
BLACK HOLE INFORMATION AND THERMODYNAMICS

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ABSTRACT

THESE LECTURE NOTES ARE BASED ON A MASTER COURSE ON BLACK HOLES, THEIR THERMODYNAMICS AND THE BLACK HOLE INFORMATION PROBLEM, WHICH WAS GIVEN BY DIETER LÜST DURING THE SUMMER TERM 2017 AT THE LUDWIG-MAXIMILIANS-UNIVERSITÄT IN MUNICH. THE NOTES WERE PREPARED AND WRITTEN BY WARD VLEESHOUWERS. THEY PROVIDE A SHORT INTRODUCTION TO GENERAL RELATIVITY AND DISCUSS BLACK HOLE SOLUTIONS, PENROSE DIAGRAMS, BLACK HOLE THERMODYNAMICS AND ENTROPY, THE UNRUH EFFECT AND HAWKING RADIATION, THE BLACK HOLE INFORMATION PROBLEM, BLACK HOLES IN SUPERGRAVITY AND STRING THEORY, THE BLACK HOLE MICROSTATE COUNTING IN STRING THEORY, AND SOME ASPECTS OF CURRENT RESEARCH ON ASYMPTOTIC SYMMETRIES IN GENERAL RELATIVITY WITH EMPHASIS ON BLACK HOLES AND SOME ASPECTS OF A PARTICULAR QUANTUM MODEL FOR BLACK HOLES.

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1 Special relativity

In non-relativistic settings, the symmetry group of space-time is the *Galilean group*, which consists of rotations and translations. These transformations leave spatial distances (as well as temporal intervals) invariant. For example, in two spatial dimensions with coordinates (x, y) , the squared distance $s^2 = (\Delta x)^2 + (\Delta y)^2$ is invariant under rotations, which are of the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The invariance of spatial (Euclidean) distance is then given by $s'^2 = (\Delta x')^2 + (\Delta y')^2 = s^2$.

In the context of special relativity, our notion of invariant distance changes. Namely, if we consider a $(3 + 1)$ -dimensional space i.e. three spatial and one temporal dimension parametrized by (t, x, y, z) , we define the *Minkowski distance*:

$$s^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \quad (1)$$

where c is the speed of light. The group that leaves this distance invariant is called the *Poincaré group*, which is the addition of boosts to the Galilean group. Distances in special relativity are no longer positive semi-definite, namely, we distinguish:

1. $s^2 > 0$ (III) Space-like distance
2. $s^2 = 0$ (II) Light-like distance
3. $s^2 < 0$ (I) Time-like distance

If we consider a line segment parametrized in (t, x) , the Roman numerals above correspond to the regions in the space-time given below

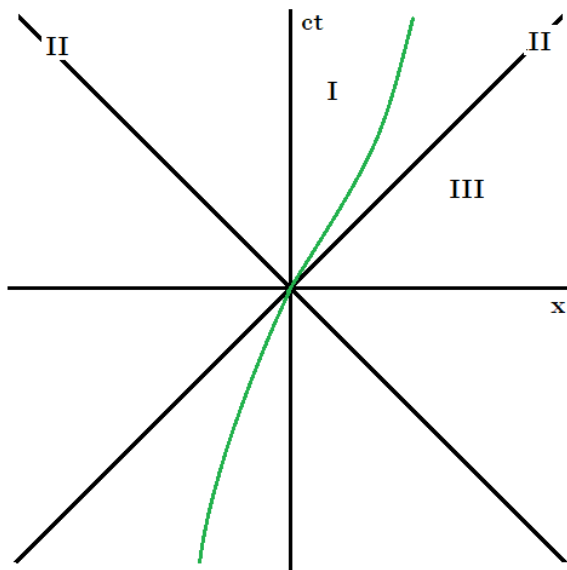


Figure 1: An example of a space-time diagram. The region indicated by Roman numeral *I* (*III*) consists of points at time-like (space-like) distance from the origin i.e. $s^2 < 0$ ($s^2 > 0$). *II* indicates light-like lines, which separate time-like and space-like regions. The green line indicates an example of a time-like trajectory with $x(t = 0) = 0$.

For intervals with $s^2 < 0$, the (squared) *proper time* τ is given by:

$$\tau^2 = -\frac{s^2}{c^2} \quad (2)$$

Henceforth we will use *four-vector notation* $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and denote spatial coordinates as $x^i = (x^1, y^2, z^3)$. The *Minkowski metric* is then given by:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3)$$

The Minkowski distance can then be conveniently be written as:

$$s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \quad (4)$$

We also employ *Einstein summation convention*:

$$x_\mu x^\mu := \sum_\mu x_\mu x^\mu$$

I.e. we omit the summation symbol for repeated indices. The symmetry transformations that make up the Poincaré group are then written as:

1. $x^\mu \mapsto \tilde{x}^\mu = x^\mu + a^\mu$ (translations)
2. $x^\mu \mapsto \tilde{x}^\mu = \Lambda_\nu^\mu x^\nu$ (boosts & rotations)

From the invariance of Minkowskian distance under boosts and rotations, we find:

$$\begin{aligned} s^2 &= (\Delta x)^T \eta \Delta x \\ &= (\Delta \tilde{x})^T \eta \Delta \tilde{x} \\ &= (\Delta \tilde{x})^T \Lambda^T \eta \Lambda \Delta \tilde{x} \quad , \quad \eta' := \Lambda^T \eta \Lambda \end{aligned} \quad (5)$$

The condition above is written in tensor notation as:

$$\eta_{\rho\sigma} = \Lambda_\rho^\mu \Lambda_\sigma^\nu \eta_{\mu\nu} \quad (6)$$

We thus conclude that $\Lambda \in O(3, 1)$, the (3 + 1)-dimensional orthogonal group. The Poincaré group is the semi-direct product of the *Lorentz group* and (3 + 1)-dimensional translations. Elements of the Lorentz group include:

1. Rotations (in x, y -plane): $\Lambda_\nu^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $0 \leq \theta < 2\pi$
2. Boosts (in x -direction): $\Lambda_\nu^\mu = \begin{pmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $-\infty < \phi < \infty$

One can easily check that these matrices satisfy $\eta = \Lambda^T \eta \Lambda$. We see that a boost transforms our coordinates as

$$\begin{aligned} \tilde{t} &= t \cosh \phi - x \sinh \phi \\ \tilde{x} &= x \cosh \phi - t \sinh \phi . \end{aligned} \quad (7)$$

For paths with $\tilde{x} = 0$, the speed in our original reference frame is:

$$v = \frac{x}{t} = \frac{\sinh \phi}{\cosh \phi} = \tanh \phi \Rightarrow \phi = \tanh^{-1} v$$

so that we find:

$$\begin{aligned} \tilde{t} &= \gamma(t - vx) \\ \tilde{x} &= \gamma(x - vt) \quad , \quad \gamma := \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned} \tag{8}$$

We see that $s^2 = 0$ for a trajectory with $v = c$. From now on we set $c = 1$, so that light-like trajectories are given by $x = \pm t$. Such trajectories are invariant under Lorentz transformations. Time-like trajectories have $v < c$, while space-like trajectories have $v > c$. Hypothetical particles which have $v > c$ are called *tachyons*, which are generally considered not to exist as propagating degrees of freedom.

2 Riemannian geometry

We consider $(d+1)$ -dimensional *smooth manifolds* \mathcal{M} , which are topological manifold which looks locally like \mathbb{R}^n i.e. they are locally flat. \mathcal{M} can be covered by open sets $U_i, i \in I$, where I is some indexing set. The *charts* which are then defined as bijective maps $\phi : U_i \rightarrow \mathbb{R}^{1,d}$ with the requirement that for $U_i \cap U_j \neq \emptyset$, the *transition function* $\phi_i \circ \phi_j^{-1}$ is C^∞ . An *Atlas* is a collection of charts which cover all of \mathcal{M} .

At each point $p \in \mathcal{M}$, we can define a *tangent space* $T_p\mathcal{M}$. The basis of $T_p\mathcal{M}$ is written as $\{\hat{e}_{(\mu)}\}$, so that any vector can be written as $V = V^\mu(x^\nu) \hat{e}_{(\mu)}$. As we saw in the previous lecture, Lorentz transformation acts as: $x^\mu = \Lambda^\mu_\nu x^\nu$. Since $\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$ and $V^\mu \partial_\mu = V^{\mu'} \partial_{\mu'}$, it follows that $V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$. We thus see that that $\hat{e}_{(\mu)} = \partial_\mu$, i.e. the tangent space is spanned by partial derivatives defined at p . The dual space to the tangent space is called the *cotangent space*, which is denoted by $T_p^*\mathcal{M}$. It is dual to $T_p\mathcal{M}$ in the sense that its basis covectors $\hat{\theta}^{(\mu)}$ satisfy $\hat{\theta}^{(\mu)} \hat{e}_{(\nu)} = \delta^\mu_\nu$. Covectors are then expressed as $w = w_\mu(x^\nu) \hat{\theta}^{(\mu)}$. We see that $\hat{\theta}^{(\mu)} = dx^\mu$, the differential forms. The disjoint union of all $T_p\mathcal{M}$ over all $p \in \mathcal{M}$ is called the *tangent bundle* over \mathcal{M} and is denoted by $T\mathcal{M}$. The collection of all $T_p\mathcal{M}$ over all $p \in \mathcal{M}$ is then called the *cotangent bundle* over \mathcal{M} , denoted by $T^*\mathcal{M}$.

2.1 Tensors

Tensors are, roughly speaking, higher-dimensional generalizations of vectors and coverctors. A (k,l) -tensor is denoted as $T_{(l)}^{(k)} = T_{\mu_1, \mu_2, \dots, \mu_l}^{\mu_1, \mu_2, \dots, \mu_k} \partial_{\mu_1} \otimes \partial_{\mu_2} \dots \partial_{\mu_l} \otimes dx^{\nu_1} \otimes dx^{\nu_2} \dots dx^{\nu_l}$. These are elements of $\underbrace{T_p\mathcal{M} \otimes \dots \otimes T_p\mathcal{M}}_k \otimes \underbrace{T_p^*\mathcal{M} \otimes \dots \otimes T_p^*\mathcal{M}}_l$.

Important examples include the Minkowski metric we encountered before, as well as the Riemann and Ricci tensors, to be considered in due time.

2.2 Differential forms

Differential forms are $(0,p)$ -tensors i.e. co-vectors with completely antisymmetrized indices. They are written as: $A_{(p)} = A_{[\mu_1 \mu_2 \dots \mu_p]} dx_1 \wedge dx_2 \wedge \dots \wedge dx_p$

The lowest-dimensional examples of p-forms are:

1. 0-form: scalar $\phi(x^\mu)$
2. 1-form: vectors $A_{(1)} = A_\mu dx^\mu$ e.g. the electromagnetic potential
3. 2-form: $F_{(2)} = F_{\mu\nu} dx^\mu \wedge dx^\nu$ e.g. the electromagnetic field strength.

We can construct $(p+1)$ -forms out of p-forms by applying the *exterior derivative* $F_{(p+1)} = dA_{(p)}$, which is how we construct the electromagnetic field strength out of the vector potential: $F_{(2)} = dA_{(1)} = \partial_\mu A_\nu dx^\mu \wedge dx^\nu$

Closed forms satisfy $dF_{(p)} = 0$, *exact forms* satisfy $F = dA$. It is easy to show that $ddA = 0$ for any A because of the antisymmetrization of the differential forms. Hence dA is automatically closed for any A . An important question is whether there exist closed forms which are not exact. Forms with this property form the basis of the cohomology group of our manifold.

3 Introduction to general relativity

A p-form $F_{(p)}$ is topologically equivalent to $F_{(n)}$ if they differ by an *exact* form. Space of closed p-forms is denoted by $Z^p(M) = \{F^{(p)} | dF^{(p)} = 0\}$ Space of exact p-forms is denoted by $B^p(M) = \{F^{(p)} | F^{(p)} = dA^{(p-1)}\}$. The *De Rham cohomology* is then defined as $H^p(M) = Z^p(M)/B^p(M) = \{F^{(p)} | \text{closed but not exact}\}$.

We consider an example of a physical application of de Rham cohomology. We saw that the electromagnetic field strength $F_{\mu\nu}$ can be written in form notation as: $F^{(2)} = F_{\mu\nu} dx^\mu \wedge dx^\nu$, of which the matrix form is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

The Bianchi identity is then given by $dF^{(2)} = 0$. Writing the gauge potential in form notation, $A^{(1)} = A_\mu dx^\mu$, we see that $F^{(2)} = dA^{(1)} = \partial_\mu A_\nu dx^\mu \wedge dx^\nu$, hence $F^{(2)}$ is exact in the vacuum. The Bianchi identity is written in terms of the vector potential as $dF^{(2)} = d^2 A^{(1)} = \partial_\rho \partial_\mu A_\nu dx^\mu \wedge dx^\nu \wedge dx^\rho = 0$. Gauge transformations are parametrized by a scalar $\chi^{(0)}$, gauge equivalent potentials: $A'_\mu \sim A_\mu - \partial_\mu \chi$. In form notation, this is written as $A^{(1)} = A^{(1)} - d\chi^{(0)}$. For a non-trivial manifold we can have two interesting situations.

1. Magnetic monopole: configuration with finite magnetic charge density $\rho(x^\mu \rightarrow \vec{B}(\vec{x}) \sim \frac{\rho}{r^2}$, so that $dF^{(2)} = \rho(\vec{x})$ (Gauss' law). This configuration cannot be covered by a single chart due to the presence of a *Dirac string* where $F^{(2)} \neq dA^{(1)}$. We therefore have to introduce multiple coordinate patches so that $F^{(2)} = dA^{(1)}$ on each patch, but we cannot introduce a single patch that covers the entire space such that $F^{(2)} = dA^{(1)}$ holds.
2. Aharonov-Bohm effect: The relevant manifold is $\mathcal{M}^{(1,3)} = \mathbb{R}^{(1,3)}/\text{cylinder}$, where the cylinder is the site of a solenoid. Outside this solenoid the magnetic field is zero, inside the solenoid the magnetic field need not be zero. Then: $A^{(1)} = d\chi^{(0)}$, but, by Stokes' theorem, $\oint_{\mathcal{C}} d\chi^{(0)} = \int_{\mathcal{S}} F = \Phi$, where \mathcal{C} is a closed curve which encloses the cylinder, \mathcal{S} is a surface such that $\mathcal{C} = \partial\mathcal{S}$, and Φ is the magnetic flux through \mathcal{S} . The wave function of a charged particle is given by $\psi(x) = e^{i\frac{e}{\hbar}\chi(x)}\psi_0(x)$. A gauge transformation then acts as $A^\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \tilde{\chi}$, $\psi \rightarrow \psi' = e^{i\frac{e}{\hbar}(\chi(x)+\tilde{\chi})}\psi$. We demand the wave function is single valued, which gives the *Dirac quantization condition*.

We can define a *dual electromagnetic field tensor* $F_{\mu\nu}^* = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. Then, $dF_{\mu\nu}^* = 4\pi j_e^*$, with j_e^* the magnetic current. This gives electric-magnetic duality.

3.1 General relativity and local coordinate transformations

GR can be (loosely) seen as the gauge theory of local coordinate transformations. The basis of GR is the *equivalence principle*, which states that physics should be independent of the choice of coordinate systems. This can already be seen in Newtonian dynamics, the inertial mass of a particle is equal to its gravitational mass.

Einstein considered a thought experiment where we have an observer in a box. The observer should not be able to distinguish between gravitational and inertial acceleration by means of local experiments i.e. if the box is small enough the observer cannot distinguish between these two types of acceleration. This is known as the weak equivalence principle. However, if we put two observers in a large enough box, they will be able to distinguish between gravitational and inertial acceleration since they can measure a gradient in the acceleration in the case of gravitational acceleration, but not in the case of inertial acceleration (assuming that the metric is independent of time). This of course constitutes a non-local experiment.

We now consider changes of the coordinate system due to constant accelerations. Assume the trajectory of an observer \mathcal{O} is given by a path in Minkowski space given by $x^\mu(\tau)$, where τ is the proper time of the observer. Hence we have the velocity vector $u(\tau) = \frac{d}{d\tau} x^\mu$ and acceleration $a(\tau) = \frac{d^2}{d\tau^2} x^\mu$. We then have $a_\mu u^\mu = \eta_{\mu\nu} a^\mu u^\nu = 0$. In the case of constant acceleration, we have $a^\mu = (0, a, 0, 0)$, where a is some constant. Taking boundary condition $x^\mu(\tau = 0) = (0, a^{-1}, 0, 0)$, a simple calculation then gives

$$x^\mu(\tau) = (a^{-1} \sinh(a\tau), a^{-1} \cosh(a\tau), 0, 0)$$

This is referred to as ‘fake gravity’, since the gravitational force is entirely due to the inertial acceleration of the particle and not due to a mass distribution. We can obtain the same result by performing a coordinate transformation on Minkowski space-time. Starting from flat Minkowski coordinates (x^0, x^1, x^2, x^3) we go to hyperbolic coordinates, which we write as (η, ρ, x^2, x^3) . The coordinate systems are related as

$$\begin{aligned}x^0 &= \rho \sinh(\eta) \\x^1 &= \rho \cosh(\eta) .\end{aligned}\tag{9}$$

The line element is then written as

$$\begin{aligned}ds^2 &= -(dx^0)^2 + d\vec{x}^2 \\&= -\rho^2 d\eta^2 - d\rho^2 + (dx^2)^2 + (dx^3)^2 .\end{aligned}\tag{10}$$

This space is known as Rindler space [1]. As stated above, we require that the physics is invariant under such coordinate transformations.

We now consider a physical gravitational force. Starting from some metric denoted as $g_{\mu\nu}(x)$, the line element is given by $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Since the metric is symmetric, we have $\frac{d}{2}(d+1)$ independent components in d dimensions, i.e. 10 independent components in $(3+1)$ dimensional space. In the case of physical gravitation (i.e. induced by matter distributions) there does not exist a coordinate transformation of the form $x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu$ such that we retrieve the flat Minkowski metric.

The strong equivalence principle states that particles move on the geodesics of a curved space-time manifold \mathcal{M} independent of the nature of the considered particle (in the limit where we ignore the back-reaction of the particle energy on the metric). This can be stated in words by saying that gravity is equivalent to curvature of space-time. Next time, we will see that, in the Einstein formulation of gravity, the geometry is invariant under local coordinate transformations.

4 General relativity - continued

4.1 Equivalence principles

We distinguish the following equivalence principles:

1. Newton's equivalence principle (which we retroactively denote as such for consistency in nomenclature): Inertial mass = gravitational mass
2. Weak equivalence principle: Gravitational and inertial acceleration are locally indistinguishable
3. Strong equivalence principle: Gravity corresponds to geodesics on a curved manifold (ignoring back-reaction on the metric of particles traveling along geodesics).

As an example we again consider Rindler space, given by:

$$\begin{aligned} t &= \rho \sinh(a\eta) \\ x &= \rho \cosh(a\eta) \end{aligned} \tag{11}$$

Where $\eta = \pm \frac{1}{a} \operatorname{arctanh}\left(\frac{t}{x}\right)$ $\rho = \pm \sqrt{x^2 - t^2}$. The plus and minus in ρ correspond to the two 'external' regions of the corresponding Penrose diagram, which are connected by space-like geodesics. We will return to this point in a later lecture. The above coordinate transformation gives:

$$ds^2 = (-a^2 \rho^2 d\eta^2 + d\rho^2) + (dx^2)^2 + (dx^3)^2$$

Observers at rest in Rindler space (i.e. at constant ρ) are constantly accelerated (and are thus hyperbolae) in Minkowski space. In a curved space-time, there does not exist a globally defined coordinate transformation which transforms $g_{\mu\nu}(x) \rightarrow \eta_{\mu\nu}$, namely, one cannot define a global chart for a curved manifold. The light cones that emanate from the origin are called *Rindler horizons*. They are given by: $\rho = 0, \eta = \infty$ and $\rho = 0, \eta = -\infty$. Hence they are never reached by locally inertial observers in Rindler coordinates i.e. observers at rest in Rindler space.

4.2 Curved manifolds

From the Riemannian metric $g_{\mu\nu}(x)$ we construct the Christoffel connection $\Gamma_{\mu\nu}^\rho$ and from that the Riemann and Ricci tensor as well as the Ricci scalar. The Christoffel connection gives us covariant derivative for parallel transport, Ricci tensor is the field strength of our connection, Ricci scalar gives a measure of curvature.

In general gauge theories (e.g. electromagnetism), we demand that the action is invariant under local transformations $\psi \rightarrow e^{ie\Lambda(x)}\psi(x)$. This requires us to replace $\partial_\mu \rightarrow D_\mu := \partial_\mu - ieA_\mu(x)$ with $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\Lambda(x)$. The field strength is then given by $[D_\mu, D_\nu]$. In general relativity, the 'gauge transformation' is of the form $x^\mu \rightarrow x^\mu + \xi^\mu(x)$ i.e. a local transformation of our coordinates. We then have $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda$ and $[\nabla_\mu, \nabla_\nu] \sim R_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor, as we will see below. For usual gauge theories, we consider gauge fields A_μ^i , where i is a group index. In GR we have an object $\Gamma_{\mu\nu}^\rho$, where the group is local $SO(1,3)$. We can raise one index on Γ to get $\Gamma_{\mu}^{\rho\nu}$, the indices ρ and ν together form a group index of $SO(1,3)$.

We consider S^2 , which is an example of a curved space. Starting from $ds^2 = dx^2 + dy^2 + dz^2$, we embed S^2 in \mathbb{R}^3 : $x = r \cos \theta \sin \varphi$, $y = r \sin \theta \sin \varphi$, $z = r \cos \theta$. This gives:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

To get S^2 , we fix $r = 1$. The unit normal vector on S^2 is $\vec{x}_n(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. We now construct the tangent vector. Two basis vectors of tangent space $T_p S^2$ are:

$$\begin{aligned} \vec{u}_1 &= \partial_\theta \hat{x} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta) \\ \vec{u}_2 &= \partial_\varphi \hat{x} = (-\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0) \end{aligned}$$

One can easily check by taking the inner product that these are orthogonal to the normal vector. We see from the line element that the metric is given by:

$$g_{\mu\nu}(\theta, \varphi) = \vec{u}_\mu \vec{u}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$$

We would like to define covariant derivatives which are intrinsic to manifold i.e. which can be found without referring to the embedding of our manifold in some other so. We first calculate:

$$\begin{aligned} \partial_\theta \vec{u}_1 &= (-\sin \theta \cos \varphi, -\sin \theta \sin \varphi, -\cos \theta) = -\vec{x}_n \\ \partial_\varphi \vec{u}_1 &= \partial_\theta \vec{u}_2 = (-\cos \theta \sin \varphi, \cos \theta \sin \varphi, 0) = \cot \theta \vec{u}_2 \\ \partial_\varphi \vec{u}_2 &= (-\sin \theta \cos \varphi, -\sin \theta \sin \varphi, 0) = -\sin^2 \theta \vec{x}_n - \cos \theta \sin \theta \vec{u}_1 \end{aligned}$$

We now drop the normal components to the derivatives of the vectors we just calculated i.e.

$$\nabla_{u_1} \vec{u}_1 = 0 \quad , \quad \nabla_{u_1} \vec{u}_2 = \nabla_{u_2} \vec{u}_1 = \cot \theta \vec{u}_2 \quad , \quad \nabla_{u_2} \vec{u}_2 = -\cos \theta \sin \theta \vec{u}_1$$

From $D_{u_\mu} u_\lambda = u_\nu \Gamma_{\mu\lambda}^\nu$, we find:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \cot \theta \quad , \quad \Gamma_{22}^1 = -\cos \theta \sin \theta$$

so that:

$$\begin{aligned} \nabla_\mu V^\nu &= \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda \\ \nabla_\mu w_\nu &= \partial_\mu w_\nu - \Gamma_{\mu\nu}^\lambda w_\lambda \end{aligned} \tag{12}$$

This is the general action of the covariant derivative ∇_μ on a vector V^ν and a form w_ν

4.2.1 Parallel transport, geodesics, and curvature

If V^λ transform as a vector under local Lorentz transformations, then $\nabla_\mu V^\lambda$ transforms as a tensor under coordinate transformations while $\partial_\mu V^\lambda$ does not. We define the *torsion tensor* as $T_{\mu\nu}^\lambda := \Gamma_{[\mu\nu]}^\lambda$. A connection is called *torsion-free* if $T_{\mu\nu}^\lambda = 0$. Further, a connection is *metric-compatible* if $\nabla_\rho g_{\mu\nu}(x) = 0$. The *Christoffel connection* is given by: $\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$, this connection is torsion-free and metric-compatible.

Parallel transport of a vector V^μ along an infinitesimally small path will give the Riemann tensor. We use $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu$, so that parallel transport is given by $\frac{d}{d\lambda} V^\mu + \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\lambda} V^\rho = 0$. As an example, we parallel V^μ along a curve given by $x^\nu(\lambda)$, which is a closed curve from $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$ back to $(1, 0, 0)$. The path is divided up into smooth sections A, B, C as:

$$x_A(\lambda) = (\cos \lambda, \sin \lambda, 0) \quad , \quad x_B(\lambda) = (0, \sin \lambda, -\cos \lambda) \quad , \quad x_C(\lambda) = (-\sin \lambda, 0, -\cos \lambda) \quad \text{with } \lambda_A \in \left[0, \frac{\pi}{2}\right] .$$

Infinitesimally, we have:

$$[\nabla_\mu, \nabla_\nu] V^\rho = R_{\sigma\mu\nu}^\rho V^\sigma - \Gamma_{\mu\nu}^\lambda \nabla_\lambda V^\rho$$

The second term will be zero because of antisymmetry for Riemannian connection, as we will see next time.

