, which we will do before losing a few words on what this definition actually means.

Definition 3.2.2 *We call a unital, associative algebra $\mathfrak{A}$ over $\mathbb{C}$ a **quasi-triangular quasi-Hopf algebra** (QTQHA) if it is quasi-Hopf, and there exists an invertible element $\mathcal{R}$ (the universal R-Matrix) that obeys*

$$\Delta^{op}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \qquad \forall x \in \mathfrak{A} \tag{3.22}$$

$$(\Delta \otimes id)(\mathcal{R}) = \Phi^{(312)}\mathcal{R}_{13}\,(\Phi^{(132)})^{-1}\mathcal{R}_{23}\,\Phi^{(123)} \tag{3.23}$$

$$(id \otimes \Delta)(\mathcal{R}) = (\Phi^{(231)})^{-1}\,\mathcal{R}_{13}\,\Phi^{(213)}\mathcal{R}_{12}\,(\Phi^{(123)})^{-1}, \tag{3.24}$$

where we use the shorthand $\Phi^{(klm)} = \sum_i \phi_i^{(k)} \otimes \phi_i^{(l)} \otimes \phi_i^{(m)}$ for the sake of brevity.

Two things should be noted here: First and most obviously, in the limit $\Phi = 1$, $\alpha = \beta = 1$, we obviously recover the defining features of a Hopf algebra, as introduced in (2.2.2), and also the conditions for quasi-triangularity resume their familiar form (2.7). Moreover, as we will show in an instance, $\mathcal{R}$ now satisfies a *generalized Yang-Baxter equation*, which explicitly reads

$$\mathcal{R}_{12}\,\Phi^{(312)}\mathcal{R}_{13}(\Phi^{(132)})^{-1}\mathcal{R}_{23}\,\Phi^{(123)} = \Phi^{(321)}\mathcal{R}_{23}\,(\Phi^{(231)})^{-1}\mathcal{R}_{13}\,\Phi^{(213)}\mathcal{R}_{12}. \tag{3.25}$$

Proof. Set $\mathcal{R} := x_i \otimes y_i \in \mathfrak{A} \otimes \mathfrak{A}$ with summation over i understood. We calculate

$$\mathcal{R}_{12}(\Delta \otimes id)(x_i \otimes y_i) = \mathcal{R}\Delta(x_i) \otimes y_i \underset{(3.22)}{=} \Delta^{\mathrm{op}}(x_i)\mathcal{R} \otimes y_i = (\Delta^{\mathrm{op}} \otimes id)(x_i \otimes y_i)\mathcal{R}_{12}$$

By reinserting $\mathcal{R} = x_i \otimes y_i$ on the L.H.S., and using (3.23), we get

$$\mathcal{R}_{12}(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{12}\,\Phi^{(312)}\mathcal{R}_{13}(\Phi^{(132)})^{-1}\mathcal{R}_{23}\,\Phi^{(123)},$$

while doing the same for the R.H.S. (note that Δ^{op} flips spaces $1 \leftrightarrow 2$) gives us

$$((\Delta^{\mathrm{op}} \otimes id)\mathcal{R})\mathcal{R}_{12} = \Phi^{(321)}\mathcal{R}_{23}\,(\Phi^{(231)})^{-1}\mathcal{R}_{13}\,\Phi^{(213)}\mathcal{R}_{12}.$$

The proof can be repeated in the exact same manner for the 'standard' YBE of quasi-triangular Hopf algebras (2.8). ■

That is an impressive arsenal, and it is not necessary to carry the entire collection around with us all the time. Rather than focusing on the most general case, we will be interested in quasi-Hopf algebras that are obtained by suitably 'twisting' an ordinary Hopf algebra. Let us see how this is done.

3.2.2 Generating quasi-Hopf algebras

Definition 3.2.3 *Using the notation from def.s* (3.2.1) *and* (3.2.2), *let* $\mathcal{F} \in \mathfrak{A} \otimes \mathfrak{A}$ *be an invertible element that satisfies* $(id \otimes \epsilon)\,\mathcal{F} = 1 = (\epsilon \otimes id)\,\mathcal{F}$. *The object* $\mathcal{F}$ *is called a* ***Drinfel'd twist*** *(sometimes also* ***twistor****).*

One of the fundamental insights in this field of research is that a Drinfel'd twist allows us to generate a new quasi-triangular quasi-Hopf algebra (QTQHA) from an already existing one. The algorithm for this was first introduced and proven to work by V. Drinfel'd [1]:

Theorem 3.2.4 *Let* $\mathfrak{A} := (\mathfrak{A}, \Phi, \Delta, \epsilon, S, \alpha, \beta, \mathcal{R})$ *be a QTQHA as defined in* (3.2.2). *Writing*

$$\mathcal{F}_{12} = \sum_i v_i^{(1)} \otimes v_i^{(2)} \quad , \quad \mathcal{F}_{12}^{-1} = \sum_i w_i^{(1)} \otimes w_i^{(2)},$$

and setting

$$\tilde{\Delta}(x) = \mathcal{F}_{12}\,\Delta(x)\mathcal{F}_{12}^{-1} \quad \forall x \in \mathfrak{A} \tag{3.26}$$

$$\tilde{\mathcal{R}} = \mathcal{F}_{12}\mathcal{R}_{12}\,\mathcal{F}_{12}^{-1} \tag{3.27}$$

$$\tilde{\Phi} = (\mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F})\,(\Phi(\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}))^{-1} \tag{3.28}$$

$$\tilde{\alpha} = \sum_i S(w_i^{(1)})\alpha\, w_i^{(2)} \quad \textit{and} \quad \tilde{\beta} = \sum_i v_i^{(1)}\beta\, S(v_i^{(2)}), \tag{3.29}$$

we obtain a second QTQHA $\tilde{\mathfrak{A}} := (\mathfrak{A}, \tilde{\Phi}, \tilde{\Delta}, \epsilon, S, \tilde{\alpha}, \tilde{\beta}, \tilde{\mathcal{R}})$.

As promised earlier on, we are mostly interested in a special case, namely where we start from a Hopf algebra (which has $\alpha = \beta = \Phi = 1$), and require that the twistor $\mathcal{F}$ be dependent on a parameter λ that lies in some abelian subalgebra $\mathfrak{h}$ (not necessarily the Cartan subalgebra) of $\mathfrak{A}$. In fact, we will limit ourselves even more. Fix some notation first: By $\{h_i\}$ ($\{h^i\}$), we will denote the (dual) basis of $\mathfrak{h}$, $\lambda = \sum_i \lambda_i h^i$, and $\lambda + h^{(k)}$ is shorthand for $\sum_i (\lambda_i + h_i^{(k)})h^i$, where h acts on the k^{th} space. Then:

Definition 3.2.5 *The subclass of Drinfel'd twists $\mathcal{F}$ that satisfies the* ***shifted cocycle condition***

$$\mathcal{F}_{12}(\lambda)(\Delta \otimes id)(\mathcal{F}(\lambda)) = \mathcal{F}_{23}(\lambda + h^{(1)})(id \otimes \Delta)(\mathcal{F}(\lambda)) \tag{3.30}$$

bears the name ***Gervais-Neveu-Felder (GNF) twist****.*

Note that the restriction on GNF twists of Hopf algebras also simplifies the co-associator $\tilde{\Phi}$ significantly, which can now be expressed as $\Phi^{(123)} = \mathcal{F}_{23}(\lambda)\mathcal{F}_{23}(\lambda + h^{(1)})^{-1}$. In the same vein, we can also simplify the generalized YBE (3.25) by inserting the definition for Φ under a GNF twist, and using (3.28) multiple times to find what is called the *dynamical YBE*:

$$\tilde{\mathcal{R}}_{12}(\lambda + h^{(3)})\tilde{\mathcal{R}}_{13}(\lambda)\tilde{\mathcal{R}}_{23}(\lambda + h^{(1)}) = \tilde{\mathcal{R}}_{23}(\lambda)\tilde{\mathcal{R}}_{13}(\lambda + h^{(2)})\tilde{\mathcal{R}}_{12}(\lambda) \tag{3.31}$$

Later on, we will develop a better understanding of the difference between the YBEs (2.8) and (3.31). The most important take-away for now is the shift in λ as we go from L.H.S. to R.H.S., which generally leads to a change in parameters (such as the deformation parameter q). This shift carries over to the RLL relations that define the algebra, so that upon reversing the order of two generators, said parameter(s) will no longer remain static, hence the name 'dynamical'.

3.3 Drinfel'd twists applied: The $\mathcal{R}$-Matrix of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$

Equipped with these notions, let us now turn to the question of actually constructing the all-important $\mathcal{R}$-Matrix. There are in fact two different types of elliptic solutions to the YBE, connected with so-called vertex and face type algebras, the second of which we will not consider in here[3]. For the vertex type, here is the storyline: We construct a twistor as a certain infinite product of the universal R-Matrix. We will see that it can be viewed as the unique solution to a difference equation, and that said twistor satisfies the shifted cocycle condition (3.30), hence is a GNF twistor. When applying this twistor to a Hopf algebra (such as $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$), we should therefore find that it satisfies the dynamical YBE (3.31) - which is indeed the case. Finally, to make the connection with the previous section, we will explain how, starting with the aforementioned difference equation and a suitable representation morphism, we can find a concrete realization of the R-Matrix. This, as we already saw for $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$, can then be used to alternatively define the algebra via the FRT formalism.

3.3.1 Algebraic twistors for $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$

The construction of the twistor for the vertex type algebra is achieved by constructing an automorphism that acts on the Chevalley generators in a very precise fashion. Let $\mathfrak{h}$ be the Cartan subalgebra of $\widehat{\mathfrak{sl}}_N$, and denote its basis by $\{h_0, \dots, h_{N-1}, d\}$, where the element d - the derivation introduced in section 2.1.2 - is chosen such that

$$[d, e_i] = \delta_{i,0} e_i, \qquad [d, f_i] = -\delta_{i,0} f_i.$$

This is usually called the *homogeneous grading*. The inner product defined on the generators naturally gives rise to a dual basis $\{\Lambda_0, \dots, \Lambda_{N-1}, c\}$, which is comprised of the fundamental weights Λ_i first encountered in section 2.1.1, and c is the central charge.

For any Chevalley generator $x_i := \{e_i, f_i, h_i\}$, let τ be the automorphism that reflects the symmetry of the extended Dynkin diagram of $\widehat{\mathfrak{sl}}_N$: $\tau(x_i) = x_{i+1 \bmod N}$. It is manifestly of order N, i.e. $\tau^N = \mathrm{id}$. It acts on the fundamental weights as

$$\tau(\Lambda_i) = \Lambda_{i+1 \bmod N} - \frac{n-1-2i}{2n} c$$

and leaves invariant the sum $\rho := \sum_{i=0}^{N-1} \Lambda_i$. Due to the isomorphic nature of $\mathfrak{h}$ and its dual counterpart $\mathfrak{h}^\star$ (a consequence of them being finite dimensional), we can also determine the commutator of ρ and the Chevalley generators, which reads

$$[\rho, e_i] = e_i, \qquad [\rho, f_i] = -f_i.$$

and gives the *principal grading*.

Based on τ, let us introduce another automorphism for a complex $r \in \mathbb{C}$ by

$$\varphi_r = \tau \circ \mathrm{Ad}(q^{\frac{2(r+c)}{N}\rho})$$

[3]It is worth remarking that so-called face type twistors can be found for any $\mathcal{U}_q(\widehat{\mathfrak{g}})$, while the vertex type solution requires us to set $\mathfrak{g} = \widehat{\mathfrak{gl}}_N$. See [35].

where $\mathrm{Ad}(X)Y := XYX^{-1}$. Furthermore, set

$$T = \frac{1}{N}(\rho \otimes c + c \otimes \rho - \frac{N^2-1}{12} c \otimes c)$$

and define the **vertex type twistor** as

$$\mathcal{F}(r) = \prod_{k \geq 1}^{\curvearrowleft} \mathcal{F}_k(r) = \prod_{k \geq 1}^{\curvearrowleft} (\varphi^k \otimes \mathrm{id})(\widehat{\mathcal{R}}^{-1}) \tag{3.32}$$

with $\mathcal{R} = q^T \mathcal{R}[\mathcal{U}_q(\widehat{\mathfrak{sl}}_N)]$. The infinite product $\overset{\curvearrowleft}{\prod}$ should be understood in the sense that it is carried out right to left, as in $\mathcal{F}(r) = \ldots \mathcal{F}_3(r)\mathcal{F}_2(r)\mathcal{F}_1(r)$. It is important not to confuse r with the spectral parameter as it appears in (2.19); the latter only appears after choosing a representation for the universal objects from the algebra.
In a landmark article, Jimbo et al. [35] were able to show that the twistor (3.32) satisfies a shifted cocycle condition, which in this case reads

$$\mathcal{F}_{12}(r)(\Delta \otimes id)(\mathcal{F}(r)) = \mathcal{F}_{23}(r + c^{(1)})(id \otimes \Delta)(\mathcal{F}(r)),$$

($c^{(1)} = c \otimes \mathbb{I} \otimes \mathbb{I}$) and also fulfills

$$(id \otimes \epsilon)\mathcal{F}(r) = 1 = (\epsilon \otimes id)\mathcal{F}(r).$$

We will not repeat the proof here, which is quite technical, and simply state the final result.

