Arnold Sommerfeld Center for Theoretical Physics
Ludwig-Maximilians-Universität München

Prof. Erwin Frey
B. Nettuno, T. Roth, D. Toffenetti, K. Wharam

Nonequilibrium Field Theories and Stochastic Dynamics
Sheet 6

Exercise 17 – Path integrals and the saddlepoint approximation

Path integrals are an alternative formulation of stochastic processes and can be used to express conditional probabilities. For instance, from the Fokker-Planck equation we can derive the following formula for a one-dimensional random variable subject to a deterministic drift $A(x)$ and a diffusion term $B(x)$

$$p(x_f, t_f|x_0, t_0) = \int_{x(t_0)=x_0}^{x(t_f)=x_f} \mathcal{D}[x, q] e^{-\mathcal{S}[x,iq]} \ , \tag{1}$$

with the action $\mathcal{S}[x, q] = \int_{t_0}^{t_f} \mathrm{d}\tau \ \left[iq\left(\partial_\tau x - A(x)\right) - \frac{1}{2} iqB(x)iq\right]$. However, unlike the other representations we have discussed so far (Master equations, Fokker-Planck equations and Langevin equations) path integrals are not very practical to directly calculate these probabilities. Their strengths are rather their physical interpretation, the derivation of exact relations, and the possibility to treat them with perturbation theory and the renormalization group, e. g. to investigate phase transitions and critical behavior.

In this exercise, we will discuss one of the few approximation methods to analytically compute a path integral – the *saddlepoint approximation*. Despite being an approximation, the result in the particular case we study below will be exact. The system we consider is a single Brownian particle in a one-dimensional harmonic potential:

$$A(x) = \mu F(x) = -\mu \partial_x U(x) \quad \text{and} \quad U(x) = \frac{1}{2} K x^2 \tag{2}$$

a) Start by showing that the general formula for the path integral, Eq. (1), reduces to the following, slightly simpler path integral for a process with purely additive noise, i. e. for $B(x) \equiv B = \text{const.}$

$$p(x_f, t_f|x_0, t_0) = \int_{x(t_0)=x_0}^{x(t_f)=x_f} \mathcal{D}[x] e^{-\mathcal{G}[x]} \ , \tag{3}$$

with the so-called *Onsager-Machlup functional*

$$\mathcal{G}[x] = \frac{1}{4D} \int_{t_0}^{t_f} \mathrm{d}\tau \left(\dot{x}(\tau) - A(x)\right)^2 \ . \tag{4}$$

What is D and how are the two measures $\mathcal{D}[x, q]$ and $\mathcal{D}[x]$ related? Give a physical interpretation of this path integral (where does noise enter the expression?). Could we have arrived at the reduced path integral in a simpler way, i. e. without starting from $\mathcal{S}[x, iq]$?

b) The central idea of the saddlepoint approximation is to find the most likely path (often called the stationary point or saddle point[1]) of the process and perform an expansion in small deviations from this path. The largest contribution inside the path integral comes from the maximum of the integrand $e^{-\mathcal{G}[x]}$ or, equivalently, from the minimum of the Onsager-Machlup functional $\mathcal{G}[x]$.

Given the form of $\mathcal{G}[x]$, how would you intuitively calculate the most likely path starting from x_0? Can you use your intuition to find the most likely path between x_0 and a given x_f covered within $t_0 = 0$ and t_f?

c) The problem of finding the most likely path is completely analogous to the principle of least action we know from classical mechanics. There we derived the *Euler-Lagrange equations* which have to be satisfied for the solution that minimizes the classical action – or in our case the most likely trajectory, which minimizes $\mathcal{G}[x]$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \ . \tag{5}$$

[1] This method is closely related to Laplace's method or the stationary-phase method. The name derives from applying the approximation to complex functions where each extremum in the real and imaginary part is automatically a saddle point, which follows from the Cauchy-Riemann equations.

