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# Nonequilibrium Field Theories and Stochastic Dynamics
# Sheet 5

## Exercise 14 – Poisson process on a ring: continuous limit

Consider a particle performing a stochastic motion on a discrete ring with $N$ states, labeled by $i = 1, \ldots, N$. At each time step of duration $\mathrm{d}t$, the particle can jump forward (to state $i+1$) with probability $\nu\,\mathrm{d}t$ and backward (to state $i-1$) with probability $\epsilon\,\mathrm{d}t$. We assume periodic boundary conditions ($i + N \equiv i$). Let $p_i(t)$ be the probability of the particle being at site $i$ at time $t$. This is the same system we already analyzed in exercise 9. In this exercise we want to take the continuous-space limit and write down the corresponding Fokker-Planck equation.

**a)** Write down the master equation for the evolution of $p_i(t)$.

**b)** In the limit $N \to \infty$, the discrete site index becomes a continuous angular coordinate $\theta \in [0, 2\pi)$ with spacing $\delta\theta = 2\pi/N$. Keeping terms to order $1/N^2$ allows to derive a Fokker–Planck equation starting from the discrete master equation derived in part a). Find the Fokker-Planck equation for this process, identifying the explicit expressions for the drift and diffusive term.
*Hint: T*he continuous limit can be obtained in two different ways. Either you identify the discrete operators and then take the continuous limit

$$\frac{p_{i+1}(t) - p_{i-1}(t)}{2\delta\theta} \longrightarrow \frac{\partial p(\theta,t)}{\partial\theta}, \frac{p_{i+1}(t) - 2p_i(t) + p_{i-1}(t)}{(\delta\theta)^2} \longrightarrow \frac{\partial^2 p(\theta,t)}{\partial\theta^2},$$

or you Taylor expand

$$p(\theta \pm \delta\theta, t) = p(\theta, t) \pm \delta\theta\,\frac{\partial p}{\partial\theta} + \frac{(\delta\theta)^2}{2}\,\frac{\partial^2 p}{\partial\theta^2} + \mathcal{O}((\delta\theta)^3).$$

In both cases, the appropriate mesoscopic limit for the parameters must be identified.

**c)** Write down the Langevin equation corresponding to the Fokker–Planck equation obtained in part (b).

**d)** Compute the mean and variance of $\theta(t)$ given the Langevin dynamics in part (c). Compare your result with the equations obtained in exercise 9, part c).

## Exercise 15 – Fokker–Planck Dynamics for Motility-Induced Phase Separation

Active Brownian particles (ABPs) are a minimal model for self-propelled motion in two dimensions. Each particle moves with a density-dependent self-propulsion speed and undergoes both translational and rotational diffusion. In this exercise we will calculate the condition when the system exhibits *motility-induced phase separation* (MIPS), i.e. when the system separates into domains of high particle density, surrounded by domains of low particle density.
The system is described by the overdamped Langevin dynamics of a *single* ABP:

$$\dot{\mathbf{x}}(t) = v(\rho)\,\mathbf{n}(\theta(t)) + \sqrt{2D_t}\,\boldsymbol{\eta}(t), \tag{1}$$

$$\dot{\theta}(t) = \sqrt{2D_r}\,\xi(t), \tag{2}$$

where:

- $\mathbf{x}(t) \in \mathbb{R}^2$ is the particle's position,
- $\theta(t) \in [0, 2\pi)$ is the orientation of the particle's polarity vector $\mathbf{n}(\theta) = (\cos\theta, \sin\theta)$,
- $D_t$ is the translational diffusion constant,
- $D_r$ is the rotational diffusion constant,

- $\boldsymbol{\eta}(t)$ and $\xi(t)$ are independent Gaussian white noises with zero mean and unit variance,
- $v(\rho)$ is a speed that depends on the local particle density $\rho(\mathbf{x}, t)$, and decreases monotonically with $\rho$.