Definition 3.3.1 *The* ***vertex type algebra*** $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ *(with $p = q^{2r}$) is comprised of the set $\{\mathcal{U}_q(\widehat{\mathfrak{sl}}_N), \Delta_r, \epsilon, S, \alpha_r, \beta_r, \Phi(r), \mathcal{R}(r)\}$, where the action of S and ϵ is given by def. 2.2.1. By fixing $\sum_i d_i \otimes e_i = \mathcal{F}(r)^{-1}$, $\sum_i f_i \otimes g_i = \mathcal{F}(r)$, the bi-algebra structure has the form*

$$\Delta_r(x) = \mathcal{F}_{12}(r)\Delta(x)\mathcal{F}_{12}(r)^{-1} \quad \forall x \in \mathfrak{A} \tag{3.33}$$

$$\mathcal{R}(r) = \mathcal{F}_{21}(r)\mathcal{R}\,\mathcal{F}_{12}(r)^{-1} \tag{3.34}$$

$$\Phi(r) = (\mathcal{F}_{23}(r)\mathcal{F}_{23}(r + c^{(1)})^{-1} \tag{3.35}$$

$$\alpha_r = \sum_i S(d_i)e_i \quad \textit{and} \quad \beta_r = \sum_i f_i\, S(g_i) \tag{3.36}$$

with the twistor $\mathcal{F}(r)$ given by (3.32)*. $\mathcal{R}$ and Δ are the universal $\mathcal{R}$-Matrix and co-product of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_N)$, respectively.*

Obviously, $\mathcal{R}(r)$ will satisfy a dynamical YBE, which takes the form

$$\tilde{\mathcal{R}}_{12}(r + c^{(3)})\tilde{\mathcal{R}}_{13}(r)\tilde{\mathcal{R}}_{23}(r + c^{(1)}) = \tilde{\mathcal{R}}_{23}(r)\tilde{\mathcal{R}}_{13}(r + c^{(2)})\tilde{\mathcal{R}}_{12}(r). \tag{3.37}$$

Note how the parameter shift spares all the generators of the Cartan subalgebra but the central charge; compare this to eq. (3.21).

3.3.2 Determining the R-Matrix of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$

While the results obtained by Jimbo et al. are aesthetically very satisfactory, the universal $\mathcal{R}$-Matrix (3.34) comes with the defect of being built upon infinite products. For practical purposes, one would therefore be interested in finding a concrete matrix representation. In the generic case, no such construction has yet been successfully finalized. The exception to the rule, scarcely surprising, is the Drinfel'd-twisted version of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$, whose explicit realization we will sketch here.

The first thing to remark is that the twistor (3.32) can alternatively be described as the (unique) solution to the difference equation

$$\mathcal{F}(p^{1/N}q^{2c^{(1)}/N}z, r) = (\tau \otimes \mathrm{id})^{-1}(\mathcal{F}(z,r)) \cdot \tilde{\mathcal{R}}((p^{1/N}q^{2c^{(1)}/N}z) \tag{3.38}$$

with initial condition $\mathcal{F}(0,r) = 1$. In here, we have chosen

$$\begin{aligned} \tilde{\mathcal{R}}(z) &= (\mathrm{Ad}(z^\rho) \otimes \mathrm{id})(q^T \mathcal{R}) \\ \mathcal{F}(z,r) &= (\mathrm{Ad}(z^\rho) \otimes \mathrm{id})\mathcal{F}(r). \end{aligned} \tag{3.39}$$

Sketch of the proof. Set $t = p^{1/N}$, $s = q^{2c^{(1)}/N}$, where $c^{(1)}$ refers to the tensor component, and look at (we suppress the dependence on r)

$$\mathcal{F}(t^2s^2z) = (\tau \otimes \mathrm{id})^{-1}\mathcal{F}(tsz) \cdot \tilde{\mathcal{R}}(tsz) = (\tau \otimes \mathrm{id})^{-2}\mathcal{F}(z) \cdot (\tau \otimes \mathrm{id})^{-1}\tilde{\mathcal{R}}(tsz) \cdot \tilde{\mathcal{R}}(t^2s^2z).$$

We can repeat this procedure for $\mathcal{F}(t^k s^k z)$ with $k \in \mathbb{N}$, and since $|t| < 1$, we will find

$$1 = \mathcal{F}(0) = \lim_{k\to\infty} \mathcal{F}(t^k s^k z) = \mathcal{F}(z) \prod_{k=1}^{\infty} (\tau^k \otimes \mathrm{id})\tilde{\mathcal{R}}(t^k s^k z)$$

Inverting this relation and reinserting (3.39), we recover the expression (3.32). ■

Given (3.38), there are now two ways to determine the R-Matrix: one is to directly calculate the image of the twistor (3.32), as it has been done in [35], the second option consists in using the difference equation to solve directly for the entries of the Drinfel'd twistor. We will pursue the latter way here.

We need a representation first; the best candidate is the principal gradation evaluation introduced in (2.15), in which the central charge vanishes. Applying this to the L.H.S. of (3.38), we find that from the form of (2.16) and $\mathbb{Z}_2$-symmetry (inherited from the Belavin-Baxter Matrix (3.7)) that

$$(\pi_z \otimes \pi_z)\mathcal{F}(p^{1/2}, r) = \begin{pmatrix} a_F(p^{1/2}z) & 0 & 0 & d_F(p^{1/2}z) \\ 0 & b_F(p^{1/2}z) & c_F(p^{1/2}z) & 0 \\ 0 & c_F(p^{1/2}z) & b_F(p^{1/2}z) & 0 \\ d_F(p^{1/2}z) & 0 & 0 & a_F(p^{1/2}z) \end{pmatrix}. \tag{3.40}$$

The R.H.S. is a bit more tricky; we need to determine what to do with $\pi \circ \tau$. This is best done on the generators e_i, f_i and h_i, and we find that

$$\pi \circ \tau = \mathrm{Ad}(\sigma_x) \circ \pi \quad \text{with} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Performing the calculations explicitly, one finds

$$R.H.S. = \begin{pmatrix} b_F(z) & 0 & 0 & c_F(z) \\ 0 & a_F(z) & d_F(z) & 0 \\ 0 & d_F(z) & a_F(z) & 0 \\ c_F(z) & 0 & 0 & b_F(z) \end{pmatrix} \cdot R(pz^2),$$

where the R-Matrix calculated in (2.16) appears. Multiplying it out and comparing the coefficients, we find the following two relations

$$\begin{aligned} a_F(p^{1/2}z) \pm d_F(p^{1/2}z) &= \rho(pz^2)(b_F(z) \pm c_F(z)) \\ b_F(p^{1/2}z) \pm c_F(p^{1/2}z) &= \rho(pz^2) \left(\frac{q(1 \pm p^{1/2}q^{-1}z)}{1 \pm p^{1/2}qz} \right) (a_F(z) \pm d_F(z)), \end{aligned}$$

with the prefactor that we calculated earlier on,

$$\rho(z^2) = q^{-1/2} \frac{(q^2z^2; q^4)_\infty (q^2z^2; q^4)_\infty}{(z^2; q^4)_\infty (q^4z^2; q^4)_\infty}. \tag{3.41}$$

It is not too hard to see that we can reformulate the problem by iteration in order to obtain a relation that only depends on the quantity $a_F(z) \pm d_F(z)$ (or $b_F(z) \pm c_F(z)$). In fact, using the initial condition $F(0) = 1$, we find

$$\begin{aligned} 1 = \lim_{n\to\infty} (a_F(p^n z) \pm d_F(p^n z)) &= \prod_{m=1}^{\infty} \tilde{\rho}(p^m z^2) \left(\frac{q(1 \pm p^{1/2+m}q^{-1}z)}{1 \pm p^{1/2+m}qz} \right) (a_F(z) \pm d_F(z)) \\ &= \frac{(pq^2z^2; p, q^4)_\infty^2}{(pz^2; q^4, p)_\infty (pq^4z^2; q^4, p)_\infty} \cdot \frac{(\mp p^{1/2}q^{-1}z; p)_\infty}{(\mp p^{1/2}qz; p)_\infty} (a_F(z) \pm d_F(z)) \end{aligned} \tag{3.42}$$

with $\tilde{\rho}(z^2) = q^{-1/2}\rho(z^2)$. We find a similar pattern for $b_F(z) \pm c_F(z)$, which fully spelled out reads

$$1 = \frac{(pq^2z^2; p, q^4)_\infty^2}{(pz^2; q^4, p)_\infty (pq^4z^2; q^4, p)_\infty} \cdot \frac{(\mp pq^{-1}z; p)_\infty}{(\mp pqz; p)_\infty} (b_F(z) \pm c_F(z)).$$

Two more steps need to be performed before we obtain the R-Matrix of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$. First we will do the twisting, that is, we calculate

$$\tilde{R}\,[\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)](z) = F_{21}(z^{-1}) R\,[\mathcal{U}_q(\widehat{\mathfrak{gl}}_2)](z) F_{12}(z)^{-1}.$$

More explicitly, this reads

$$\tilde{R}\,[\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)](z) = q^{-1/2}\Upsilon(z^2) \begin{pmatrix} a^+(z) & 0 & 0 & d^+(z) \\ 0 & b^+(z) & c^+(z) & 0 \\ 0 & c^+(z) & b^+(z) & 0 \\ d^+(z) & 0 & 0 & a^+(z) \end{pmatrix} \tag{3.43}$$

with coefficients

$$
\begin{aligned}
a^+(z) \pm d^+(z) &= \frac{(\mp p^{1/2} q z^{-1}; p)_\infty (\mp p^{1/2} q^{-1} z; p)_\infty}{(\mp p^{1/2} q^{-1} z^{-1}; p)_\infty (\mp p^{1/2} q z; p)_\infty} \\
b^+(z) \pm c^+(z) &= q \left(\frac{1 \pm q^{-1} z}{1 \pm q z} \right) \frac{(\mp p q z^{-1}; p)_\infty (\mp p q^{-1} z; p)_\infty}{(\mp p q^{-1} z^{-1}; p)_\infty (\mp p q z; p)_\infty}
\end{aligned} \tag{3.44}
$$

and normalization factor

$$
\Upsilon(z^2) = \frac{(q^2 z^2; p, q^4)_\infty^2 (p z^{-2}; p, q^4)_\infty (p q^4 z^{-2}; p, q^4)_\infty}{(p q^2 z^{-2}; q^4, p)_\infty^2 (z^2; q^4, p)_\infty (q^4 z^2; p, q^4)_\infty}.
$$

These results were first derived by Frønsdal [33, 34].