The Lagrangian density $\mathcal{L}$ is defined through $\mathcal{G}[x] = \int \mathrm{d}\tau \mathcal{L}(x, \dot{x})$ and is read as a function of two independent variables x and $\dot{x}$. Show that the equation of motion that the most likely path is given by

$$x^*(\tau) = x_0 \cosh(\gamma(\tau - t_0)) + \chi \sinh(\gamma(\tau - t_0)) \ , \tag{6}$$

with the shorthand notations $\chi := [x_f - x_0 \cosh(\gamma T)] / \sinh(\gamma T)$, $\gamma := \mu K$, and $T := t_f - t_0$. Also compute the corresponding value of the functional $\mathcal{G}[x^*]$.

d) The most likely path $x^*(\tau)$ can now be used as an expansion point. Since we are dealing with functionals, we are interested in the variation $\delta\mathcal{G}[x]$, defined by

$$\mathcal{G}[x^* + \delta x] = \mathcal{G}[x^*] + \delta\mathcal{G}[x] \ . \tag{7}$$

The function $\delta x(\tau)$ is the small pertubation around the saddle point, given the initial and final condition. It thus has to satisfy $\delta x(t_0) = \delta x(t_f) = 0$. Show that the variation $\delta\mathcal{G}$ has the same quadratic form as $\mathcal{G}$ itself, i. e. that

$$\delta\mathcal{G}[x] = \mathcal{G}[\delta x] \ . \tag{8}$$

Why is this relationship special? What property of the system does it correspond to on the level of (differential) equations?

e) We can use the property of $\mathcal{G}$ to pull the saddlepoint contribution in front of the path integral, as it is not integrated over:[2]

$$p(x_f, t_f | x_0, t_0) = e^{-\mathcal{G}[x^*]} \int_{\delta x(t_0)=0}^{\delta x(t_f)=0} \mathcal{D}[\delta x] e^{-\mathcal{G}[\delta x]} \ , \tag{9}$$

Finally, using this expression, compute the above path integral $\mathcal{I} := \int \mathcal{D}[\delta x] e^{-\mathcal{G}[\delta x]}$.
Hint: Which property of $p(x_f, t_f | x_0, t_0)$ could we use to obtain an explicit expression for $\mathcal{I}$?

Exercise 18 – Inertial effects for heterogeneous diffusion.

In the lecture you have learned that a stochastic differential equation like

$$\mathrm{d}x = A(x,t)\mathrm{d}t + C(x,t) \circ_\alpha \mathrm{d}W_t \tag{10}$$

is only specified when its correct interpretation is known (to say this in van Kampen's words, Eq. (10) "is really a meaningless string of symbols" without stating the value of α! See "Ito versus Stratonovich", Journal of Statistical Physics, Vol. 24, 175–187 (1981), for a fun read).
The use of a certain interpretation is explicitly denoted by "$\circ_\alpha$" in Eq. (10).

Eq. (10) can be rewritten as

$$\mathrm{d}x = A(x,t)\mathrm{d}t + \alpha C(x,t) \frac{\partial C(x,t)}{\partial x} \mathrm{d}t + C(x,t)\mathrm{d}W_t \tag{11}$$

where $C(x,t)\mathrm{d}W_t$ is to be understood in the Ito sense.

a) For which special case is Eq. (11) independent of the interpretation we use?
For all other cases the value of α is not a priori clear! It has to be obtained from the physics of the system at study, e.g. with experiments or other means. One example where we don't need a lab to reveal the value of α we will see in the following.
Consider a particle that moves along a one-dimensional line subject to an external force $F(x) = -\nabla U(x)$, and friction with a space-dependent friction coefficient $\gamma(x)$. We assume that the dynamics is described by the Langevin equation

$$m\ddot{x} = F(x) - \gamma(x)\dot{x} + \gamma(x)\sqrt{2D(x)}\Lambda, \tag{12}$$

where Λ is delta-correlated white noise (with zero mean) and $\langle \Lambda(t)\Lambda(t') \rangle = \delta(t - t')$. The noise amplitude (diffusion coefficient) $D(x)$ is related to the friction coefficient by the Stokes–Einstein relation $D(x) = k_B T / \gamma(x)$. This second order stochastic differential equation can be rewritten as a set of two first order stochastic differential equations:

$$\mathrm{d}x = v\mathrm{d}t, \tag{13a}$$

[2] Another way to think about this is that we make a variable transformation $x = x^* + \delta x$. The only difference is that x and δx are functions instead of real variables.

$$\mathrm{d}v = \frac{F(x)}{m}\mathrm{d}t - \frac{k_B T}{mD(x)}v\mathrm{d}t + \frac{k_B T\sqrt{2}}{m\sqrt{D(x)}} \circ_\alpha \mathrm{d}W_t\,. \tag{13b}$$

b) Why is this a case in which all interpretations are equivalent? (i.e. why can we just drop $\circ_\alpha$?)

c) Assume that we can safely take the overdamped limit (i.e. $m \to 0$). Write down a SDE for $\mathrm{d}x$ (which should be independent of v) by setting $m = 0$. You should arrive at

$$\mathrm{d}x = \frac{FD(x)}{k_B T}\mathrm{d}t + \sqrt{2D(x)} \circ_\alpha \mathrm{d}W_t. \tag{14}$$

d) Why is the interpretation now important? Express Eq. (14) in the spirit of Eq. (11).