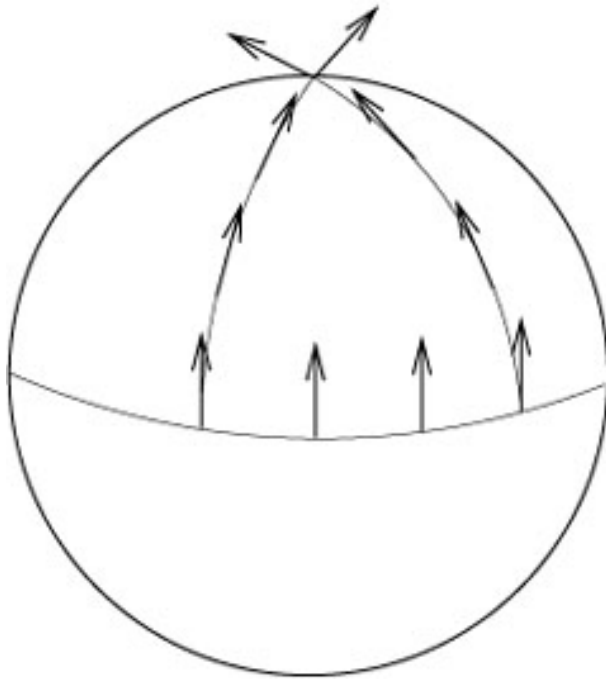


Figure 2: Parallel transport of a vector along a sphere, as described in the accompanying text. Image adapted from [2]

5 General relativity and gravitation

5.1 Christoffel connection

The Christoffel connection is associated to a covariant derivative acting on tensors. In familiar gauge theories, the partial derivative is replaced by a covariant derivative as: $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu(x)$. In general relativity, the covariant derivative acts as $\partial_\mu \rightarrow \nabla_\mu = (\partial_\mu + \Gamma_{\mu\lambda}^\nu)$, where \circ indicates that multiplication is tensorially non-trivial, see 12. As we saw, the Christoffel connection is given by $\Gamma_{\mu\nu}^\sigma = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$.

5.1.1 Parallel transport

Parallel transport of a vector V^μ satisfies

$$\frac{d}{d\lambda}V^\mu + \Gamma_{\rho\sigma}^\mu \frac{dx^\sigma}{d\lambda}V^\rho = 0 .$$

A special case of parallel transport involves the geodesics on \mathcal{M} . Take a tangent vector on \mathcal{M} given by $V^\mu = \frac{dx^\mu}{d\lambda}$, we find:

$$\frac{d^2x(\lambda)}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \quad (13)$$

This is known as the *geodesic equation*.

5.1.2 Curvature: the ‘field strength’ of Γ

As we saw, in a typical gauge theory the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -i[D_\mu, D_\nu]$. In general relativity, the equivalent object is the curvature (Ricci tensor), given by:

$$[\nabla_\mu, \nabla_\nu]V^\rho = \underbrace{R_{\sigma\mu\nu}^\rho V^\sigma}_{\text{curvature}} - \underbrace{\Gamma_{\mu\nu}^\lambda \nabla_\lambda V^\rho}_{\text{torsion}}$$

Usually we take the torsion to equal zero. Working out the terms gives:

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda \quad (14)$$

The Riemann tensor becomes easier to handle if we lower an index using the metric: $R_{\rho\sigma\mu\nu} = g_{\rho\lambda}R_{\sigma\mu\nu}^\lambda$. The symmetry properties of this tensor are

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma} \\ R_{\rho\sigma\mu\nu} &= R_{[\rho\sigma][\mu\nu]} . \end{aligned} \quad (15)$$

The Bianchi identity is given by

$$\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = 0 . \quad (16)$$

Note that the Riemann tensor transforms non-trivially under coordinate transformations e.g. Lorentz transformations. We then construct the *Ricci tensor* as

$$R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda , \quad R_{\mu\nu} = R_{\nu\mu} . \quad (17)$$

The *Ricci scalar* is then defined as

$$R = R^\mu{}_\mu . \quad (18)$$

We combine these with the metric to construct the *Einstein tensor*:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \quad , \quad \nabla^\mu G_{\mu\nu} = 0 \quad (19)$$

Where the second equality is the defining property of the Einstein tensor.

Example: S^2

$$ds^2 = a^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad , \quad a = \text{radius}$$

For this metric:

$$R_{\rho\sigma\mu\nu} = -\frac{1}{a^2} (g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

Space-times with a metric that can be written in this form are called *maximally symmetric space-times* or *Einstein spaces*. They are easily seen to satisfy $R_{\mu\nu} \propto g_{\mu\nu}$, e.g. for S^2 we have $R = \frac{2}{a^2}$

5.2 Einstein equations

The Einstein equations tell us how matter or energy affects the curvature of a space-time, as well as how matter or energy is affected by the curvature of space-time. This is reminiscent of the equivalence between inertial mass and gravitational mass. Newton's equation of gravitation can be written as $m_I a = -m_G \nabla \Phi$, where Φ is the gravitational potential and m_I and m_G are the inertial and gravitational mass, respectively. Consider the geodesic equation of a massive particle with $|\frac{dx^i}{d\tau}| \ll |\frac{dt}{d\tau}|$ i.e. in the limit of small velocities. Then:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{00}^\mu \left(\frac{dt}{d\tau} \right)^2 = 0$$

Using $\Gamma_{00}^\mu = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}$ and making a *weak field approximation*: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, we find:

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \left(\frac{dt}{d\tau} \right)^2$$

Since $\frac{dt}{d\tau} \approx 1$, we have:

$$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2}\partial_i h_{00}$$

For small Φ , we have $g_{00} = -(1 + 2\Phi)$, so that we retrieve the Newtonian limit

$$\frac{d^2 x^i}{d\tau^2} = -\nabla \Phi$$

Introducing a matter density $\rho(\vec{x})$, we find the Poisson equation

$$\nabla^2 \Phi = 4\pi G\rho$$

Where G is Newton's constant. On the left hand side, we have a geometric quantity akin to curvature, and on the right hand side we have an matter distribution. Einstein's equations of general relativity can be seen as a fully covariant generalization of this expression. Observe that $\rho = T^{00}$, where $T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}$ is the *energy-momentum tensor*. Here, $g = \det g_{\mu\nu}$, the determinant of the metric, and S_{matter} is specific to the matter system we are considering e.g. the standard model.

We can thus rewrite

$$\nabla^2 h_{00} = -8\pi G T_{00} .$$

The *Einstein field equations* are then given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \frac{1}{2} \Lambda g_{\mu\nu} \quad (20)$$

Where Λ is know as the *cosmological constant*, which constitutes a constant energy density in our space-time. From energy-momentum conservation, $\nabla^\mu T_{\mu\nu} = 0 = \nabla^\mu G_{\mu\nu}$, which explains the relevance of the Einstein tensor.

A natural question is which action gives rise to the Einstein field equations. The answer is the *Einstein-Hilbert action*, which is given by:

$$S_{EH} = \frac{1}{8\pi G} \int d^4x \sqrt{-g} (R - \Lambda) + \mathcal{L}_{\text{matter}} \quad (21)$$

Using

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

and demanding $\frac{\delta S_{EH}}{\delta g_{\mu\nu}} = 0$, one can derive the Einstein equation. Since the energy-momentum tensor depends on the metric, the equations we end up with are typically very complicated.

5.3 Remarks

Einstein equations are a system of coupled non-linear second order partial differential equations. This means that analytic solutions are rather difficult to find, and very few are known. To find a solution we have to choose initial conditions. In Newton, we choose $(p_i, x_i)(t_0)$, where t_0 is some initial time. In GR, we choose a space-like hypersurface Σ , which is called a *Cauchy surface* if the union of its past and future domains of dependence cover the entire space-time. For such a Σ we can formulate a well-defined initial value problem.

Approximate solutions include two important classes

1. Gravitational waves: solutions of the vacuum Einstein equations in the limit $|h_{\mu\nu}| \ll 1$, $v = c$. This gives the following equation

$$\square h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \square h = -16\pi G T_{\mu\nu}$$

Since we are in a vacuum, $T_{\mu\nu} = 0$, so that

$$\square h_{\mu\nu} = 0 . \quad (22)$$

This describes a fully relativistic wave with two polarization, namely h_{++} and h_{--} , so that this wave corresponds to a spin-2 particle.

2. Newtonian limit: solutions of the vacuum Einstein equations in the limit $|h_{\mu\nu}| \ll 1$, $v \ll c$. For an object of mass m at position $\vec{x} = 0$, we then have:

$$h_{00} = -2\Phi \quad , \quad \rho(\vec{x}) = m\delta(\vec{x}) \Rightarrow \Phi = -\frac{GM}{r} \quad (23)$$

Then:

$$\begin{aligned} ds^2 &= -(1 + 2\Phi) dt^2 + (1 - 2\Phi) (dx^2 + dy^2 + dz^2) \\ &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) (dr^2 + r^2 d\Omega^2) \end{aligned} \quad (24)$$

Next lecture we will look at the Schwarzschild black hole solution, which will be reminiscent of the metric given above.

6 Black holes

6.1 Schwarzschild solution

The metric of a star of mass M which we derived last time is the large r , small M limit of the Schwarzschild solution, which we consider here. We look for an exact solution of $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu}$ under the following assumptions:

1. Vacuum solution: $T_{\mu\nu} = 0$
2. Spherical symmetry: $\partial_\phi g_{\mu\nu} = 0$
3. $\Lambda \leq 0$ i.e. the metric is asymptotically flat or asymptotically anti de Sitter. Today we consider $\Lambda = 0$

Birkhoff's theorem then tells us that there is a unique spherically symmetric solution of Einstein's equations, which reads:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + \underbrace{r^2(d\theta^2 + \sin^2\theta d\varphi^2)}_{r^2 d\Omega^2} \quad (25)$$

This is known as the Schwarzschild metric. We have one parameter, mass M , while electric charge Q and angular momentum L are zero. We see that for large r , we can expand $\left(1 - \frac{2M}{r}\right)^{-1} \approx 1 + \frac{2M}{r}$, which gives the metric for a star derived last time. For $r \rightarrow \infty$ and/or $M \rightarrow 0$, we retrieve Minkowski space.

We now compute the curvature of this geometry. In this metric, we have $R_{\mu\nu} = 0 = R$ i.e. this space is so-called *Ricci flat*. Hence, we consider another curvature invariant; the *Kretschmann scalar*, which gives for the Schwarzschild metric

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6} \xrightarrow{r \rightarrow 0} \infty.$$

We see that there is a *space-time singularity* at $r = 0$, namely, a singularity which cannot be resolved by a coordinate transformation. This is different from the *coordinate singularity* at $r = 2M$, where the Kretschmann scalar is finite. This coordinate transformation can be removed e.g. by going to Eddington-Finkelstein coordinates, as we will see later.

6.2 Event horizon

The event horizon has the topology of S^2 and is located at $r = 2M$. Here, the metric becomes divergent. However, all components of $R_{\mu\nu\rho}^\lambda$ are finite. Hence we see (as stated before) that this is not a true (space-time) singularity but merely a coordinate singularity. However, in spite of this singularity being removable there is a lot of interesting physics taking place at this point.

For $r > 2M$, called *region I* or the *exterior* of the black hole, the metric has signature $(-, +, +, +)$.

For $r < 2M$, called *region II* or the *interior* of the black hole, the metric has signature $(+, -, +, +)$.

At $r = 2M$, we have the event horizon, which is a *null surface* since it has signature $(0, 0, +, +)$.

We thus see that the event horizon is the point at which x and t exchange their respective (positive and negative) signatures. In region *II* we have to move forward in r i.e. to $r = 0$, which means one inevitably hits the singularity. This is reminiscent of particles always moving forward in time (after setting our conventions of time direction) in region *I*.

An outside observer will never see a signal reach the event horizon since signals close to the event horizon get infinitely red-shifted i.e. signals slow down until they become stationary close to the event horizon. Infinite redshift means that light loses all its energy i.e. it becomes completely dark, which is why black hole is an appropriate name for this object. We can see this from the light-cone completely 'closing up' when approaching the horizon. At $r = 2M$, the light-cone flips since metric signature changes from $(-, +, +, +)$ to $(+, -, +, +)$.

We now consider a co-moving observer that is propagating toward a black hole. Such an observer crosses the event horizon in finite proper time τ and does not experience anything special when crossing the event horizon. This observer also reaches the space-time singularity in finite time. Consider a radial null geodesic i.e. $ds^2 = 0$ and $d\theta = 0 = d\varphi$. This gives:

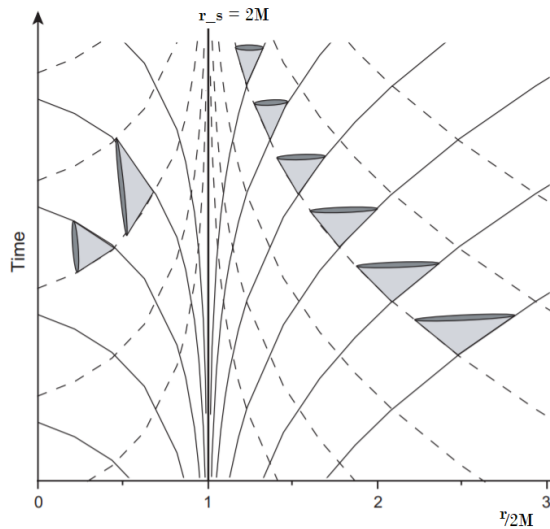


Figure 3: Space-time diagram of the near-horizon region of the Schwarzschild metric. We see that the future directed light cones, given by the shaded areas, ‘close up’ when we approach the Schwarzschild radius r_s from $r > r_s$. At $r = r_s$, the direction of the light cones changes discontinuously as r and t exchange (positive and negative) signature. Image adapted from [3].

$$\frac{dr}{dt} = \pm \left(1 - \frac{2M}{r}\right) \quad (26)$$

This tells us that the light cone indeed ‘closes’ ‘up’ when approaching the event horizon. We now change to a coordinate system which does not contain the coordinate singularity we have in the Schwarzschild solution. We first go to *tortoise coordinates*. We first solve 26, which gives:

$$t = \pm \left[r + 2M \ln \left(\frac{2r}{2M} - 1 \right) \right]$$

We then introduce $r^* = r + 2M \ln \left(\frac{2r}{2M} - 1 \right) \xrightarrow{r \rightarrow 2M} -\infty$. Our metric then becomes:

$$ds^2 = \left(1 - \frac{2M}{r}\right) (-dt^2 + (dr^*)^2) + r^2 d\Omega^2 \quad (27)$$

This metric is conformally equivalent to the Minkowski metric, which means that light cones are given by $r^* = \pm t$. However, this metric is not quite what we are looking for since $g_{tt}, g_{rr} \xrightarrow{r \rightarrow 2M} 0$.

We then introduce $v = t + r^*$ and $u = t - r^*$ such that $v = \text{constant}$ and $u = \text{constant}$ describe outgoing and infalling radial null curves, respectively. We can then express our metric as follows:

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \quad (28)$$

This is the so-called *Eddington-Finkelstein* metric, which we mentioned above. For a null curve, we have $ds^2 = 0$. For $d\Omega = 0$ (i.e. radial null curves) this gives: $ds^2 = - \left(1 - \frac{2M}{r}\right) dv^2 + 2drdv = 0$

The solutions are :

$$\frac{dv}{dr} = \begin{cases} 0 \\ \frac{2}{1 - \frac{2M}{r}} \end{cases}$$

Hence for $r < 2M$ all paths are in the direction of decreasing r , which means that a signal at $r < 2M$ will inevitably hit the space-time singularity at $r = 0$.

In the next lecture, we introduce the *Kruskal-Szekeres* coordinates, which helps us completely get rid of coordinate singularities. This will make our space-time *geodesically complete*, but we also get additional regions *III* and *IV* that are copies of *I* and *II*.

7 Kruskal-Szekeres coordinates and geodesics of the Schwarzschild black hole

Short summary of last lecture:

The Schwarzschild solution is characterized by a *space-time/curvature singularity* at $r = 0$ and a *metric singularity* at $r = \frac{2GM}{c^2}$, where we re-introduced the gravitational constant G and speed of light c . The space-time has an exterior and an interior region characterized by $r > 2M$ and $r < 2M$ and denoted by I and II , respectively. Today we will consider a new coordinate system which will allow us to double the space-time and find copies of regions I and II . So far we have introduced the following coordinate systems:

1. Schwarzschild coordinates: (t, r, θ, φ)
2. Tortoise coordinates: $(t, r^*, \theta, \varphi)$ with $r^* = r + 2M \ln|\frac{r}{2M} - 1|$
3. Eddington-Finkelstein coordinates: (v, u, θ, φ) , where u, v are light-cone coordinates given by $v = t + r^*$, $u = t - r^*$.

We now introduce *Kruskal-Szekeres coordinates* by introducing:

$$\left. \begin{aligned} R &= |\frac{r}{2M} - 1|^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \\ T &= |\frac{r}{2M} - 1|^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \end{aligned} \right\} \text{region } I$$

$$\left. \begin{aligned} R &= |\frac{r}{2M} - 1|^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \\ T &= |\frac{r}{2M} - 1|^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \end{aligned} \right\} \text{region } II$$

The metric is then given by

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} (-dT^2 + dR^2) + r^2 d\Omega_{(2)}^2. \quad (29)$$

A useful expression is $T^2 - R^2 = (1 - \frac{r}{2M}) e^{r/2M}$, which holds in both regions I and II .

Kruskal-Szekeres coordinates are perhaps the most useful coordinates for describing a black hole since they are geodesically complete and they do not exhibit a metric singularity. They cover the entire space-time, in fact the double cover of our original space-time, as we will see. Some remarks are in order:

1. We still have a curvature singularity at $r = 0 \leftrightarrow T^2 = R^2 + 1$.
2. However, the metric is regular at $r = 2M \leftrightarrow T = \pm R$.
3. Radial null curves are given by $T = R + \text{constant}$

We can allow T and R to range over $-\infty < T < \infty$ and $-\infty < R < \infty$, which will give a second copy of our space.

The four regions of the Kruskal-Szekeres diagram are:

- $I : r > 2M$: exterior of black hole
 - $II : r < 2M$: interior of black hole
 - $III : r > 2M$: parallel universe, causally disconnected from I
 - $IV : r > 2M$: white hole
- } our original space-time .
} a copy of our original space-time .

The *white hole* is a region of space-time from which signals can reach any point in space-time, but not the other way around. The white hole is shielded by a horizon, as per the *cosmic censorship conjecture*. This holds that all curvature singularities are shielded by an event horizon i.e. there are no so called *naked singularities*.

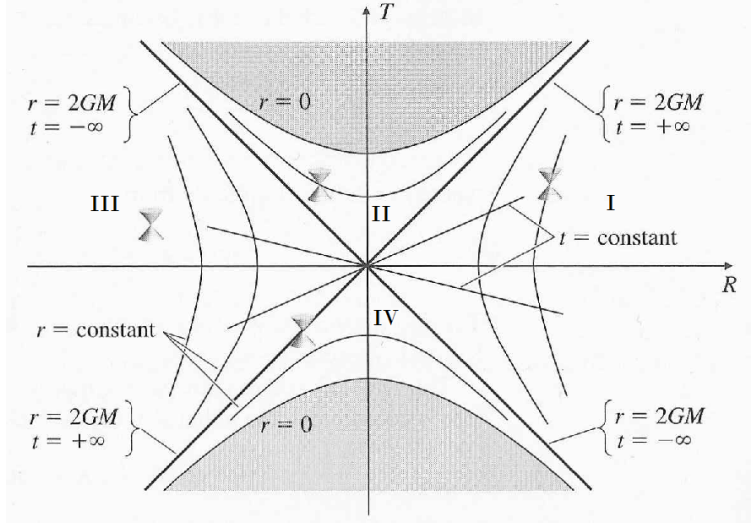


Figure 4: A diagram of the Schwarzschild black hole in Kruskal-Szekeres coordinates, where null curves travel at 45 degrees. Note that the even horizons are also at 45 degrees, which signals the fact that they are null-like hypersurfaces. We have four regions indicated by Roman numerals as described in the text. Image adapted from [4]

7.1 Geodesics and effective potential of the Schwarzschild black hole

We consider paths outside the horizon and ask ourselves whether we can escape the gravitational pull of the black hole as long as we are outside the event horizon. We will see that the answer is yes. Any timelike path can in principle accelerate away from the horizon before reaching it. We look at the geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau}$$

The symmetries of our system allow us to isolate a single differential equation with an effective potential. The metric is spherically symmetric as well as symmetric under time translations. This gives two *Killing vectors*, namely

$$\begin{aligned} K_\phi^\mu &= (0, 0, 0, 1) \\ K_t^\mu &= (1, 0, 0, 0) \end{aligned}$$

These satisfy the *Killing equations*: $K_\mu \frac{dx^\mu}{d\tau} = \text{constant}$. This entails that angular momentum $L = r^2 \frac{d\phi}{d\tau}$ and energy $E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$ are conserved along geodesics, corresponding to spherical symmetry and time invariance, respectively. The geodesic equation then gives:

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + V(r) = \frac{1}{2} E^2 \quad (30)$$

Setting $\theta = \frac{\pi}{2}$, the effective potential $V(r)$ is given by:

$$V(r) = \frac{1}{2} \left(\epsilon - \frac{L^2}{r^2} \right) \left(1 - \frac{2M}{r} \right) \quad (31)$$

with $\epsilon = +1, -1, 0$ for time-like, space-like, and null geodesics, respectively. We look for roots in the derivative of the effective potential:

$$\frac{dV(r)}{dr} = \epsilon \frac{M}{r^2} - \frac{L^2}{r^2} + \frac{3ML^2}{r^4} = 0$$

We only find a solution for $L > 12M^2$. We have circular orbits at:

$$r_{\pm} = \frac{L^2}{2M} \left(1 \pm \sqrt{L^2 - 12M^2} \right) \quad (32)$$

A more detailed derivation can be found in the lecture notes on General relativity by M. Blau.

7.2 Gravitational energy

The gravitational energy is the energy which gives rise to the gravitational field. We look at a matter system, where we can use the standard definition of energy and Einstein's equations to find the energy in a space-time:

$$E = \int_{\Sigma} d^3x T_{00} = \frac{1}{8\pi G} \int_{\Sigma} d^3x G_{00}$$

Where $G_{\mu\nu}$ is the familiar Einstein tensor and Σ is an achronal surface which covers our entire space. We perform a weak field approximation: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a perturbation around our background metric $\bar{g}_{\mu\nu}$, which we set to $\eta_{\mu\nu}$. At a linearized level, we have:

$$\begin{aligned} G_{00} &= -\frac{1}{2}\Delta(\delta^{ik}h_{ik}) + \frac{1}{2}\partial_i\partial_k h^{ik} \\ &= \partial_i \left(-\frac{1}{2}\partial^i h_k^k + \frac{1}{2}\partial_k h^{ik} \right) \end{aligned}$$

We can then use Stokes'theorem to find

$$E = \frac{1}{16\pi G} \oint_{S_{\infty}^2} dS_i (\partial_k h^{ik} - \partial^i h_k^k) .$$

Covariantizing gives

$$E = \frac{1}{16\pi G} \oint_{S_{\infty}^2} dS^i (\nabla^k h_{ik} - \nabla_i h) . \quad (33)$$

This quantity is know as the ADM mass (energy) of a gravitational system. We look at the ADM mass for the Schwarzschild metric. We have $\partial_k h_{ik} - \partial_i h_{kk} = \frac{4GM}{r^3} x_i$.

$$\oint dS^i x^i = \oint d\Omega r^2 n_i x^i = 4\pi r^3$$

We can then calculate:

$$E_{ADM} = \lim_{r \rightarrow \infty} \frac{1}{16\pi G} \frac{4GM}{r^3} 4\pi r^3 = M . \quad (34)$$

We thus find that the energy of the Schwarzschild space-time is equal to the mass of the black hole, as we would naturally expect.

8 Conformal compactifications and Penrose diagrams

Typical space-time metrics are infinite in coordinate extension, e.g. $\mathbb{R}^{1,3}$ or Schwarzschild space. This means that there are boundaries of our space-time at infinite coordinate distance in this coordinate system. To make such space-times more manageable we perform so-called *conformal compactifications*, which is a transformation of our original coordinate system such that:

1. Infinity boundaries are mapped to lines, points, or hypersurfaces at finite distances
2. The conformal structure is easily visible, in particular, we require that light rays travel at 45 degrees. This can be expressed as $da = \pm db$, with a and b our new conformal coordinates.

8.1 Examples

8.1.1 Two-dimensional flat space \mathbb{R}^2

We have $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\phi^2$ with $x = r \cos \phi$, $y = r \sin \phi$, $0 \leq r < \infty$. We then transform as $r = \tan \frac{\theta}{2}$, $0 \leq \theta < \pi$. The metric then becomes:

$$ds^2 = \frac{1}{\cos^4 \frac{\theta}{2}} (d\theta^2 + \sin^2 \theta d\phi^2) = f(\theta) (d\theta^2 + \sin^2 \theta d\phi^2) =: f(\theta) d\tilde{s}^2$$

This conformal rescaling allows us to relate the metric of a non-compact space (ds^2) to the metric of a compact space ($d\tilde{s}^2$). Generally, a conformal rescaling is a coordinate transformation of the form: $g_{\alpha\beta}(x) \rightarrow \tilde{g}_{\alpha\beta}(x) = \Omega(x)^2 g_{\alpha\beta}$. Since it is simply a local rescaling of the metric, the causal structure remains the same i.e. light rays still travel at 45 degrees. The Penrose map is a combination of a coordinate transformation that maps infinity to finite coordinate distance and a conformal rescaling.