Presented in this way, our elliptic R-Matrix (3.43) is still quite cumbersome to handle; it would be very advantageous to devise a method for the disentanglement of the matrix entries (3.44). The good news is that such a method exists, as explained in detail in appendix B. In abridged form, what happens is that we rewrite the identities (3.44) using Jacobi Theta functions - possibly the most famous type of elliptic functions out there, and the reason why $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ is called an *elliptic* quantum algebra -, and then take advantage of a very powerful theorem that allows us to transform sums of (functions of) Theta functions into a product of Theta functions (cf. appendix[4] A). Carrying out the calculations explicitly here would take up to much space; the interested reader is invited to consult the example in appendix B. We will simply quote the final result, which reads:

$$
\tilde{R}(z) = \frac{1}{\kappa_2(z^2)} \frac{(p^2; p^2)_\infty}{(p; p)_\infty^2} \frac{\Theta_{q^4}(q^2 z^2)}{\Theta_{q^4}(z^2)} \begin{pmatrix} \tilde{a}(z) & 0 & 0 & \tilde{d}(z) \\ 0 & \tilde{b}(z) & \tilde{c}(z) & 0 \\ 0 & \tilde{c}(z) & \tilde{b}(z) & 0 \\ \tilde{d}(z) & 0 & 0 & \tilde{a}(z) \end{pmatrix} \tag{3.45}
$$

with entries

$$
\begin{aligned}
\tilde{a}(z) &= \frac{\Theta_{p^2}(p z^2) \Theta_{p^2}(p q^2)}{\Theta_{p^2}(p q^2 z^2)} & \qquad \tilde{c}(z) &= z \frac{\Theta_{p^2}(p z^2) \Theta_{p^2}(q^2)}{\Theta_{p^2}(q^2 z^2)} \\
\tilde{b}(z) &= q \frac{\Theta_{p^2}(z^2) \Theta_{p^2}(p q^2)}{\Theta_{p^2}(q^2 z^2)} & \qquad \tilde{d}(z) &= -\frac{p^{\frac{1}{2}}}{q z^2} \frac{\Theta_{p^2}(z^2) \Theta_{p^2}(q^2)}{\Theta_{p^2}(p q^2 z)}
\end{aligned} \tag{3.46}
$$

and normalization factor

$$
\frac{1}{\kappa(z^2)} = \frac{(q^4 z^{-2}; p, q^4)_\infty (q^2 z^2; p, q^4)_\infty (p z^{-2}; p, q^4)_\infty (p q^2 z^2; p, q^4)_\infty}{(q^4 z^2; p, q^4)_\infty (q^2 z^{-2}; p, q^4)_\infty (p z^2; p, q^4)_\infty (p q^2 z^{-2}; p, q^4)_\infty}., \tag{3.47}
$$

This corresponds to the $N = 2$ case of expression (3.7). It should be noted that the typesetting choice, using $\tilde{R}(z)$ rather than $R(z)$, is deliberate and corresponds to the rescaling (3.6). However, the matrix *defining* $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$ via the RLL relations is really the expression (3.45) that was derived from $\mathcal{U}_q(\widehat{\mathfrak{gl}}_2)$ using Drinfel'd twistors.

[4] To recap, the Jacobi Theta function for a parameter p is defined as $\Theta_p(x) := (x, p)_\infty (p x^{-1}, p)_\infty (p, p)_\infty$, where $(x, p)_\infty := \prod_{l=0}^\infty (1 - x p^l)$.

3.3.3 Evaluation representation for $N > 2$

At this point, it is worthwhile to pause for a moment to understand why this is so hard to scale up. Indeed, a significant part of this thesis was concerned with finding a method to repeat that calculations for the twistor of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_3)$, which we will use here as an illustrative example. Again, we represent the universal $\mathcal{R}$-Matrix using the principal gradation to find

$$(\pi_z \otimes \pi_z)\mathcal{F}(p^{1/3}, r) = \\ = \begin{pmatrix} a_F(z_p) & 0 & 0 & 0 & 0 & d_F(z_p) & 0 & e_F(z_p) & 0 \\ 0 & b_F(z_p) & 0 & f_F(z_p) & 0 & 0 & 0 & 0 & g_F(z_p) \\ 0 & 0 & c_F(z_p) & 0 & h_F(z_p) & 0 & j_F(z_p) & 0 & 0 \\ 0 & j_F(z_p) & 0 & c_F(z_p) & 0 & 0 & 0 & 0 & h_F(z_p) \\ 0 & 0 & e_F(z_p) & 0 & a_F(z_p) & 0 & d_F(z_p) & 0 & 0 \\ g_F(z_p) & 0 & 0 & 0 & 0 & b_F(z_p) & 0 & f_F(z_p) & 0 \\ 0 & 0 & f_F(z_p) & 0 & g_F(z_p) & 0 & b_F(z_p) & 0 & 0 \\ h_F(z_p) & 0 & 0 & 0 & 0 & j_F(z_p) & 0 & c_F(z_p) & 0 \\ 0 & d_F(z_p) & 0 & e_F(z_p) & 0 & 0 & 0 & 0 & a_F(z_p) \end{pmatrix}$$

where we set $z_p = p^{1/3}z$ to keep things legible. The form of this matrix is motivated by the general structure of the Belavin-Baxter matrix (3.7) and the requirement of $\mathbb{Z}_N$-symmetry. Then, we again used the difference equation (3.38) to find a relation between the entries. It is possible to eliminate all but three functions, yielding, for example, the following system:

$$\begin{aligned} c_F(p^{4/3}z) &= \left[A(pz)B(p^{2/3}z)B(p^{1/3}z) + p^{4/3}zB(pz)A(p^{1/3}z)\right] j_F(p^{1/3}z) \\ &\quad + \left[A(pz)B(p^{2/3}z)A(p^{1/3}z) + p^2z^2B(pz)B(p^{1/3}z)\right] f_F(z) + A(pz)A(p^{2/3}z)c_F(p^{1/3}z) \\ j_F(p^{4/3}z) &= \left[B(pz)B(p^{2/3}z)B(p^{1/3}z) + A(pz)A(p^{1/3}z)\right] j_F(p^{1/3}z) \\ &\quad + \left[B(pz)B(p^{2/3}z)A(p^{1/3}z) + p^{2/3}zA(pz)B(p^{1/3}z)\right] f_F(z) + B(pz)A(p^{2/3}z)c_F(p^{1/3}z) \\ f_F(pz) &= pzA(p^{2/3}z)B(p^{1/3}z)j_F(p^{1/3}z) + A(p^{2/3}z)A(p^{1/3}z)f_F(z) + pzB(p^{2/3}z)c_F(p^{1/3}z) \end{aligned}$$

The function $A(z)$ and $B(z)$ are defined as

$$A(z) = \frac{q(1 - pz^3)}{1 - pq^2z^3} \quad \text{and} \quad B(z) = \frac{p^{1/3}z(1 - q^2)}{1 - pq^2z^3}.$$

Unfortunately, it gets very messy right here. At the very best, what we found was a difference equation of 3^{rd} degree for $f_F(z)$ that has the form $f_F(t^3z) \sim f_F(t^2z) + f_F(tz) + f_F(z)$, with coefficients that would stretch along multiple lines - and this is still only 1 out of the 9 equations we would need! Unlike the case of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$, we did not find a way to solve the above system as we did in (3.42)–(3.43) by discovering *some* linear combination of those functions so that it would be proportional to a shifted version of itself. The project was eventually abandoned.

We will conclude this chapter by emphasizing once more that $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ is a dynamical algebra, where the parameter p changes upon inverting the order of two generators L_{ij}, L_{kl}.

This follows from (3.37) upon applying a (partial) representation $(\pi_{z_1} \otimes \pi_{z_2} \otimes \mathbb{I})$. Since under the evaluation representation, the central charge equals zero, only $c^{(3)}$ gives an extra contribution, which is captured by $p^\star$ appearing on the R.H.S. of (3.21). By also applying a representation to the third space, $c^{(3)}$ will vanish likewise, leading to the result that the YBE (3.11) sees no parameter shift.

There is one more drawback caused by the FRT formalism, used in conjunction with the Belavin-Baxter matrix (3.1): It gives us only $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, whereas we would actually prefer to deal with $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$. We have encountered the same story before, when setting up $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ with the help of the FRT formalism, and the remedy will be the same in both cases: We need to divide out the parts we do not want. The method is quite similar to the definition of the special linear group $SL(N, \mathbb{R})$ as a subgroup of $GL(N, \mathbb{R})$; it consists of all matrices in $GL(N, \mathbb{R})$ with determinant equal to 1. Similarly, a so-called *quantum determinant* can be constructed for $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, and be factored out after showing that it constitutes in the center of the algebra. While the precise procedure has long been known for $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$, this is not true for the elliptic algebra. This construction constitutes the heart and soul of this thesis, and will be the topic of the following chapter.

Chapter 4

The quantum determinant

An important step towards a successful classification of quantum elliptic algebras is to be trace them back to some underlying, (simple) Lie algebra. In our case, this quest may be rephrased in the following way: Are there any non-trivial subalgebras of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ that preserve its elliptic structure? A good starting point to look for such subalgebras is to check if there are ideals other than $\{0\}$ and the algebra itself. More precisely, we will discover a non-trivial center of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, generated by an object known as the *quantum determinant*. How to construct it, and prove that not only does it commute with every object in $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, but also spans the entire center of the algebra, will be described in detail in this section. In analogy to, for example, the special and general Lie group, this determinant will be used to distinguish $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$.
The following results were first published in [56].

We shall soon see that the quantum determinant is constructed from the Lax operators as introduced in (3.20), where $\mathfrak{S}_N$ is the set of permutations of N objects:

$$\text{q-det } L(z) = \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) L_{1\sigma(1)}(z) L_{2\sigma(2)}(\frac{z}{q}) \dots L_{N\sigma(N)}(zq^{1-N}), \tag{4.1}$$

From the above expression, it should be evident why this object bears the name of a determinant. In fact, this is not the only possible presentation. To pick one example, in the case $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$, there are four (equivalent) ways to express the quantum determinant:

$$\begin{aligned} \text{q-det} L(z) =& L_{11}(q^{-1}z)L_{22}(z) - L_{21}(q^{-1}z)L_{12}(z) \\ =& L_{22}(q^{-1}z)L_{11}(z) - L_{12}(q^{-1}z)L_{21}(z) \\ =& L_{11}(z)L_{22}(q^{-1}z) - L_{12}(z)L_{21}(q^{-1}z) \\ =& L_{22}(z)L_{11}(q^{-1}z) - L_{21}(z)L_{12}(q^{-1}z) \end{aligned} \tag{4.2}$$

And in fact, for the special case of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$, it has in fact been known for a long time that (4.2) generates the center [25] of the algebra. However, no such proof has yet been devised for higher dimensions, which is why we set out to close that gap.

The general idea of this proof is somewhat involved, and makes it easy to loose track. So let us briefly present the structure here: After introducing the quantum determinant, we set out to show that it can be recast as a partial trace. To this end, we will have to prove the veracity of an exchange relation in the $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ algebra for the Lax generators $L_{ij}(z)$. This in turn allows us to manipulate the original expression for the determinant; or at least it does so after invoking the properties of the antisymmetrizer $A^{(N)}$. Having established the trace expression for the quantum determinant, we will demonstrate that the commutator of it with the Lax matrix $L(z)$ vanishes. For this last step, however, we also need to show that the determinant takes a peculiar form in the fundamental representation, which constitutes the penultimate step. At the very end, what is left to show is that the center is not bigger than what the quantum determinant generates. This will be achieved by exploiting the properties of an algebra homomorphism between $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$.

4.1 The quantum determinant is central in $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$

Without further ado, let us state what is the fundamental result of this section:

Theorem 4.1.1 *Let $A^{(N)}_{1\dots N}$ be the antisymmetrizer of N spaces $\mathbb{C}^N$. One has the following identity*

$$L_1(z)\dots L_N(zq^{1-N})\,A^{(N)}_{1\dots N} = q\text{-}detL(z)\,A^{(N)}_{1\dots N} \tag{4.3}$$

where q-det $L(z)$, called the quantum determinant, is a scalar function that lies in the center of the $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ algebra. It can be rewritten as

$$\begin{aligned} q\text{-}detL(z) &= \mathrm{tr}_{1\dots N}\Big(L_1(z)\dots L_N(zq^{1-N})A^{(N)}_{1\dots N}\Big) && (4.4)\\ &= \sum_{\sigma\in\mathfrak{S}_N}\mathrm{sgn}(\sigma)L_{1\sigma(1)}(z)L_{2\sigma(2)}(\frac{z}{q})\dots L_{N\sigma(N)}(zq^{1-N})\,. && (4.5)\end{aligned}$$

Moreover, for generic values of the parameters p, q and of the central charge c, the quantum determinant lies in the center of the $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ algebra.

As already pointed out, to the best of our knowledge, the case $N > 2$ was not studied yet. We shall see that the proof of centrality for the case $N > 2$ is different from the $N = 2$ case [25]. The latter mimics the usual proofs done for Yangians[1] of quantum affine algebras.
Before we begin to prove theorem 4.1.1, let us remark that the relation (4.3) uses the (undeformed) antisymmetrizer, contrarily to the expression found for the homogeneous gradation that is based on a q-antisymmetrizer [30]. We will discuss this in detail below.

[1]Similar to $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$, Yangians are a class of infinite-dimensional Hopf algebras based on the so-called rational solutions to the Yang-Baxter equation. The notion was introduced by Drinfel'd in [16]. See also [30].

