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# Nonequilibrium Field Theories and Stochastic Dynamics
# Sheet 11

## Exercise 30 – Pattern Formation in Fluids with Active Stress

In this exercise, we study the pattern formation in a conserved scalar field which interact with a surrounding viscous fluid. In fact, in systems such as the actomyosin cortex of animal cells, molecular regulators (e.g., myosin motors) modulate active stresses and are themselves transported by the resulting flows, leading to feedback-driven pattern formation. If you like have a look at the studies by Bois, Jülicher and Grill from 2011 in the journal PRL. We consider a conserved density field $c(x,t)$, whose dynamics is influenced by hydrodynamic interactions with a surrounding viscous fluid. In turn, the field $c(x,t)$ acts as a regulator of active stress, thereby affecting the fluid flow.

The concentration field $c(x,t)$ evolves according to a conservation law:

$$\partial_t c = -\nabla \cdot \mathbf{j}, \qquad \mathbf{j} = \mathbf{v}c - D\nabla c, \tag{1}$$

where $\mathbf{v}(x,t)$ is the velocity field and $D$ is the diffusion coefficient.

The fluid velocity is governed by the Navier–Stokes equation:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{\text{ext}}, \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress tensor, and $\mathbf{f}_{\text{ext}}$ includes external forces. These are not necessarily gradient fields and thus can break energy conservation. In our case, we assume frictional forces $\mathbf{f}_{\text{ext}} = -\xi \mathbf{v}$, and a constitutive relation for the stress tensor of the form:

$$\boldsymbol{\sigma} = \eta \nabla \mathbf{v} + f(c)\,\mathbb{I}, \tag{3}$$

where $\eta$ is the viscosity and $f(c)$ represent the active stress, depending on the concentration. At microscopic scales, inertial terms in the Navier–Stokes equation are negligible:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \approx 0, \tag{4}$$

which corresponds to the low Reynolds number limit.

**a)** Rewrite the equations for $c(x,t)$ and $\mathbf{v}(x,t)$ in one spatial dimension under low Reynolds number limit. Show that the constitutive equation becomes:

$$\partial_x \left(\eta \partial_x v + f(c)\right) = \xi v. \tag{5}$$

**b)** We introduce the transformation $u = \ln(c)$ for convenience. What condition does $c$ have to fulfill? Is this met for a physical particle density? In the stationary state, where $\partial_t = 0$, derive the relation

$$\partial_x^2 u = \ell^{-2} u + \frac{1}{D\xi}(\sigma_0 - f(e^u)), \quad \ell = \sqrt{\eta/\eta}, \tag{6}$$

and interpret the role of $\ell$

**c)** Interpret the equation for $u(x)$ as the equation of motion of an anharmonic oscillator in a potenial $V(u)$, where $x$ has the role of time in that analogy. Find the explicit value of $V(u)$ and the "energy-like" conserved quantity of the anharmonic oscillator. Assume there are only two stable extrema $c_-, c_+$ of the steady-state concentration profile. How is the function $V(u)$ related to the extremum value $c_-, c_+$ of a non-homogeneous pattern?

**d)** Discuss the implications of the shape of $V(u)$ for pattern formation. What happens if $V(u)$ is convex?

**e)** Perform a linear stability analysis of the dynamic system of $c(x,t)$ and $v(x,t)$. Derive a condition for instability of the homogeneous state. Compare your result with the condition from part (d).

**f)** Interpret physically the instability condition derived above. Under which conditions will a pattern form?

# Exercise 31 – Field Theory for Population Dynamics

Consider the *diffusive epidemic process* (DEP) defined as

$$A + B \xrightarrow{\lambda} 2B \quad \text{and} \quad B \xrightarrow{\delta} A. \tag{7}$$

**a)** Show that the Lagrangian for the *Kramers-Moyal* path integral is given by

$$\mathcal{L} = \mathrm{i}(\tilde{n}_A\,\partial_t n_A + \tilde{n}_B\,\partial_t n_B) - \lambda\, n_A\, n_B \left(e^{\mathrm{i}(\tilde{n}_B - \tilde{n}_A)} - 1\right) - \delta\, n_B \left(e^{\mathrm{i}(\tilde{n}_A - \tilde{n}_B)} - 1\right), \tag{8}$$

where $n_A$ and $n_B$ denote the fields representing the $A$ and $B$ particle numbers, respectively, and $\tilde{n}_A$ and $\tilde{n}_B$ the corresponding response fields.

*Hint: S*tart from the general formula derived in class and consider how it can be generalized to multiple species and multiple reactions in light of its derivation.

**b)** Next, expand to quadratic order in the dual fields to show that

$$\begin{aligned}\mathcal{L}_{\text{low noise}} = \mathrm{i}\tilde{n}_A\left(\partial_t n_A + \lambda n_A n_B - \delta n_B\right) + \mathrm{i}\tilde{n}_B\left(\partial_t n_B - \lambda n_A n_B + \delta n_B\right) \\ - \frac{1}{2}\left(\mathrm{i}\tilde{n}_A - \mathrm{i}\tilde{n}_B\right)^2\left(\lambda n_A n_B + \delta n_B\right).\end{aligned} \tag{9}$$

Derive the corresponding Langevin equation in the Itô interpretation and discuss the nature of the multiplicative noise.

**c)** Show that the Lagrangian for the *coherent state path integral* is given by

$$\mathcal{L} = \mathrm{i}(\tilde{a}\,\partial_t a + \tilde{b}\,\partial_t b) - \lambda ab(\mathrm{i}\tilde{b} + 1)(\mathrm{i}\tilde{b} - \mathrm{i}\tilde{a}) - \delta b(\mathrm{i}\tilde{a} - \mathrm{i}\tilde{b}), \tag{10}$$

where the fields $a = x_a$ and $\tilde{a} = iq_a$ (and analogously $b$ and $\tilde{b}$) are derived from the coherent state path integral representation of the stochastic process.

Also derive the Langevin equation in the Itô interpretation and compare it with the above result in the low-noise limit. What is the difference, and why does it matter?

Your solutions should be handed in by uploading them to Moodle by **Wednesday, 16$^{\text{th}}$ July 2025, 10:00 am**.