8.1.2 (1+3)-dimensional Minkowski space $\mathbb{R}^{1,3}$

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{(2)}^2 = -dudv + \left(\frac{v-u}{4}\right)^2 d\Omega_{(2)}^2, \quad 0 \leq r < \infty \quad (35)$$

with $v = t + r$, $u = t - r$. We now define $V := \arctan v$, $U := \arctan u$. The range of the coordinates is $-\frac{\pi}{2} < u \leq v < \frac{\pi}{2}$. We can also define $T := U + V$, $R = V - U$, which then gives $|T| + R < \pi$, $0 \leq R < \pi$. T and R thus cover a triangle i.e. a compact space. We then have

$$ds^2 = \frac{1}{4 \cos^2 V \cos^2 U} \left(-4dUdV + \sin^2(V-U)d\Omega_{(2)}^2 \right) = \frac{1}{4 \cos^2 V \cos^2 U} d\tilde{s}^2$$

or

$$d\tilde{s}^2 = \underbrace{-dT^2 + dR^2}_{\text{flat metric}} + \sin^2 R d\Omega_{(2)}^2.$$

This is the metric of the Einstein static universe, its topology is given by $I \otimes S^3$, with I a finite interval.

We have:

- i^+ : *future timelike infinity* i.e. $t = \infty$, fixed r_0
- i^- : *past timelike infinity* i.e. $t = -\infty$
- i^0 : *spatial infinity* i.e. $r = \infty$
- \mathcal{I}^+ : *future null infinity* i.e. $v = \infty$, $u = \text{fixed}$: asymptotics of outgoing radial null curves
- \mathcal{I}^- : *past null infinity* i.e. $v = \text{fixed}$, $u = \infty$: asymptotics of ingoing radial null curves

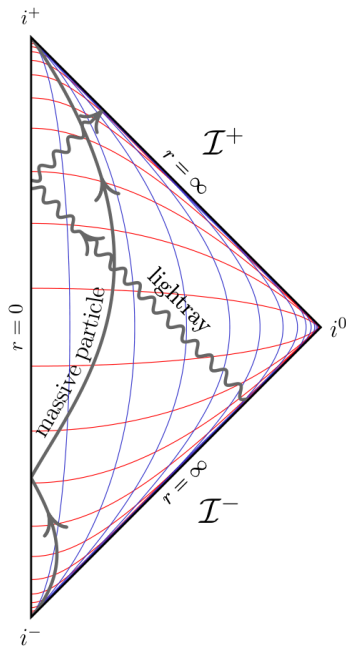


Figure 5: The conformal diagram of Minkowski space with time-, null-, and space-like coordinate infinities indicated as explained below. Image adapted from [5].

We make the following remarks

1. Radial null geodesics i.e. light cones are at ± 45 degrees.
2. All points in the Penrose diagram represent a 2-sphere except i^0 and i^\pm .
3. i^0 and i^\pm are really points
4. \mathcal{I}^\pm are null hypersurfaces $\mathbb{R}xS^2$
5. All infinitely extended timelike geodesics begin at i^- and end at i^+ .
6. All infinitely extended spacelike geodesics begin at i^0 , are reflected at $r = 0$ and come back i^0 .
7. All infinitely extended null geodesics begin at \mathcal{I}^- , are reflected at $r = 0$, and at \mathcal{I}^+ .

Any time-like geodesic observer will eventually be able to see all of Minkowski space i.e. at i^+ the past light-cones of all observers cover all of Minkowski space. The past and future light-cones of any two events have a non-empty intersection. In particular, any two events in Minkowski were causally connected at some point in the past. This entails that there is no horizon in Minkowski space. We will see that this will not be true for the following example.

8.1.3 Two-dimensional Rindler space: Accelerated observer

We start from the $(1 + 1)$ -dimensional Minkowski metric given by $ds^2 = -dt^2 + dx^2$, $-\infty < t < \infty$, $-\infty < x < \infty$. We define $x^\pm := t \pm x$ and $X^\pm := \arctan x^\pm = \tan\left(\frac{1}{2}(T \pm X)\right)$. We then have

$$d\tilde{s}^2 = -dX^+dX^- \quad , \quad \frac{\pi}{2} \leq X^+, X^- \leq \frac{\pi}{2}$$

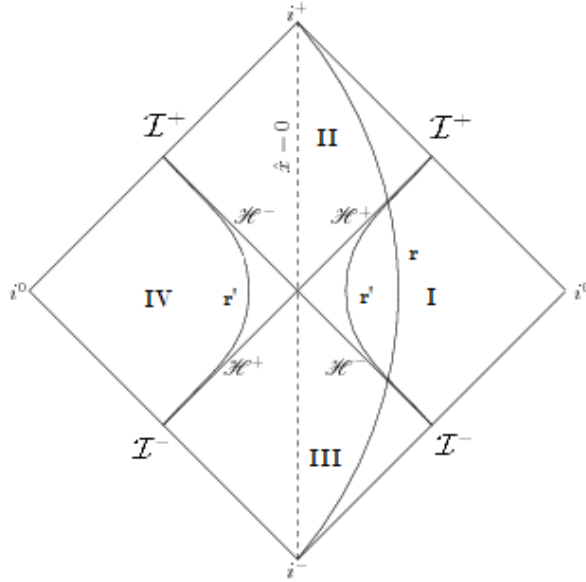


Figure 6: The maximally extended conformal (Penrose) diagram of Rindler space with Rindler horizons at $X^\pm = 0$ are indicated by \mathcal{H}^\pm and Rindler wedges indicated by Roman numerals $I - IV$. The curved lines indicated by r and r' indicate locally inertial frames in Minkowski and Rindler coordinates, respectively. The other symbols have the same meaning as in Minkowski space. Figure adapted from [6].

9 Penrose diagrams (continued) and charged & rotating black holes

9.1 Penrose diagram for Schwarzschild black hole

As we saw, Penrose diagrams exhibit the causal structure of space-time. Penrose diagrams are found in two steps:

- A) Choose coordinates that map boundaries of space-time to finite coordinate distance.
- B) Conformal rescaling of metric i.e. get rid of divergent part of metric, namely the conformal factor.

We treated Minkowski space $\mathbb{R}^{3,1}$, which is geodesically complete, as well as Rindler space, which is not geodesically complete

We will now discuss the Penrose diagram of the Schwarzschild space. We focus on region I i.e. the first outer region of the black hole corresponding to $r > 2M$. We transform:

$$(r, t) \rightarrow (u, v) ; \quad u = t - r^* \quad , \quad v = t + r^*$$

Note that $-\infty \leq t \leq \infty$ and $-\infty \leq r^* \leq \infty$. The metric is given by:

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dudv + \dots \tag{36}$$

We have the following boundaries:

$$\begin{aligned} \mathcal{I}^+ & \quad u = \text{finite} \quad , \quad v \rightarrow \infty \\ \mathcal{I}^- & \quad v = \text{finite} \quad , \quad u \rightarrow -\infty \end{aligned}$$

We have horizons:

$$\begin{aligned} H^+ & \quad : \quad u \rightarrow \infty \quad , \quad v = \text{finite} \\ H^- & \quad : \quad v \rightarrow -\infty \quad , \quad u = \text{finite} \end{aligned}$$

Step A: define $u =: \tan \tilde{U}$, $v =: \tan \tilde{V}$, $|\tilde{U}| \leq \frac{\pi}{2}$, $|\tilde{V}| \leq \frac{\pi}{2}$

We now go to Kruskal coordinates (R_k, T_k) with metric:

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} (dT_k^2 - dR_k^2) + r^2 d\Omega_{(2)}^2 \tag{37}$$

We successively redefine as: $u_k := T_k - R_k := \tan \hat{U}$, $v_k := T_k + R_k := \tan \hat{V}$, so that $\hat{T} = \hat{U} + \hat{V}$, $\hat{R} = \hat{V} - \hat{U}$. This gives the fully extended Penrose diagram, which has the following regions:

1. Region I : Exterior of black hole
2. Region II : Interior of black hole
3. Region III : Parallel universe
4. Region IV : White hole

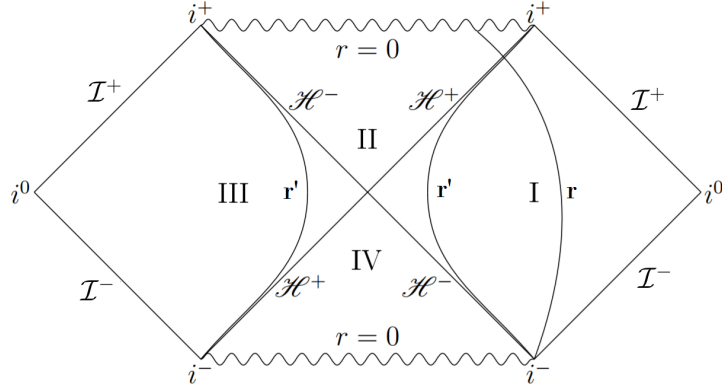


Figure 7: The maximally extended conformal (Penrose) diagram of the Schwarzschild black hole. The curved lines indicated by r and r' indicate locally inertial frames in Kruskal-Szekeres and Schwarzschild coordinates, respectively. Note the difference with Rindler space indicated in figure 6, namely, the requirement that $r > 0$ ‘cuts off’ the top and bottom halves of regions II and IV , respectively. Figure adapted from [6].

9.2 Charged and rotating black holes

9.2.1 Reissner-Nordström black holes (charged)

We consider the following action: $S = \int d^4x \sqrt{-g} R - \underbrace{\frac{1}{4} g^{\alpha\mu} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}}_{\text{matter action}}$, where $F_{\mu\nu}$ is the electromagnetic field strength tensor. The electromagnetic field equations are $\partial_{[\alpha} F_{\beta\gamma]} = 0$ and $\partial_\mu (\sqrt{-g} F^{\mu\nu}) = -\sqrt{-g} J^\nu$, where J^ν is the electromagnetic current which satisfies $\partial_\nu (\sqrt{-g} J^\nu) = 0$. By demanding that the variation with respect to the metric vanishes, we find that the energy-momentum tensor is given by:

$$\frac{\delta S}{\delta g_{\mu\nu}} \Rightarrow T_{\mu\nu} = -F_{\mu\alpha} F_\nu^\alpha + \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - J_\alpha A^\alpha \right) g_{\mu\nu} \quad (38)$$

For a point-like electric charge, the field strength is given by $F_{rt} = \frac{Q}{r^2}$, where Q is the electric charge. The unique closed form solution to $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$ is given by:

$F_{rt} = \frac{Q}{r^2}$, $Q =$ electric charge. The unique closed form solution is:

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + d\Omega_{(2)}^2 \quad (39)$$

This is known as the *Reissner-Nordström* metric, which is characterized by parameters (M, Q) . We again have a curvature singularity (characterized by diverging Kretschmann scalar) at $r = 0$. However, we now have two horizons, located at $r = r_\pm = M \pm \sqrt{M^2 - Q^2}$. The metric signatures of the Reissner-Nordström solution are given by

1. $r > r_+$: $(-, +, +, +)$
2. $r_- < r < r_+$: $(+, -, +, +)$
3. $r < r_-$: $(-, +, +, +)$.

From the metric signatures we can see that that time-like geodesics do not necessarily end at $r = 0$ since they are forced to move in the direction of decreasing r only for $r_- < r < r_+$. The singularity at $r = 0$ forms a time-like line and can thus be avoided, as opposed to uncharged (Schwarzschild) black holes, where $r = 0$ is space-like. The Reissner-Nordström metric has two interesting limits

1. $Q \rightarrow 0$: In this limit we retrieve the Schwarzschild radius. Then $r_+ = 2M$, $r_- = 0$.
2. Extremal black hole: $Q^2 = M^2$, so that $r_- = r_+ = M$.

Since, for the extremal black hole, $r_+ = r_-$, the signature of the metric is $(-, +, +, +)$ everywhere in our space-time. The metric is

$$ds_{Extremal}^2 = - \left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_{(2)}^2 . \quad (40)$$

Hence we see directly that the metric does not change signature at the horizon at $r = M$. Extremal black hole can be coupled to fermions such that it preserves *supersymmetry*, hence it is often (somewhat erroneously) referred to as a *supersymmetric black hole*. $M^2 = Q^2$ is a limiting case in the sense that we require $M^2 \geq Q^2$. For $M^2 < Q^2$, we get a naked singularity, thus violating the cosmic censorship conjecture. That we have a naked singularity can be seen from the fact that the metric is completely regular for $r \rightarrow 0$, while we still have a space-time singularity at $r = 0$. A bound such as $M^2 \geq Q^2$ is generally referred to as a *BPS-bound*, we will return to this point in the lectures on black holes in string theory.

10 Rotating black holes and black hole mechanics

In this lecture, we consider rotating black holes as well as black hole mechanics and its similarities with thermodynamics.

10.1 Rotating (Kerr) black hole

So far, we considered spherically symmetric black holes. We will now consider a black hole with some non-zero angular velocity along azimuthal angle ϕ . The metric will then depend on θ in a non-trivial way, but it will not depend on ϕ i.e. it will be axially symmetric.

Remark: A slowly rotating body at large r has the following metric:

$$ds^2 = \underbrace{ds_0^2}_{\text{non-rotating metric}} + \frac{4J}{r} \sin^2 \theta dt d\phi$$

Where J is the angular momentum of the rotating body. We now check that this is reproduced by the full solution, like we did in the case for the metric of a spherical body at large r . The full solution is:

$$ds^2 = - \left(1 - \frac{2Mr}{r^2 + a^2 \cos^2 \theta} \right) (du + a \sin \theta d\phi)^2 + 2 (du + \sin^2 \theta d\phi) (dr + a \sin^2 \theta d\phi) + (r^2 a^2 \cos^2 \theta) \underbrace{(d\theta^2 + \sin^2 \theta d\phi^2)}_{d\Omega_{(2)}^2} \quad (41)$$

Where, again $u = t - r - 2M \log \left| \frac{r}{2M} - 1 \right|$. This solution was found by Kerr in 1963. The fact that it took almost 50 years since Einstein published his gravitational equations for someone to discover this solution is probably due to the fact that it has a rather complicated mixing between t and ϕ . If we redefine our coordinates as: $t \rightarrow t - 2M \int \frac{r dr}{r^2 - 2Mr + a^2}$, $\phi \rightarrow -\phi - a \int \frac{r dr}{r^2 - 2Mr + a^2}$ the metric becomes:

$$ds^2 = - \left(1 - \frac{2Mr}{\rho^2} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2 \quad (42)$$

with $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$, $a = \frac{J}{M}$.

For a more detailed derivation we refer to the lecture notes by Stefan Vandoren.

We remark on two limits of parameters a and M :

1. $a \rightarrow 0$. Then $\rho \rightarrow r$, $\Delta = r^2 - 2Mr$, this gives the Schwarzschild metric.

2. $M \rightarrow 0$. This gives flat Minkowski in so-called *oblate spheroidal coordinates*.

The Kerr metric does not depend on t or ϕ and is invariant under the combination $t \rightarrow -t$ & $\phi \rightarrow -\phi$. The horizons are located at the points where $g_{rr} \rightarrow \infty$ i.e. where $\Delta \rightarrow \infty$. We thus find two horizons, located at:

$$r_{\pm} = M \pm \sqrt{M^2 - a^2} \quad (43)$$

As for the Reissner-Nordström black hole, the metric signature is $(+, -, +, +)$ for $r_- < r < r_+$, and $(-, +, +, +)$ for $r < r_-$ and $r > r_+$. We require $M^2 \geq a^2 \sim M^2 \geq |J|$. If this bound is violated the solution is again unphysical since we have a naked singularity. The BPS-saturated (extremal) black hole is given by $M^2 = |J|$. We can generalize the Kerr metric to include electric charge Q and magnetic charge P by simply replacing $2Mr$ with $2Mr - (Q^2 + P^2)$, the result is known as the *Kerr-Newman metric*.

10.2 Laws of black hole thermodynamics (mechanics)

We will now compare black hole mechanics with thermodynamics. In particular, we will see during the next lectures that black holes are an objects with a finite temperature that lose energy via thermal (Hawking) radiation. We will see that for a classical black hole ($\hbar = 0$), $T = 0$ and there is no radiation. That black holes emit thermal radiation is thus an inherently quantum-mechanical phenomenon. .

10.2.1 Zero'th law

We refer to notes by Vandoren and Townsend for further details on the laws of black hole mechanics. Recall the definitions of null-hypersurfaces and Killing horizons. Take a set of hypersurfaces $\{S\}$ i.e. of 3d achronal submanifolds of 4d space-time characterized by $S(x^\mu) = 0$, where S is some function. A vector field normal to S is given by: $\ell^\mu = N(x)\partial^\mu S(x)$, where $N(x)$ gives a normalization. A null hypersurface Σ is then characterized by

$$\ell^2 \Big|_\Sigma = \ell^\mu \ell_\mu \Big|_\Sigma = 0$$

i.e. its normal vector ℓ is orthogonal to itself. We then consider the *Lie derivative*, which is the derivative of a tensor along a vector field. Namely, the Lie derivative tells us how a space-time tensor field changes when we take an infinitesimal step along a vector field defined in that space-time. The Lie derivative of the metric along a *Killing vector field* K vanishes i.e. $(\mathcal{L}_K g)_{\mu\nu} = 0$, with $(\mathcal{L}_K g)_{\mu\nu} = K^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu K^\lambda g_{\nu\lambda} + \partial_\nu K^\lambda g_{\mu\lambda}$. A *Killing horizon* is then given by a null hypersurface Σ for which there exists a Killing vector field that is normal to it i.e. for some $\xi^\mu = f(x)\ell^\mu(x)$ defined at Σ , we have $\xi^2 \Big|_\Sigma = f^2 \ell^2 \Big|_\Sigma = 0$. One can show that:

$$\xi^\sigma \nabla_\sigma \xi^\mu \Big|_\Sigma = \kappa \xi^\mu$$

where κ is the *surface gravity*, as you will compute in a problem set. For the Schwarzschild metric, we have $\kappa = \frac{1}{4M}$. The meaning of κ is as follows: For an asymptotically flat space-time, κ gives the acceleration of a static observer at the Killing horizon. Another useful formula is:

$$\kappa^2 = -\frac{1}{2} (\nabla^\alpha \xi^\beta) (\nabla_\alpha \xi_\beta) \Big|_\Sigma .$$

A shortcut for computing κ for Schwarzschild goes as follows: Take Euclidean Schwarzschild i.e. let $t \rightarrow i\tau$ (Wick rotation). The metric is then:

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_{(2)}^2$$

This is equivalent to considering our system at finite temperature. The periodicity of τ gives the temperature. We now examine near-horizon region of black hole by introducing $\frac{x^2}{8M} = r - 2M$. This gives:

$$ds^2 = (\kappa x)^2 d\tau^2 + dx^2 + \frac{1}{4M} d\Omega_{(2)}^2$$

where $\kappa = \frac{1}{4M}$. This is the metric of \mathbb{R}^2 in angular coordinates. τ thus has the periodicity $\tau \sim \tau + \frac{2\pi}{\kappa} =: \tau + \beta$, from which we see that $\beta = T^{-1} = \frac{2\pi}{\kappa}$.

11 Black hole mechanics and thermodynamics

In the 1970's, Bekenstein realized that a black hole must carry entropy [7]. This follows from the fact that if we throw an object with some entropy into a black hole, the entropy of the total system may not decrease, hence the entropy of a black hole must grow when we throw an object into it if the second law of thermodynamics is to be satisfied. We state the laws of black hole mechanics and the corresponding laws of thermodynamics:

Law	Thermodynamics	BH mechanics
0	T constant in equilibrium	$\kappa = \text{constant}$
1	$\delta E = T\delta S$	$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J + \Phi\delta Q$
2	$\delta S \geq 0$	$\delta A \geq 0$
3	Cannot let $T \rightarrow 0$ in a finite number of steps	Cannot let $\kappa \rightarrow 0$ in a finite number of steps

During the next few lectures we will look at the laws of black hole mechanics and their similarities to the laws of thermodynamics.

11.1 First law of black hole mechanics

The *first law of black hole mechanics* tells us how the mass of a black hole changes with its area, charge, and angular momentum. It reads

$$\delta M = \frac{\kappa}{4\pi}\delta A + \Omega\delta J + \Phi\delta . \tag{44}$$

We will show that this is the correct expression for stationary, axisymmetric, and asymptotically flat space-times. We need our space-time to have these properties so that mass and angular momentum are well-defined. To find an expression for the mass of a black hole, we first consider the example of electromagnetism. The electric charge inside a volume V is given by

$$Q(V) = \int \rho dV = \int \nabla \cdot \vec{E} dV = \int_{\partial V} \vec{E} \cdot \hat{n} dS ,$$

where \hat{n} is the normal vector to ∂V . The covariant generalization of this expression is

$$Q(V) = \int_V dS_\mu j^\mu = \int_V dS_\mu D_\nu F^{\mu\nu} = \int_{\partial V} dS_{\mu\nu} F^{\mu\nu} .$$

We now consider general relativity. The *Komar integral* associated to a Killing vector ξ is given by

$$Q_\xi(V) = \frac{c}{4\pi} \int_{\partial V} \nabla^\mu \xi^\nu dS_{\mu\nu} ,$$

where c is some constant. Comparing this expression with that of electromagnetism, we note that $\nabla^\mu \xi^\nu = -\nabla^\nu \xi^\mu$, similar to $F^{\mu\nu} = -F^{\nu\mu}$. We now use $\nabla_\nu \nabla_\mu \xi^\nu = R_{\mu\nu} \xi^\nu$, which holds for any Killing vector ξ^ν , to rewrite

$$Q_\xi(V) = \frac{c}{4\pi} \int_V R^\nu{}_\mu \xi^\mu dS_\nu .$$

Remark: If $\nabla^a T_{ab} = 0$ i.e. if energy and momentum are conserved, then $T_{ab} \xi^b = 0$. From this it follows that Q_ξ is a conserved quantity. Hence, the Komar mass i.e. the Komar integral associated with the time-like Killing vector ξ is:

$$M_\xi(V) = -\frac{1}{4\pi} \oint_{\partial V} \nabla^\mu \xi^\nu dS_{\mu\nu} \tag{45}$$

Where we have taken $c = -1$ because it reproduces the right result for familiar space-times e.g. the Schwarzschild metric. The mass M is subject to the *positive energy theorem*, due to Witten & Yau, which reads:

If T_{ab} satisfies the dominant energy condition i.e. every observer observes strictly positive energy, then $M_{ADM} \geq 0$, and $M_{ADM} = 0$ only for Minkowski space-time. The dominant energy condition entails that $T_{ab}u^a \geq 0$ for any u^a that is a tangent vector to a general future-directed causal curve.

If, instead, we consider axial Killing vector ϕ , then the Komar integral gives the angular momentum:

$$J_\phi(V) = \frac{1}{8\pi} \oint_{\partial V} \nabla^\mu \phi^\nu dS_{\mu\nu} \quad (46)$$

11.2 Smarr's formula

We use B to refer to the interior of a black hole, and Σ to refer to the exterior. We assume ∂B is a Killing horizon for $K = \xi - \Omega_H \phi$, where we take into account possible non-zero rotation of the black hole i.e. $\Omega_H \neq 0$ for Kerr(-Newman) black holes. Then:

$$\begin{aligned} M_k(S_\infty^2) &= -\frac{1}{4\pi} \int_{S_\infty^2} \nabla^\mu \xi^\nu dS_{\mu\nu} \\ &= -\frac{1}{4\pi} \int_\Sigma R^\mu{}_\nu K^\nu dS_\mu + \frac{1}{4\pi} \int_{\partial B} \nabla^\mu K^\nu dS_{\mu\nu} \\ &= \int_\Sigma 2(T_\nu^\mu - T\delta_\nu^\mu) K^\nu dS_\mu + 2\Omega_H J + \frac{1}{4\pi} \int_{\partial B} \nabla^\mu \xi^\nu dS_{\mu\nu} \end{aligned}$$

Using $\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu$, we can rewrite the last term in the last expression to find Smarr's law:

$$M = \Phi Q + 2\Omega_H J + \frac{\kappa}{4\pi} A \quad (47)$$

Where Φ is the electromagnetic potential between $r = \infty$ and $r = 2M$. There are two ways to use Smarr's law to arrive at the first law of black hole mechanics. We will first derive it via the more complicated method. Assuming $Q = 0$, we have

$$\delta M = \frac{1}{4\pi} (\delta\kappa + \kappa\delta A) + (\delta\Omega_H J + \Omega\delta J) .$$

We also have $-\delta M = \frac{1}{4\pi} \delta\kappa + 2\delta\Omega_H J$, which one can derive. Combining the two gives the first law of black hole mechanics

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega\delta J + \Phi\delta Q . \quad (48)$$

The other derivation of the first law goes as follows. We use the uniqueness theorem, which states that the surface area of a black hole is a (non-trivial) function of only the mass, angular momentum, and charge. Since we set $Q = 0$, this can be written as $A = A(M, J)$, akin to $M = M(A, J)$. Now note that A and J are both proportional to the (Komar mass)² from dimensional reasons. We thus have $\alpha M = M(\alpha^2 A, \alpha^2 J)$ i.e. $M(A, J)$ is a homogeneous function of degree $\frac{1}{2}$. Hence $A \frac{\partial M}{\partial A} + J \frac{\partial M}{\partial J} = \frac{1}{2} M = \frac{\kappa}{8\pi} A + \Omega_H J$, so that

$$A \left(\frac{\partial M}{\partial A} - \frac{\kappa}{8\pi} \right) + J \left(\frac{\partial M}{\partial J} - \Omega_H \right) = 0 .$$

Since A and J are independent, the object multiplying them in the above expression are both equal to zero. We thus find:

$$\begin{aligned}
\delta M &= \frac{\partial M}{\partial A} \delta A + \frac{\partial M}{\partial J} \delta J \\
&= \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J
\end{aligned} \tag{49}$$

11.3 Kerr-Newman black hole

The metric of a charged, rotating black hole is given by:

$$ds^2 = - \left(1 - \frac{2Mr - Q^2}{\rho^2} \right) dt^2 - \frac{2a(2Mr - Q^2)^2 \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{(2Mr - Q^2)a^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\varphi^2 \tag{50}$$

Where $a := \frac{J}{M}$, $\rho = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 + a^2 - 2Mr + Q^2$. For this metric, we have

$$dM = \frac{1}{16\pi} \frac{r_+ - r_-}{r_+^2 + a^2} dA + \underbrace{\frac{a}{r_+^2 + a^2}}_{\Omega_H} dJ + \underbrace{\frac{Qr_+}{r_+^2 + a^2}}_{\Phi} dQ . \tag{51}$$

Where $\Omega_H = \left. \frac{d\phi}{dt} \right|_{r=r_H} = \frac{a}{r_+^2 + a^2}$ can be interpreted as the angular velocity.