4.1.1 Exchange relations for Lax generators

The first milestone on our way to a successful demonstration of theorem 4.1.1 is the following equality:

Lemma 4.1.2 *In the $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ algebra, it holds true that*

$$L_{ij}(z)L_{kl}(\frac{z}{q}) - L_{il}(z)L_{kj}(\frac{z}{q}) = L_{kl}(z)L_{ij}(\frac{z}{q}) - L_{kj}(z)L_{il}(\frac{z}{q}) \quad \forall i,j,k,l = 1,\dots,N. \tag{4.6}$$

In particular, we have

$$L_{ij}(z)L_{il}(\frac{z}{q}) = L_{il}(z)L_{ij}(\frac{z}{q}) \quad \forall i,j,l = 1,\dots,N. \tag{4.7}$$

Proof: Here and in the following, we will use the R-Matrix (3.7) rather than $\tilde{R}(z)$, as this saves us from some confusion notation, and the results naturally carry over to the case of $\tilde{R}(z)$. We consider the RLL relations (3.21) for $w = z/q$ and project them onto an arbitrary element $e_{i,j} \otimes e_{k,l}$. This leads to the following equation, valid $\forall i,j,k,l = 1,\dots,N$:

$$\sum_{n,m=1}^{N} R_{i,k}^{n,m}(q) L_{n,j}(z) L_{m,l}(\frac{z}{q}) = \sum_{n,m=1}^{N} R^{*}{}_{m,n}^{j,l}(q) L_{k,n}(\frac{z}{q}) L_{i,m}(z), \tag{4.8}$$

(Recall that $R^{*}{}_{m,n}^{j,l}(q)$ is a matrix element with $p \to p^{\star}$.) We will refer to this equation as $X_{i,k}^{j,l}(z)$.

Before continuing, note an additional symmetry of the matrix entries $R_{a,c}^{b,d}(z)$ when evaluated at $z = q$:

$$R_{i,k}^{j,l}(q) = R_{i,k}^{l,j}(q)\,. \tag{4.9}$$

Moving on, we look at the difference $X_{i,k}^{j,l}(z) - X_{i,k}^{l,j}(z)$. Focusing first on the R.H.S., we observe that

$$\sum_{n,m=1}^{N} \left(R^{*}{}_{m,n}^{j,l}(q) - R^{*}{}_{m,n}^{l,j}(q) \right) L_{k,n}(\frac{z}{q}) L_{i,m}(z) = 0$$

as a consequence of (4.9). In other words, $X_{i,k}^{j,l}(z) - X_{i,k}^{l,j}(z)$ does **not** depend on $p^{\star}$. Finally, the L.H.S. gives us

$$\sum_{n,m=1}^{N} R_{i,k}^{n,m}(q) \left(L_{n,j}(z) L_{m,l}(\frac{z}{q}) - L_{n,l}(z) L_{m,j}(\frac{z}{q}) \right) = 0. \tag{4.10}$$

Note that the indices j and l do not play any role in these relations, so if we can solve (4.10) for one pair j,l, we can do it for any. We thus consider the equations for fixed indices j and l, and omit them to ease the notation.

Let us define a new quantity for the expression in brackets,

$$M_{i,k}(z) \equiv M_{i,k}^{j,l}(z) = L_{i,j}(z) L_{k,l}(\frac{z}{q}) - L_{i,l}(z) L_{k,j}(\frac{z}{q}).$$

The system (4.10) has N^2 equations (labeled by the pair (i,k)) and N^2 unknown $M_{i,k}(z)$. However, due to the property $R^{n,m}_{i,k}(q) \neq 0$ only when $i+k = m+n \pmod N$, we can decompose it into N subsystems of N equations each, with N unknowns $M_{i,k}$ and $i+k$ fixed:

$$\sum_{\substack{n,m=1 \\ n+m=i+k}}^{N} R^{n,m}_{i+\alpha,k-\alpha}(q)\, M_{n,m}(z) = 0 \qquad \forall\, \alpha = 0, \dots N-1 \tag{4.11}$$

where all indices should be understood modulo N.
We thus fix the sum $i+k$, and focus on a subsystem. By virtue of (4.9), we find that some of the coefficients appearing in any given equation of (4.11) coincide, and that, in fact, the system contains fewer than N unknowns $M_{i,k}(z)$. To see that, we define yet another quantity

$$T_{i,k}(z) = M_{i,k}(z) + M_{k,i}(z),$$

which obviously satisfies $T_{i,k}(z) = T_{k,i}(z)$ and $T_{i,i}(z) = 2M_{i,i}(z)$. Remark that relation (4.6) rewrites $T_{i,k}(z) = 0$. Then, we may rewrite (4.11) as

- **When N is odd, we have $x = \frac{N+1}{2}$ variables.** The system reads

$$R^{n_1,n_1}_{i+\alpha,k-\alpha}(q)\, M_{n_1,n_1}(z) + \sum_{s=2}^{x} R^{n_s,m_s}_{i+\alpha,k-\alpha}(q)\, T_{n_s,m_s}(z) = 0, \tag{4.12}$$

  where $s = 1, ..., x$ labels the solutions to $n_s + m_s = i+k$ modulo N, $s = 1$ corresponding to the solution with $n = m$.

- **When N is even and $i+k$ is even, we have $x = \frac{N+2}{2}$ variables.** They obey

$$R^{n_1,n_1}_{i+\alpha,k-\alpha}(q)\, M_{n_1,n_1}(z) + R^{n_2,n_2}_{i+\alpha,k-\alpha}(q)\, M_{n_2,n_2}(z) + \sum_{s=3}^{x} R^{n_s,m_s}_{i+\alpha,k-\alpha}(q)\, T_{n_s,m_s}(z) = 0, \tag{4.13}$$

  where again $s = 1, ..., x$ labels the solutions to $n_s + m_s = i+k$ modulo N and $s = 1, 2$ correspond to the two solutions $n = m$ (with obviously $|n_1 - n_2| = \frac{N}{2}$).

- **When N is even and $i+k$ is odd, we have $x = \frac{N}{2}$ variables.** In that case,

$$\sum_{s=1}^{x} R^{n_s,m_s}_{i+\alpha,k-\alpha}(q)\, T_{n_s,m_s}(z) = 0, \tag{4.14}$$

  with still $s = 1, ..., x$ and $n_s + m_s = i+k$ modulo N.

In all cases, the index α runs from 1 to N. Obviously, $T_{i,k}(z) = 0$ is a solution to this system. Since we have a homogeneous system of linear equations, to prove that $T_{i,k}(z) = 0$ is the only solution, it is sufficient to show that the determinant of a sub-system of size x

(the number of variables) is non-vanishing. For convenience, we pick in equations (4.12), (4.13) or (4.14) the first x equations. Then, the determinant of this subsystem reads

$$\det(R') = \sum_{\sigma\in\mathfrak{S}_x} \left(\mathrm{sgn}(\sigma) \prod_{\alpha=1}^{x} R_{i+\alpha,k-\alpha}^{n_{\sigma(\alpha)},m_{\sigma(\alpha)}}(q) \right). \tag{4.15}$$

Instead of trying to simplify this expression directly, we examine it in the limiting case $p \to 0$. Indeed, looking at the matrix entries in (4.46), one gets easily

$$R_{i+\alpha,k-\alpha}^{n,m}(q)\Big|_{p=0} = 0 \qquad \text{unless } n = i+\alpha, m = k-\alpha \text{ or } n = k-\alpha, m = i+\alpha.$$

Entries of the form $R_{a,c}^{c,a}(q)$ or $R_{a,c}^{a,c}(q)$, by contrast, evaluate to a finite (non-zero), well-defined function of q, with in addition $R_{a,c}^{c,a}(q) = R_{a,c}^{a,c}(q)$, see (4.9). This shows that for fixed i, k, α, there is at least one and at most two values of n ($n = i+\alpha$ or $k-\alpha$) such that $R_{i+\alpha,k-\alpha}^{n,m}(q)\Big|_{p=0} \neq 0$. Running over the different values of α, one gets

$$\det(R')\Big|_{p=0} = \prod_{\alpha=1}^{x} R_{i+\alpha,k-\alpha}^{i+\alpha,k-\alpha}(q) \neq 0. \tag{4.16}$$

To go from (4.16) to the claim (4.6), we note that the matrix elements $R_{a,c}^{b,d}$ are continuous in p, so that there is an open set around $p = 0$ for which $\det(R') \neq 0$. By virtue of Cramer's rule, we infer $T_{i,k}^{j,l}(z) = 0$ for all allowed values of i, j, k and l. The relation (4.7) is just a consequence of (4.6) for $k = i$. ■

4.1.2 Properties of the antisymmetrizer

The real power of (4.7) is revealed when we use this insight in conjunction with the antisymmetrizer appearing in theorem 4.1.1. Since a good deal of the proof takes advantage of the properties of the antisymmetrizer, we will review those first.
To this and, we want to spell out the antisymmetrizer explicitly. We first define the permutation operator $\mathcal{P} = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}$, satisfying $\mathcal{P}^2 = \mathbb{I}$, which acts on a tensor product of vectors as $\mathcal{P}(v_1 \otimes v_2) = v_2 \otimes v_1$, with $v_1, v_2 \in \mathbb{C}^N$. Let us further define $\mathcal{P}_{s_i} = \mathcal{P}_{i,i+1}$, acting in the i^{th} and $(i+1)^{st}$ copies of $\mathbb{C}^N$. Since any permutation $\sigma \in \mathfrak{S}_N$ can be written as a succession of transpositions s_i, we will write $\mathcal{P}_\sigma = \mathcal{P}_{s_{i_1}} \mathcal{P}_{s_{i_n}} \dots$ for $\sigma = s_{i_1} \circ \dots \circ s_{i_n}$. Then the antisymmetrizer reads

$$A^{(N)} = \frac{1}{N!} \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) \mathcal{P}_\sigma \tag{4.17}$$

The antisymmetrizer is a rank 1 projector. The eigenvector corresponding to the eigenvalue 1 reads:

$$w = \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(N)}.$$

$A^{(N)}$ projects any given vector v on w. If we express v as

$$v = \sum_{1 \le i_1, \dots, i_N \le N} v_{i_1,\dots,i_N}\, e_{i_1} \otimes \dots \otimes e_{i_N},$$

it will be antisymmetrized by $A^{(N)}$ in the following way:

$$\begin{aligned}
A_N v &= \frac{1}{N!} \sum_{i_r \neq i_s \forall r,s=1,\dots,N} v_{i_1,\dots,i_N} \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) e_{\sigma(i_1)} \otimes \dots \otimes e_{\sigma(i_N)} \\
&= \frac{1}{N!} \sum_{\sigma,\mu \in \mathfrak{S}_N} v_{\mu(1),\dots,\mu(N)} \operatorname{sgn}(\sigma) e_{\sigma\circ\mu(1)} \otimes \dots \otimes e_{\sigma\circ\mu(N)} \\
&= \frac{1}{N!} \sum_{\sigma',\mu \in \mathfrak{S}_N} \operatorname{sgn}(\sigma' \circ \mu^{-1}) v_{\mu(1),\dots,\mu(N)} e_{\sigma'(1)} \otimes \dots \otimes e_{\sigma'(N)} \\
&= \sum_{\mu \in \mathfrak{S}_N} \operatorname{sgn}(\mu) v_{\mu(1),\dots,\mu(N)} \sum_{\sigma' \in \mathfrak{S}_N} \operatorname{sgn}(\sigma') e_{\sigma'(1)} \otimes \dots \otimes e_{\sigma'(N)}.
\end{aligned}$$

The last equality shows that the following condition holds:

$$A^{(N)} v = \langle w, v \rangle\, w \qquad \forall v \in (\mathbb{C}^N)^{\otimes N}, \tag{4.18}$$

where $\langle w, v \rangle = \sum_{1 \le i_1,\dots,i_N \le N} v_{i_1,\dots,i_N} w_{i_1,\dots,i_N} = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) v_{\sigma(1)\dots\sigma(N)}$. This will be used momentarily.

4.1.3 Explicit expression for the quantum determinant

Due to the equality (4.18), to get an expression for the quantum determinant, it is enough to compute

$$L_1(z) \dots L_N(zq^{1-N}) w = \sum_{i_1,\dots,\imath_N=1}^{N} \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) L_{i_1,\sigma(1)}(z) \dots L_{i_N,\sigma(N)}(q^{1-N}z)(e_{i_1} \otimes \dots \otimes e_{i_N}) \tag{4.19}$$

We first prove that all the indices $i_1, \dots, i_N$ in (4.19) must be different. In other words, we prove that terms with identical indices vanish. This is achieved by recursion on the 'distance' between two identical indices.

