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Nonequilibrium Field Theories and Stochastic Dynamics
Sheet 9

Exercise 24 – State dependent Onsager matrices - Model A

In this exercise, we revisit the derivation of the Fokker–Planck equation from the Langevin equations for spatially extended fields, following the general approach outlined in Section 5.3.2 of the lecture notes. The goal is to understand how different interpretations of the stochastic integral affect the resulting dynamics and to demonstrate that only the Hänggi (also known as kinetic or transport) interpretation yields a Fokker–Planck equation whose stationary solution corresponds to the correct Gibbs-Boltzmann distribution. This establishes the Hänggi interpretation as the only thermodynamically consistent one for field theories with multiplicative noise.

In the next exercise, we extend the derivation from Model A dynamics, which describes non-conserved fields and which we examine in this exercise, to Model B dynamics, where the order parameter is conserved. This generalization highlights how conservation laws modify the structure of the Fokker–Planck equation and the associated fluctuation–dissipation relation.

To begin, we consider a non-conserved field $\phi(\mathbf{x},t)$. The evolution is governed by a stochastic Langevin equation of the form

$$\mathrm{d}\phi(\mathbf{x},t) = -L(\phi(\mathbf{x},t))\mu(\mathbf{x},t)\mathrm{d}t + C(\phi(\mathbf{x},t))\mathrm{d}W(\mathbf{x},t) \; , \tag{1}$$

where $L(\phi(\mathbf{x},t))$ is a local, state-dependent Onsager coefficient and $C(\phi(\mathbf{x},t))$ gives the coefficient of the multiplicative noise and is related to the noise amplitude by $N(\phi(\mathbf{x},t)) = \frac{1}{2}C(\phi(\mathbf{x},t))^2$. Further, we have the usual relations for the Wiener increment

$$\langle \mathrm{d}W(\mathbf{x},t)\rangle = 0 \; , \tag{2}$$

$$\langle \mathrm{d}W(\mathbf{x},t)\mathrm{d}W(\mathbf{x}',t')\rangle = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\,\mathrm{d}t\,\mathrm{d}t' \; . \tag{3}$$

Note that in Eq. (3) we mean $\delta(0)\mathrm{d}t = 1$.

a) *Discretizing the Langevin equation in space*

First, we rewrite the stochastic differential equation (SDE) for the field into a coupled set of SDEs by discretizing space. To this end, we introduce a lattice, where each lattice point is labeled by an index i. Using this, we define the discretized fields as the fields evaluated at the lattice points $\phi_i \equiv \phi(\mathbf{x}_i)$ and the discretized noise $\mathrm{d}W_i \equiv \mathrm{d}W(\mathbf{x}_i)$.

Write down the Langevin equation in the discretized space setting. How does this change the noise amplitude, when we define $\langle \mathrm{d}W_i(t)\mathrm{d}W_i(t)\rangle = \delta_{ij}\delta(t-t')\mathrm{d}t\mathrm{d}t'$?

Hint: Use the discretized form of the delta distribution

$$\delta(\mathbf{x}_i - \mathbf{x}_j) \to \frac{1}{a^d}\delta_{ij} \; ,$$

where a is the lattice constant of our discretized space and d the dimension of the system.

b) *Rewrite the SDE in Itô interpretation*

As you have learned in the lecture, an SDE is only fully defined if you know the correct interpretation. You can find the discussion in the lecture notes in Section 4.3. Assume that the SDE is given in any interpretation defined by a general variable α. Meaning if we assume $\alpha = 0$, our SDE would be interpreted in Itô sense, for $\alpha = \frac{1}{2}$ it would be interpreted in Stratonovich sense and for $\alpha = 1$ in Hänggi sense.