12 Black hole mechanics and thermodynamics - continued

12.1 Last lecture: Zero'th and first laws

During the last lecture, we discussed black hole thermodynamics and mechanics. The zero'th law states that surface gravity is constant over an event horizon. For our discussion of the first law we considered Komar quantities. For some surface $S \subset M$ with volume element dS_μ with boundary ∂S with volume element $dS_{\mu\nu}$ and ξ a Killing vector, we have general Komar quantity:

$$Q_k = -\frac{1}{4\pi} \int_{\partial S} dS_{\mu\nu} \nabla^\mu \xi^\nu . \quad (52)$$

E.g. for $k \sim \partial_t$ or $\tilde{k} \sim \partial_\phi$, we find the Komar mass or angular momentum, respectively:

$$\begin{aligned} M_k &= -\frac{1}{4\pi} \int_{\partial S} dS_{\mu\nu} \nabla^\mu k^\nu \\ J_k &= -\frac{1}{8\pi} \int_{\partial S} dS_{\mu\nu} \nabla^\mu \tilde{k}^\nu \end{aligned} \quad (53)$$

Total mass inside some S :

$$M = \Phi Q + 2\Omega_H J + \frac{\kappa}{4\pi} A \quad (54)$$

Combining these gives the first law:

$$dM = \frac{\kappa(M, J, Q)}{8\pi} dA + \Omega(M, J, Q) dJ + \Phi(M, J, Q) dQ \quad (55)$$

Where κ is the acceleration at the horizon, Ω is the angular velocity at the horizon, and Φ is the electric potential at the horizon.

12.2 Second law

A natural question concerning black holes is whether there are restrictions on the possible charges, areas, angular momenta, and masses of black holes for a given set of boundary conditions. This question is relevant for matter that collapses to a black hole, or generally for dynamical problems. The answer is that there are indeed restrictions, namely we have $\frac{dA}{dt} \geq 0$ (Hawking area theorem). The assumptions underlying this theorem are: our space is asymptotically flat, we have cosmic censorship i.e. no naked singularities, and the weak energy condition holds i.e. $T_{\mu\nu} v^\mu v^\nu \geq 0$, where v^μ is an arbitrary time-like vector.

Proof (relies on causality): A *causal curve* is a curve that is nowhere space-like. We define the *causal past* of some surface S as $J^-(S) := \{x \in \mathcal{M} \mid \exists s \in S, x \leq s\}$, where $x \leq s$ means that the time coordinate of x is less than or equal to s . The *causal future* is then defined as $J^+(S) := \{x \in \mathcal{M} \mid \exists s \in S, x \geq s\}$. We now consider *chronological curves* which are defined to be everywhere time-like. The *chronological past* and *future* are then given by $I^- := \{x \in \mathcal{M} \mid \exists s \in S, x < s\}$ and $I^+(S) := \{x \in \mathcal{M} \mid \exists s \in S, x > s\}$. The boundary of the causal past is $\partial J^-(S) = J^-(s)/I^-(S)$: $\partial J^-(S)$ is generated by a set of null geodesics, which are referred to as the *null geodesic generators* of $\partial J^-(S)$.

There is a lemma due to Penrose (1967) which states that in any subset $S \subset M$, a null geodesic generator of $\partial J^-(S)$ cannot have future endpoints on $\partial J^-(S)$. In other words, the area of a trapped surface cannot decrease. A result that then easily follows is the Hawking area theorem, which reads:

For an asymptotically flat black hole space-time, the area of the event horizon $\partial J^-(\mathcal{I}^+)$ cannot decrease in time if the weak energy condition and the cosmic censorship conjecture are satisfied.

12.2.1 Consequences for coalescing black holes

We consider the limits of mass-energy conversion of a black hole collision (such as observed by LIGO). Consider two black holes with masses M_1 and M_2 , respectively. They coalesce to form a third black hole, which has mass M_3 . During this process, part of the total energy is converted to gravitational waves; this fraction of the total energy is given by $M_1 + M_2 - M_3$. We can then define an *efficiency coefficient* $\eta := \frac{M_1 + M_2 - M_3}{M_1 + M_2} = 1 - \frac{M_3}{M_1 + M_2}$. Black hole surface area is $A = 16\pi M^2$, so Hawking's area theorem tells us that $A_3 = 16\pi M_3^2 \geq 16\pi (M_1^2 + M_2^2)$, hence $M_3 \geq \sqrt{M_1^2 + M_2^2}$ i.e.

$$\eta \leq 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2} = 1 - \frac{1}{\sqrt{2}}. \quad (56)$$

One then easily sees that a single black hole cannot split up into two separate black holes. The exceptions are BPS black holes, which can freely split up without decreasing total entropy.

12.3 Third law of black hole mechanics

The third law of black hole mechanics states that it is impossible to let the surface gravity κ go to zero in a finite number of operations. We give this statement without proof, but it can be seen from the fact that we need to let $M \rightarrow \infty$ or $(a^2 + Q^2 + P^2) \rightarrow M^2$. We thus need to add an infinite amount of (BPS-saturated) matter. We thus see that extremal (BPS) black holes cannot be formed continuously since they have $\kappa = 0$. This is related to supersymmetry, since BPS black holes are supersymmetric while the non-BPS black holes are not.

13 Black holes and entropy

So far, we have considered the laws of black hole mechanics and briefly discussed their correspondence to the laws of thermodynamics. We now wish to consider the laws of black hole mechanics as thermodynamical statements. This means that we will define a temperature and an entropy for our black holes. In thermodynamics, we may consider e.g. the grand canonical ensemble, where the system is characterized by quantities μ, V, T , namely chemical potential, volume, and temperature. The first law of thermodynamics is then:

$$dE = TdS - pdV + \mu dN$$

The first law of black hole mechanics is

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$$

By comparing, we see that $-pdV + \mu dN$ is analogous to $\Omega dJ + \Phi dQ$, so that $\frac{\kappa}{8\pi} dA$ is the entropy term. Moreover, Hawking's area theorem tells us that $\frac{dA}{dt} \geq 0$ which further solidifies the analogy between A and S . The precise relation will involve \hbar , signalling the importance of quantum effects. The full expression is Bekenstein-Hawking area law:

$$S = \frac{c^3 A}{4G_N \hbar}. \quad (57)$$

Hence S diverges for $\hbar \rightarrow 0$. We can rewrite this as

$$S = \frac{\text{area}}{4\pi L_p^2} \sim \frac{1}{4\pi} \frac{\text{area}}{(10^{-33})^2}.$$

Where L_p is the Planck length $\sim 10^{-33}$ cm. E.g. for a solar mass black hole with $r_s \sim 3$ km, $S = 10^{77}$.

In statistical physics, entropy is interpreted as information. The von Neumann entropy is given by $S = -\sum_n p_n \log p_n$, where p_n are probabilities satisfying $\sum_n p_n = 1$. Entropy is related to information (quantum bits). One naturally wonders what the quantum bits of black holes are i.e. what are the carriers of black hole information. People have conjectured that the quantum bits are 'distributed' over the horizon, since $S \propto A$. This area law has inspired the principle of holography, which is the idea that gravitational degrees of freedom are somehow dual to degrees of freedom on a holographic 'screen' which has one dimension lower than the gravitational system. An explicit example of holography is the AdS/CFT correspondence [8], where the gravitational bulk and the holographic screen are the AdS space and the boundary CFT, respectively.

14 Hawking and Unruh radiation

To investigate Hawking radiation [9] we take a semiclassical approach, which means that we treat the gravitational background as classical, while we treat the fields that are defined on this gravitational background quantum-mechanically. We can make an estimate of the Hawking temperature by using Wien's law $\lambda = \frac{\hbar c}{k_B T}$. If we then take λ to be the Schwarzschild radius $\lambda = r_s = \frac{2GM}{c^2}$, we find $T = \frac{\hbar c^3}{2GM}$, which is a factor 4π larger than the real value. We see that the temperature goes to zero when \hbar goes to zero, signalling that Hawking radiation is a quantum-mechanical effect. The heuristic picture of Hawking radiation is that we have pair production sufficiently close to the horizon that one of the particles passes through the horizon and falls inward to the singularity. The other particle of the pair is maximally entangled with the first, and as a consequence it propagates outward to radial infinity. The outward propagating particles constitute Hawking radiation, at Hawking temperature $T_H = \frac{\hbar c^3}{8\pi GM k_B}$. In Rindler space-time we have Unruh radiation associated to the fact that there is a horizon in Rindler space. We have the corresponding Hawking radiation in Schwarzschild space-time. In both cases the existence of a horizon obscures our notion of the vacuum as well as particle number, as we will see below. We first review a few standard ideas of QFT in Minkowski space.

14.1 Free scalar field in Minkowski space

We have $ds^2 = -dt^2 + d\vec{x}^2$ and $S[\phi] = -\frac{1}{2} \int d^4x [\eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi^2]$. The equation of motion is

$$\partial_t^2 \phi - \nabla^2 \phi + m^2 \phi = 0 .$$

We perform a Fourier transformation: $\phi(t, \vec{x}) = \int \frac{d^3 \vec{k}}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{x}} \phi_k(t)$, which gives

$$\ddot{\phi}_k + (k^2 + m^2) \phi_k = 0 .$$

We now perform canonical quantization, using $\pi(t, \vec{x}) = \partial_t \phi(t, \vec{x})$ and promoting ϕ and π to operators $\hat{\phi}$ and $\hat{\pi}$. The equal time commutators are then

$$\left[\hat{\pi}(t, \vec{x}), \hat{\phi}(t, \vec{y}) \right] = i\hbar \delta(\vec{x} - \vec{y}) \quad , \quad \left[\hat{\phi}, \hat{\phi} \right] = 0 = \left[\hat{\pi}, \hat{\pi} \right] . \quad (58)$$

We make the ansatz that $\hat{\phi}(\vec{x}, t) = \int \frac{d^3 \vec{k}}{(2\pi)^{3/2}} \left[v_{\vec{k}}^*(t) \hat{a}_{\vec{k}}^- e^{i\vec{k} \cdot \vec{x}} + v_{\vec{k}}(t) \hat{a}_{\vec{k}}^+ e^{-i\vec{k} \cdot \vec{x}} \right]$ i.e. that we can split the scalar field into positive and negative frequencies. Plugging this into equation 58 gives

$$\left[\hat{a}_{\vec{k}}^-, \hat{a}_{\vec{p}}^+ \right] = \delta(\vec{k} - \vec{p}) \quad , \quad \left[\hat{a}^-, \hat{a}^- \right] = 0 = \left[\hat{a}^+, \hat{a}^+ \right] \quad , \quad \dot{v}_{\vec{k}}(t) v_{\vec{k}}^*(t) - \dot{v}_{\vec{k}}^*(t) v_{\vec{k}}(t) = 2i \quad : \quad \text{Wronski condition.} \quad (59)$$

We implicitly define the vacuum $|0\rangle$ as $\hat{a}_{\vec{k}}^- |0\rangle = 0$. The mode functions $v_{\vec{k}}$ are not completely defined by the commutation relations. The operators $\hat{a}_{\vec{k}}^-$ and $\hat{a}_{\vec{k}}^+$ are also not unambiguously defined. The extra condition we require is that the vacuum is the state of lowest energy of the Hamiltonian, which is given by:

$$\hat{H}(t) = \frac{1}{2} \int d^3 k \left[\dot{v}_{\vec{k}}^2 + \omega_k v_{\vec{k}}^2 \right] \hat{a}_{\vec{k}}^+ \hat{a}_{-\vec{k}}^+ + h.c. + (|\dot{v}_{\vec{k}}|^2 + \omega_k |v_{\vec{k}}|^2) \left(2\hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}}^- + \delta(0) \right)$$

Where the $\delta(0)$ gives an infinite contribution to the energy. The vacuum expectation value of $H(t)$ is then

$$\langle 0 | \hat{H} | 0 \rangle = \int d^3 \vec{k} (|\dot{v}_{\vec{k}}|^2 + \omega_k |v_{\vec{k}}|^2) .$$

The energy density is therefore $\epsilon = |\dot{v}_{\vec{k}}|^2 + \omega_k |v_{\vec{k}}|^2$. A solution of the energy minimization is

$$v_k(t) = \sqrt{\frac{1}{2\omega_k}} e^{i\omega_k t} , \quad \omega_k = \text{frequencies} . \quad (60)$$

We define the annihilation (creation) operators corresponding to positive (negative) frequency modes. Plugging the solution back into Hamiltonian gives:

$$\hat{H}(t) = \int d^3k \omega_k \hat{a}_k^+ \hat{a}_{-\vec{k}}^-$$

The momentum operator is given by $p^i = -\int d^3\vec{x} \pi \partial_i \phi = -\int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p} a_p^+ a_{-p}^-$, excited states with energy ω_k and momentum \vec{k} are given by $\hat{a}_{\vec{k}}^+ |0\rangle$. The *Fock space* is then spanned by

$$|n_1, n_2, \dots\rangle = \frac{1}{\sqrt{n_1! n_2! \dots}} \left(\hat{a}_{\vec{k}_1}^+\right)^{n_1} \left(\hat{a}_{\vec{k}_2}^+\right)^{n_2} \dots |0\rangle . \quad (61)$$

For QFT in flat space, the vacuum, particle numbers, and momenta/energies are all well-defined quantities. This statement still holds after considering Poincaré transformations. We will see that this no longer holds for curved spaces.

15 Quantum field theory in curved space-time backgrounds

This part of the lecture notes is based on the lecture notes by Mukhanov and Winitzki.

15.1 Last time: Minkowski space with free scalar field

The scalar field action is given by $S[\phi] = -\frac{1}{2} \int d^4x [\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2]$. We promote the field $\phi(t, \vec{x})$ to an operator $\hat{\phi}(t, \vec{x})$ with associated creation and annihilation operators (which we can make time-dependent):

$$\left. \begin{aligned} \hat{a}_{\vec{k}}^+ &\rightarrow \hat{a}_{\vec{k}}^+(t) = e^{i\omega_{\vec{k}} t} \hat{a}_{\vec{k}}^+ \\ \hat{a}_{\vec{k}}^- &\rightarrow \hat{a}_{\vec{k}}^-(t) = e^{-i\omega_{\vec{k}} t} \hat{a}_{\vec{k}}^- \end{aligned} \right\} \text{Dispersion relation: } \omega_{\vec{k}} = c|\vec{k}|, \quad m = 0$$

The vacuum is given by $\hat{a}_{\vec{k}}^- |0_M\rangle = 0$, the Fock space is given by:

$$|n_1, n_2, \dots\rangle = \frac{1}{\sqrt{n_1! n_2! \dots}} \left(\hat{a}_{\vec{k}_1}^+\right)^{n_1} \left(\hat{a}_{\vec{k}_2}^+\right)^{n_2} \dots |0_M\rangle$$

Remarks

1. The vacuum energy is divergent: $E_0 = \sum_{\vec{k}} \frac{1}{2} \omega_{\vec{k}}$, $\frac{E_0}{V} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \omega_{\vec{k}} \sim \int dk k^3$. We thus introduce a UV cut-off $k_{max} \sim M_{Planck}$ so that $\frac{E_0}{V} \approx \frac{M_{Planck}^4}{V} \approx 10^{94} g/cm^3$. The cosmological constant would then be $\Lambda \approx M_{Planck}^4$, which is much higher than the observed cosmological constant which corresponds to $\frac{E_0}{V} \approx 10^{55} g/cm^3$. This is referred to as the *cosmological fine-tuning problem*, which is one of the main open problems in contemporary theoretical physics.
2. In Fock space language, the out-states are related to in-states by a scattering (S -) matrix as $|\text{out}\rangle = S |\text{in}\rangle$. In QFT, we require S to be unitary. We will see that a black hole appears to give rise to non-unitarity. In Minkowski space, $|0\rangle_{in} \sim |0\rangle_{out}$. We will see that this changes when we go to curved space-times, where the in-states (out-states) are defined on \mathcal{I}^- (\mathcal{I}^+)
3. Poincaré transformations act as $\hat{\phi}(t, \vec{x}) \rightarrow \phi'(t', \vec{x}')$ e.g. for a Lorentz boost: $\vec{k}\vec{x} - \omega t = \vec{k}'\vec{x}' - \omega' t'$. This induces $a_{\vec{k}}^- \rightarrow b_{\vec{k}'}^- \simeq a_{\vec{k}}^-$ and $|0_M\rangle = |0'_M\rangle$. This means that all inertial observers in Minkowski space will agree on the number of particles in Minkowski space, i.e. QFT in flat space-time has unambiguously defined vacuum and particle states. This will be markedly different when we go to curved space-times.

In the context of general relativity there is no preferred coordinate system. Hence, if one observer sees well-defined particles with respect to a set of positive and negative frequency modes a^+ and a^- , another (general) observer will see a different number of particles corresponding to new modes b^+ and b^- . We will also see that $|0\rangle_a \neq |0\rangle_b$ i.e. different observers will have different corresponding vacuum states. In general, we expect that b -modes are related to a -modes by *Bogoliubov transformations* i.e. $a_k = c_1 b_{\vec{k}}^- + c_2 b_{\vec{k}}^+$, $c_1, c_2 \in \mathbb{C}$.

In curved space-time, we have squeezing/rotation of light-cone as well as $x \leftrightarrow t$ when we cross horizon. This indicates that we will mix positive and negative frequency modes. This is relevant in two instances:

1. Accelerated coordinates in Minkowski space-time, this will give rise to the Unruh effect.
2. Curved space-time, in particular black holes. Here we will see the Hawking effect, the curved space analogue of the Unruh effect.

This discussion also applies in cosmology e.g. the FLRW space-time and particle-antiparticle creation at the big bang. The relevant vacuum here is the so-called *Bunch-Davies vacuum*.

15.2 Unruh effect

We start from two-dimensional Minkowski space, with inertial observer corresponding to the metric $ds^2 = -dt^2 + dx^2$. We now go to Rindler space with velocity $u^\alpha = \frac{dx^\alpha}{d\tau} = (\dot{t}(\tau), \dot{x}(\tau))$ and constant acceleration a given by $a^\alpha(\tau) = \dot{u}^\alpha(\tau)$.

Again, we will see that the notion of particles and the definition of positive and negative frequency modes depends on the observer. Namely, for the inertial observer, the modes are defined with respect to t , while for the accelerated (co-moving) observer, the modes are defined with respect to τ . We will denote the latter observer by ξ_0 . We now compare in 3 steps:

1. Determine the trajectory of the accelerated observer
2. Define a new accelerated coordinate system (co-moving coordinate system), which we will refer to as Rindler coordinates
3. Solve the wave equation of the scalar particle in both coordinate systems and compare corresponding vacua and modes.

These steps are performed in the following fashion:

1. We go to light-cone coordinates given by $u = t - x$, $v = t + x$, so that $ds^2 = -dudv$. For a Lorentz boost, we transform as: $u \rightarrow u' = \alpha u$, $v \rightarrow v' = \frac{1}{\alpha} v$. The trajectory of the accelerated observer is given by $u(\tau) = -\frac{1}{a} e^{-a\tau}$, $v(\tau) = \frac{1}{a} e^{a\tau}$ corresponding to $t(\tau) = \frac{v+u}{2} = \frac{1}{a} \sinh a\tau$, $x(\tau) = \frac{1}{a} \cosh a\tau$.
2. The Rindler coordinates \tilde{u}, \tilde{v} are then implicitly defined by $u = -\frac{1}{a} e^{-a\tilde{u}}$, $v = \frac{1}{a} e^{a\tilde{v}}$. Then: $ds^2 = -dudv = -e^{a(\tilde{v}-\tilde{u})} d\tilde{u}d\tilde{v} = -e^{2a\xi^1} \left(-(d\xi^0)^2 + (d\xi^1)^2 \right)$, where $\xi^0 = \frac{\tilde{u}+\tilde{v}}{2}$, $\xi^1 = \frac{\tilde{u}-\tilde{v}}{2}$. However, (ξ^0, ξ^1) is not a complete coordinate system i.e. it does not cover all of Minkowski space. We refer to figure 6, where the curve indicated by r' has constant ξ^1 .
3. We now introduce quantum fields on our space-time. We will see that we will find different particle numbers for different observers. t and ξ^0 are related in a non-trivial way, which implies that positive frequency modes with respect to t will be a superposition of positive and negative frequency modes with respect to ξ^0 .

Consider again the scalar field action

$$S[\phi] = -\frac{1}{2} \int g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \sqrt{-g} d^2x .$$

Conformal transformations, i.e. transformations of the form $g_{\alpha\beta} \rightarrow \tilde{g}_{\alpha\beta} = \Omega^2(x) g_{\alpha\beta}$, leave our action invariant. This is easy to see since $\sqrt{-\tilde{g}} = \Omega^2 \sqrt{-g}$, $\tilde{g}^{\alpha\beta} = \Omega^{-2} g^{\alpha\beta}$. Hence, we can see that going to Rindler space i.e. to coordinates (ξ^0, ξ^1) simply corresponds to a conformal rescaling. We have

$$S = -2 \int \partial_u \phi \partial_v \phi \, dudv = -2 \int \partial_{\tilde{u}} \phi \partial_{\tilde{v}} \phi \, d\tilde{u}d\tilde{v} .$$

The field equations are $\partial_u \partial_v \phi = 0 = \partial_{\tilde{u}} \partial_{\tilde{v}} \phi$, with solutions given by

$$\phi(u, v) = A(u) + B(v) \quad , \quad \phi(\tilde{u}, \tilde{v}) = \tilde{A}(\tilde{u}) + \tilde{B}(\tilde{v}) .$$

The modes that make up $A(u)$ ($B(v)$) is referred to as right-moving (left-moving). In the following we will focus on the right-moving modes. In Minkowski space: $\phi(u) \sim e^{-i\omega u} = e^{-i\omega(t-x)}$, while in Rindler space we have $\phi(\tilde{u}) \sim e^{-i\Omega \tilde{u}} = e^{-i\Omega(\xi^0 - \xi^1)}$. Namely, ω is the Minkowski frequency and Ω is the Rindler frequency. We consider the domain where $x > |t|$, corresponding to the Rindler wedge where both our Rindler and Minkowski coordinates are well-defined. We thus have that $\hat{\phi} = \int \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} (e^{-i\omega u} \hat{a}_\omega^- + e^{i\omega u} \hat{a}_\omega^+) = \int \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} (e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+)$. This is easy to see since the definition of our quantum field cannot depend on our choice of coordinate system. We have two sets of modes with commutation relations

$$[\hat{a}_\omega^-, \hat{a}_{\omega'}^+] = \delta(\omega - \omega') \quad (\text{Minkowski}) \quad \text{and} \quad [\hat{b}_\Omega^-, \hat{b}_{\Omega'}^+] = \delta(\Omega - \Omega') \quad (\text{Rindler}) .$$

Correspondingly, we have two vacua, $|0_M\rangle$ and $|0_R\rangle$, which satisfy

$$\hat{a}_\omega^- |0_M\rangle = 0 = \hat{b}_\Omega^- |0_R\rangle .$$

Next time we will compare these two vacua and derive the Unruh effect.