We first consider the terms with $i_k = i_{k+1}$. Without loss of generality, we can check what happens for $k = N-1$: The reasoning naturally translates to all other possible pairs of adjacent indices. Focusing on the coefficient of $e_{i_1} \otimes \dots \otimes e_{i_{N-2}} \otimes e_{i_N} \otimes e_{i_N}$ only, we write (all indices arbitrary, but fixed):

$$\begin{aligned}
&\sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) L_{i_1,\sigma(1)}(z) \dots L_{i_N,\sigma(N-1)}(q^{2-N}z) L_{i_N,\sigma(N)}(q^{1-N}z) \\
= &\sum_{\sigma' \in \mathfrak{S}_N} \operatorname{sgn}(\sigma' \circ s_{N,N-1}) L_{i_1,\sigma'(1)}(z) \dots L_{i_N,\sigma'(N)}(q^{2-N}z) L_{i_N,\sigma'(N-1)}(q^{1-N}z) \\
= &\frac{1}{2} \sum_{\sigma' \in \mathfrak{S}_N} \operatorname{sgn}(\sigma') L_{i_1,\sigma'(1)}(z) \dots L_{i_{N-2},\sigma'(N-2)}(q^{3-N}z) \\
&\times \Big(L_{i_N,\sigma'(N-1)}(q^{2-N}z) L_{i_N,\sigma'(N)}(q^{1-N}z) - L_{i_N,\sigma'(N)}(q^{2-N}z) L_{i_N,\sigma'(N-1)}(q^{1-N}z) \Big) = 0
\end{aligned} \tag{4.20}$$

where the last equality is done by virtue of (4.7).
Suppose now that the terms where $i_k = i_{k+n}$ with $1 \le n \le m$ have zero contribution and consider the term where $i_k = i_{k+m+1}$:

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) L_{i_1,\sigma(1)}(z)\dots L_{i_k,\sigma(k)}(q^{1-k}z)\dots L_{i_k,\sigma(k+m+1)}(q^{-m-k}z)\dots L_{i_N,\sigma(N)}(q^{1-N}z)\\
&= \frac{1}{2}\sum_{\sigma'\in\mathfrak{S}_N} \mathrm{sgn}(\sigma') L_{i_1,\sigma'(1)}(z)\dots L_{i_{k-1},\sigma'(k-1)}(q^{2-k}z)\\
&\times\Big(L_{i_k,\sigma(k)}(q^{1-k}z)L_{i_{k+1},\sigma(k+1)}(q^{-k}z) - L_{i_k,\sigma(k+1)}(q^{1-k}z)L_{i_{k+1},\sigma(k)}(q^{-k}z)\Big)\\
&\times L_{i_{k+2},\sigma'(k+2)}(q^{3-k}z)\dots L_{i_k,\sigma(k+m+1)}(q^{-m-k}z)\dots L_{i_N,\sigma(N)}(q^{1-N}z) \qquad (4.21)\\
&= -\frac{1}{2}\sum_{\sigma'\in\mathfrak{S}_N} \mathrm{sgn}(\sigma') L_{i_1,\sigma'(1)}(z)\dots L_{i_{k-1},\sigma'(k-1)}(q^{2-k}z)\\
&\times\Big(L_{i_{k+1},\sigma'(k)}(q^{1-k}z)L_{i_k,\sigma'(k+1)}(q^{-k}z) - L_{i_{k+1},\sigma'(k+1)}(q^{1-k}z)L_{i_k,\sigma'(k)}(q^{-k}z)\Big)\\
&\times L_{i_{k+2},\sigma'(k+2)}(q^{3-k}z)\dots L_{i_k,\sigma'(k+m+1)}(q^{-m-k}z)\dots L_{i_N,\sigma'(N)}(q^{1-N}z) \qquad (4.22)\\
&= \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) L_{i'_1,\sigma(1)}(z)\dots L_{i'_k,\sigma(k)}(q^{1-k}z)\dots L_{i'_k,\sigma(k+m)}(q^{-m-k}z)\dots L_{i'_N,\sigma(N)}(q^{1-N}z).
\end{aligned}$$

Here is a brief explanation of what happened: To get (4.21), we have used the same trick as in the calculation of (4.20). Then, to go from (4.21) to (4.22), we have used the exchange relation (4.6). In the last equality, we have introduced new indices $i'_\ell = i_\ell$ for $1 \le \ell \le k-1$, $i'_k = i_{k+1}$ and $i'_\ell = i_{\ell-1}$ for $k < \ell \le N$. This last expression vanishes due to the recursion hypothesis.

Since all indices i_r are different, we can replace the sum on $i_1,\dots,i_N$ by a sum over permutations $\mu \in \mathfrak{S}_N$. We pick one such permutation and examine the coefficient of $e_{\mu(1)}\otimes\cdots\otimes e_{\mu(N)}$:

$$\begin{aligned}
\chi_\mu &:= \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) L_{\mu(1),\sigma(1)}\dots L_{\mu(N),\sigma(N)}\\
&= \frac{1}{2}\sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) L_{\mu(1),\sigma(1)}\dots L_{\mu(k-1),\sigma(k-1)}\big\{L_{\mu(k),\sigma(k)}L_{\mu(k+1),\sigma(k+1)} - L_{\mu(k),\sigma(k+1)}L_{\mu(k+1),\sigma(k)}\big\}\\
&\quad\times L_{\mu(k+2),\sigma(k+2)}\dots L_{\mu(N),\sigma(N)}
\end{aligned}$$

But we can also look at a different permutation $\mu' = \mu \circ s_k$. In this case, we find that

$$\begin{aligned}
\chi_{(\mu\circ s_k)} = \frac{1}{2}\sum_{\sigma\in\mathfrak{S}_N} &\mathrm{sgn}(\sigma) L_{\mu(1),\sigma(1)}\dots L_{\mu(k-1),\sigma(k-1)}\\
&\times\big\{L_{\mu(k+1),\sigma(k)}L_{\mu(k),\sigma(k+1)} - L_{\mu(k+1),\sigma(k+1)}L_{\mu(k),\sigma(k)}\big\}L_{\mu(k+2),\sigma(k+2)}\dots L_{\mu(N),\sigma(N)}
\end{aligned}$$

Once more, condition (4.6) shows that $\chi_{(\mu\circ s_k)} = -\chi_\mu$. This allows us to conclude that in fact, for any $\sigma,\mu \in \mathfrak{S}_N$, we have $\chi_\mu = \mathrm{sgn}(\sigma)\chi_{(\mu\circ\sigma)}$. In particular, $\chi_\mu = \mathrm{sgn}(\mu)\chi_{\mathrm{id}}$, and we finally arrive at the result

$$\begin{aligned} L_1(z)\dots L_N(zq^{1-N})w &= \frac{1}{N!}\sum_{\mu\in\mathfrak{S}_N}\chi_\mu\, e_{\mu(1)}\otimes\cdots\otimes e_{\mu(N)} \\ &= \frac{1}{N!}\sum_{\mu\in\mathfrak{S}_N}\mathrm{sgn}(\mu)\,\chi_{\mathrm{id}}\, e_{\mu(1)}\otimes\cdots\otimes e_{\mu(N)} = \chi_{\mathrm{id}}\, w \end{aligned}$$

From this, we directly infer that the quantum determinant is χ_{id}, which proves the equality (4.1). ■

Remark that in this way we have proved that

$$L_1(z)\dots L_N(zq^{1-N})\,A^{(N)}_{1\dots N} = \mathbb{M}(z)\,A^{(N)}_{1\dots N}\,, \tag{4.23}$$

where $\mathbb{M}(z)$ is scalar in the spaces 1,...,N and given by (4.1). It remains to prove that $\mathbb{M}(z)$ is central in $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$.

4.1.4 Evaluation representation of quantum determinant

There is more to be learned about the form of $\mathbb{M}(z)$ as introduced in (4.23) before proceeding to the computation of the commutator in the next subsection. In particular, we will have to show that for a particular representation - the fundamental map, which represents the Lax generators $L_{ij}(z)$ by blocks of the R-Matrix (3.7) - it is proportional to the identity, and the factor of proportionality ceases to depend on the parameter p.

Lemma 4.1.3 *The R-Matrix* (3.7) *for the* $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ *algebra obeys the following relation*

$$\tilde{R}_{10}\,(z)\dots\tilde{R}_{N0}\left(zq^{1-N}\right)A^{(N)}_{1\dots N} = A^{(N)}_{1\dots N}. \tag{4.24}$$

Proof: We apply the evaluation maps π_j: $L_j(z)\to\tilde{R}_{j0}(z)$, $j=1,...,N$ to the equality (4.23):

$$\tilde{R}_{10}\,(z)\dots\tilde{R}_{N0}\left(zq^{1-N}\right)A^{(N)}_{1\dots N} = \pi\big(\text{q-det }L(z)\big)\,A^{(N)}_{1\dots N} = M_0(z)\,A^{(N)}_{1\dots N}, \tag{4.25}$$

where $\pi=\pi_1\otimes\pi_2\otimes...\otimes\pi_N$, and $M_0(z)$ is a matrix $M(z)$ (yet to be determined) acting on the space 0 only. From the explicit form of the fundamental map,

$$L_{ij}(z)\longmapsto\eta(z)\sum_{k=1}^{N}S^{j}_{i,k}(z)e_{l,i+k-j}, \tag{4.26}$$

with matrix elements $S^b_{a,c}(z)$ and normalization factor $\eta(z)$ as defined in (3.7), we can express $M(z)$ as

$$M(z)=\prod_{j=0}^{N-1}\eta\left(\frac{z}{q^j}\right)\sum_{\sigma\in\mathfrak{S}N}\mathrm{sgn}(\sigma)\sum_{k=1}^{N}S^{\sigma(1)}_{1,k}(z)\dots S^{\sigma(N)}_{N,k+\sum_{i=1}^{N-1}(i-\sigma(i))}\left(\frac{z}{q^{N-1}}\right)e_{kk}\equiv\sum_{k=1}^{N}m_k(z)\,e_{kk}, \tag{4.27}$$

which is indeed a diagonal matrix. Remark that this uses the matrix $R(z)$, rather than $\tilde{R}(z)$, a problem we will rectify soon.
Simplifying this expression might look like a hopeless task, but it is not impossible. In fact, the product that appears in (4.27) will render a number of terms more compact. In particular, we note the following identities:

$$\begin{aligned}
&\prod_{k=0}^{N-1} \kappa_N(\frac{z^2}{q^{2k}}) = (-1)^{N+1} \frac{q^{N^2-N}}{z^{2N-2}} \frac{\Theta_p(q^{2-2N} z^2)}{\Theta_p(z^2)} \\
&\prod_{k=1}^{N} \tau_N(q^{\frac{2k-1}{2}} z^{-1}) = (-1)^{N+1} \\
&\prod_{j=0}^{N-1} \eta\left(\frac{z}{q^j}\right) = -\frac{(p^N, p^N)_\infty^{3N}}{(p,p)_\infty^{3N}} \frac{\Theta_p^N(q^2)}{q^{N-1}} \frac{\Theta_p(z^2)}{\Theta_p(q^2 z^2)}
\end{aligned} \tag{4.28}$$

Similarly, powers of z, q and p that appear through the matrix elements $S_{a,c}^b(z)$ can be simplified. When the dust has settled, the preliminary result one obtains is

$$\begin{aligned}
m_k(z) = (-1)^N &\frac{(p^N, p^N)_\infty^{3N}}{(p,p)_\infty^{3N}} \frac{q^{2k-2N} \cdot \Theta_p^N(q^2) \Theta_p(z^2)}{\Theta_p(q^2 z^2)} \\
&\times \Big(\sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) \prod_{l=1}^{N} \frac{\Theta_{p^N}(p^{N+k+x_l} q^{4-2l} z^2)}{\Theta_{p^N}(p^{N+l-\sigma(l)} q^2) \Theta_{p^N}(p^{N+k+x_{l+1}+1} q^{2-2l} z^2)} \Big),
\end{aligned} \tag{4.29}$$

where we defined $x_l := \sum_{i=1}^{l-1}(i - \sigma(i)) - l$. Another, somewhat more symmetric way to present this result is

$$\begin{aligned}
m_k(z) = (-1)^N &\frac{(p^N, p^N)_\infty^{3N}}{(p,p)_\infty^{3N}} \frac{q^{2k-2N} \cdot \Theta_p^N(q^2) \Theta_p(z^2) \Theta_{p^N}(p^{N+k-1} q^2 z^2)}{\Theta_p(q^2 z^2) \Theta_{p^N}(p^k q^{2-2N} z^2)} \\
&\times \Big(\sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) \prod_{l=1, m=2}^{N} \frac{\Theta_{p^N}(p^{N+k+x_m} q^{2-2m} z^2)}{\Theta_{p^N}(p^{N+l-\sigma(l)} q^2) \Theta_{p^N}(p^{N+k+x_m+1} q^{2-2m} z^2)} \Big).
\end{aligned} \tag{4.30}$$