Use this general α to write down the equivalent SDE interpreted in Itô sense.

c) *Deriving the Fokker-Planck equation*

What is the Fokker-Planck equation for the probability density function $P\left(\{\phi_i\}_{i\in[1,\ldots,d]},t\right)$ corresponding to the discretized set of coupled Langevin equations? Write down the three different Fokker-Planck equations for Itô ($\alpha = 0$), Stratonovich ($\alpha = \frac{1}{2}$) and Hänggi ($\alpha = 1$) and simplify the result as far as possible.

d) In order to obtain the result you already know from the lecture and to obtain the Fokker-Planck equation describing the dynamics of the original Model A Langevin equation we need to go back to the continuous equations. Show that for a general function $f(\phi_i)$, when taking the continuum limit, we obtain

$$\partial_{\phi_i} f(\{\phi\}_i) \to \frac{\delta f[\phi]}{\delta \phi} \mathrm{d}^d x \tag{4}$$

Hint: Write the derivative as a limit and introduce a small quantity $\epsilon = \Delta\phi a^d$, where $\Delta\phi$ is the variation in the field ϕ. You can use that the functional derivative of a general function is given by

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon\to 0} \frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon}$$

e) Use this to obtain the continuum space Fokker-Planck equation for the three different interpretations Itô, Stratonovich and Hänggi. For which can one guess that the Boltzmann distribution will be a stationary state distribution for the Fokker-Planck equation?

Hint: Recall the definition of the chemical potential μ, and the equilibrium relation for the stationary probability distribution to the (minimized) free energy:

$$\mu(\mathbf{x}) = \frac{\delta \mathcal{F}[\phi(\mathbf{x})]}{\delta \phi(\mathbf{x})}\,, \tag{5}$$

$$\ln(P_{\mathrm{stat}}[\phi(\mathbf{x})]) = -\beta\mathcal{F}[\phi(\mathbf{x})] + const. \tag{6}$$

f) Choose the correct interpretation and assume the Boltzmann distribution

$$P_{\mathrm{stat}}[\phi(\mathbf{x})] \propto \exp\left(-\beta\mathcal{F}[\phi(\mathbf{x})]\right)\,,$$

as the stationary distribution of this problem.

With this, derive the *Einstein Onsager relation* (a neat version of the fluctuation dissipation theorem) from the lecture.

Exercise 25 – State dependent Onsager matrices - Model B

In this exercise, we will redo the same calculation as in the previous exercise but for model B dynamics in d spatial dimensions.

To this end, we consider a conserved field $\phi(\mathbf{x}, t)$ governed by model B dynamics of the form

$$\mathrm{d}\phi(\mathbf{x}, t) = \nabla\Big(L(\phi(\mathbf{x}, t))\nabla\mu(\mathbf{x}, t)\Big)\mathrm{d}t + \nabla\Big(C(\phi(\mathbf{x}, t))\mathrm{d}W(\mathbf{x}, t)\Big)\,, \tag{7}$$

where again $L(\phi(\mathbf{x}, t))$ is a local, state-dependent Onsager coefficient and $C(\phi(\mathbf{x}, t)) \in \mathbb{R}^d$ is a vector to compensate for the divergence operation in front of the term. We have

$$\langle \mathrm{d}W(\mathbf{x}, t)\rangle = 0\,, \tag{8}$$

$$\langle \mathrm{d}W(\mathbf{x}, t)\mathrm{d}W(\mathbf{x}', t')\rangle = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\,\mathrm{d}t\,\mathrm{d}t'\,. \tag{9}$$

a) *Discretizing the Langevin equation in space*

We again start by discretizing space, turning the model B dynamics into coupled stochastic differential equations describing the field at every point $\mathbf{x}_i$ in discrete space.

Derive an SDE describing the evolution of $\phi(\mathbf{x}_i, t)$. To this end, use the convention that the discretized derivative at point $\mathbf{x}_i$ of a space dependent quantity f is given by

$$\partial_{x^n} f(\mathbf{x}_i) \to \frac{f(\mathbf{x}_i) - f((\mathbf{x}_i - a\mathbf{e}_n)}{a}\,,$$

where a is the lattice constant x^n is the n-coordinate of the vector $\mathbf{x}$ and $\mathbf{e}_n$ is the unit vector corresponding to the n-th component.