16 Unruh effect

We express a quantum scalar field in Minkowski and Rindler space as:

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} [e^{-i\omega u} \hat{a}_\omega^- + e^{i\omega u} \hat{a}_\omega^+] = \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} [e^{-i\Omega \tilde{u}} \hat{b}_\Omega^- + e^{i\Omega \tilde{u}} \hat{b}_\Omega^+]$$

In Minkowski space, u are light-cone coordinates, ω are frequencies, \hat{a}_ω^+ , \hat{a}_ω^- are creation and annihilation operators, $|0_M\rangle$ is the vacuum. \tilde{u} and Ω are the coordinates and frequencies for Rindler space, which has $|0_R\rangle$ as its vacuum state. We are naturally led to ask what is the ‘correct’ or ‘true’ vacuum. The answer is that the vacuum depends on the particular observer. Namely, if a particle detector (observer) is accelerated, its correct vacuum is $|0_R\rangle$, if it is not being accelerated the correct vacuum is $|0_M\rangle$. With respect to $|0_R\rangle$, $|0_M\rangle$ contains infinitely many excited states. We now calculate the relation between $|0_M\rangle$ and $|0_R\rangle$ i.e. between a^\pm and b^\pm . We use the ansatz that they are related by a *Bogoliubov transformation*, which is of the form:

$$\hat{b}_\Omega^- = \int_0^\infty d\omega [\alpha_{\Omega\omega} \hat{a}_\omega^- - \beta_{\Omega\omega} \hat{a}_\omega^+]$$

Note that there does not exist an inverse Bogoliubov transformation since Rindler space covers only half of Minkowski space. From the commutation relations for b_Ω , we find

$$\int_0^\infty d\omega (\alpha_{\Omega\omega} \alpha_{\Omega'\omega}^* - \beta_{\Omega\omega} \beta_{\Omega'\omega}^*) = \delta(\Omega - \Omega') .$$

This gives: $\frac{1}{\sqrt{\omega}} e^{-i\omega u} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}} (\alpha_{\Omega'\omega} e^{-i\Omega' \tilde{u}} - \beta_{\Omega'\omega}^* e^{i\Omega' \tilde{u}})$. Multiplying this with $e^{\pm i\Omega \tilde{u}}$ gives

$$\begin{aligned} \alpha_{\Omega\omega} (\beta_{\Omega\omega}) &= \int_{-\infty}^\infty e^{\pm i\omega u + i\Omega \tilde{u}} d\tilde{u} = \pm \frac{1}{2\pi} \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^\infty (-au)^{-i\frac{\Omega}{a}-1} e^{\mp i\omega u} du \\ &= \pm \frac{1}{2\pi} \sqrt{\frac{\Omega}{\omega}} \exp\left(\frac{i\Omega}{a} \ln \frac{\omega}{a}\right) \Gamma\left(-\frac{i\Omega}{a}\right) \end{aligned} \quad (62)$$

and

$$|\alpha_{\Omega\omega}|^2 = e^{2\pi\Omega/a} |\beta_{\Omega\omega}|^2 , \quad (63)$$

where a is the acceleration parameter. We now go to the Rindler frame i.e. the accelerated observer and we compute the occupation number of Rindler states in the Minkowski vacuum. The expectation value of the occupation number: $\hat{N}_\Omega = \hat{b}_\Omega^+ \hat{b}_\Omega^-$ is

$$\begin{aligned} \langle \hat{N}_\Omega \rangle &= \langle 0_M | \hat{b}_\Omega^+ \hat{b}_\Omega^- | 0_M \rangle \\ &= \left\langle 0_M \left| \int d\omega [\alpha_{\omega\Omega}^* \hat{a}_\omega^+ - \beta_{\omega\Omega}^* \hat{a}_\omega^-] \int d\omega' [\alpha_{\omega'\Omega} \hat{a}_{\omega'}^- - \beta_{\omega'\Omega} \hat{a}_{\omega'}^+] \right| 0_M \right\rangle \\ &= \int d\omega |\beta_{\Omega\omega}|^2 . \end{aligned} \quad (64)$$

Normalization condition with $\Omega = \Omega'$ is then: $\int d\omega (|\alpha_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2) = \delta(0)$. Hence:

$$\langle N_\Omega \rangle = \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1 \right]^{-1} \delta(0) . \quad (65)$$

This is divergent due to $\delta(0)$, which signals that we are considering a space of infinite volume. Hence we compute instead the particle density, given by:

$$n_\Omega = \frac{\langle N_\Omega \rangle}{V} = \frac{1}{\exp\left(\frac{2\pi\Omega}{a}\right) - 1} \quad (66)$$

Where V is the volume of our space-time. This expression gives the massless particles detected by an accelerated observer in the Minkowski vacuum. We see that it obeys Bose-Einstein statistics, which shows that it corresponds to a thermal bath at temperature:

$$T_{Unruh} = \frac{a}{2\pi} . \quad (67)$$

Reintroducing the constants that we previously set to zero gives

$$T_{Unruh} = \frac{\hbar a}{2\pi k_B c} . \quad (68)$$

We see that the Unruh temperature goes to zero if we let \hbar go to zero, signalling its quantum-mechanical origin. If we take $a \approx \frac{c}{\text{micrometer}}$, $T_{unruh} \approx 2K$ i.e. the Unruh temperature is typically very low and thus Unruh radiation is very hard to detect.

The interpretation of this phenomenon is as follows. Unruh radiation describes a thermal bath of particles seen by the Rindler observer in the Minkowski vacuum. Hence, the accelerated detector is coupled to the quantum fluctuations of the Minkowski vacuum. These 'couplings' are supplied by an 'agent' who is responsible for the acceleration of the observer. The temperature (energy) which is detected by the Rindler observer is borrowed from its own acceleration.

16.1 Hawking Radiation

A result due to Hawking states that black hole emit a thermal spectrum of particles. Hawking radiation, which was discovered before Unruh radiation, was rather unexpected, since it was previously believed that particles can only be produced in non-static gravitational fields. We will see that particles with positive and negative frequencies can be produced near the horizon of a static black hole. We do not go through this derivation in great detail since it is similar to that of Unruh radiation. Indeed, by the equivalence principle, inertial and gravitational acceleration are locally indistinguishable, so that the corresponding particle creation should be locally indistinguishable as well.

The first coordinate system is the Schwarzschild space-time in tortoise coordinates. These contain a coordinate singularity at the horizon. These coordinates are analogous to the Rindler coordinates in the derivation of the Unruh effect. The quantity analogous to the Rindler acceleration a will be the surface gravity κ . The tortoise coordinate is given by $r^*(r) = r + r_s \ln\left(\frac{r}{r_s} - 1\right)$, with $r_s = 2M$, the Schwarzschild radius. We see that $r^* \rightarrow \infty$ for $r \rightarrow r_s$ i.e. we are considering one external region of the black hole corresponding to a single Rindler wedge. The coordinate range is $r_s \leq r \leq \infty$ corresponding to $-\infty \leq r^* \leq \infty$. Going to light-cone coordinates $\tilde{u} = t - r^*$ and $\tilde{v} = t + r^*$, the two-dimensional tortoise coordinates are of the form

$$ds_{r^*}^2 = - \left(1 - \frac{r_s}{r(\tilde{u}, \tilde{v})}\right) d\tilde{u}d\tilde{v} . \quad (69)$$

This is conformally equivalent to a flat metric. We see that it is singular at $r = r_s$, hence these coordinates only cover region I .

The other coordinate system are the Kruskal-Szekeres (KS) coordinates, corresponding to a free falling observer. These coordinates are non-singular at the horizon, hence a locally inertial observer in these coordinates should not notice anything special at the horizon. This coordinate system is analogous to the Minkowski coordinates for the Unruh effect, since they cover all four coordinate wedges of Rindler or Schwarzschild space and are non-singular everywhere. We go to coordinates $u = -2r_s \exp\left(-\frac{\tilde{u}}{2r_s}\right)$, $v = 2r_s \exp\left(\frac{\tilde{v}}{2r_s}\right)$, which gives the following metric:

$$ds_{KS}^2 = -\frac{r_s}{r(u,v)} \exp\left[1 - \frac{r(u,v)}{r_s}\right] dudv . \quad (70)$$

This metric is also conformally equivalent to a flat metric, so that it is regular at $r = r_s$ and indeed covers coordinate wedges $I - IV$. We now introduce the 2-dimensional scalar field theory action

$$S[\phi] = -\frac{1}{2} \int g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \sqrt{-g} d^2x . \quad (71)$$

Note that this action is conformally invariant. We can expand in both bases, giving $\phi(\tilde{u}, \tilde{v}) = \phi(u, v)$. $\phi(\tilde{u}, \tilde{v})$ corresponds to $|0_B\rangle, \hat{b}_\Omega^-, \hat{b}_\Omega^+$, while $\phi(u, v)$ corresponds to $|0_{KS}\rangle, \hat{a}_\omega^-, \hat{a}_\omega^+$. We use similar notation to the one we used for the Unruh effect to signal that *Boulware vacuum* $|0_B\rangle$ (with associated creation and annihilation operators) is analogous to the Rindler vacuum and *Kruskal-Szekeres vacuum* $|0_{KS}\rangle$ (with associated creation and annihilation operators) is analogous to the Minkowski vacuum. Consider an observer, A, at constant r^* i.e. constant r , which corresponds to the accelerated Rindler observer, and another observer, B, at non-constant r^* which is an inertial observer. What is the particle spectrum seen by A in the Kruskal-Szekeres vacuum $|0_{KS}\rangle$?. Analogous to our previous computation, we find that

$$\langle \hat{N}_\Omega \rangle = \langle 0_{KS} | \hat{b}_\Omega^+ \hat{b}_\Omega^- | 0_{KS} \rangle = \frac{1}{\exp\left(\frac{2\pi\Omega}{\kappa} - 1\right)} \delta(0) . \quad (72)$$

We thus find a thermal spectrum at Hawking temperature:

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} = \frac{\hbar c^3}{8\pi G k_B M} . \quad (73)$$

Where we introduce all constants of nature at the last equality. Note again that $T \rightarrow 0$ when $\hbar \rightarrow 0$, signalling the quantum-mechanical origin of Hawking radiation as it did for Unruh radiation.

17 Information loss paradox

The results of the last few lectures can be summarized as follows:

Phenomenon:	Unruh effect	Hawking radiation
Origin:	Accelerated coordinate systems	Gravitational background of a black hole
True vacuum:	$ 0_M\rangle$ - Inertial vacuum of Minkowski space	$ 0_{KS}\rangle$ - KS-vacuum (free-falling observer)
Singular vacuum:	$ 0_R\rangle$ - Rindler vacuum of accelerated observer	$ 0_B\rangle$ - Boulware vacuum (observer at $r = \infty$)
Temperature:	a - acceleration	κ - surface gravity i.e. acceleration at horizon
Coordinates:	$u = -a^{-1} \exp(-a\tilde{u}), v = -a^{-1} \exp(a\tilde{v})$	$u = -\kappa^{-1} \exp(-\kappa\tilde{u}), v = -\kappa^{-1} \exp(\kappa\tilde{v})$

Hawking (Unruh) radiation is found by calculating the occupation $\langle 0|b_{\Omega}^{\dagger}b_{\Omega}^{-}|0\rangle = \left[\exp\left(\frac{2\pi\Omega}{a(\kappa)}\right) - 1\right]^{-1} \delta(0)$, where b_{Ω}^+ , b_{Ω}^- are the Boulware (Rindler) modes and $|0\rangle$ is the KS- (Minkowski) vacuum.

Remarks:

1. For a black hole in a thermal equilibrium with an external heat bath with temperature T_H , both modes are present. Namely, the black hole emits and absorbs particles at the same rate.
2. For a black hole in empty space, the black hole only emits particles, hence it will evaporate, $\delta M < 0$, and it disappears within finite time.
3. Recall: $dM = \frac{1}{8\pi M} d\left(\frac{A}{4}\right) = T_H dS, \frac{\kappa}{2\pi}, T_H = \hbar \frac{\kappa}{2\pi}$. Hence: $S = \frac{A}{4\hbar} \xrightarrow{\hbar \rightarrow 0} \infty$.

Hawking radiation is thermal, which signals the loss of information regarding the initial matter state that formed the black hole. For example, presume a black hole is formed from highly energetic muons, schematically: $\mu^+ \mu^- \rightarrow$ ‘black hole’ \rightarrow ‘gravitons+electrons+photons+etc’. This schematic process is meant to show that we completely lose information about the fact that the black hole here was formed by muons, since the emitted Hawking radiation consists of many different particles.

More explicitly: In quantum mechanics, we have a time evolution operator which we write as $S(t) = \exp\left(\frac{iHt}{\hbar}\right)$, where H is our Hamiltonian. In the context of QFT, S is referred to as the S-matrix. If we know $|\psi(t_0)\rangle$, then $|\psi(t)\rangle = S(t) |\psi(t_0)\rangle$. We can generalize this using the *quantum density matrix*, which we can write as:

$$\hat{\rho} = \sum_i c_i |\psi_i\rangle \langle \psi_i| \tag{74}$$

with $|\psi_i\rangle \in \mathcal{H}, c_i \in \mathbb{C}$, and $tr \hat{\rho} = 1$. When $\hat{\rho}^2 = \hat{\rho}$, $\hat{\rho}$ corresponds to a *pure state*. For $\hat{\rho}^2 \neq \hat{\rho}$, $\hat{\rho}$ corresponds to a *mixed state*. Hawking asserted that due to the thermal nature of Hawking radiation, the outgoing particles are always described by a mixed state. Namely, the corresponding density matrix is $\rho_H = \sum_n e^{-\beta E_n} |n\rangle$, where $|n\rangle$ are energy eigenstates with energy E_n . Hence, we go from a pure state to a mixed state, which signals that we break unitarity. Hawking thus introduced the non-unitary *dollar matrix* [10], which relates ρ^{final} to $\rho^{initial}$ as $\rho^{final} = \$\rho^{initial}$, as an ad hoc substitute for the usual S-matrix.

17.1 Possible solutions to the information problem

Solutions to the information loss paradox that can be found in the literature include:

1. Black holes violate Quantum mechanics (no longer very popular)
2. Information stays inside the black hole. This requires one to stop Hawking radiation artificially so that black hole remnant remains, since Hawking temperature increases when the black hole shrinks. This does not seem very plausible
3. Information goes somewhere else, e.g. region II or some (other) parallel universe.
4. Quantum mechanics is ok - Information comes out with the Hawking radiation. This is typically considered most plausible, hence we will make a few remarks on this possibility.

So far, we did not consider the back-reaction of infalling matter on the black hole. Furthermore, we did not treat gravity in a quantum-mechanical way. One may hope that taking these effects into account will restore black hole unitarity. This would imply that the Hawking quanta will not be thermal anymore. Another possibility is that the information is stored on the horizon, so that the microscopic details of Hawking radiation are determined by a holographic principle. The most explicit holographic principle known to date is the AdS/CFT correspondence, which states that the gravitational AdS bulk is dual to a non-gravitational conformal field theory (CFT) at the boundary of the AdS space. Since CFT's undergo unitary time evolution, so should the corresponding AdS spaces, even when they contain a black hole.

Another proposed (partial) solution is called *black hole complementarity*, which gave rise to the firewall picture. Another proposed picture is the so-called *fuzz-ball*, and yet another picture is the idea that a black hole is a graviton condensate at the quantum critical point. We will now comment on black hole complementarity and the firewall picture.

17.1.1 Black hole complementarity (Susskind, Thorlacius, 't Hooft)

According to black hole complementarity, information is simultaneously reflected and passed through the horizon. The reflected information can be perceived by the outside observer, while the information that passes through can be perceived by the freely falling observer, but no single observer can confirm both pictures at the same time. This gives rise to the notion of the stretched horizon [11], which is a thin 'membrane' with a size of the order of the Planck length. According to the external observer, the infalling and reflected information gets 'heated up' at the horizon, yet the infalling observer does not see anything special. This picture works if there is entanglement between the infalling and reflected information.

17.1.2 Firewall paradox

In 2012, the AMPS (Almheiri, Marolf, Polchinski, Sully) paper was published [12], which is nicely reviewed in [13]. AMPS pointed out a flaw in black hole complementarity. They argued that the following three statements cannot be simultaneously true.

1. Hawking radiation is in a pure state.
2. Information is emitted from a region (e.g. the stretched horizon) near the horizon, where an effective description of gravity in terms of GR is correct.
3. Infalling observer encounters nothing special when crossing the horizon.

The proposed solution by AMPS is to give up the third statement i.e. the infalling observer will be 'burned' at the horizon, hence this is known as the firewall paradox.

18 Solitons in String Theory

The next few lectures are based on the source found at [14]. They consider the construction of black holes in string theory and the verification of the Bekenstein-Hawking entropy via string microstate counting, found by Vafa and Strominger [?].

18.1 Review of electrodynamics in Minkowski space

The vacuum Maxwell's equations are

$$\begin{aligned} d * F &= 0 \\ dF &= 0 . \end{aligned} \tag{75}$$

Where F is the field strength in form notation and $*F$ is its *Hodge dual*. Generally, the Hodge dual of an $(r-d)$ -form $\omega_{\mu_1\mu_2\dots\mu_{r-d}}$ is given by $(*\omega)_{\mu_1\mu_2\dots\mu_r} = \frac{\sqrt{-g}}{r!(r-d)!} \epsilon^{\mu_{r+1}\dots\mu_d} \omega_{\mu_{r+1}\dots\mu_d}$. We then have $d * F = *j_e$, where j_e is the electromagnetic current. We denote $*F$ as \tilde{F} , following standard notation. We have the following expression for the electric charge:

$$e = \oint_{S^2} d\vec{n} \vec{E} = \oint_{S^2} *F = \int_{B^3} d * F = \int_{B^3} *j_e$$

The expression $dF = 0$ is known as the *Bianchi identity*. This is an identity since we can write $F = dA$, where A is the electromagnetic field in form notation. The Poincaré identity then tells us that $dF = d^2A = 0$. Electromagnetism in form notation allows us to easily define a magnetic current and charge as follows. We have $dF = j_g$; $F = 2(?)dA + \omega$, where $d * \omega = 0$ and $d\omega = j_g$, where j_g is the magnetic current. We then define magnetic charge as

$$g = \int_{\partial\mathcal{M}} F = \int_{\mathcal{M}} j_g . \tag{76}$$

Where \mathcal{M} is some three-dimensional volume in our $(3+1)$ -dimensional Minkowski space. We still have the following gauge freedom: $A \rightarrow A + d\Lambda$, where Λ is a 0-form i.e. a scalar. We are now going to use these techniques in higher-dimensional spaces, particularly for solitonic solutions in string theory. We will be considering d -dimensional spaces and $(p+1)$ -form potentials of the form $A^{(p+1)} = A_{\mu_1\dots\mu_{p+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}$. We then have gauge freedom $A^{(p+1)} \rightarrow A^{(p+1)} + d\Lambda^{(p)}$, our field strength is of the form $F^{(p+1)} = (p+2)dA^{(p+1)} + \omega^{(p+2)}$. The ω -term is typically absent in electromagnetism, except when we introduce magnetic charges by hand. The charges corresponding to F and $*F$ are given by:

$$e = \int_{\mathcal{M}} *j_e^{(d-p-1)} , \quad g = \int_{\tilde{\mathcal{M}}} j_g^{(p+3)} \tag{77}$$

Where \mathcal{M} and $\tilde{\mathcal{M}}$ are $(d-p-1)$ -dimensional and $(p+3)$ -dimensional subspaces, respectively. In analogy with classical electromagnetism, we assume the charges that give rise to $*j_e^{(d-p-1)}$ to be localized in p spatial dimensions. The magnetic objects corresponding to j_g extends $d-p-4$ dimensions.

18.2 Dirac quantization

If we have take an electric charge along some path that encloses a magnetic charge, the phase of the wave function has to change by $2\pi n$, $n \in \mathbb{Z}$. This means that:

$$eg = 2\pi n , \quad e = \frac{2\pi n}{g} \tag{78}$$

Namely, we see that the electric charge is quantized due to the presence of a magnetic charge.

To summarise, the form degree of the potential gives us the dimension of the electric object. From this, we can obtain the dimension of the magnetic object, that is, $\tilde{F}^{(d-p-2)} = d\tilde{A}^{(d-p-3)}$, so that we see that our magnetic object is $(d-p-3)$ -dimensional. We will now apply these ideas to string theory.

18.3 Supergravity and p-branes

We present the massless degrees of freedom of the following superstring theories in a table.

Model	Potential	Field strength	\mathbf{p}_e	\mathbf{p}_m
Het., II_A , II_B	$B^{(2)} = A^{(1+1)}$	$F^{(2)}$	Fundamental string	5 (NS5-brane)
II_A	$A^{(1)}, A^{(3)}$	$F^{(2)}, F^{(4)}$	D-particle, 2-brane	6-brane, 4-brane
II_B	$A^{(0)}, A^{(2)}, A^{(4)}$	$F^{(1)}, F^{(3)}, F^{(5)}$	1-brane, D-string, 3-brane	D7-brane, D5-brane, 3-brane
M-theory (d=11)	$A^{(3)}$	$F^{(4)}$	M2-brane	M5-brane

We try to find solutions to the field equations corresponding to the SUGRA action, corresponding to the massless degrees of freedom of the superstring theories listed above:

$$S_{eff}^E = \frac{1}{2K^2} \int d^{10}x \sqrt{-G^E} \left(R(G^E) - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - \sum \frac{1}{2n!} e^{a\Phi} F^{\mu_1 \dots \mu_n} F_{\mu_1 \dots \mu_n} + \text{fermionic part} \right) \quad (79)$$

Here, $R(G^E)$ corresponds to the graviton and Φ is the dilaton. We have $a = -1$, $n = 3$ for the universal sector, and $a = \frac{5-n}{2}$ for the RR-sector. From now on, we will ignore the \sum_n -term in the action and consider only one term in the sum. This gives the following equations of motion:

$$\delta G_{\mu\nu}^E : R_{\mu\nu}^E = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2(n-1)!} e^{a\Phi} \left(F_{\mu\rho_1 \dots \rho_{n-1}} F_{\nu}^{\rho_1 \dots \rho_{n-1}} - \frac{n-1}{n(d-2)} F^2 G_{\mu\nu} \right) \quad (80)$$

$$\delta \Phi : \partial_\mu \partial^\mu \Phi = \frac{a}{n!} e^{a\Phi} F^2 . \quad (81)$$

We make the ansatz that x_μ with $\mu \in \{0, \dots, p\}$ are the coordinates of $(p+1)$ -dimensional charged objects that are invariant under the Poincaré group $P(1, p)$, and y_M with $M \in \{p+1, \dots, d-1\}$ are coordinates of a space with an $SO(d-p-1)$ isometry. We then define $r = \sqrt{y_M y^M}$. We then make the ansatz that the metric is of the form:

$$ds^2 = e^{2A(r)} dx^\mu dx_\mu + e^{2B(r)} dy_M dy^M \quad (82)$$

Where the μ and M -contractions are performed with a flat metric. We then make the ansatz that the field strength is of the form

$$F_{M\mu_2 \dots \mu_n}^e = \epsilon_{\mu_2 \dots \mu_n} \partial_M e^{C(r)} . \quad (83)$$

Plugging this into the equations of motion gives

$$\begin{aligned} e^{2A(r)} &= H(r)^{\frac{-4d}{\Delta(d-2)}} \\ e^{2B(r)} &= H(r)^{\frac{-4d}{\Delta(d-2)}} \\ e^{2C(r)} &= \frac{2}{\sqrt{\Delta}} H(r)^{-1} \\ e^{\Phi(r)} &= H(r)^{\frac{2a}{\Delta}} \\ \partial_M \partial^M H(r) &= 0 . \end{aligned} \quad (84)$$

We then introduce sources to find non-trivial solutions to the last equation. These are of the form $H(r) = 1 + \frac{\alpha}{r^{\tilde{d}}}$, $\alpha > 0$. The symbols in the above expressions are $\tilde{d} = d - p - 3$, $\Delta = a^2 + \frac{2(p-1)\tilde{d}}{2}$, $\zeta = \pm 1$, with $+1$ for the electric solution and -1 for the magnetic solution, and α fixed by the charge for the electric solution and $\alpha = \frac{q\sqrt{\Delta}}{2\tilde{d}}$ for magnetic solutions.

To summarise, we found solutions to the low energy effective field theory of string theory. They are extended and carry charge and mass. Additionally, they are BPS saturated, from which we conclude that they also exist in the quantum regime and remain BPS. This entails that there is some residual supersymmetry for these particular solutions. We will use this later to count the number of states.

19 Brane solutions

19.1 Explicit examples of solitonic solutions to type II SUGRA: p-branes

In the (NS,NS)-sector, we have the explicit solution given by the fundamental string for $d = 10$, $n = 3$:

$$ds_E^2 = \left(1 + \frac{\alpha}{r^6}\right)^{-3/4} dx_\mu^2 + \left(1 + \frac{\alpha}{r^6}\right)^{1/4} dy_n^2, \quad e^\Phi = \left(1 + \frac{\alpha}{r^6}\right)^{-1/2} \sim g_s \quad (85)$$

NS5-brane with $n = 7$

$$ds_E^2 = \left(1 + \frac{\alpha}{r^6}\right)^{1/4} dx_\mu^2 + \left(1 + \frac{\alpha}{r^6}\right)^{3/4} dy_n^2, \quad e^\Phi = \left(1 + \frac{\alpha}{r^6}\right)^{1/2} \sim \frac{1}{g_s} \quad (86)$$

Here, n is the dimension of the field strength tensor under consideration. The fact that we have r^6 comes from the fact that the codimension of the object under consideration is 6. These objects are dual to each other, as we can see from the string couplings for the two objects, namely, g_s and $\frac{1}{g_s}$, respectively. For the 3-brane, we have $n = 5$ i.e. the object is self-dual. The corresponding metric is

$$ds_E^2 = \left(1 + \frac{\alpha}{r^4}\right)^{-1/2} dx_\mu^2 + \left(1 + \frac{\alpha}{r^4}\right)^{1/2} dy_n^2, \quad e^\Phi = 1. \quad (87)$$

Lastly, the 5-brane has the same metric as the NS5-brane, given above, but with $\Phi = \frac{1}{\Phi_{NS5}}$.

19.2 D-branes

For the remainder of this lecture, we will consider D-branes, which are originally introduced as Dirichlet boundary conditions for open strings. Dirichlet boundary conditions are given by:

$$\partial_\tau X^i \Big|_{\sigma=0,\pi} = 0$$

This is necessary for open string to be consistent with T-duality, as we will discuss later. From T-duality, we will see that

$$X^i \Big|_{\sigma=0,\pi} = c, \text{ for some } c \in \mathbb{C}.$$

In general, we call the following set-up a Dp-brane (where p indicates the spatial dimensionality of the brane): we have Neumann boundary conditions in p dimensions i.e.

$$\partial_\tau X^\mu \Big|_{\sigma=0,\pi} = 0, \quad \mu = 0, \dots, p,$$

and Dirichlet boundary conditions in $q - p$ dimensions i.e.

$$\partial_\tau X^i \Big|_{\sigma=0,\pi} = 0, \quad i = p + 1, \dots, q.$$

It may seem as if the D-branes are rigid objects, but we will see during the remainder of the lecture that this is not the case. We will see that they are dynamical objects with a tension $T_p \sim 1/g_s$. Hence, at $g_s \ll 1$, $T_p \gg 1$, so that the D-branes appear static. At $g_s \gg 1$, the opposite is true, and it will be easy to see the dynamics of the D-brane, which hints at the fact that D-branes are indeed solitonic (non-perturbative) objects in our string theory.