From (4.28), we see that the difference between using the matrices $R(z)$ and $\tilde{R}(z)$ is merely a swap of signs; not enough to cause a headache at some later stage.
This presentation is still very involved, but we can use some general properties of the R-Matrix to derive further restrictions. Here, we will rely only on the invariance property (3.16), $[h \otimes h, R(z)] = 0$. It is easy to show that

$$\begin{aligned}
&h_0 h_1 \cdots h_N \, \pi\big(\text{q-det} L(z)\big) \, A_{1 \ldots N}^{(N)} = h_0 h_1 \cdots h_N \, \tilde{R}_{10}(z) \ldots \tilde{R}_{N0}\left(z q^{1-N}\right) \, A_{1 \ldots N}^{(N)} \\
&= \tilde{R}_{10}(z) \ldots \tilde{R}_{N0}\left(z q^{1-N}\right) h_0 h_1 \cdots h_N \, A_{1 \ldots N}^{(N)} = \pi\big(\text{q-det} L(z)\big) \, h_0 h_1 \cdots h_N \, A_{1 \ldots N}^{(N)} \\
&= \pi\big(\text{q-det} L(z)\big) \, A_{1 \ldots N}^{(N)} \, h_0 h_1 \cdots h_N .
\end{aligned} \tag{4.31}$$

Due to the expression (4.25), this implies $[h\,,\, M(z)] = 0$. From $h_{ij} = \delta_{i,j+1}$ and the fact that $M(z)$ is diagonal, we conclude $m_k(z) = m_{k+1}(z)$, that is to say, $M(z) = m(z)\,\mathbb{I}$.

But this is not all: In fact, using the properties of Jacobi Theta functions as stated in (A.5), it is a simple exercise to show that $m(z; qp^{N/2}, p) = m(z; q, p)$, where the dependence on the parameters p and q is now made explicit. Then, because $|p| < 1$ by definition, we infer

$$m(z; q, p) = m(z; qp^{\ell N/2}, p) = \lim_{\ell\to\infty} m(z; qp^{\ell N/2}, p) = \lim_{q'\to 0} m(z; q', p), \quad \forall\, p, q.$$

But this is really just a formal way of saying that neither does $m(z; q, p)$ depend on q! And as chance would have it, this is much more powerful than independence of z. Fixing $q = 1$, we see from expression (4.29) that only the term linked to the identity permutation contributes, since it alone can cancel out the N-fold root stemming from $\lim_{q\to 1} \Theta_p^N(q^2)$. Finally, a quick calculation shows that $m(z, 1, p) = 1$ for generic values of z, p and N.
Let us remark that we also showed $m(z) = 1$ for $N = 2$ or $N = 3$ explicitly during our work on this topic. Especially the latter is quite remarkable in light of the form of the matrix, whereas the result $N = 2$ could always be due to some lucky twist of fate (recall that the R-Matrix becomes symmetric in this case). Additionally, for generic q and N, we checked that the $\lim_{p\to 0} m(z; q, p) = 1$, corresponding to a presentation of $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ which we will discuss in more detail below. This reassures us of the validity of the above calculation. We will now see how this result is used to complete the proof.

4.1.5 Computing the commutator

With all the machinery developed so far, we now wish to show that

$$[\text{q-det}L(z), L_{ij}(w)] = 0 \qquad \forall i, j = 1, \ldots N, z, w \in \mathbb{C}. \tag{4.32}$$

This will be achieved by commuting $L_0(w)$ through the expression (4.4) for the quantum determinant:

$$\begin{aligned}
&\text{q-det}L(z)\, L_0(w) = \ \mathrm{tr}_{1\ldots N}\left[L_1(z)\ldots L_N(zq^{1-N})A^{(N)}_{1\ldots N}\right] L_0(w) \\
&= \mathrm{tr}_{1\ldots N}\left[L_1(z)\ldots L_N(zq^{1-N})L_0(w)A^{(N)}_{1\ldots N}\right] \\
&= \mathrm{tr}_{1\ldots N}\left[L_1(z)\ldots L_{N-1}(zq^{2-N})\tilde{R}^{-1}_{N0}\left(\frac{z}{w}q^{1-N}\right)L_0(w)L_N(zq^{1-N})\tilde{R}^{\star}_{N0}\left(\frac{z}{w}q^{1-N}\right)A^{(N)}_{1\ldots N}\right] \\
&= \mathrm{tr}_{1\ldots N}\left[\tilde{R}^{-1}_{N0}\left(\frac{z}{w}q^{1-N}\right)L_1(z)\ldots L_{N-1}(zq^{2-N})L_0(w)L_N(zq^{1-N})\tilde{R}^{\star}_{N0}\left(\frac{z}{w}q^{1-N}\right)A^{(N)}_{1\ldots N}\right] \\
&= \mathrm{tr}_{1\ldots N}\left[\tilde{R}^{-1}_{N0}\left(\frac{z}{w}q^{1-N}\right)\ldots\tilde{R}^{-1}_{10}\left(\frac{z}{w}\right)L_0(w)L_1(z)\ldots L_N(zq^{1-N})\right. \\
&\quad\left.\times\, \tilde{R}^{\star}_{10}\left(\frac{z}{w}\right)\ldots\tilde{R}^{\star}_{N0}\left(\frac{z}{w}q^{1-N}\right)A^{(N)}_{1\ldots N}\right] \\
&= \mathrm{tr}_{1\ldots N}\left[\widehat{R}^{-1}_{N0}\left(\frac{z}{w}q^{1-N}\right)\ldots\tilde{R}^{-1}_{10}\left(\frac{z}{w}\right)L_0(w)\,\mathrm{q}-\det L(z)\,A^{(N)}_{1\ldots N}\right],
\end{aligned}$$

where we used the RLL relations (3.21) and the fact that generators acting in different subspaces commute. The last equalities are due to lemma 4.1.3 and definition 4.3.

Next, applying the fact that the quantum determinant is a scalar in the spaces 0, 1, 2..., N, we get

$$\begin{aligned}\text{q-det } L(z)\, L_0(w) &= \mathrm{tr}_{1\ldots N}\left[\tilde{R}_{N0}^{-1}\left(\frac{z}{w}q^{1-N}\right)\ldots\tilde{R}_{10}^{-1}\left(\frac{z}{w}\right)A_{1\ldots N}^{(N)}L_0(w)\right]\text{q-det}L(z)\\ &= L_0(w)\mathrm{tr}_{1\ldots N}\left[A_{1\ldots N}^{(N)}\right]\text{q-det}L(z)\\ &= L_0(w)\,\text{q-det}L(z),\end{aligned}$$

where we used that $L_0(w)$ and the antisymmetrizer commute as they live in different spaces, applied the inverse of (4.24), and finally traced over the antisymmetrizer. As a consequence, we can infer that (4.32) indeed holds, and thus that the quantum determinant lies in the center of the algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ as desired.

4.2 Centrality of the quantum determinant and connection with $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$

The attentive reader will have noticed that was has been said so far was only half of the story. Specifically, while we showed that the quantum determinant *lies* in the center of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, we have yet to prove that it in fact *constitutes* the center, i.e. that there are no elements in $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ that are central, but cannot be written in the form (4.1).

As pointed out in chapter 2, the algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ is rather well understood. Back then, we had mostly worked in what is called the *principal gradation*, simply by stating ad hoc how the generators are represented in this particular case. But it is not the only possibility there is, and we need to invest some time here to understand how it connects to two other presentations.

Before doing so, let us recall how $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ is set up in the FRT formalism: The generators $L_{ij}^{\pm}[\mp n]$, where $n \in \mathbb{Z}_{\geq 0}$, $1 \leq i, j \leq N$ and $L_{ij}^{+}[0] = L_{ji}^{-}[0] = 0$ for $i > j$, are coded in formal generating functions $L_{ij}^{\pm}(z)$, themselves encapsulated into matrices $L^{\pm}(z)$:

$$L^{\pm}(z) = \sum_{i,j=1}^{N} L_{ij}^{\pm}(z)\, e_{ij} \quad \text{and} \quad L_{ij}^{\pm}(z) = \sum_{n=0}^{\infty} L_{ij}^{\pm}[\mp n]\, z^{\pm n}\,. \tag{4.33}$$

The following exchange relations hold:

$$R_{12}(\frac{z_{\pm}}{w_{\pm}})L_1^{\pm}(z)L_2^{\pm}(w) = L_2^{\pm}(w)L_1^{\pm}(z)R_{12}(\frac{z_{\pm}}{w_{\pm}})\,, \tag{4.34}$$

$$R_{12}(\frac{z_{\pm}}{w_{\mp}})L_1^{+}(z)L_2^{-}(w) = L_2^{-}(w)L_1^{+}(z)R_{12}(\frac{z_{\mp}}{w_{\pm}})\,, \tag{4.35}$$

where $z_{\pm} = zq^{\pm c/2}$, $w_{\pm} = wq^{\pm c/2}$, and c is the central charge. In what follows below, we will investigate possible forms of the R-Matrix in more detail.

4.2.1 Homogeneous gradation

Let e_i, f_i $(0 \le i \le N-1)$ and h_i $(0 \le i \le N)$ denote the generators of $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ in the Serre–Chevalley basis and let $\mathcal{R}$ be the universal $\mathcal{R}$-Matrix of $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ (see e.g. [26]). The R-Matrix (4.38) is obtained from $\mathcal{R}$ by calculating its image $R(z/w) = (\pi_z \otimes \pi_w)\mathcal{R}$ in the N-dimensional evaluation representation π_z such that $(1 \le i \le N)$

$$\pi_z(e_i) = e_{i,i+1}\,, \quad \pi_z(f_i) = e_{i+1,i}\,, \qquad \pi_z(h_i) = e_{ii}\,, \tag{4.36}$$

$$\pi_z(e_0) = z e_{N1}\,, \quad \pi_z(f_0) = z^{-1} e_{1N}\,, \quad \pi_z(h_0) = e_{NN} - e_{11}\,. \tag{4.37}$$

It takes the form

$$R^{(h)}(z) = \rho_N(z) \left[\sum_i e_{ii} \otimes e_{ii} + \frac{q(1-z)}{1-q^2 z} \sum_{i \ne j} e_{ii} \otimes e_{jj} + \frac{(1-q^2)}{1-q^2 z} \left(\sum_{i<j} + z \sum_{i>j} \right) e_{ij} \otimes e_{ji} \right]. \tag{4.38}$$

The normalization factor $\rho_N(z)$ can be expressed as

$$\rho_N(z) = q^{\frac{1}{N}-1} \frac{(q^2 z; q^{2N})_\infty (q^{2N-2} z; q^{2N})_\infty}{(z; q^{2N})_\infty (q^{2N} z; q^{2N})_\infty}\,. \tag{4.39}$$

This defines the so-called homogeneous gradation.

The quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ is endowed with the following co-product structure (compare this to (2.24)):

$$\Delta\big(L^{\pm}_{ij}(z)\big) = \sum_{k=1}^{N} L^{\pm}_{kj}(z q^{\mp c^{(2)}/2}) \otimes L^{\pm}_{ik}(z q^{\pm c^{(1)}/2})\,,$$

where $c^{(1)} = c \otimes 1$ and $c^{(2)} = 1 \otimes c$.
The quantum determinant is given in the homogeneous gradation by

$$\operatorname{qdet} L(z) = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma)\, q^{\ell(\sigma)} L^{+}_{1,\sigma(1)}(z) \dots L^{+}_{N,\sigma(N)}(z q^{2-2N})\,, \tag{4.40}$$

where $\ell(\sigma)$ denotes the length of the permutation σ and $\operatorname{sgn}(\sigma) = (-1)^{\ell(\sigma)}$.
Finally, thanks to the RLL relations, the action of the finite Cartan generators on the Lax matrices is given by $(1 \le i,j,k \le N)$

$$q^{h_i} L^{\pm}_{jk}(w) = L^{\pm}_{jk}(w)\, q^{h_i + \delta_{ik} - \delta_{ij}}\,. \tag{4.41}$$

4.2.2 Principal gradation

Recall from section 3.3.1 that this gradation is defined on the Chevalley generators as

$$[\rho, e_i] = e_i, \qquad [\rho, f_i] = -f_i, \quad (i = 0, \dots, N)$$

where $\rho := \sum_{i=0}^{N-1} \Lambda_i$, and Λ_i are the fundamental weights. The R-Matrix in said principal gradation reads

$$R^{(p)}(z) = \rho_N(z^2)\Big[\sum_i e_{ii}\otimes e_{ii} + \frac{q(1-z^2)}{1-q^2z^2} e_{ii}\otimes e_{jj} + \frac{z(1-q^2)}{1-q^2z^2}\left(\sum_{i<j} z^{(2j-2i-N)/N} + \sum_{i>j} z^{(2j-2i+N)/N}\right) e_{ij}\otimes e_{ji}\Big]. \tag{4.42}$$

The two matrices (4.38) and (4.42) are related by a gauge transformation

$$\mathsf{R}(z/w) = V(z)\otimes V(w)\, R(z^2/w^2)\,(V(z)\otimes V(w))^{-1}$$

with $V(z) = \displaystyle\sum_{i=1}^{N} z^{(N+1-2i)/N}\, e_{ii}$.