**a)** Let $P(\mathbf{x}, \theta, t)$ be the probability density of finding a particle at position $\mathbf{x}$ with orientation $\theta$ at time $t$. Show that the Fokker-Planck equation—corresponding to the overdamped Langevin equations—is given by:

$$\partial_t P = -\nabla_{\mathbf{x}} \left[ v(\rho)\mathbf{n}(\theta)\, P \right] + D_t \nabla_{\mathbf{x}}^2 P + D_r \partial_\theta^2 P. \tag{3}$$

From hereon, we will drop the subscript $\mathbf{x}$ at $\nabla$.
Define the marginal density field:

$$\rho(\mathbf{x}, t) = \int_0^{2\pi} P(\mathbf{x}, \theta, t)\, d\theta, \tag{4}$$

and the first angular moment (polarization density):

$$\mathbf{p}(\mathbf{x}, t) = \int_0^{2\pi} \mathbf{n}(\theta)\, P(\mathbf{x}, \theta, t)\, d\theta. \tag{5}$$

**b)** Interpret what information is contained in the marginal density field $\rho$ and the polarization field $\mathbf{p}$. Which one do you expect to be a relevant order parameter for a mixture of Active Brownian particles?

**c)** Show that integrating the Fokker–Planck equation yields a continuity equation for $\rho(\mathbf{x}, t)$ of the form:

$$\partial_t \rho = -\nabla \cdot \left[ v(\rho)\, \mathbf{p} \right] + D_t \nabla^2 \rho. \tag{6}$$

Then, derive the analogous equation for the dynamics of $\mathbf{p}$ by multiplying the Fokker–Planck equation by $\mathbf{n}$, and then integrating over the angle. In these kind of problems, a closure relation is needed to simplify the analysis. Find the adiabatic solution for $\mathbf{p}$, in the limit of dominating diffusion. This is $\partial_t \mathbf{p} \approx \nabla^2 \mathbf{p} \approx 0$.
*Hint: use the approximation:*

$$\int_0^{2\pi} d\theta\, n_i(\theta)\, n_j(\theta)\, P(\mathbf{x}, \theta, t) \approx \frac{1}{2}\, \rho(\mathbf{x}, t)\, \delta_{ij}, \tag{7}$$

*where $i$ and $j$ are 1 and 2.*

**d)** Using the adiabatic result from part (c), show that the density dynamics can be written as a nonlinear diffusion equation:

$$\partial_t \rho = \nabla \cdot \left[ D_{\text{eff}}(\rho)\, \nabla \rho \right], \tag{8}$$

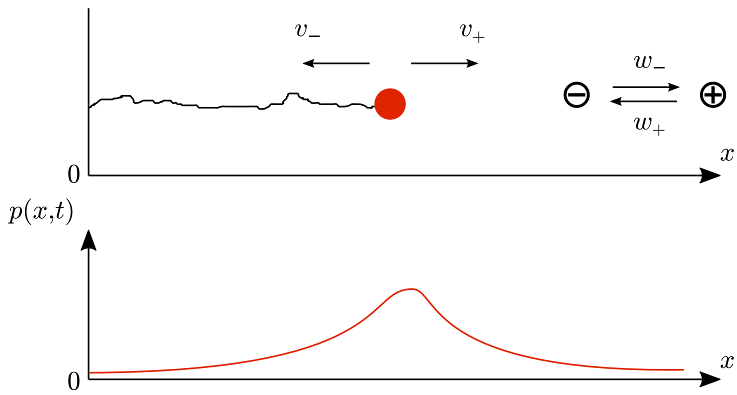
and determine the effective diffusivity $D_{\text{eff}}(\rho)$ in terms of $v(\rho)$, $D_r$, and $D_t$.

**e)** Perform a linear stability analysis of the homogeneous density state, i.e. insert $\rho(\mathbf{x}, t) = \rho_0 + \delta\rho(\mathbf{x}, t)$ and keep only orders of $\delta$ in the dynamics.
Derive a criterion on $D_{\text{eff}}(\rho)$ under which small fluctuations grow over time. Interpret why the derivative of the velocity field needs to be negative such that the system undergoes phase separation.

**f)** Interpret the expression of the effective mobility $D_{\text{eff}}(\rho)$ for the case where the velocity is constant, $v(\rho) = v_0$.