19.3 T-duality for the closed string

We compactify our string theory on a circle or torus, and look at the spectrum of our theory. If we consider compactification on a circle of radius R , we know that the spectrum is classified by (momentum, winding number) = $(\frac{n}{R}, m)$, $m, n \in \mathbb{Z}$. One can show that this theory is dual to a theory compactified on a circle with radius $\frac{\alpha'}{R}$ and a spectrum classified by $(m, \frac{n}{R})$ via T-duality. Note that momentum and winding number are exchanged under T-duality. T-duality maps X_R to $-X_R$ and X_L to X_L .

19.4 T-duality for the open string

For the open string, we cannot define a meaningful winding number, hence we have only momentum. Via T-duality, we map to a theory without momentum and with only winding number, since we fix the end-points of our open string. We again map X_R to $-X_R$ and X_L to X_L . For the open string, we also exchange von Neumann and Dirichlet boundary conditions. A general feature of T-duality along one of the dimensions of our Dp-brane, we go from a Dp-brane to a D(p-1)-brane, whilst if we perform T-duality along one of the dimensions normal to the Dp-brane, we go to a D(p+1)-brane.

19.5 Massless spectrum of open string

The massless states are those without winding. One can show that these are given by $\alpha_{-1}^\mu |0\rangle$ and $\alpha_{-1}^9 |0\rangle$. $\alpha_{-1}^\mu |0\rangle$ correspond to $U(1)$ -gauge bosons in (8+1)-dimensional target space, while $\alpha_{-1}^9 |0\rangle$ corresponds to a scalar in target space. Generalizing this to higher-dimensional tori, the massless states will correspond to $U(1)$ -gauge boson in $(p+1)$ dimensions and $(9-p)$ scalars in $(p+1)$ dimensions, which describe the position of the Dp-brane. Since these states do not have any momentum orthogonal to the D-brane, we can see that the gauge theory lives on the world volume of the D-brane. Due to supersymmetry, the gauge bosons are in the same multiplet as the scalar, and they share this multiplet with a spinor.

19.6 Generalization: several D-branes

We now extend to a situation where we have N D-branes, $N \in \mathbb{N}$, we will see that this will give rise to super Yang-Mills theories. Namely, if one associates a charge to the end of our open strings, one can show that the string scattering satisfies a super Yang-Mills algebra. The charges are referred to as *Chan-Paton factors*. This introduces a non-abelian structure from which one can find a non-abelian gauge potential. Namely, we will find a gauge field A_9 of the form $A_9 = \frac{1}{2\pi R} \text{diag}(\theta_1, \theta_2, \dots, \theta_n)$, where $\theta_i \in \mathbb{R}$. Introducing a gauge connection changes our canonical momentum and the bosonic field in the compactified dimension as:

$$p = \frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R}, \quad X_{(i,j)}^9 = (\sigma, \tau) = C_{(i,j)} = + \left(2L + \frac{\theta_j - \theta_i}{\pi} \right) R_D \sigma + \text{oscillators}.$$

Setting $C_{(i,j)} = \theta_i R_D$, we have $X_{(i,j)}^9 = 2\pi L R_D + \theta_j R_D$

Remark: Since D-branes break $P(1,9)$, i.e. the Poincaré group of $\mathbb{R}^{1,9}$ they also break 10d SUSY.

19.7 Dynamics of D-branes

We will see now that D-branes are in fact dynamical: they interact with closed and open strings, they couple to gravity, the dilaton, and RR-fields, and they fluctuate in position and shape. The perturbative degrees of freedom are the open strings. We consider two D-branes with a closed string propagating between them, following a calculation done by Polchinski. Depending on how we choose our time direction, this process can either be seen as an open string vacuum amplitude, or a closed string exchange. Matching the amplitudes for these processes is expressed as:

$$\left\langle 0 \left| e^{-\pi^2 (L_0 + \bar{L}_0 - 2)/2} \right| 0 \right\rangle_{\text{tree}} = \mathcal{A} = \left\langle 0 \left| e^{-2t(L_0^D - 1)} \right| 0 \right\rangle_{\text{one-loop}}$$

Here, L_0^D is the L_0 associated to the D-brane. One can show that this amplitude is zero. This means that there is no force between the parallel D-branes, which means that they are BPS-saturated, i.e. the attractive force from gravity and the dilaton equals the repulsive force from the RR-fields. The result for this calculation is

$$\lim_{r \rightarrow \infty} \mathcal{A} = V_{p+1}(1-1)2\pi(4\pi^2\alpha')\Delta_{(q-p)}(r^2) \quad (88)$$

Here, $\Delta_{(q-p)}(r^2)$ is the propagator in $(q-p)$ -dimensional subspace with r the distance from the D-brane. The $+1$ and -1 contributions come from the gravity+dilaton and the RR-fields, respectively. We now use this expression to find the tension T_p and the charge density ρ_p by comparing this result with the one obtained from the low energy effective action:

$$S_{eff}^{II} = \frac{1}{2\kappa^2} \int d^{10}X \left[\underbrace{\sqrt{-G} \left(R(G) + \frac{1}{2} (d\Phi)^2 + \frac{1}{12} e^{-\Phi} (dB)^2 \right)}_{\text{Universal part}} + \sum_p \frac{1}{2(p+2)!} e^{\frac{3-p}{2}\Phi} (dA^{(p+1)})^2 \right]. \quad (89)$$

The effective action on the D-brane is

$$S_{eff}^D = T_p \int_{\mathcal{M}^{(p+1)}} d^{p+1}\xi e^{\frac{3-p}{4}\Phi} \sqrt{-\det(\Pi^* G + \Pi^* B + 2\pi\alpha' F)} \quad (90)$$

where Π^* is the pull-back from 10-dimensional target space onto the D-brane. This is known as the *Dirac-Born-Infeld action*. Lastly, we will need the coupling of the Dp-brane to the RR-field $A_{\mu_1 \dots \mu_p}^{(p+1)}$:

$$S_{eff}^D = \rho_p \int_{\mathcal{M}^{(p+1)}} \Pi^* A^{(p+1)} \quad (91)$$

20 Dimensional reduction and black holes

We start from the effective action

$$S_{eff}^{II} + S_{eff}^D + S_{eff}^{WZ} = \int d^{10}x \left[\frac{1}{2\kappa^2} \left(\sqrt{-G}R + \frac{1}{2} (d\phi)^2 \frac{1}{2(p+2)!} \left(dC^{(p+1)} \right)^2 \right) + \left(T_p \delta^{(1-p)}(x_\perp) \right) \left(-\frac{p-3}{4} \phi + \sqrt{-G} \right) + \rho_p C^{(p+1)} \delta^{(1-p)}(x_\perp) + \dots \right].$$

Dropping all coupling and higher order terms, since we are only interested in the long range behaviour, we find:

$$\ln Z_{vac} = 2V_{p+1} \kappa^2 (\rho_p^2 - T_p^2) \Delta_{(q-p)}(p^2) \stackrel{!}{=} 0, \quad \rho_p^2 = T_p^2 = \frac{\pi}{\kappa^2} (4\pi^2 \alpha')^{3-p} \quad (92)$$

The last expression tells us that the charge of a Dp-brane is equal to its tension. We now consider Dirac-Zwanziger quantization. We move a Dp-brane around its magnetic dual, which is a D(6-p)-brane, we have:

$$\rho_p \rho_{6-p} = \frac{\pi n}{\kappa^2}$$

This is a charge quantization, akin to the Dirac quantization condition in electromagnetism. Instead of calculating $e \oint_C A$ for a closed curve around a monopole and requiring the expression gives a multiple of 2π , we now calculate $\rho_p \oint_{\mathcal{M}_{\sqrt{+}\infty}} C^{(p+1)}$, to give the expression above. This shows that the D-branes are states with minimal RR-charge, hence we call them elementary.

To summarize: Introducing these D-branes breaks half of our original supersymmetry (as we found for the p-branes). We find that $\rho_p = T_p$ and $\rho_p \rho_{6-p} = \frac{\pi}{\kappa^2}$. The low energy limit is a SUGRA p-brane solution. In this picture, our perturbative degrees of freedom are massless open string states ending on D-branes.

20.1 Black holes in string theory

Our strategy for constructing and analyzing string black holes will follow the following steps:

1. Construct black hole via dimensional reduction of p-brane solutions (extremal black holes in 5d: $T = 0$, no Hawking radiation).
2. Use D-brane description of the p-branes (i.e. relating strongly and weakly coupled limits of our theory) to count microstates N , calculate entropy as $S_{stat} = \log N$
3. Compare to $S_{BH} = \frac{A}{4}$

20.1.1 Dimensional reduction of effective action

The string frame action in terms of R and ϕ reads

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-G} e^{-2\phi_D} (R + 4\partial_M \phi \partial^M \phi_D) \quad , \text{ take } x^M = (x^\mu, x) \quad , \quad \text{with } x \sim x + 2\pi R n, \quad n \in \mathbb{Z}. \quad (93)$$

We perform a metric ansatz of the form

$$G_{MN} = \begin{pmatrix} \tilde{G}_{\mu\nu} + e^{2\sigma} A_\mu A_\nu & e^{2\sigma} A_\mu \\ e^{2\sigma} A_\nu & e^{2\sigma} \end{pmatrix}. \quad (94)$$

This corresponds to considering only massless states, since we assume A_μ to be independent of x , the coordinate of our compactified dimension. Here, $\tilde{G}_{\mu\nu}$ is the $(D-1)$ -dimensional metric in string frame, A_μ is a Kaluza-Klein gauge field,

and σ is the Kaluza-Klein scalar. We have $\sqrt{-G} = e^\sigma \sqrt{-\bar{G}}$, hence the geodesic length ρ of our compactified dimension is related to the parametric length R of the compactified dimension as

$$2\pi\rho = 2\pi R e^{\langle\sigma\rangle}. \quad (95)$$

We will see that $\langle\sigma\rangle$ is not fixed by the equations of motion. We thus define a new dilaton in our $(D-1)$ -dimensional theory

$$\bar{\phi}_{D-1} = \phi_D - \frac{\sigma}{2}. \quad (96)$$

We then find

$$S = \frac{1}{2\kappa_{D-1}^2} \int d^{D-1}x \sqrt{-\bar{G}} e^{-2\bar{\phi}_{D-1}} \left(\bar{R} + 4\partial_\mu \bar{\phi}_{D-1} \partial^\mu \phi_{D-1} - \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} \right), \quad \frac{1}{\kappa_{D-1}^2} = \frac{2\pi R}{\kappa_D^2}. \quad (97)$$

The massive states in this theory (which we ignore here) are charged with respect to the $U(1) \simeq S^1$ gauge theory on the compactified circle with charges $Q \sim \frac{n}{R}$. The gauge coupling depends on the modulus and the dilaton field. In the full effective action of type II_A and II_B , there are also tensor fields which have to be dimensionally reduced.

20.1.2 Dimensional reduction of p-branes

We have two ways of performing dimensional reduction.

1. Compactify along world volume directions of the p-brane. We then go from a p-brane in D dimensions to a $(p-1)$ -brane in $(D-1)$ dimensions
2. We compactify along directions transverse to the p-brane. This gives rise to a p-brane in $(D-1)$ dimensions. The transverse directions are not isometry directions of the p-brane, but we can use the no-force property of BPS to construct a periodic array of p-branes.

We consider the second of the dimensional reduction constructions. We now split off the the transverse directions as $\vec{x} = (\bar{x}, x)$. At all $x = 2\pi Rn$, $n \in \mathbb{Z}$, we have

$$H = 1 + \sum_{-\infty}^{\infty} \frac{Q}{|\vec{x} - \vec{x}_n|^{D-p-3}}, \quad \text{with } \vec{x}_n = (0, 2\pi nR) \quad (98)$$

Through some rather involved calculations, one can show that:

$$H = 1 + \frac{Q}{R|x|^{D-p-4}} + \mathcal{O}\left(e^{-\frac{|x|}{R}}\right) \quad (99)$$

One says that the p-brane is delocalized or ‘smeared’ along the circle. This state is invariant under rotations of the circle.

20.1.3 Interlude: Tangherlini black hole

This is a 5-dimensional version of the extremal Reissner-Nordström black hole solution, it is of the form:

$$ds_E^2 = -H^{-2} dt^2 + H (dr^2 + r^2 d\Omega_3^2) \quad (100)$$

where H is a function that is harmonic with respect to the transverse coordinates, i.e. $H = 1 + \frac{Q}{r^2}$, where Q is the electric charge of the black hole. In these coordinates, the event horizon is at $r = 0$. The area of the event horizon and corresponding is then

$$A = 2\pi^2 \lim_{r \rightarrow 0} \left(r^3 H^{3/2} \right) = \underbrace{2\pi^2}_{\text{Area of } S^3} Q^{3/2} \quad , \quad S_{BH} = \frac{A}{4} = \frac{\pi^2}{2} Q^{3/2} \quad (101)$$

20.2 Dimensional reduction of the D1-brane

We now try to construct a 5-dimensional black hole by dimensional reduction of a 1-brane, starting from an expression from last lecture for the string frame action:

$$ds_{SF}^2 = H_1^{-1/2} (dt^2 + dy^2) + H_1^{1/2} (dx_1^2 + \dots + dx_8^2) \quad , \quad e^{-2b_{10}} = H_1^{-1} \quad , \quad H_1 = 1 + \frac{Q_1^{(10)}}{r^6} \quad (102)$$

We now compactify along $x_4, x_5, \dots, x_9 = y$. x_9 is a direction along our D-brane, i.e. compactifying along this dimension reduces the dimensionality of our p-brane to a (p-1)-brane, according to the first of the two methods of compactification outlined above. This gives the metric

$$ds_{SF}^2 = -H_1^{1/2} dt^2 + H_1^{1/2} (dx_1^2 + \dots + dx_4^2) \quad , \quad H_1 = 1 + \frac{Q_1^{(5)}}{r^2} \quad . \quad (103)$$

Where $Q_1^{(5)}$ can be calculated from $Q_1^{(10)}$ in the original string frame action. Additionally, the dilatons are related as: $e^{-2\phi_5} = e^{-2\phi_{10}} \sqrt{G_{internal}} = H_1^{-1/4}$.

We now go from the string frame solution to the Einstein frame solution, which reads:

$$ds_E^2 = H_1^{-2/3} dt^2 + H_1^{1/3} (dr^2 + r^2 d\Omega_3^2) \quad (104)$$

We then find for the area/Bekenstein-Hawking entropy [?].

$$A = 2\pi^2 \lim_{r \rightarrow 0} \left(r^3 \sqrt{\frac{Q_1}{r^2}} \right) = 0 \Rightarrow S_{BH} = \frac{A}{4} = 0 \quad . \quad (105)$$

The solution does not have an event horizon. This is due to the Kaluza-Klein scalars σ which become singular at the event horizon and at spatial infinity. This signals that these solutions don't make sense as solutions to the lower-dimensional theory. In our case, the toroidal radii behave as $R_{5,6,7,7} \rightarrow \infty$, $R_9 \rightarrow 0$ when we approach the horizon. We are only interested in regular solutions, so we have to look further. Finding such regular solutions is typically referred to as *stabilization of the moduli*. The generic method for stabilization is to employ ratios of harmonic functions, i.e. we consider not just H_1 but ratios of harmonic functions.

20.3 Solution: p-brane superposition

We are interested in a superposition of p-branes of different kinds, where preserve part of the original supersymmetry. We consider a Dp-Dp'-system which are either parallel or perpendicular in all dimensions. For such a BPS-solution, the dimensionality of the mutual transverse directions is $4k$, $k \in \mathbb{N}$. In these directions, we either (N,D) or (D,N) boundary conditions, i.e. on one brane we have Neumann and on the other brane we have Dirichlet boundary conditions.

21 Black holes in string theory from p/D-branes

21.1 p-branes

p-Branes are p-dimensional extended solutions of 10-dimensional (super)string theory, particularly supergravity i.e. the massless sector, coupled to n-form gauge fields. If our 10-dimensional target space is parametrized by (t, x_1, \dots, x_9) , then e.g. a 5-brane can be parametrized by (t, x_1, \dots, x_5) , where we set the transverse coordinates (x_6, \dots, x_9) to zero. The metric is a harmonic function associated to some charge Q , then:

$$H(r) = 1 + \frac{Q}{r^\Delta} \quad , \quad \Delta \text{ depends on the codimension of the brane} \quad (106)$$

these solutions look similar to black holes, in particular they can have a horizon, hence they are sometimes referred to as black branes. The charge Q characterizes the coupling of the p-brane to a (p+1)-form gauge field.

1. p=0: particle (when we look at it from a distance) coupled to a 1-form field $A^{(1)}$. This describes a black hole
2. p=1: string coupled to $A^{(2)}$.

21.2 D-branes

D-branes are hypersurfaces in 10-dimensional space-time, on which we define boundary conditions for open strings. Namely, we fix the endpoints of the open strings fixed on the D-brane i.e. we define Dirichlet boundary conditions for open strings on the D-brane. If we take the D-brane to be p-dimensional, we refer to it as a Dp-brane. Polchinski discovered that there exists a duality between D-branes and p-branes, namely

The picture of a black hole in string theory is either as:

1. A bound state of p-branes
2. A bound state of D-branes

The former picture is valid in the regime where the string coupling $g_s \sim \mathcal{O}(1)$, while the latter picture is valid when $g_s \rightarrow 0$. We can interpolate between the two pictures by employing supersymmetry. The entropy will then depend on the charges $\{Q_i\}$ of the branes, but not on g_s . We can thus compute the entropy in the two pictures and compare the expressions. In picture I, we compute the entropy from the area of the horizon, namely $S_{BH} = S_{BH}(Q_1, \dots, Q_n)$. In picture II, we count the degeneracies of the open strings attached to the D-branes, namely $S_{microscopic} = S(Q_1, \dots, Q_n) = \log(\#\text{states})$. We will later see that these two expressions agree, thus validating this picture. The number of D-branes will be seen to correspond to the charges $\{Q\}$. In this lecture we will focus picture I. We will construct a 5-dimensional black hole since these are easier to construct.

21.3 5-dimensional charged (Tangherlini) black hole

The metric of the black hole (i.e. p-brane with p=0) in the Einstein frame (the Einstein frame and the string frame are conformally equivalent) is given by:

$$ds_E^2 = -H(r)^{-2} dt^2 + H(r) \left(dr^2 + r^2 d\Omega_{(3)}^2 \right) \quad , \quad H(r) = 1 + \frac{Q}{r^2} \quad (107)$$

Here, $r = \sqrt{y_1^2 + y_2^2 + y_3^2}$ is the radius of S^3 . In this picture, the horizon is at $r = 0$, which is simply due to our choice of coordinate system. The area is then computed as:

$$A = 2\pi^2 \lim_{r \rightarrow 0} \left(r^3 H^{3/2} \right) = 2\pi^2 Q^{3/2} \quad (108)$$

Hence, we have:

$$S_{BH} = \frac{A}{4} = \frac{\pi^2}{2} Q^{3/2} \quad (109)$$

We will now reproduce this formula in two different ways: as a bound state of p-branes, and as D-branes with open strings attached

21.4 10-dimensional gravity

We choose

$$\mathcal{M} = \mathbb{R}^{1,4} \otimes T^5$$

i.e. we compactify 5 of the 10 dimensions. We have

1. Double-dimensional reduction: one wraps certain world-volume directions of the p-brane around n-dimensional cycle of T^5 . This gives $p' = p - n$
2. Normal dimensional reduction: the world volume of the p-brane is in $\mathbb{R}^{1,4}$. This gives $p' = p$

We consider three cases:

21.4.1 A: D1-brane

The D1-brane is parametrized by the coordinates (t, y_1) , and T^5 is parametrized by (y_1, \dots, y_5) , i.e. we wrap one dimension of our D1-brane on T^5 to give a 0-dimensional black hole-like object.

For a D1-brane, we have:

$$ds_{\text{string}}^2 = H_1^{-1/2} (-dt^2 + dy^2) + H_1^{1/2} (dx_1^2 + \dots + dx_8^2) \quad , \quad H_1 = 1 + \frac{Q_1}{r^6} \quad (110)$$

The dimension $\Delta = 6$ is given by the codimension D1-brane.

We have the dilaton $e^{-2\phi_{10}} = H_1(r)^{-1}$, i.e. we have to rescale the metric by $e^{-2\phi_{10}}$ to find the metric in Einstein frame:

$$ds_E^2 = H_1^{-2/3} dt^2 + H_1^{1/3} (dr^2 + r^2 d\Omega_{(3)}^2) \quad (111)$$

This is the metric for the (1+4)-dimensional uncompactified space. The area is once again computed as:

$$A = 2\pi^2 \lim_{r \rightarrow 0} \left(r^3 \sqrt{\frac{Q_1}{r^2}} \right) = 0 \quad (112)$$

We see that this is not the black hole we are looking for. This follows from the fact that this is a $\frac{1}{2}$ -BPS object i.e. we have 16 supersymmetry generators in our (4+1)-dimensional target space. We will see that we need to further reduce supersymmetry by a factor 4.

21.4.2 B: D1, D5 - system

We now have a D1-brane in the same configuration as in the previous example. We also have a D5-brane, of which we compactify all 5 dimensions on T^5 so that we again recover a 0-dimensional object in target space.

The computation of the metric requires so called *superposition rules*, which we do not present here. Heuristically, we need to superimpose the harmonic functions $H_1(r)$ and $H_5(r)$ associated to Q_1 and Q_5 , which is possible since we are considering BPS objects. Both the D1-brane and the D5-brane have SUSY $\frac{1}{2}$ i.e. they are $\frac{1}{2}$ -BPS objects, which means

we further reduce the number of target space SUSY generators by a factor 2 so that the resulting target space has 8 SUSY generators. This gives the following expression for the metric

$$ds_E^2 = -(H_1 H_5)^{-2/3} dt^2 + (H_1 H_5)^{1/3} \left(dr^2 + r^2 d\Omega_{(3)}^2 \right) , \quad H_1 = 1 + \frac{1}{r^6} , \quad H_5 = 1 + \frac{Q_5}{r^2} . \quad (113)$$

Then

$$A = (2\pi)^2 \lim_{r \rightarrow 0} \left(r^3 \sqrt{\frac{Q_1 Q_5}{r^4}} \right) = 0 . \quad (114)$$

21.4.3 C: D1, D5 , pp-wave configuration

This configuration has charges Q_1 , Q_5 , and Q_k , where Q_k corresponds to momentum in the D1-direction. The associated harmonic functions are H_1 , H_5 , and H_k , the resulting configuration is $\frac{1}{8}$ SUSY i.e. we have $\frac{32}{8} = 4$ SUSY generators in target space. We then have the metric:

$$ds_E^2 = -(H_1 H_5 H_k)^{-2/3} dt^2 + (H_1 H_5 H_k)^{1/3} \left(dr^2 + r^2 d\Omega_{(3)}^2 \right) \quad (115)$$

The area is then

$$A = 2\pi^2 \lim_{r \rightarrow 0} \left(r^3 \sqrt{\frac{Q_1 Q_5 Q_k}{r^6}} \right) = 2\pi^2 \sqrt{Q_1 Q_5 Q_k} . \quad (116)$$

Hence:

$$S_{BH} = \frac{\pi^2}{2} \sqrt{Q_1 Q_5 Q_k} . \quad (117)$$

We see that this agrees with the entropy of the Tangherlini black hole when we set $Q_1 = Q_5 = Q_k$. One caveat is that this formula for S_{BH} only holds for large Q . Next time, we will derive the same expression by counting string microstates.

22 Black hole microstate counting

The D-brane configuration under consideration consists of Q_5 D5-branes, Q_1 D1-branes, and Q_k quanta of light-like (conventionally) left-moving momentum along the common compactified dimension (similar to Kaluza-Klein modes). This is an excited BPS state. The statistical entropy is given by the number of distinct ways in which we can distribute the total momentum on the available excited states of the system. Since we have light-like momenta, we have to look for massless excitations. We use the BPS nature of our configuration to go to another point in parameter space which facilitates the calculation of the number of excited states. We consider the picture where $Ng_s \ll 1$, while our calculation of the entropy in the previous lecture was performed at $Ng_s \gg 1$. Here, N is the number of branes, and g_s is the string coupling constant. The fact that $Ng_s \ll 1$ allows us to use string perturbation theory, while $Ng_s \gg 1$ gives non-perturbative solutions. In the former picture, which we will use today, the D-branes will be used as boundary conditions for open strings.

We compactify on the five torus $T^5 = (S^1)^5$; we proceed to make one of these circles S^1 large. The D5-branes wrap around all five copies of S^1 in T^5 , while the D1-branes wraps only around the large S^1 . We thus have internal momentum flowing along the large circle in the picture presented in the previous lecture. We thus have the following situation:

	0	1	2	3	4	5	6	7	8	9
D5	x					x	x	x	x	x
D1	x									x

where ‘x’ indicates that the D-brane extends into this dimension. As we did previously, we compactify dimensions 5-8 on T^4 , and we compactify the 9th dimension on the large copy of S^1 . After dimensional reduction on T^4 , the D-brane system is 1+1-dimensional with compact space direction. At low energies, the effective world volume theory of Q_p D-branes is a $U(Q_p)$ super Yang-Mills (SYM) theory. In our case, we thus obtain a 2d SYM theory with $\mathcal{N} = (4, 4)$ SUSY and gauge group $U(Q_1) \otimes U(Q_5)$. We thus have two kinds of light-like excitations.