Further, introduce a new spatially discrete noise term $\mathrm{d}\tilde{W}(\mathbf{x}_i, t)$ with

$$\langle \mathrm{d}\tilde{W}(\mathbf{x}_i, t)\rangle = 0\,, \quad \langle \mathrm{d}\tilde{W}(\mathbf{x}_i, t)\mathrm{d}\tilde{W}(\mathbf{x}_j, t')\rangle = \delta_{ij}\delta(t-t')\mathrm{d}t\mathrm{d}t'$$

Use this to rewrite the noise term in the discrete space case as

$$\mathrm{d}\phi(\mathbf{x}_i,t) = (\dots)\mathrm{d}t + \sum_k \tilde{C}_{ik}\Big(\{\phi(\mathbf{x}_l,t)\}_l\Big)\mathrm{d}\tilde{W}(\mathbf{x}_k,t)\,,$$

where i and k refer to the i^{th} and k^{th} point, where the point k is separated from the point i by just a length of a on the lattice. The sum over k thus represents the derivative.
The noise amplitude relates these points

$$N_{ij} = \frac{1}{2}\sum_k \tilde{C}_{ik}\Big(\{\phi(\mathbf{x}_l,t)\}_l\Big)\tilde{C}_{jk}\Big(\{\phi(\mathbf{x}_l,t)\}_l\Big)\,.$$

Since knowing N_{ij} is sufficient to be able to determine the Fokker-Planck equation you do not need to find an explicit expression for $\tilde{C}_{ik}\Big(\{\phi(\mathbf{x}_l,t)\}_l\Big)$.

b) *Moving to the Fokker-Planck equation*
Inspired by the success of interpreting the SDE describing model A dynamics in Hänggi interpretation we immediately want to adapt this interpretation also for the model B dynamics. Using the coupled stochastic differential equations derived in the previous part, write down the corresponding Fokker-Planck equation by applying the Hänggi interpretation. Here you may use without proof that in this case the Fokker-Planck equation using the Hänggi interpretation is of the form

$$\partial_t P\Big(\{\phi(\mathbf{x}_l)\}_l,t\Big) = \sum_i \partial_{\phi(\mathbf{x}_i)}\Big(-A_i\Big(\{\phi(\mathbf{x}_l)\}_l\Big)P\Big(\{\phi(\mathbf{x}_l)\}_l,t\Big) + \sum_j N_{ij}\partial_{\phi(\mathbf{x}_j)}P\Big(\{\phi(\mathbf{x}_l)\}_l,t\Big)\Big)$$

where $A_i\Big(\{\phi(\mathbf{x}_l)\}_l\Big)$ is the deterministic part of the discretized SDE (in Ito interpretation) corresponding to $\phi(\mathbf{x}_i)$.[1]

c) *Taking the continuum limit*
As for model A in the previous exercise, we are now interested in taking the continuum limit to arrive at the Fokker-Planck equation describing the time evolution of the probability density functional P as a function of the configuration of the spatially continuous field ϕ. To this end, move from regular to functional derivatives in a similar manner as for model A and pay careful attention on how to properly reintroduce the spatial derivatives.

d) Finally, again using the ansatz

$$P_{\text{stat}}[\phi(\mathbf{x})] \propto \exp\left(-\beta\mathcal{F}[\phi(\mathbf{x})]\right)\,,$$

derive the *Einstein Onsager relation* for model B dynamics and compare to the one from model A.

Exercise 26 – Path Integral Formulation of Model C

The goal of this exercise is to show how a two-component stystem of equations can be reduced to a one-component one by integrating out a field. Consider a passive Model C system with two fields: a non-conserved order parameter field $\phi(\vec{x},t)$ and a conserved density field $\rho(\vec{x},t)$. These types of models are actively being used in research, for example to describe crystallization processes of complex melts and critical phenomena of quantum chromodynamics. For our purposes, however, think of Model C as just a combination of Model A and Model B.
The equations of motion are:

$$\partial_t\phi(\vec{x},t) = -\frac{\delta\mathcal{F}[\phi,\rho]}{\delta\phi(\vec{x},t)} + \eta_\phi(\vec{x},t), \tag{10a}$$

$$\partial_t\rho(\vec{x},t) = \nabla^2\left(\frac{\delta\mathcal{F}[\phi,\rho]}{\delta\rho(\vec{x},t)}\right) + \nabla\cdot\boldsymbol{\eta}_\rho(\vec{x},t), \tag{10b}$$

where η_ϕ and η_ρ are white Gaussian noises with the correlations:

$$\langle\eta_\phi(\vec{x},t)\eta_\phi(\vec{x}',t')\rangle = 2D_\phi\,\delta(\vec{x}-\vec{x}')\delta(t-t'),$$