# Exercise 16 – Dynamic instability of microtubules

*All exercise parts marked with a star (*) are optional and do not need to be solved to obtain the full points for the bonus.*
In this exercise we will study the dynamic instability, which is a minimal model for the self-assembly/disassembly of process of microtubules. In this model, we consider a single filament in one dimension which can be either in a growing ("+") phase in which it grows with speed $v_+$ or in a shrinking phase ("−") phase in which it shrinks with speed $v_-$. The filament randomly switches between growing and shrinking phases with rate $+ \xrightarrow{\omega_+} -$ and $- \xrightarrow{\omega_-} +$. Here we are interested in the statistics of the filament tip position, i. e., we are interested in the time evolution of the probability to find the filament tip at position $x$ at time $t$ in either the growing or shrinking phase $p_\pm(x, t)$. The probability to find the tip at position $x$, *regardless of its state*, is then $p(x, t) = p_+(x, t) + p_-(x, t)$. We assume that the tip position is confined to $x \in [0, \infty)$ as indicated in the sketch below.




**a)** Show that, and argue why, the master equation for this process is given by

$$\partial_t p_+(x,t) = -v_+ \partial_x p_+(x,t) - \omega_+ p_+(x,t) + \omega_- p_-(x,t) \,, \tag{9}$$
$$\partial_t p_-(x,t) = v_- \partial_x p_-(x,t) + \omega_+ p_+(x,t) - \omega_- p_-(x,t) \,. \tag{10}$$

**b)** We denote the probability of the microtubule to be in the growing phase by $p_+(t)$ and the probability to be in the shrinking phase by $p_-(t)$. Note that the tip of the filament can be at either position $x \in [0,\infty)$ with probability $p_\pm(x,t)$, so we can marginalize the overall probabilities as $p_\pm(t) = \int_0^\infty \mathrm{d}x\, p_\pm(x,t)$. Solve for the differential equation for $p_\pm(t)$.
Your final results for $p_\pm$ should read

$$p_\pm(t) = \frac{\omega_\mp}{\omega} + \left[p_\pm(0) - \frac{\omega_\mp}{\omega}\right] e^{-\omega t} \,, \tag{11}$$

with $\omega = \omega_+ + \omega_-$.

**c)** Use the aboved derived expression for $p_\pm(t)$ to derive an expression for the average velocity of the filament in the steady state. The average velocity $\langle v(t)\rangle$ for $t \to \infty$ we will call it $V \equiv \langle v(\infty)\rangle$.

**d)** Show that the time evolution of $p_\pm(x,t)$ is governed by the decoupled partial differential equations

$$\partial_t^2 p_\pm(x,t) + \omega \partial_t p_\pm(x,t) = -\Delta v \partial_t \partial_x p_\pm(x,t) + \omega\, D \partial_x^2 p_\pm(x,t) - \omega V \partial_x p_\pm(x,t) \,, \tag{12}$$

with $\Delta v = v_+ - v_-$ and $D = v_+ v_- / \omega$. Why does the probability $p(x,t) = p_+(x,t) + p_-(x,t)$ to find the filament tip at position $x$ (in either state) satisfy the same equation? Consider the possibility of an equilibrium distribution $(\partial_t p^{\mathrm{eq}}(x,t) = 0)$ and show that the equilibrium distribution has to satisfy a balance equation between "diffusive" and "advective" fluxes:

$$D \partial_x^2 p^{\mathrm{eq}}(x) - V \partial_x p^{\mathrm{eq}}(x) = 0 \,. \tag{13}$$

Find an expression for the equilibrium distribution and the average filament length (if possible) and interpret your result.
*Hint: Consider the cases $V > 0$ and $V < 0$ separately.*

**e)** We now want to study the statistics of the filament dynamics beyond the stationary state. Consider the case were the filament is initially in phase $\pm$ with their equilibrium probability $p_\pm(\infty)$. Show that in this case the initial condition for the tip-position probability $p(x,t)$ has to satisfy

$$\partial_t p(x,t)\Big|_{t=0} = -V \partial_x p(x,0) \,. \tag{14}$$

In the following we will consider the case $p(x,0) = \delta(x - x_0)$.
*Note: $x_0$ must be greater than zero, otherwise the probability to be initially in the shrinking state has to vanish.*

**f)** Instead of directly solving the Fokker-Planck equation (Eq. (12)) we aim to characterize the dynamics of $p(x,t)$ by studying its moments, defined as

$$m_n(t) := \int_{-\infty}^{\infty} \mathrm{d}x\, (x - x_0)^n p(x,t) \,. \tag{15}$$