1. Excitations which start and end on the same D-brane stack, e.g. those on start and on the D1-brane. These form a vector multiplet.
2. Excitations which start and end on different D-brane stacks. These form a hypermultiplet i.e. a supersymmetric multiplet.

States in the vector multiplet transform under the adjoint representation of $U(Q_1) \otimes U(Q_5)$, while states in the hypermultiplet transform under the (anti)fundamental representation of $U(Q_1)$ ($U(Q_5)$), or under the (anti)fundamental representation of $U(Q_5)$ ($U(Q_1)$). Here, the former (latter) case corresponds to string that start (end) at the D1-brane stack. To identify massless excitations, we have to find all flat directions of the scalar potential. This potential has a complicated valley structure with two main branches:

1. Coulomb branch, named such since it corresponds to a non-zero VEV of the scalar field in the vector multiplet.
2. Higgs branch, named such since it is characterized by a non-zero VEV of the scalar in the hypermultiplet, which couples to the gauge fields in a way that is reminiscent to the Higgs effect.

If we are in the Coulomb branch, the scalars of the hypermultiplet become massive. Conversely, in the Higgs branch, the scalars of the vectormultiplet become massive. Hence, the two branches are mutually exclusive. We are interested in the branch with the largest number of available states i.e. which maximizes the entropy. In the Coulomb branch, the gauge group is spontaneously broken down to $U(1)^{Q_1} \otimes U(1)^{Q_5}$. The only string states that remain are those that start and end on the same brane, since those that stretch between different branes have a non-zero mass. The number of massless states we end up with is thus $Q_1 + Q_5$, which are the Cartan directions of our gauge group. In the Coulomb branch, the branes are moved away from each other so that they do not form a bound state. In the Higgs branch, the gauge group is broken to $U(1)$. Since the VEV of the scalars of the vectormultiplet vanish along the Higgs branch, all the branes stay on top of each other, hence they form a bound state. In this picture, the $U(1)$ gauge freedom parametrizes overall translations along the large copy of S^1 .

Via a rather involved analysis of the system, one can show that the potential $4Q_1Q_5$ flat directions. Heuristically, the factor 4 arises as follows: we can have strings going from the D1-stack to the D5-stack, namely, the states correspond to $Q_1 \otimes \bar{Q}_5$ and $\bar{Q}_1 \otimes Q_5$, where \bar{Q}_i corresponds to the antifundamental representation. Additionally, we have a complex scalar. The complex scalar is the supersymmetric partner of two Weyl spinors, which contributes another factor of Q_1Q_5

two our microstates. If we increase the size of our large S^1 , we can reduce the energy carried by individual excitations. We can then use the infrared limit of the effective theory of massless modes.

The $\mathcal{N} = (4, 4)$ SUSY implies that the infrared fixed point of the theory is a superconformal sigma model with a hyper-Kähler target space. This entails that the central charge of the theory is that of a free theory, which is $c = 1$ for a boson field and $c = 1/2$ for a Majorana-Weyl fermion field, so that the total central charge is

$$c_{\text{tot}} = \frac{3}{2} 4Q_1 Q_5 = 6Q_1 Q_5 . \quad (118)$$

We can then use Cardy's formula for the asymptotic number of states $N(E)$ for a system with total energy E in a 2d CFT with compact support:

$$N(E) = \exp \sqrt{\pi c_{\text{tot}} E L / 3} \quad (119)$$

Here, L is the volume of space. Note that this formula holds only when $E \gg 1$. We thus find

$$S_{\text{stat}} = \ln N(E) = \sqrt{\pi c_{\text{tot}} E L / 3} = \sqrt{\pi^2 Q_1 Q_5 \frac{Q_k}{R} \frac{2\pi R}{R}} = 2\pi \sqrt{Q_1 Q_5 Q_k} = S_{BH} . \quad (120)$$

Where we used the fact that $E = \frac{Q_k}{R}$ and $L = 2\pi R$. As indicated above, this matches the result of the Bekenstein-Hawking entropy we derived in the previous lecture. Note that we need the fact that the black hole is in a BPS state to consistently compare the black hole picture and the string microstate picture as presented in the two previous lectures.

A possible improvement to the calculation presented above is to consider a 4d-example, where we use D2-brane and D6-brane stacks, an NS5-brane, and a pp-wave. There, the result is:

$$S_{\text{stat}} = 2\pi \sqrt{Q_2 Q_6 Q_5 Q_k} \quad (121)$$

Where, in this case Q_5 is the charge of the NS5-brane. One can also extend this calculation to the non-extremal case, such that our black hole has a finite temperature. This means that we also include right-moving momenta. In this picture, open strings from the left-moving and right-moving sector combine to create closed strings, which constitute Hawking radiation. Oddly enough, there is no in-moving closed string that corresponds to the antiparticle partner of the outgoing Hawking radiation. Since this picture is not in BPS, the states we consider are not protected, so it is not clear whether this calculation is consistent.

23 Asymptotic symmetries in general relativity and black hole charges/hair

A good review on this topic can be found on at [5]

23.1 Introduction

We consider asymptotic symmetries, which characterize the infrared (IR) structure of general relativity. Asymptotic symmetries provide conserved charges for a gravitational system with an asymptotic region ($r \rightarrow \infty$) or a boundary (horizon). In a general relativistic space-time, these asymptotic symmetries were first considered by Bondi, van der Burg, Metzner, and Sachs [15] [16]. They are an infinite extension of the standard Poincaré symmetry of asymptotically Minkowski space-times. Asymptotic symmetries are special kinds of diffeomorphisms which can be subdivided into *supertranslations* and *superrotations*. These are also present in gauge theories such as QED and QCD, where they are related to so-called *large gauge transformations*, namely those that do not die off at infinity [?]. An important point is that these asymptotic symmetries act as global symmetries despite the fact that they are constructed from gauge symmetries i.e. even though they act non-trivially on the Hilbert space of the system.

23.1.1 Classical picture

Classically, BMS transformations lead to an infinite class of space-time metrics for one particular space-time geometry (manifold), which are nevertheless physically distinct. A BMS transformation acts as

$$\delta_{BMS} g_{\mu\nu}(x^\mu) \rightarrow \tilde{g}_{\mu\nu}(x^\mu) \quad , \quad g \text{ and } \tilde{g} \text{ are physically distinct.} \quad (122)$$

This construction can be used for asymptotically flat space-times, such as Minkowski space, but also for black hole space-times. The difference between g and \tilde{g} can be measured by the *gravitational memory effect*.

23.1.2 Quantum picture

Quantum-mechanically, we say that the metric describes a particular state in the Hilbert space i.e. we describe the metric $g_{\mu\nu}$ as some ket $|g_{\mu\nu}\rangle$. The variation δ_{BMS} is now promoted to an operator with associated charges \hat{Q}_{BMS} . These charges satisfy an algebra, which is only known for particular cases such as AdS spaces. In the quantum picture, the variation of the metric is written as

$$\hat{Q}_{BMS} |g_{\mu\nu}\rangle = |\tilde{g}_{\mu\nu}\rangle \quad . \quad (123)$$

We thus find an (almost) degenerate set of vacua, where each vacuum is characterized by a metric. They are almost degenerate since we might get slightly different values for physical parameters after applying the BMS transformation as above, which we will see later. This degeneracy of the vacuum that follows from the set of BMS-transformations is then related to the entropy of a given space-time geometry. The BMS transformation transforms a metric to a different metric, which entails that the BMS symmetry group is spontaneously broken. This gives rise to Goldstone modes, which are massless particles corresponding to fluctuations in the ‘flat’ directions in the potential.

23.2 The interpretation of the Goldstone particles

The Goldstone particles that arise due to the spontaneous breaking of the BMS group are soft gravitons, which appear in any gravitational scattering amplitude (even those at tree level). This is the gravitational construction analogous to work done on QED by Kulish and Faddeev in the 1970’s.

Consider a four point graviton amplitude, with momenta p_1, p_2, p_3, p_4 . Such amplitudes are generically IR-divergent, i.e. they diverge when we let $\{p_i\}$ go to zero. According to Weinberg, such IR divergences are cancelled by the inclusion of infinitely many soft gravitons which ‘dress’ the final state of the scattering process i.e. the final state includes an infinite cloud of soft particles. This is called the *soft theorem*. Schematically, the final state of our four-point ($2 \rightarrow 2$) scattering process is given by

$$|\text{final}\rangle = |\text{two-particle state}\rangle \otimes |\text{coherent state of soft gravitons}\rangle .$$

This is expressed by the so-called BMS Ward identity, as recognized by Strominger. The meaning of the BMS Goldstone modes is thus that the action of \hat{Q}_{BMS} creates a soft graviton. An amplitude with n soft modes is related to an amplitude with $n + 1$ soft modes by a BMS transformation. The soft gauge bosons (e.g. gravitons), asymptotic symmetries, and the memory effect (to be discussed later) are then related as follows:

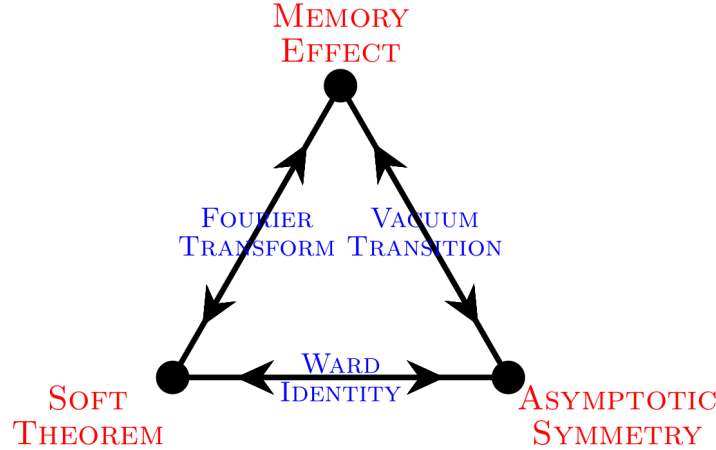


Figure 8: A schematic summary of the relation between soft gauge bosons, asymptotic symmetries, and the memory effect. The remainder of this lecture series is devoted to elucidating these three concepts and exactly how they are related. Figure taken from [5].

The BMS group provides a possible solution to the BMS information puzzle, posed very recently by Strominger in [17]. Consider a black hole scattering process, where we start from an initial state which is pure. The final state of this scattering process are Hawking modes, which are in a mixed state.

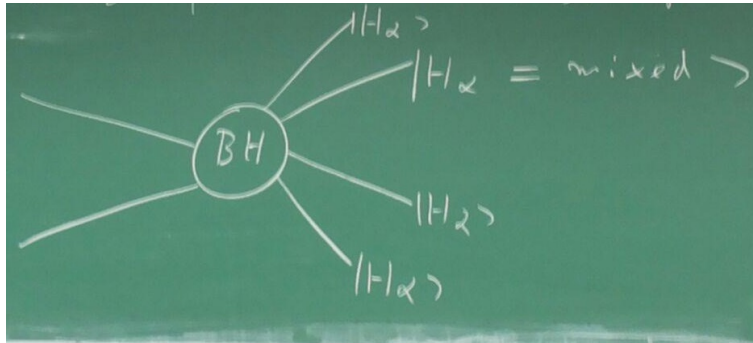


Figure 9: A schematic illustration of the Hawking radiation, which are hard modes characterized by a mixed density matrix.

Hawking modes have an energy (\sim temperature) $T_H = \frac{1}{8\pi M} > 0$, which we consider to be *hard*. Recall that the spectrum of the Hawking modes is $n(\omega) = \frac{1}{e^{\omega/T_H} - 1}$. The final state is then given by some density matrix $\rho = \sum_{\alpha} \rho_{\alpha} |H_{\alpha}\rangle \langle H_{\alpha}|$. Strominger argued that this picture is not complete, since each Hawking quantum has to be dressed by an infinite number of soft modes. He then proposed that the soft modes purify the final state due to their entanglement with the hard Hawking modes H_{α} . The full quantum process is then given by a unitary S-matrix as

$$S |\psi_{\text{initial}}\rangle = \sum_{\alpha, \beta} |H_{\alpha}\rangle |S_{\beta}\rangle = |\psi_{\text{final}}\rangle . \tag{124}$$

Where $|\psi_{\text{final}}\rangle$ is a pure state. Page has another proposal for the purification of Hawking radiation, namely that the early

Hawking radiation is purified by the late Hawking modes [18]. We will not comment further on this proposal; interested readers are encouraged to have a look at the literature.

23.3 Asymptotic BMS transformations - supertranslations and superrotations

We investigate the asymptotic symmetries by following these steps:

1. Assume the space-time is asymptotically flat
2. Expand the metric around spatial infinity
3. Consider a special class of diffeomorphisms, namely those that leave the asymptotic structure and boundary conditions at infinity invariant. This entails that the resulting metric is also asymptotically flat

23.3.1 Expansion of the metric around spatial infinity

We now introduce so-called *Bondi coordinates*, which are often used to describe asymptotic symmetries. However, this are not the only possible choice; their common usage is probably due to historical reasons. We have lightcone coordinates u and v which parametrize \mathcal{I}^+ and \mathcal{I}^- , respectively. We then use complex coordinates z and \bar{z} to parametrize the two-sphere for a particular value of u or v . The *retarded* and *advanced* Bondi coordinates thus given by (u, r, z, \bar{z}) and (v, r, z, \bar{z}) , respectively. We have $z = i \cot(\theta/2)e^{i\phi}$, so that the spherical metric is given by

$$ds_{S^2}^2 = d\theta^2 + \sin^2 \theta d\phi^2 = 2\gamma_{z\bar{z}} dz d\bar{z} \quad , \quad \gamma_{z\bar{z}} = \frac{1}{(1 + z\bar{z})^2} . \quad (125)$$

For example Minkowski space is given by

$$ds^2 = -dt^2 + d\vec{x}^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}} dz d\bar{z} . \quad (126)$$

We will continue our derivation of the asymptotic symmetries during next lecture.

24 Asymptotic symmetries of 4D space-time geometries

BMS transformations are related to the emission of soft gravitons, which play the role of Goldstone bosons of spontaneously broken BMS symmetry. In this lecture, we will look more closely at the BMS group of asymptotic symmetries, in particular supertranslations.

24.1 Supertranslations

Supertranslations are one component of the asymptotic symmetries of gravity in asymptotically flat space-times, i.e. space-times with asymptotic boundaries \mathcal{I}^\pm . The derivation of the asymptotic symmetry group is summarized as follows:

1. We use the retarded BMS coordinates (u, r, x^A) , where x^A are spherical coordinates. This corresponds to the symmetry group BMS^+ . There are also the advanced BMS coordinates, where u is replaced by v ; this corresponds to BMS^- .
2. We choose Bondi gauge i.e. we fix the local diffeomorphisms by setting: $g_{rr} = 0 = g_{rA}$. This gives the metric

$$ds^2 = -Udu^2 - e^{2\beta}dudr + g_{AB} \left(d\theta^A + \frac{1}{2}U^A du \right) \left(d\theta^B + \frac{1}{2}U^B du \right) , \quad \text{gauge condition: } \partial_r \det \left(\frac{g_{AB}}{r^2} \right) . \quad (127)$$

3. We go to asymptotically flat space-times, write $x^A = (z, \bar{z})$, $z = \cot(\theta/2)e^{i\varphi}$, and expand the metric around infinity in $\frac{1}{r}$. This gives

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + \frac{2m_B}{r}du^2 + \frac{1}{r} \left[\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{\bar{z}\bar{z}}) \right] dudz + \text{c.c.} + \mathcal{O}(r^{-2}) . \quad (128)$$

We now have three different functions: C_{zz} , m_B , N_z , which are functions of u , z , \bar{z} , but not of r .

- m_B is called the *Bondi mass aspect*. We can integrate it as $\int_{S^2} m_B(z, \bar{z})dzd\bar{z}$ to give the total Bondi mass, which equals the ADM mass in the case of a black hole.
- N_z is called the *angular momentum aspect*. We can integrate it as $\int_{S^2} N_z V^z(z, \bar{z})dzd\bar{z}$ to give the total angular momentum.
- C_{zz} and $C_{\bar{z}\bar{z}}$ are gauge potentials of a gravitational wave, akin to (electromagnetic) vector potentials.
- $N_{zz} := \partial_u C_{zz}$ is called the *Bondi news*. N_{zz} and $N_{\bar{z}\bar{z}}$ correspond to the two helicity components of a spin-2 gravitational wave. This quantity is similar to the electromagnetic field strength.

The supertranslations are diffeomorphisms that keep the Bondi gauge and the asymptotic structure of the metric invariant. These are residual large (global) gauge transformations. One could naively expect that the only residual symmetries are the Poincaré group, which is the finite-dimensional symmetry group of the Minkowski metric. Surprisingly, however, BMS_\pm turns out to be infinite-dimensional and contains the Poincaré group as a subgroup. Hence, general relativity does not simply reduce to special relativity at large radial distances and weak fields. This implies that there is a large space of degenerate metrics (vacua), even in the case of Minkowski space.

More concretely: consider diffeomorphisms which are given by some vector $\xi^\mu(u, r, z, \bar{z})$, which we will relate to some $\xi^\mu(z, \bar{z})$. We have $\xi^u, \xi^r \sim \mathcal{O}(1)$, ξ^z and $\xi^{\bar{z}} \sim \mathcal{O}(\frac{1}{r})$. Then:

$$\begin{aligned}
(\mathcal{L}_\xi g)_{ur} &= -\partial_u \xi^u + \mathcal{O}\left(\frac{1}{r}\right) \\
(\mathcal{L}_\xi g)_{zr} &= r^2 \gamma_{z\bar{z}} \partial_r \xi^{\bar{z}} - \partial_z \xi^u + \mathcal{O}\left(\frac{1}{r}\right) \\
(\mathcal{L}_\xi g)_{z\bar{z}} &= r \gamma_{z\bar{z}} [2\xi^r + r D_z \xi^z + r D_{\bar{z}} \xi^{\bar{z}}] + \mathcal{O}(1) \\
(\mathcal{L}_\xi g)_{uu} &= -2\partial_u \xi^u - 2\partial_u \xi^r + \mathcal{O}\left(\frac{1}{r}\right).
\end{aligned} \tag{129}$$

Where D_z and $D_{\bar{z}}$ are covariant derivatives on S^2 . The solution to these equations is:

$$\xi = f \partial_u + \frac{1}{r} (D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + D^z D_z f \partial_r \quad , \quad f = f(z, \bar{z}) . \tag{130}$$

This is the generator of BMS supertranslations. It is given in terms of an arbitrary function $f(z, \bar{z})$, hence we have an infinite family of supertranslations.

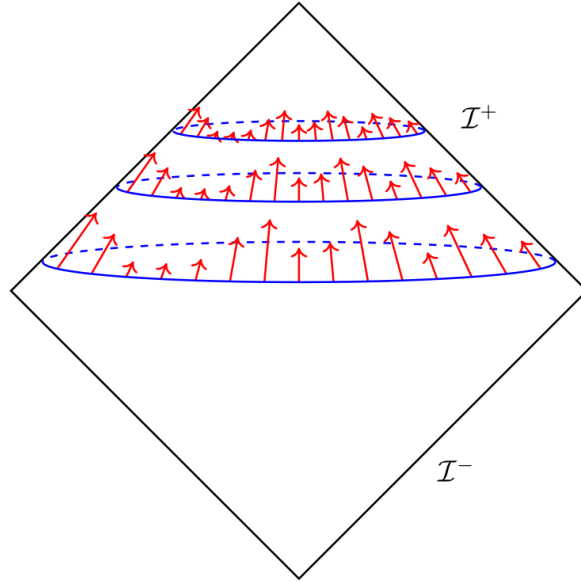


Figure 10: Under the action of a supertranslation in BMS_+ , \mathcal{I}^+ undergoes a translation in u that is a meromorphic function of the angular coordinates (z, \bar{z}) . Adapted from [?].

As seen in the picture, a supertranslation is nothing but an angle-dependent shift of the retarded time at \mathcal{I}^+ . This is a generalization (localization) of the four Poincaré translations. If we take $f(z, \bar{z})$ to be constant, we find u -translations. If we take $f(z, \bar{z})$ to be an $\ell = 1$ harmonic function, this corresponds to a spatial translation. All supertranslations act non-trivially i.e. they transform physically inequivalent metric into each other. This gives a measurable effect, which is known as the *gravitational memory effect*. This is a gravitational analogue of the Aharonov-Bohm effect. We now look at the action of a supertranslation parametrized by some function f on the quantities given above, which is given by

$$\begin{aligned}
\mathcal{L}_f m_B &= f \partial_u m_B + \frac{1}{4} [N^{zz} D_z^2 f + z D_{\bar{z}} N^{zz} D_z f + \text{c.c.}] \\
\mathcal{L}_f N_{zz} &= f \partial_u N_{zz} \quad , \quad \text{field strength} \\
\mathcal{L}_f C_{zz} &= f \partial_u C_{zz} - 2D_z^2 f(z, \bar{z}) \quad , \quad \text{potential} .
\end{aligned} \tag{131}$$

Note that we have an infinite number of choices for $f(z, \bar{z})$, hence we have an infinite family of metrics.

24.2 Example: Minkowski space

In Minkowski space: $m_B = 0$, $N_{zz} = 0$, and $C_{zz} = 0$, where the last equality holds in the standard Minkowski metric $ds^2 = -dt^2 + d\vec{x}^2$. We now perform a BMS supertranslation given by some $f(z, \bar{z})$, which will result in another metric which we denote by Minkowski'. One sees from 131 that the result is given by:

$$m'_B = 0 \quad , \quad N'_{zz} = 0 \quad , \quad C'_{zz} = 2D_z^2 f(z, \bar{z}) \quad (132)$$

One can easily show that the curvature components will still be equal to zero i.e. $R_{\mu\nu\rho\sigma} = 0$, which is generally the case if C_{zz} and $C_{\bar{z}\bar{z}}$ equal to the second derivative of some function. Hence, for Minkowski space, we get an infinite family of Minkowski metrics. The manifold C of flat vacua (similar to a moduli space) is given by:

$$C = \{C_{zz} \mid C_{zz} = D_z^2 \Phi(z, \bar{z})\} \quad (133)$$

Hence, the space of flat vacua is isomorphic to the space of functions $\Phi(z, \bar{z})$ on S^2 . Namely, a point in C corresponds to a point in S^2 . A BMS supertranslation transforms Φ as:

$$\Phi(z, \bar{z}) \rightarrow \Phi(z, \bar{z}) - 2f(z, \bar{z})$$

Minkowski space in Minkowski coordinates in Bondi coordinates is

$$ds^2 = -du^2 - dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} \quad , \quad \Phi(z, \bar{z}) = 0 \quad .$$

After a BMS transformation, the metric is of the form

$$ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} + rC_{zz} dz^2 + rC_{\bar{z}\bar{z}} d\bar{z}^2 \quad .$$

This is still Minkowski space, i.e. $R_{\mu\nu\rho\sigma} = 0$ iff $C_{zz} = 2D_z^2 f(z, \bar{z})$.

24.3 Quantum language

In the quantum picture, we write these results as

Classically	Quantum mechanically
$C_{zz} = D_z^2 \Phi$	$ \Phi\rangle \hat{=} C_{zz}\rangle$
$\Phi \rightarrow \Phi - 2f$	$ \Phi\rangle \rightarrow \hat{Q}_f \Phi\rangle = \Phi - 2f\rangle \quad .$

(134)

The generator \hat{Q}_f does not annihilate the vacuum i.e. the BMS group is spontaneously broken, as mentioned during the last lecture. This gives rise to massless Goldstone modes. Classically, the Goldstone modes are just the variations of the metric of the form $D_z^2 f \sim \mathcal{L}_f g$. We will make this a bit more precise later by considering the gravitational memory effect.

Consider two classical vacua at some initial time u_i and some final time u_f , which differ by a BMS supertranslation. We construct an interpolating space-time that brings us from the vacuum at u_i to the vacuum at u_f . At u_i , we take $C_{zz}^i = D_z^2 \Phi(z, \bar{z})$, and at u_f , we have $C_{zz}^f = D_z^2 \Phi^f(z, \bar{z})$, where $\Phi^i - \Phi^f = 2f$. We now look for an interpolating metric, which is described by a function $C_{zz}^{\text{int}}(u, z, \bar{z})$ such that $C_{zz}^{\text{int}}(u_i, z, \bar{z}) = C_{zz}^i$ and $C_{zz}^{\text{int}}(u_f, z, \bar{z}) = C_{zz}^f$. The interpolating metric must have some non-trivial time dependence. This gives a gravitational wave (pulse), since $N_{zz}^{\text{int}} = \partial_u C_{zz}^{\text{int}} \neq 0$. The picture is as follows: if we send out a gravitational wave past \mathcal{I}^+ and turn it off, this will shift \mathcal{I}^+ in a way that is measurable. This is known as the gravitational memory effect.