[1] Proving this statement is rather involved as it requires us to compute the $C_{jk}\Big(\{\phi(\mathbf{x}_l,t)\}_l\Big)$ as functions of the $C_n(\phi(\mathbf{x}_i,t))$ and then carefully performing the transition from Hänggi to Ito interpretation and simplifying the corresponding Fokker-Planck equation.

$$\langle \eta_\rho^i(\vec{x},t)\eta_\rho^j(\vec{x}',t')\rangle = 2D_\rho\,\delta^{ij}\,\delta(\vec{x}-\vec{x}')\delta(t-t').$$

a) *The Janssen–de Dominicis path integral formulation for this system*

Use the Martin–Siggia–Rose–Janssen–de Dominicis (MSRJD) formalism to write down the path integral action corresponding to the stochastic dynamics above. Define response fields $\tilde{\phi}$ and $\tilde{\rho}$ and write the action $S[\phi,\tilde{\phi},\rho,\tilde{\rho}]$ explicitly in terms of the noise strengths D_ϕ and D_ρ.

b) *Integrating out the response field $\tilde{\phi}$ and interpret the result*

Perform the functional integral over $\tilde{\phi}$ in the partition function. Show that in the limit $D_\phi \to 0$, this enforces the deterministic equation of motion for ϕ.

c) *Fourier-transform the action in space and time*

Fourier transform all fields and response fields as:

$$\phi(\vec{x},t) = \int \frac{d^dq}{(2\pi)^d}\frac{d\omega}{2\pi}\, e^{i(\vec{q}\cdot\vec{x}-\omega t)}\,\phi(\vec{q},\omega) \quad \text{(and similarly for the others)}$$

Rewrite the action in momentum-frequency space.

Hint: You don't have to explicitly Fourier-transform the variation of the free energy in the two fields ϕ,ρ.

d) *Special case: linear dynamics and the Green's function representation*

Suppose the free energy is such that the equation of motion for ϕ is linear:

$$\mathcal{F}[\phi,\rho] = \int d^dx\,\left(-\rho(\vec{x})^2 + \rho(\vec{x})^4 + \kappa(\nabla\rho(\vec{x}))^2 - a\phi(\vec{x})^2 + b\phi(\vec{x})\rho(\vec{x})\right) \tag{11}$$

where a and b are constants. Calculate the equation of motion for this explicit Free energy and calculate the Fourier Transform. Use the result from part (b), transformed in Fourier Space, to show that the action obtained in part (c) can be reduced to an action for the field ρ only.

Hint: For the nonlinear convolution of ρ in the Janssen-de Dominicis action you should find:

$$\int_{\vec{q}_1,\vec{q}_2,\vec{q}_3,\omega_1,\omega_2,\omega_3} 3\,\vec{q}_1\cdot\vec{q}_3\,\tilde{\rho}(\vec{q}_3,\omega_1)\rho(\vec{q}_1,\omega_1)\rho(\vec{q}_2,\omega_2)\rho(-\vec{q}_3-\vec{q}_2-\vec{q}_1,-\omega_3-\omega_2-\omega_1)\,. \tag{12}$$

Finally, comment on the following questions very briefly: Can the same reduction be done when the deterministic part of the time evolution for ϕ is non-linear? Is the result analogous to solving the deterministic equation for ϕ in Fourier space and then deriving the MSRJD path integral?

Your solutions should be handed in by uploading them to Moodle by **Wednesday, 2[nd] July 2025, 10:00 am**.