Laplace transform Eq. (12). Then multiply the resulting equation by $(x-x_0)^n$ and integrate over $x$ to show that all moments obey the following recurrence relation for $n \geq 2$ (in Laplace space):

$$
\begin{aligned}
\tilde{m}_n(s) &= n\frac{s\Delta v + \omega V}{s(s+w)}\ \tilde{m}_{n-1} + n(n-1)\frac{\omega D}{s(s+\omega)}\ \tilde{m}_{n-2} \\
\tilde{m}_1(s) &= \frac{V}{s^2} \\
\tilde{m}_0(s) &= \frac{1}{s}
\end{aligned}
\tag{16}
$$

*Hint: Recall that the Laplace transform is given by* $\tilde{f}(s) = \int_0^\infty \mathrm{d}t\, e^{-st} f(t)$. *The moments in Laplace space are then given by* $\tilde{m}(s) = \int_{-\infty}^{\infty} \mathrm{d}x\, (x-x_0)^n \tilde{p}(x,s)$.

**g)** Using Eq. (16) show that the mean and variance are given by

$$E[x](t) = m_1(t) = V\ t, \tag{17}$$

$$\mathrm{Var}[x](t) = m_2(t) - m_1(t)^2 = 2D_{\mathrm{app}} \left\{ t - \frac{1}{\omega}(1-\exp(-\omega\ t)) \right\}, \tag{18}$$

with

$$D_{app} = D - \frac{V^2}{\omega}\left[1 - \frac{\Delta v}{V}\right] . \tag{19}$$

Give an interpretation of Var[x](t) in the limit of $\omega\ t \ll 1$ and $\omega\ t \gg 1$ and for the apparent diffusion constant $D_{\mathrm{app}}$.

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*The following problem parts are optional. Nevertheless, we strongly encourage you to give them a try since they contain valuable methods for solving PDE's.*

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**h*)** *Diffusion limit of the dynamic instability*

To further characterize the dynamics of the process we proceed by studying the dynamic instability in the diffusion limit. Argue why it makes sense to call the limit $\omega \to \infty$ diffusion limit. Show that the probability $p(x,t) = p_+(x,t) + p_-(x,t)$ and the probability flux $j(x,t) := v_+p_+ - v_-p_-$ obey

$$\partial_t p(x,t) + \partial_x j(x,t) = 0 \tag{20}$$

$$\partial_t j(x,t) + \Delta v \partial_x j(x,t) + \omega\ j(x,t) = -\omega D \partial_x\ p(x,t) + \omega\ V\ p(x,t)\ . \tag{21}$$

**i*)** Show that the formal solution for $j(x,t)$ can be written as

$$j(x,t) = j(x-\Delta v\ t, 0)\mathrm{e}^{-\omega\ t} + \int_{-\infty}^{\infty}\int_0^t \delta(x-x'-\Delta v(t-t'))\ \omega \mathrm{e}^{-\omega(t-t')} \left\{-D\partial_x\ p(x,t) +\ V\ p(x,t)\right\}\ . \tag{22}$$

Consider the limit $\omega \to \infty$ and show that the probability current obeys Fick's first law in the presence of a drift

$$j(x,t) = V\ p(x,t) - D\ \partial_x p(x,t)\ . \tag{23}$$

Finally show that $p(x,t)$ obeys the Fokker-Planck equation

$$\partial_t p(x,t) = -V\partial_x p(x,t) + D\partial_x^2 p(x,t)\ . \tag{24}$$

*Hint:* $\lim_{\omega\to\infty} \omega \exp(-\omega(t-t')) \to \delta(t-t')$

**j*)** We will now make use of the simpler form of Eq. (24). Consider an absorbing boundary at $x = 0$. Find the survival probability, i.e., the probability that a filament has not touched the the boundary $x = 0$ at time $t$. Consider the limit $t \to \infty$ for $V < 0$ and $V > 0$.
*Hint:*

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\ \sigma} \exp\left(-\frac{(a+x)^2}{2\sigma^2}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{a}{\sqrt{2}\sigma}\right)\ , \tag{25}$$

were $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the complementary error function. You can use $\mathrm{erfc}(x) \approx 1 - \mathrm{sgn}(x)$ for large $x$.

Your solutions should be handed in by uploading them to Moodle by **Wednesday, 4th of June 2025, 10:00 am**.