25 Asymptotic symmetries - continued

We are discussing a family of asymptotically flat metrics (which includes Minkowski space). We do this in the Bondi gauge using Bondi coordinates, as presented during the previous lecture. The Cauchy data (or parameters) divide into the following subclasses:

1. Mass: Bondi mass aspect $m_B(u, z, \bar{z})$
2. Angular momentum: Bondi angular momentum aspects N_z and $N_{\bar{z}}$
3. Radiation (e.g. gravitational waves): Gauge potentials $C_{zz}(u, z, \bar{z})$ and $C_{\bar{z}\bar{z}}(u, z, \bar{z})$
4. Field strength: Bondi news $N_{zz} = \partial_u C_{zz}$ and $N_{\bar{z}\bar{z}} = \partial_u C_{\bar{z}\bar{z}}$

What are the symmetries that keep the asymptotic form of the metric invariant? These are the asymptotic killing vectors (AKV's), which are of the form

$$\xi_{f(z, \bar{z})}^\mu = (\xi^u, \xi^z, \xi^{\bar{z}}, \xi^r) = \left(f(z, \bar{z}), -\frac{1}{r} D^z f(z, \bar{z}), -\frac{1}{r} D^{\bar{z}} f(z, \bar{z}), D_z D^z f(z, \bar{z}) \right). \quad (135)$$

Where $f(z, \bar{z})$ are functions on S^2 . Note that these AKV's keep the asymptotic form of the metric invariant but act non-trivially on the interior of our space-time. Namely, the variation with respect to an AKV given by some function $f(z, \bar{z})$ is given by

$$\delta_{f(z, \bar{z})} C_{zz} = f(z, \bar{z}) \underbrace{\partial_u C_{zz}}_{N_{zz}} - 2D_z^2 f(z, \bar{z}). \quad (136)$$

The second term gives rise to an infinite family of supertranslated metrics. Namely, Minkowski space corresponds to m_B, N_z, N_{zz} all equal to zero. The family of Minkowski vacua is then given by

$$C = \{C_{zz} \mid C_{zz} = -2D_z^2 \Phi(z, \bar{z})\}. \quad (137)$$

A supertranslation then acts as $\Phi(z, \bar{z}) \rightarrow \Phi(z, \bar{z}) + f(z, \bar{z})$.

Remarks:

1. The soft quanta (Goldstone bosons of broken BMS symmetry) have zero energy. These given by:

$$\lim_{\omega \rightarrow 0} \Delta C_{zz} = \lim_{\omega \rightarrow 0} \int_{u_i}^{u_f} du e^{i\omega u} \partial_u C_{zz}(u, z, \bar{z}) = C_{zz}(u^f, z, \bar{z}) - C_{zz}(u^i, z, \bar{z}) = \delta_f C_{zz} = \underbrace{2D_z^2 f(z, \bar{z})}_{\text{Goldstone mode}} \quad (138)$$

In field theory language, this is written as

$$|\text{Goldstone mode}\rangle = |\text{Soft graviton}\rangle .$$

In words, this means that Minkowski space is equipped with an arbitrary number of soft modes.

2. The field equations are the familiar Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}^M$$

Where $T_{\mu\nu}^M$ is the energy-momentum tensor corresponding to the matter content of our space-time. For the asymptotically flat metrics, we obtain the following on-shell condition

$$\partial_u m_B = \frac{1}{4} [D_z^2 N_{zz} + D_{\bar{z}}^2 N_{\bar{z}\bar{z}}] - T_{uu} \quad , \quad T_{uu} = \frac{1}{4} N_{zz} N^{\bar{z}\bar{z}} + 4\pi G \lim_{r \rightarrow \infty} [r^2 T_{uu}^M] . \quad (139)$$

25.1 Asymptotic charges

The asymptotic charges can also be defined for the bulk but are only conserved at the null-like boundaries \mathcal{I}^\pm . These charges are Noether charges, which are typically derived in Hamiltonian formalism. A Hamiltonian corresponds to a symplectic structure, which is a symplectic form that depends on position and momentum variables $\omega(p, q)$. We will not derive the symplectic structure here, but instead present the result. The charge corresponding to the supertranslations are

$$\begin{aligned}
Q_{f(z, \bar{z})} &= \omega(C_{zz}, \underbrace{\delta_f C_{zz}}_{\mathcal{L}_f C_{zz}}) = \frac{1}{4\pi G} \int_{\mathcal{I}^\pm \cong S^2} dz d\bar{z} \gamma_{z\bar{z}} f(z, \bar{z}) m_B(u, z, \bar{z}) \\
&= -\frac{1}{4\pi G} \int_{\mathcal{I}^+ \cap \mathcal{I}^- \equiv C} dudzd\bar{z} f(z, \bar{z}) \partial_u m_B \\
&= \frac{1}{4\pi G} \int_C dudzd\bar{z} f(z, \bar{z}) \left[T_{uu} - \frac{1}{4} (D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}) \right] \\
&= -\frac{1}{8\pi G} \int_C dudzd\bar{z} C_{zz} \partial_u (\delta_f C_{zz}) \quad , \quad \delta_f C_{zz} = 2D_z^2 f . \tag{140}
\end{aligned}$$

Here, $\mathcal{I}^+ \cap \mathcal{I}^- \equiv C$ is a Cauchy surface for our space-time. We have an infinite number of charges since $f(z, \bar{z})$ is allowed to be meromorphic. We see that the charges correspond to the u -derivative of the Goldstone mode. We can then compute the charge algebra: $\{Q_f, Q_{f'}\}_{\text{P.B.}} = Q_{f+f'}$, this gives two copies of the Virasoro algebra, each of which is of the form $[L_m, L_n] = (m-n)L_{m+n}$ [?]. This tells us that we are simply performing conformal transformations on S^2 .

Remarks:

1. The charges are related to soft gravitons \equiv Goldstone bosons given by the ket $|GB\rangle$ via the standard low energy theory as follows:

$$Q|0\rangle = |0\rangle + |GB\rangle = |0'\rangle$$

2. The expectation values for the charges Q_f for static and stationary backgrounds, i.e. u -independent backgrounds, are classically zero. In quantum language:

$$\langle Q \rangle = \langle 0|Q|0\rangle = \langle 0|0'\rangle = 0$$

However, the following expression is not equal to zero:

$$\langle 0|Q|GB\rangle = \langle GB|GB\rangle \neq 0$$

25.2 Superrotations

Superrotations are generalizations of standard rotations. We saw before that the supertranslations are related to the Bondi mass aspect m_B . We will see that the superrotations are related to the angular mass aspect $N_A \equiv (N_z, N_{\bar{z}})$. The superrotations are given by functions $Y_A(z, \bar{z}) = (Y_z, Y_{\bar{z}})$. The corresponding charges are:

$$Q_{Y_A} = \frac{1}{8\pi G} \int_{\mathcal{I}^\pm \cong S^2} dz d\bar{z} \gamma_{z\bar{z}} Y^A(z, \bar{z}) N_A(u, z, \bar{z}) \tag{141}$$

They destroy asymptotic flatness, in the sense that they generate a so-called conical deficit angle in space-time. This corresponds to the creation of a cosmic string into the bulk. The charges Q_f correspond to generators $T_{(m,n)}$, while Q_Y correspond to Virasoro generators L_n . They satisfy the following algebra:

$$\begin{aligned}
[L_m, L_n] &= (m - n)L_{m+n} \\
[L_m, T_{(p,q)}] &= -pT_{(p+m,q)} \\
[T, T] &= 0
\end{aligned}
\tag{142}$$

The second expression entails that supertranslations give rise to superrotation charges.

25.3 BMS-like transformations and charges on the horizon of Schwarzschild space-time

We consider the event horizons H^\pm as boundaries of our space-time. We follow the same steps as before:

1. Expand the metric around H^+
2. Look for those diffeomorphisms that keep the asymptotic expansion invariant, which we write as BMS^H .
3. Derive a family of black hole metrics $\delta_{g(z,\bar{z})}g_{\mu\nu}^{BH} = g_{\mu\nu}^{BH'}$, where $g(z, \bar{z})$ now plays the role that $f(z, \bar{z})$ played before.
4. We find Goldstone modes which correspond to gravitational waves which pass through H^+
5. We then compute charges Q_g at the horizon. We will find that classically, for static black holes: $\langle Q_g \rangle = 0$. Quantum mechanically, we will find that the operators \hat{Q}_g corresponding to the charges Q_g will have a non-zero VEV.

26 Current research on BMS-like transformations and BMS-like charges for black holes

The BMS-like charges we consider here are also referred to as BMS soft hair, since they correspond to soft gravitons and extend the notion of black hole hair. Today we will focus on the classical picture, the next lecture will focus on the quantum model. Consider Schwarzschild geometry in Eddington-Finkelstein coordinates in advanced time $v = t + r^*$. The only difference of this form of the metric with that of the Bondi coordinates is the fact that we use r^* instead of r

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)^2 dv^2 + 2dvdr^* + r^2\gamma_{z\bar{z}}dzd\bar{z}. \quad (143)$$

We see that if we let $r_s \rightarrow 0$, we recover (flat) Bondi coordinates. We have a family of metrics

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)^2 dv^2 + 2dvdr + r^2\gamma_{z\bar{z}}dzd\bar{z} + \underbrace{\frac{r_s r}{r - r_s} C_{zz}^H(v, z, \bar{z}) dz^2 + \frac{r r_s}{r - r_s} C_{\bar{z}\bar{z}}^H(v, z, \bar{z}) d\bar{z}^2}_{\text{radiation through horizon}}. \quad (144)$$

We now consider diffeomorphisms that preserve the Eddington-Finkelstein ‘gauge’ and the asymptotic structure of the metric. The corresponding asymptotic killing vectors are given by an infinite set of functions $g(z, \bar{z})$

$$\chi_g^\mu = g(z, \bar{z})\partial_v - \frac{r_s - r}{r r_s} (D_z^2 g \partial_z + \text{h.c.}) + D^z D_z g(z, \bar{z})\partial_r. \quad (145)$$

We refer to the corresponding modes as H-modes, which will later be contrasted with A-modes, to be defined below. The gauge potentials transform as

$$C_{zz}^H \rightarrow C_{zz}^H - 2D_z^2 g(z, \bar{z}). \quad (146)$$

This transformation has the same form as the one we found in previous lectures for $f(z, \bar{z})$. We thus get a family of equivalent Schwarzschild vacua characterized by

$$C^H = \{C_{zz}^H \mid C_{zz}^H = D_z^2 \Phi(z, \bar{z})\}. \quad (147)$$

Which, again, is of the same form we found previously. One can then show that the mass of the black hole is unchanged. Additionally, we want to check if the family of metrics also correspond to vacuum solutions. Namely, if we start from some $g_{\mu\nu}^0$ and vary with respect to an AKV, we wish to check that the following holds:

$$g_{\mu\nu}^g = g_{\mu\nu}^0 + \delta_\chi g_{\mu\nu} \quad , \quad R_{\mu\nu}(g_{\mu\nu}^g) = 0. \quad (148)$$

One can show that the angular components of the Ricci tensor ($R_{\theta\phi}, \dots$) vanish on the horizon. The other components of the Ricci tensor vanish at the linearized level, which corresponds to the fact that such a perturbation corresponds to a gravitational wave. This picture is sometimes referred to as the membrane paradigm.

26.1 Soft gravitons

We have the field strengths analogues $N_{zz}^H = \partial_v C_{zz}$ and $N_{\bar{z}\bar{z}}^H = \partial_v C_{\bar{z}\bar{z}}$, corresponding to the two polarizations of a (soft) graviton. Then:

$$N_{zz}^{H,w=0} = \lim_{\omega \rightarrow 0} \int_{v_i}^{v_f} e^{i\omega v} \partial_v C_{zz}^H = \underbrace{2D_z^2 g(z, \bar{z})}_{\text{Goldstone modes of Schwarzschild space-time}}. \quad (149)$$

The supertranslation charges at the horizon are then

$$\begin{aligned}
Q^H &= \int_{I_{BH}H^\pm \cup \mathcal{I}^\pm} dv dz d\bar{z} \partial_v C_{zz}^H D_z^2 g(z, \bar{z}) \\
&= - \int_{I_{BH}} dv dz d\bar{z} C_{zz}^H \partial_v (D_z^2 g) \\
&= -\frac{1}{2} \int_{BH} dv dz d\bar{z} C_{zz}^H \partial_v (\delta_g C_{zz}^H) .
\end{aligned} \tag{150}$$

For static (stationary) black hole metrics, $Q^H = 0$, but $Q^H \neq 0$ for a collapsing shell of matter.

26.2 A-modes

We see in the H-modes that we have $\frac{r_s-r}{r_s r} = \frac{1}{r} - \frac{1}{r_s}$. The first term corresponds to the familiar BMS transformations for flat (Minkowski) metrics, while the second term is new and arises from the presence of a black hole [19]. We now consider

$$\chi'_g = g(z, \bar{z}) \partial_v + \frac{1}{r_s} (D_z^2 g \partial_z + \text{h.c.}) + D_z^2 g \partial_z , \tag{151}$$

in the context of the metric

$$ds^2 = - \left(1 - \frac{r_s}{r}\right) dv^2 + 2dvdr + r^2 \gamma_{z\bar{z}} dz d\bar{z} + r_s C_{zz}^A dz^2 + r_s C_{\bar{z}\bar{z}}^A d\bar{z}^2 . \tag{152}$$

This corresponds to $C_{zz}^A \rightarrow C_{zz}^A - 2D_z^2 g(z, \bar{z})$, so that we can write

$$\delta_{\chi_g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & \gamma \end{pmatrix} , \tag{153}$$

where

$$\begin{aligned}
\alpha &= -2 \frac{r^2}{r_s} \frac{\partial^2 g}{\partial \theta^2} \\
\beta &= -2 \frac{r^2}{r_s} \left(\frac{\partial^2 g}{\partial \theta \partial \varphi} \right) \\
\gamma &= -2 \frac{r^2}{r_s} \left(\frac{\partial^2 g}{\partial \varphi^2} + \sin \theta \cos \theta \frac{\partial g}{\partial \theta} \right) .
\end{aligned} \tag{154}$$

The A-modes are the Goldstone modes of the family of black hole space-times.

Remarks:

1. $Q^A = 0$ for static or stationary solutions.
2. Since the Goldstone modes are $\omega = 0$ gravitational waves, they are gapless modes i.e. they carry no energy and leave the mass of the black hole invariant.
3. The wavelength of the A-mode is finite, namely $\lambda_A \approx r_s$. This is consistent with the fact that they have no energy since we have an infinite redshift at the horizon.
4. A-modes correspond to gravitational waves that travel along the horizon. This is reminiscent of open strings travelling along the black hole, whose degeneracy gives rise to black hole entropy in string theory.

5. Classically, we have an infinite number of charges i.e. infinitely many Goldstone modes. We can expand $g(z\bar{z})$ in angular coordinates as:

$$g(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} b_{\ell m} Y_{\ell m}(\theta, \varphi)$$

Hence we have one charge for each $b_{\ell m}$. These charges are referred to as soft hair.

The classical entropy $S = \frac{A}{4\hbar G_N} \xrightarrow{\hbar \rightarrow 0} \infty$, but this entropy needs an infinite amount of time to be released i.e. it is not classically accessible.

26.3 Quantum picture of a Schwarzschild black hole

The physical picture considered here has been developed by Dvali and Gomez [20]. We will see that the classical picture presented above corresponds to a large N -limit of the quantum picture, in a way that is reminiscent of AdS/CFT correspondence. The black hole seems to correspond to a bound state of N gravitons, akin to a Bose-Einstein condensate. We refer to the Goldstone modes as Bogoliubov modes for reasons that will become clear in due time. These modes are the collective excitations of the graviton bound state. Schematically, we will rename $|\text{Goldstone}\rangle \equiv |\text{Bogoliubov}\rangle$. The collective mode is formed at the quantum critical point, where the gravitons condense. We have a graviton condensate which we refer to as ϕ , such that $\langle \phi \rangle \sim \langle gg \rangle$, i.e. the gravitons form a condensate corresponding to a charge \hat{Q}^A . Then: $\hat{Q}^A |\text{BH}\rangle = |\text{BH}'\rangle$

26.3.1 Toy model

In a condensed matter toy model one can calculate that $M = \sqrt{N}M_p + \mathcal{O}\left(\frac{1}{N}\right)$ and $r_s = \sqrt{N}L_p + \mathcal{O}\left(\frac{1}{N}\right)$, hence:

$$S \sim r_s^2 = N + \mathcal{O}(1) \xrightarrow{N \rightarrow \infty} \infty$$

There is a collective coupling, given by $\lambda = \alpha N$, where α is the coupling of individual gravitons. The critical point is then at $\alpha = \frac{1}{N} \sim \frac{1}{r_s^2}$ i.e. $\lambda = 1$, which is familiar from condensed matter physics.

This toy model corresponds to bosons on a circle, its spectrum can be seen in figure ??.

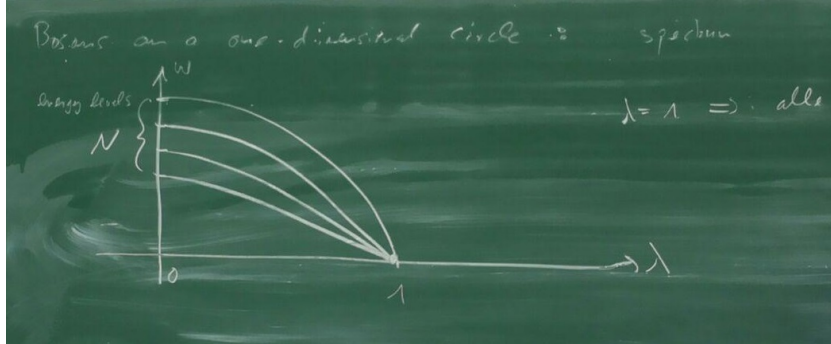


Figure 11: The spectrum of the condensed matter toy model of a black hole as a function of the coupling constant λ .

We see that $\omega_i = 0 + \mathcal{O}\left(\frac{1}{N}\right)$ at $\lambda = 1$ i.e. the non-degenerate spectrum collapses to a point where it becomes completely degenerate. The $\mathcal{O}\left(\frac{1}{N}\right)$ term signals a breakdown of degeneracy, i.e. the Goldstone modes will get masses of $\mathcal{O}\left(\frac{1}{N}\right)$, so that they become *pseudogoldstone modes*. The limit $\lambda \rightarrow 1$ corresponds to infinite redshift, so that the event horizon can be seen to correspond to the quantum critical point of the BEC.

27 The gravitational memory effect

The memory effect corresponds to a change in spatial separation between two points (on \mathcal{I}) due to the passing of gravitational radiation, which is characterized by $N_{zz} \neq 0$. Namely, if we start from two points that are separated by a distance $(s^z, s^{\bar{z}})$ and let gravitational waves pass between them, the change in s^z in retarded coordinates is given by

$$\Delta s^z = \frac{\gamma^{z\bar{z}}}{2r} \Delta C_{z\bar{z}} s^{\bar{z}} . \quad (155)$$

We start from

$$C_{zz} = -2D_z^2 C(z, \bar{z}) \quad (156)$$

for some function $C(z, \bar{z})$. From 139, we then find that

$$D_z^2 \Delta C^{zz} = 2\Delta m_B + 2 \int du T_{uu} . \quad (157)$$

The solution is given by

$$\Delta C(z, \bar{z}) = - \int d^2 w \gamma_{w\bar{w}} G(z, \bar{z}; w, \bar{w}) \int du (T_{uu}(w, \bar{w}) + \Delta m_B) , \quad (158)$$

where the Green's function is given by

$$G(z, \bar{z}; w, \bar{w}) = \frac{|z-w|^2}{\pi(1+z\bar{z})(1+w\bar{w})} \log \left[\frac{|z-w|^2}{(1+z\bar{z})(1+w\bar{w})} \right] , \quad D_z^2 D_{\bar{z}}^2 G(z, \bar{z}; w, \bar{w}) = \gamma_{z\bar{z}} \delta^{(2)}(z-w) + \dots \quad (159)$$

One can rewrite it as [21]

$$G(z, \bar{z}; w, \bar{w}) = \frac{1}{\pi} \sin^2 \frac{\Delta\Theta}{2} \log \sin^2 \frac{\Delta\Theta}{2} \quad (160)$$

where $\Delta\Theta$ is the angle on the two-sphere at \mathcal{I} between (z, \bar{z}) and (w, \bar{w}) . Note that the Green's function effect vanishes for small $\Delta\Theta$ and attains its maximum absolute value at $\Delta\Theta = \pi/2$.

28 Quantum hair and quantum black hole vacua

We have seen that there is an infinite family of classical metrics that describe black hole, particularly Schwarzschild. We transform the metric as follows:

$$\delta_\chi g_{\mu\nu}(x) + g_{\mu\nu} . \quad (161)$$

This gives rise to an infinite number of Goldstone modes. The BMS groups at radial infinity and at the event horizon are related as follows:

BMS_∞ $\omega = 0, \lambda = \infty \ (k = 0)$ $\omega = ck$	BMS_H (BMS_A) $\omega = 0, \lambda = r_s \ (k = 1/r_s)$ Infinite redshift at horizon: $\omega = g_{rr}ck$
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Let us now assume that a black hole is a bound state of N gravitons ($N \rightarrow \infty$). The individual coupling between gravitons goes as $\alpha(E) = \frac{E^2}{M_{pl}^2} = \frac{L_p^2}{r^2}$. The *collective coupling* is then given by $\lambda = \alpha(E)N = \frac{L_p^2}{r^2}N (= \lambda(r))$. At $\lambda(r) = 1$, we have quantum criticality i.e. the gravitons condensate. $\lambda(r)$ can be seen as the *holographic coupling*, namely, λ gives rise to an emergent (radial) coordinate in a renormalization type picture. Then, $\lambda = 1$ is the fixed point of our renormalization group flow. A similar picture arises in AdS/MERA correspondence. For $\lambda = 1$: $r_s^2 = N$, so that $M = \sqrt{N}M_p$ and $r_s = \sqrt{N}L_p$, and $S \sim A \sim r_s^2 \sim N$, so that N can be viewed as the number of microstates.

28.1 Collective excitations

The collective excitations are called Bogoliubov modes. These are the quantum counterparts of the Goldstone modes. Schematically:

$$|\text{Bogoliubov mode}\rangle = |\text{Goldstone boson}\rangle$$

The Bogoliubov mode has some $1/r_s$ and frequency $\Delta\omega \sim \frac{1}{r_2} = \frac{1}{N}$. $\Delta\omega$ is the first correction to the energy degeneracy at $\lambda = 1$. Classically, we have seen that the charges for superrotations are zero. In the quantum picture, we write:

$$\hat{Q} = \epsilon \partial_v C_{zz}^H \delta_{\chi_g} g_{\mu\nu} = \int \partial_v \delta_{\chi_g} g_{\mu\nu}(x) \quad (162)$$

We then have

$$\begin{aligned} \delta_{\chi_g} g_{\mu\nu} &= \sum_{m,\ell} b_{\mu\nu}^{\ell m}(r, \theta, \varphi) Y^{\ell m}(\theta, \varphi) && ; \text{ Classically} \\ \delta_{\chi_g} g_{\mu\nu} &= \sum_{m,\ell} \frac{1}{\sqrt{\omega_{\ell,m}}} \left[\hat{b}_{\mu\nu}^{\ell m}(r, \theta, \varphi) Y^{\ell m}(\theta, \varphi) e^{-iv\omega_{\ell,m}} + \hat{b}_{\mu\nu}^{\dagger \ell m}(r, \theta, \varphi) Y^{*\ell m}(\theta, \varphi) e^{+iv\omega_{\ell,m}} \right] && ; \text{ Quantum mechanically . } \end{aligned} \quad (163)$$

The expression for the charge is

$$\hat{Q} = \int \partial_v \delta_{\chi} \hat{g}_{\mu\nu}(v, \theta, \varphi) = -i\sqrt{\omega_{\ell,m}} \left(b_{m\ell} e^{-i\omega_{\ell,m}v} - b_{m\ell}^\dagger e^{i\omega_{\ell,m}v} \right) . \quad (164)$$

We see that for $\omega_{\ell m} \rightarrow 0$, $\frac{1}{N} \rightarrow 0$, hence \hat{Q} goes to zero as $\frac{1}{\sqrt{N}}$. We use a coherent state ansatz, which satisfies

$$\hat{Q} |N\rangle = |N\rangle' .$$

This state $|N\rangle$ is given by

$$|N\rangle = \exp\left(\sum_{m,\ell} n_{\ell m} (b_{\ell m} - b_{\ell m}^\dagger)\right) |0\rangle .$$

Here, $n_{\ell m} = \langle b_{\ell m}^\dagger b_{\ell m} \rangle$ i.e. the occupation number of the individual Bogoliubov modes. The VEV of the charge is then

$$\langle N | \hat{Q}_{\ell m} | N \rangle = \sqrt{\omega_{\ell m} n_{\ell m}} \approx \sqrt{\omega_{\ell m}} , \text{ for } n_{\ell m} = 0, 1 . \quad (165)$$

The approximation made in the above expression gives

$$\langle \hat{Q}_{\ell m} \rangle = \frac{1}{\sqrt{N}} \xrightarrow{N \rightarrow \infty} 0$$

We will take $\ell_{\max} \sim r_s = \sqrt{N}$, since for higher angular momenta we cannot ignore the back-reaction. For $\ell \leq m \leq \ell$, we find that

$$\#\text{modes} \sim \ell_{\max}^2 \sim r_s^2 \sim N = \#\text{charges} = Q_{\ell m} .$$

We then assume that each mode corresponds to some qubit with values $n = 0, 1$. Then, the number of states is $d = 2^N$, so that $S = \log d = N$. We can summarize the picture in the following table

Quantum picture	Geometric picture
$N = \text{finite}$	$N = \infty$
Quantum criticality	Emergence of the horizon
Bogoliubov modes	A-modes $\delta g_{\mu\nu}$
Coupling $\alpha \sim \frac{1}{N}$	$\alpha = 0$
Energy gap $\omega \sim \frac{1}{N}$	Massless modes $\omega = 0$

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