Our goal is now to find a way to determine the value of this α by taking the limit of $m \to 0$ in a proper manner. You could argue that we could just work with Eqs. (13a) and (13b) by choosing a small numerical value for m when we do the numerical integration. This also works, but comes with its own problems as you never can be really sure what value of m one can safely take. In case you are interested please have a look at "Computer simulations of Brownian motion of complex systems." Journal of Fluid Mechanics, Vol. 282, 373-403 (1995), for more information about this problem.

Before we determine the value of α in Eq. (14), let's first have a look at the stationary solution of the stochastic dynamics.

e) *Stationary state.*

Show that the Fokker-Planck equation corresponding to the system of stochastic differential equations, Eq. (13a)–(13b) is given by (for all interpretations):

$$\partial_t p(x,v,t) + v\,\partial_x\, p(x,v,t) + \frac{F}{m}\,\partial_v\, p(x,v,t) = \frac{\gamma(x)}{m}\partial_v\left[v\,p(x,v,t) + \frac{k_B T}{m}\partial_v p(x,v,t)\right]. \tag{15}$$

What would happen if the Stokes-Einstein relation $D(x) = k_B T/\gamma(x)$ were not to hold? Find a stationary solution, and show that it is consistent with a Boltzmann distribution. *Hint: to find the solution, first assume that the term in the square brackets on the rhs=0, then use this result to solve lhs=0.*

f) *"Proper" approximation for $m \to 0$.*

Now we come to the procedure to derive the limit for small m from the system of stochastic differential equations Eq. (13a)–(13b). **Be warned** that the following calculation through which we will guide you is mathematically **not** fully rigorous. It is partially also based on physical intuition and some handwaving assumptions. There is a proper way to perform this calculation rigorously (in case you are interested in that procedure of an *adiabatic elimination* please refer to "The Smoluchowski-Kramers Limit of Stochastic Differential Equations with Arbitrary State-Dependent Friction" Communications in Mathematical Physics, Volume 336, 1259–1283 (2015)).

We first have to transform from the differential formulation $\mathrm{d}x = ...$ to the integral formulation $x = x_0 + \int_0^t ...$ and back again. Proceed step by step:

- Solve Eq. (13b) for $v\mathrm{d}t$ and insert your result into Eq. (13a).
- Rewrite $\mathrm{d}v$ as $\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t$ and switch to the integral representation.
- For the only integral that contains m, you need to make a partial integration step that is a bit tricky. You obtain a boundary term that goes to 0 with $m \to 0$ and another term. Achieve that this term contains the factor v^2 by making another substitution. Then make the approximation $mv^2 = k_B T$, i.e. we replace the kinetic energy by its average value (the physical picture is that the velocity v is a fast variable that homogenizes in the $m \to 0$ limit. This is an adiabatic elimination.).
- Go back to the differential formulation. You now should have an expression that does not contain m but an additional drift term. Compare this to the result you obtained in part c). What is the correct value of α for this problem?

In summary, you have now derived an overdamped stochastic differential equation together with its interpretation from a Langevin equation that does not suffer from an intpretation problem. In essence, you have shown that the interpretation results from a proper analysis of the inertial term that intuitively leads to a "memory effect".

You can convince yourself that this is indeed the case in the next part of the part of the exercise:

g) *Numerical comparison of equations*

To check your results, we now compare the results to a direct numerical integration of the full system Eq. (13a)–Eq. (13b) for different masses to a numerical integration of Eq. (14) with different values of α. We do this in an illustrative way, namely simply by looking at a single realization of the stochastic process and the trajectory we get in the different cases. Integrate trajectories for:

- the full system Eq. (13a)–(13b), one trajectory for each value of $m \in \{1, 10, 100\}$,
- the system of equation Eq. (12) with simply m set to zero (i.e. just assume $\alpha = 0$ in Eq. (14)),
- and the approximation you derived in the previous paragraph, corresponding to Eq. (14) with the additional noise-induced drift term ($\alpha = 1$) (correct approximation for $m \to 0$).

For all examples, take the same realization of the Wiener process (with a small enough time increment) and the same initial conditions and then compare the trajectories you get in a similar manner as in Figure 1. Take $\gamma = 0.02x$, $F(x) = 0$, and $x \in [0, \infty[$. Hand in a plot similar to this one.