It follows that the Lax matrices $L^{\pm}(z)$ and $\mathsf{L}^{\pm}(z)$ that define the quantum affine algebra in the homogeneous and principal gradations respectively are related by

$$\mathsf{L}^{+}(z) = V(zq^{c/2})\, L^{+}(z^2)\, V(zq^{-c/2})^{-1}, \tag{4.43}$$

$$\mathsf{L}^{-}(z) = V(z)\, L^{-}(z^2)\, V(z)^{-1}. \tag{4.44}$$

Note that these relations ensure that equation (4.41) also holds for the Lax matrices $\mathsf{L}^{\pm}(z)$ in the principal gradation.

The quantum determinant is then given in the principal gradation by

$$\mathrm{qdet}\,\mathsf{L}(z) = \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma)\, q^{\ell(\sigma)+\frac{2}{N}\sum_{i=1}^{N} i(\sigma(i)-i)}\, \mathsf{L}^{+}_{1,\sigma(1)}(z)\dots \mathsf{L}^{+}_{N,\sigma(N)}(zq^{1-N}). \tag{4.45}$$

4.2.3 Non-elliptic presentation

The limit $p\to 0$ of the $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ algebra allows us to reveal still another presentation of the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$. Since this presentation is related to the non-elliptic limit of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, we will call it the *non-elliptic presentation*. The R-Matrix obtained in this limit reads:

$$\mathsf{R}'(z) = \rho_N(z^2)\left[\sum_i e_{ii}\otimes e_{ii} + \frac{q(1-z^2)}{1-q^2z^2}\left(\sum_{i<j} q^{(2j-2i-N)/N} + \sum_{i>j} q^{(2j-2i+N)/N}\right) e_{ii}\otimes e_{jj} + \frac{z(1-q^2)}{1-q^2z^2}\left(\sum_{i<j} z^{(2j-2i-N)/N} + \sum_{i>j} z^{(2j-2i+N)/N}\right) e_{ij}\otimes e_{ji}\right]. \tag{4.46}$$

When $N>2$, this matrix differs from the previous one (cf. (4.42)), essentially by some powers of q in the diagonal terms. Obviously, it is still $\mathbb{Z}_N$-symmetric. It can be obtained from (4.42) by a (constant, non-factorized) diagonal twist:

$$\mathsf{R}'(z) = F_{21}\, R^{(p)}(z)\, F_{12}^{-1}$$

where

$$F_{12} = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{1 \le i \ne j \le N} q^{\alpha_{ij}}\, e_{ii} \otimes e_{jj}$$

with, for $i < j$, $\alpha_{ij} = \frac{1}{2} + (i-j)/N$ and $\alpha_{ji} = -\alpha_{ij}$. We eliminate one degree of freedom still left here by setting $\alpha_{ii} = 0$ for all i. Remark that for $N = 2$, it reduces to $\alpha_{12} = 0$, so that the twist in this case is $\mathbb{I} \otimes \mathbb{I}$.

The algebra is still defined by eqs. (4.33)–(4.34) where the Lax matrices $L^{\pm}(z)$ are now replaced by $\mathsf{L}'^{\pm}(z)$.

At the universal level, the twisted R-Matrix is given by

$$\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\, \mathcal{R}\, \mathcal{F}_{12}^{-1}$$

with

$$\mathcal{F}_{12} = q^{\sum_{ij} \alpha_{ij} h_i \otimes h_j}\,, \tag{4.47}$$

and $\mathcal{R}$ being the universal object of $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$. Here h_i $(i = 1, \dots, N)$ are the Cartan generators of the finite quantum algebra $\mathcal{U}_q(\mathfrak{gl}_N)$ satisfying the following commutation relations $(j = 1, \dots, N-1)$:

$$[h_i, e_j] = (\delta_{ij} - \delta_{i,j+1}) e_j\,, \quad [h_i, f_j] = -(\delta_{ij} - \delta_{i,j+1}) f_j\,, \quad [e_j, f_j] = \frac{q^{h_j - h_{j+1}} - q^{h_{j+1} - h_j}}{q - q^{-1}}\,.$$

The universal twist (4.47) satisfies the co-cycle condition $\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F}$, ensuring that the universal R-Matrix $\mathcal{R}^{\mathcal{F}}$ satisfies the Yang–Baxter equation as long as the R-Matrix $\mathcal{R}$ does.

The relation between the corresponding Lax matrices $\mathsf{L}^{\pm}$ and $\mathsf{L}'^{\pm}$ can be expressed as

$$\mathsf{L}'^{\pm}(z) = (\pi_z \otimes \mathrm{id})\mathcal{F}_{21}\, \mathsf{L}^{\pm}(z)\, (\pi_z \otimes \mathrm{id})\mathcal{F}_{12}^{-1}\,. \tag{4.48}$$

In the evaluation representation π_z, one gets

$$(\pi_z \otimes \mathrm{id})\mathcal{F}_{12} = (\pi_z \otimes \mathrm{id})\mathcal{F}_{21}^{-1} = \sum_{i=1}^{N} q^{\sum_{j=1}^{N} \alpha_{ij} h_j}\, e_{ii}\,. \tag{4.49}$$

The twist being diagonal and depending only on the finite Cartan generators, we find that equation (4.41) also holds for the Lax matrices $\mathsf{L}'^{\pm}(z)$.

The co-product of the twisted algebra is given by $\Delta^{\mathcal{F}} = \mathcal{F}_{12}\, \Delta\, \mathcal{F}_{12}^{-1}$. A direct calculation shows that

$$\Delta^{\mathcal{F}}\big(\mathsf{L}'^{\pm}_{ij}(z)\big) = \sum_{k=1}^{N} \mathsf{L}'^{\pm}_{kj}(z q^{\mp c^{(2)}/2}) \otimes \mathsf{L}'^{\pm}_{ik}(z q^{\pm c^{(1)}/2})$$

from which it follows that the twisted algebra inherits the same co-product structure as the original algebra.

Applying the twist to the expression (4.45), and using the correspondence (4.48), we get an expression for the quantum determinant in this new presentation. It is again expressed as a sum over permutations:

$$\mathrm{qdet}\,\mathsf{L}(z) = \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma)\, q^{n_\sigma}\, \mathsf{L}'^{+}_{1,\sigma(1)}(z)\dots \mathsf{L}'^{+}_{N,\sigma(N)}(zq^{1-N})\,,$$

where $n_\sigma = \ell(\sigma) + \frac{2}{N}\sum_{i=1}^{N} i(\sigma(i)-i) + \sum_{1\leq i<j\leq N}(\alpha_{\sigma(i),\sigma(j)} - \alpha_{ij})$. A detailed analysis of n_σ, using the explicit expression of the coefficients α_{ij}, shows that it vanishes identically. Then, the quantum determinant is given in the non-elliptic limit by

$$\mathrm{qdet}\,\mathsf{L}'(z) = \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma)\, \mathsf{L}'^{+}_{1,\sigma(1)}(z)\dots \mathsf{L}'^{+}_{N,\sigma(N)}(zq^{1-N})\,. \tag{4.50}$$

Let us remark that the relation (4.50) is based on the (undeformed) antisymmetrizer, contrarily to the expressions found for the homogeneous and principal gradations that are based on q-deformed versions of it. When $N=2$, $R^{(p)}(z)$ and $\mathsf{R}'(z)$ coincide, and only the homogeneous gradation provides a deformed antisymmetrizer.
At this point, you may rightfully ask: Why all this fuss? At the end of the day, one would like to find a way to connect (a) the well-known result from $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ in homogeneous gradation (cf. (4.38)) through the non-elliptic $p\to 0$ limit (4.46) with the quantum elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$. The good news is that this can be achieved, as the next (and final) subsection shows.

4.2.4 Center of the algebra

It is known that in $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ and for generic values of q and c, the quantum determinant, as defined in (4.40), generates the center of this quantum algebra [30]. Since, as an algebra and for generic values of p, q and c, $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ is isomorphic to $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ (e.g. one can match generators one-to-one), (4.40) also describes the full center of the algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$. The same is true for the other two expressions (4.45) and (4.50), which are but different presentations of the same quantum determinant.
Moreover, we have shown that expression (4.1) also lies in the center of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, and that its limit for $p\to 0$ coincides with the quantum determinant of $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ in the non-elliptic presentation (4.50). In other words, the limit $p\to 0$ defines a surjective mapping from a set of elements (defined by the expression (4.1)) in the center of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ to a generating set (defined by (4.50)) of the same center.
Thus, it implies that (4.1) also defines a generating set of the center of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ for generic values of p, q and c. Or, to put it more bluntly, the quantum determinant (4.1) generates the center of the vertex-type quantum elliptic algebra.
Having found the center of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ to be identical with the quantum determinant, we can factor it out by setting it to the value $q^{\frac{c}{2}}$ (remember that c is the central charge), such as to obtain

$$\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N) = \mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)/\left\langle \text{q-det}L(z) - q^{\frac{c}{2}}\right\rangle. \tag{4.51}$$

This defines $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$, and concludes the task we set for ourselves at the beginning of this chapter. The same reasoning applies for $\mathcal{U}_q(\widehat{\mathfrak{sl}}_N)$, with the important difference that in this case, the result has been known for a long time.

Chapter 5

Summary and outlook

We have come a long way and touched on many different aspects of abstract algebras, starting with the well-known Lie algebra structures and working ourselves all the way up to the realm of quantum elliptic objects. As long parts of the journey have been rather technical indeed, it might be worth looking again at the bigger picture to appreciate what was achieved.

We began by familiarizing the reader with affine structures and the concepts of q-deformed algebras, in particular with the illustrative example $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$. We elucidated its fundamental nature by introducing the notion of Hopf algebras, and explained how one can alternatively just define the algebra through exchange relations obtained from matrix solutions to the Yang-Baxter equation (FRT formalism). The subsequent chapter described how $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ can be used to generate a categorically different algebra, named $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$, that is itself related to elliptic solutions of the Yang-Baxter equation. Its properties are best captured by the concept of quasi-Hopf algebras, and we explained how this notion connects to Hopf algebras via Drinfel'd twistors. The final chapter then attempted to answer the question if anything definite can be said about the centers of these algebras as defined through the FRT formalism. We discovered that we can construct an object from the generators, the quantum determinant, and show that it generates the center of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$.

Topics for future research based on these results do not easily suggest themselves, as the findings are, to a large degree at least, self-contained. That being said, there are at least two important and yet unresolved issues we came across. Number one: Can the Drinfel'd twist (3.32) be explicitly calculated in representation for $N > 2$, and shown to lead to the same R-Matrix as the Belavin-Baxter solution (3.1) of the dynamical Yang-Baxter equation (3.10)? Recall that while the construction of a quantum elliptic quasi-Hopf algebra has been carried out at the universal level [35], it remains to be shown that this $\mathcal{R}$-Matrix can be evaluated to obtain Belavin's solution in cases other than $N = 2$. Number two: Contrary to the case of $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$, for example (see [22]), a free field realization - an indispensable tool for the study of correlation functions [15] - of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ is still missing.

Lest the reader concludes this field is mostly a playground for pure mathematicians, let me point to you to some examples of how the study of quantum algebras integrates into the body of (theoretical) physics, and is indeed an object of recent research. For one thing, while it may seem as if most fields where quantum groups play a role - for example, the study of integrable models [53] - stand little chances to be experimentally relevant, the opposite is the case. Around the turn of the millennium, following significant advancements in laser-cooling techniques, former 'toy' models suddenly entered the laboratory [38, 44, 42, 43, 46, 47, 48, 52]. Exact solutions for these models can thus be tested, and many of them have been verified with high precision [54].

More specifically, quantum *elliptic* algebras provide the basic symmetry structure for many models in quantum or statistical mechanics. Examples include the Heisenberg XYZ chain alluded to in the introduction [28] and RSOS ('restricted solid-on-solid') models [14, 37]. To give one concrete example, there exists a scaling limit of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ known as the *Sklyanin algebra* [12, 13] in which the R-Matrix takes the form

$$R(z) = \mathbb{I} \otimes \mathbb{I} + \sum_{\alpha=1}^{3} W_\alpha(z) \sigma_\alpha \otimes \sigma_\alpha,$$

where σ_α are the Pauli matrices, and $W_\alpha(z)$ are elliptic functions which we will not specify here. This algebra encodes the symmetries of the XYZ spin-$\frac{1}{2}$ model [20].

There is a problem, however. While for all these models, just as in the simpler cases of the XXX and XXZ spin chains, integrability has been established as a consequence of their R-Matrices satisfying the Yang-Baxter equation (i.e. we know that there are just as many commuting charges as there are degrees of freedom), the much bigger challenge is to use this knowledge for actual calculations. The standard algebraic Bethe ansatz used to determine eigenstates and eigenvalues of the Hamiltonian, as briefly outlined in the beginning, ceases to work for those models. In very broad terms, the structure of the R-Matrix prevents us from constructing a suitable reference state - called the pseudo-vacuum - that is necessary to generate the energy spectrum. Hopefully, we will see some progress on this front in the future.

One could go on and on here, point out the connection between $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)$ and structures known as *deformed Virasoro algebra* and *Deformed* W_N *algebras* [55], and then explain how these, in turn, link to Macdonald polynomials [29, 31], extensions of AGT duality [49] or $N = 2$ superconformal gauge theories in five dimensions [50]. But we are confident that we provided the reader with enough reasons - both intrinsic and functional - that merit the study of these objects, and hope that this thesis will provide a starting point for further investigations.

Appendix A

Jacobi Theta functions

Jacobi Theta functions naturally arise in the context of elliptic quantum algebras when performing a Drinfel'd twist on the underlying quantum affine structures. Indeed, it is because of their appearance that we speak about *elliptic* algebras in the first place. Since many of the calculations performed in this thesis really heavily on the properties of those Theta functions, we will briefly review their main characteristics here.
Let $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}$ be the upper half-plane and $\Lambda_\tau = \{\lambda_1 \tau + \lambda_2 \,|\, \lambda_1, \lambda_2 \in \mathbb{Z}\,, \tau \in \mathbb{H}\}$ the lattice with basis $(1, \tau)$ in the complex plane. Furthermore, denote by $\mathbb{Z}_N \equiv \mathbb{Z}/N\mathbb{Z}$ the congruence ring modulo N with basis $\{0, 1, \ldots, N-1\}$, and set $\omega = e^{2i\pi/N}$. Finally, for any pairs $\gamma = (\gamma_1, \gamma_2)$ and $\lambda = (\lambda_1, \lambda_2)$ of numbers, we define a pairing $\langle \gamma, \lambda \rangle \equiv \gamma_1\lambda_2 - \gamma_2\lambda_1$, which is manifestly skew-symmetric, i.e. $\langle \gamma, \lambda \rangle = -\langle \lambda, \gamma \rangle$.
There are several definitions of Jacobi Theta functions. For our analysis, we rely on the following expression, defined for rational characteristics $\gamma = (\gamma_1, \gamma_2) \in \frac{1}{N}\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$ by:

$$\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau) = \sum_{m \in \mathbb{Z}} \exp\Big(i\pi(m + \gamma_1)^2 \tau + 2i\pi(m + \gamma_1)(\xi + \gamma_2) \Big) \,. \tag{A.1}$$

On can show that the functions $\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau)$ satisfy the following shift properties:

$$\vartheta \begin{bmatrix} \gamma_1 + \lambda_1 \\ \gamma_2 + \lambda_2 \end{bmatrix} (\xi, \tau) = \exp(2i\pi\gamma_1\lambda_2)\, \vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau)\,, \tag{A.2}$$

$$\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi + \lambda_1\tau + \lambda_2, \tau) = \exp(-i\pi\lambda_1^2\tau - 2i\pi\lambda_1\xi)\, \exp(2i\pi\langle\gamma, \lambda\rangle)\, \vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau)\,, \tag{A.3}$$

where $\gamma = (\gamma_1, \gamma_2) \in \frac{1}{N}\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z} \times \mathbb{Z}$.
If we give up the assumption that $\lambda = (\lambda_1, \lambda_2)$ are integers and allow them to take on arbitrary values, we can still find one other shift property:

$$\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi + \lambda_1\tau + \lambda_2, \tau) = \exp(-i\pi\lambda_1^2\tau - 2i\pi\lambda_1(\xi + \gamma_2 + \lambda_2))\, \vartheta \begin{bmatrix} \gamma_1 + \lambda_1 \\ \gamma_2 + \lambda_2 \end{bmatrix} (\xi, \tau)\,.$$

This is not yet the Jacobi Theta function that we will be working with. Instead, whenever the main text makes a reference to Theta functions, we have in mind the product

$$\Theta_p(z) = (z; p)_\infty\, (pz^{-1}; p)_\infty\, (p; p)_\infty \tag{A.4}$$

where the infinite multiple products are defined by

$$(z;p_1,\ldots,p_m)_\infty = \prod_{n_i\geq 0}(1-zp_1^{n_1}\ldots p_m^{n_m}).$$

The relation between this Jacobi Theta function and the one with rational characteristics $(\gamma_1,\gamma_2)\in\frac{1}{N}\mathbb{Z}\times\frac{1}{N}\mathbb{Z}$ as defined in (A.1) is not apparent. Still, it is possible to show that they are indeed related, more precisely through the following equation:

$$\vartheta\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}(\xi,\tau) = (-1)^{2\gamma_1\gamma_2}\, p^{\frac{1}{2}\gamma_1^2}\, z^{2\gamma_1}\,\Theta_p(-e^{2i\pi\gamma_2}p^{\gamma_1+\frac{1}{2}}z^2)\,,$$

where $p=e^{2i\pi\tau}$ and $z=e^{i\pi\xi}$.

The Jacobi $\Theta_{a^2}(z)$ function enjoys a set of properties that is of central importance to our calculations, easily verified starting with expression (A.4):

$$\Theta_{a^2}(a^2z)=\Theta_{a^2}(z^{-1})=-\frac{\Theta_{a^2}(z)}{z}\quad\text{and}\quad\Theta_{a^2}(az)=\Theta_{a^2}(az^{-1}).\tag{A.5}$$

From the definition (A.4) and these equalities, it is not hard to show that the following identities also hold:

$$\begin{aligned}\Theta_p(x)\Theta_p(-x) &= \Theta_{p^2}(x^2)\frac{(p,p)_\infty^2}{(p^2,p^2)_\infty}\\ \Theta_{p^N}(x)\Theta_{p^N}(px)\ldots\Theta_{p^N}(p^{N-1}x) &= \Theta_p(x)\frac{(p^N,p^N)_\infty^N}{(p,p)_\infty}\\ \Theta_{p^N}(p)\Theta_{p^N}(p^2)\ldots\Theta_{p^N}(p^{N-1}) &= (p,p)_\infty^2(p^N,p^N)_\infty^{N-3}\end{aligned}\tag{A.6}$$

Despite these properties, Theta functions are notoriously hard to handle; a tribute to their elliptic nature. What saves the day in most cases cases is the following

Theorem A.1 Let $f:\mathbb{H}\to\mathbb{C}$ be a function holomorph in $\mathbb{C}^*$ satisfying $f(pz)\propto f(z)/z^m$. Such a function is said to be p-quasiperiodic. Then there exist m roots $a_1,\ldots a_m$ such that $f(a_i)=0\ \forall i$, and if $a_i\neq p^n$ for all $i=1,\ldots,m$, $n\in\mathbb{Z}^*$, then $f(z)$ can be re-expressed in product form:

$$f(z)=g(z)\prod_{i=1}^{m}\Theta_p\left(\frac{z}{a_i}\right),\tag{A.7}$$

where $g(z)$ is a nome in z [58].

An important consequence of this is the

Corollary A.2 Let f be as before, in particular, let $f(pz)\propto z^{-m}f(z)$. If one succeeds in finding $m+1$ roots a_i with $f(a_1)=\cdots=f(a_{m+1})=0$, $a_i\neq p^n$ for all $i=1,\ldots,m$, $n\in\mathbb{Z}^*$, we can directly conclude that $f(z)=0$.

Appendix B

Example: Explicit application of Theta function theorem

The theorem about meromorphic functions from the previous appendix is an extremely powerful tool to tame otherwise uncontrollable sums of products of Theta functions. Many of the results we arrived at, in particular the disentangled expression for the R-Matrix after applying a Drinfel'd twist to it (cf. (3.43) and (3.45)), were due to this theorem. It is thus worthwhile to lay down how exactly this works for a concrete example. Here, we have chosen to prove that in the principal gradation of the $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$ algebra, $\pi(\text{q-det}L(z)) = 1$. Since the quantum determinant is diagonal in evaluation representation, we can begin by spelling out the first matrix entry (4.29) explicitly:

$$m_1(z^2) = \frac{(p^2,p^2)_\infty^6}{(p,p)_\infty^6} \frac{\Theta_p^2(q^2)\Theta_p(z^2)\Theta_{p^2}(p^2q^2z^2)}{q^2\Theta_p(q^2z^2)\Theta_{p^2}(pq^{-2}z^2)} \Big(\frac{\Theta_{p^2}(pz^2)}{\Theta_{p^2}(p^2z^2)\Theta_{p^2}^2(p^2q^2)} - \frac{\Theta_{p^2}(z^2)}{\Theta_{p^2}(pq^2)\Theta_{p^2}(p^3q^2)\Theta_{p^2}(pz^2)} \Big)$$

We multiply the term in brackets with the smallest common denominator. This leads to the following expression:

$$m_1(z^2) = \frac{(p^2,p^2)_\infty^2}{(p,p)_\infty^4} \left(\frac{q^2\Theta_{p^2}^2(pz^2)\Theta_{p^2}^2(pq^2) - pz^{-2}\Theta_{p^2}(z^2)\Theta_{p^2}(q^2)}{q^2\Theta_{p^2}(pq^2z^2)\Theta_{p^2}(pq^{-2}z^2)} \right)$$

It turns out that we can apply theorem (A.1) to the nominator, which we will call $f(z^2)$ for convenience: One finds that $f(p^2z^2) = p^{-2}z^{-4}f(z^2)$, and the two roots z_1^2, z_2^2 are determined to be $p^{-1}q^{\pm 2}$. This means that

$$f(z^2) = g(z^2)\Theta_{p^2}(pq^2z^2)\Theta_{p^2}(pq^{-2}z^2)$$

with $g(z^2)$ being a power of z^2. To specify its exact form, p^2-periodicity comes to the rescue. On the one hand, we have

$$f(p^2z^2) = \frac{f(z^2)}{p^2z^4} = \frac{g(z^2)\Theta_{p^2}(pq^2z^2)\Theta_{p^2}(pq^{-2}z^2)}{p^2z^4}.$$

On the other hand, we can just as well write

$$f(p^2z^2) = g(p^2z)\Theta_{p^2}(p^3q^2z^2)\Theta_{p^2}(p^3q^{-2}z^2) = \frac{g(p^2z)\Theta_{p^2}(pq^2z^2)\Theta_{p^2}(pq^{-2}z^2)}{p^2z^4}$$

Comparison yields $g(p^2z^2) = g(z^2)$, i.e. $g(z^2) = \text{const}$. Finally, to find the constant of proportionality, we compare the original expression for $f(z^2)$ with the one we found using the theorem at a specific point, say $z = 1$. This leads to

$$f(z^2) = q^2\Theta_{p^2}^2(p)\Theta_{p^2}(pq^2z^2)\Theta_{p^2}(pq^{-2}z^2)$$

and finally, using (A.6), we find $m_1(z^2) = 1$. Since all diagonal entries are identical (as shown in (4.31)), we infer $\pi(\text{q-det}L(z) = 1$. ∎

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Selbstständigkeitserklärung

Ich, Daniel Wolfgang Issing, versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Annecy-le-Vieux, den