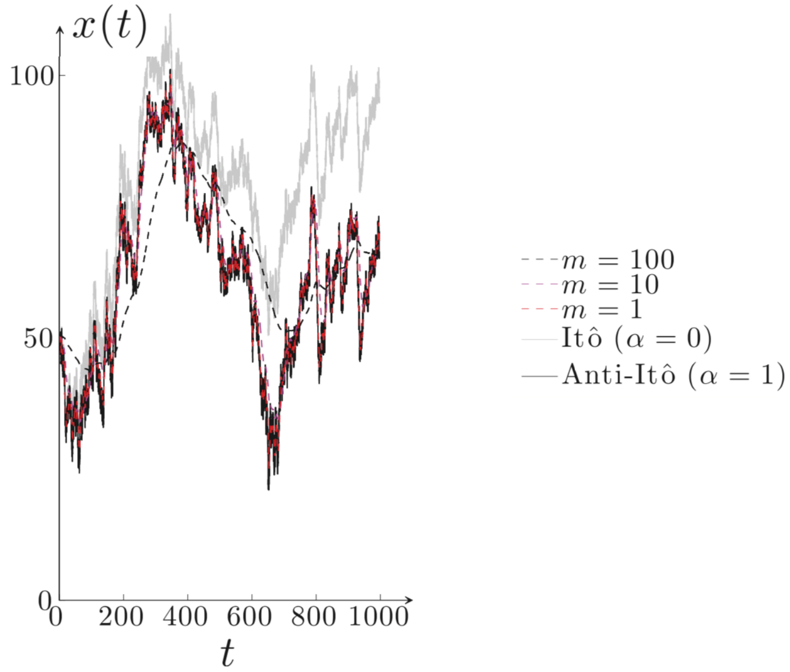

Figure 1: Realization of different SDEs for the same realization of the Wiener process

Exercise 19 – Stochastic stock market

Disclaimer: This is not financial advice. Use the models studied here at your own risk ;)

The theory of stochastic processes can be applied far beyond small particles immersed in a fluid. A prime example is the evolution of stock prices – it can be understood as stochastic processes with a deterministic term (e. g. derived from the general situation at the market), but also a random term (e. g. through events which are impossible to predict and thus appear inherently random to us). Here we want to study an explicit model of how the return of investments we make in stocks will evolve in time.[3]

For simplicity we will assume that we want to buy some stock whose price at time t we call $S(t)$. Our model for the time evolution of the stock price is formulated as a stochastic differential equation,

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\ , \tag{16}$$

with constants μ and σ, and a Wiener increment $\mathrm{d}W_t$.

a) Motivate and interpret Eq. (16). What could be reasons for assuming a constant increase in the relative return and a normally distributed noise? Does it make sense to model stock prices as Markov processes? If not, which possible sources of correlation (or *memory*) do we ignore?

b) Show that, by introducing the log-price $l(t) = \ln(S(t))$, one can derive a Fokker-Planck equation for the probability distribution of the log-price $P(l,t)$, which reads

$$\frac{\partial}{\partial t}P(l,t) = -r\frac{\partial}{\partial l}P(l,t) + \frac{\tilde{\sigma}^2}{2}\frac{\partial^2}{\partial l^2}P(l,t)\ . \tag{17}$$

What are the new prefactors r and $\tilde{\sigma}$ in terms of the original parameters? Also give the Langevin equation for the time evolution of $l(t)$ corresponding to this equation.

Hint: Apply Itō's lemma to the function $F(S(t)) = \ln(S(t))$ to derive an equivalent stochastic differential equation for $l(t)$.

c) Derive the Fokker-Planck equation for the stock price $S(t)$ itself, using Eq. (17). Interpret the drift and diffusion term for $S(t)$.

d) Calculate the expected stock price $\langle S\rangle(t)$ at a certain time t, assuming that the price was s_0 at some initial time $t_0 = 0$. Does the result surprise you?

Hint: There are multiple ways of approaching this problem. In case you want to compute the probability densities at some point, it is easier to first formulate the expectation value in terms of $l(t)$.

Your solutions should be handed in by uploading them to Moodle by **Wednesday, 11$^{\text{th}}$ June 2025, 10:00 am**.

[3] This example is not purely academic. It is the basis of many models of the stock market which are still in use today, e. g. the *Black-Scholes equation*. However, in many situations it turned out to be a rather inaccurate description of actual price fluctuations.