Übung zur Vorlesung
Mathematische Statistische Physik
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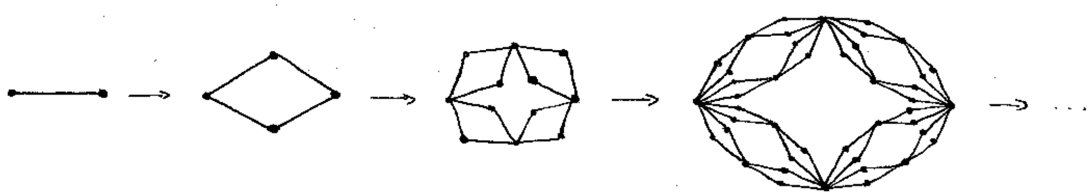
Prof. E. Frey und Prof. F. Merkl

Problem 24
It is established empirically that the critical exponents in phase transitions are identical, irrespective of whether the critical temperature is approached from above or from below. For instance, the specific heat C behaves as $C \sim |t|^{-\alpha}$ ($t = \frac{T-T_c}{T_c}$) for small $|t|$, both when $t < 0$ and $t > 0$. Suppose that the singular part of C for some ferromagnet in a magnetic field h has the scaling form

$$C_{\pm}(t,h) = |t|^{-\alpha_{\pm}} g_{\pm}(h/|t|^{\Delta_{\pm}}) \,,$$

where the equation with the plus-sign and the minus-sign holds for $t > 0$ and $t < 0$, respectively. Use the fact that there must be no singularities as one crosses the critical isotherm, $t = 0$, at nonzero h (ie suppose that C is analytic but for the critical point) to show that $\alpha_+ = \alpha_- \equiv \alpha$ and $\Delta_+ = \Delta_- \equiv \Delta$.

Problem 25
A typical obstacle to exact renormalization group transformations is the proliferation of couplings, which can even give rise to long range interactions after many renormalization steps. This difficulty can sometimes be circumvented by studying hierarchical lattices rather than the more realistic regular ones. A simple representative in this group can be constructed as follows.



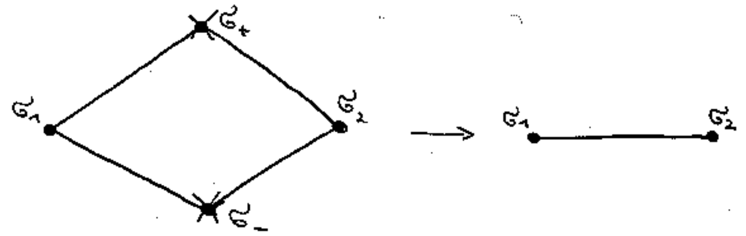
The dots denote lattice sites and the solid lines nearest neighbour bonds. One construction step consists in replacing each bond by a quadrangle. Here we have only performed the first three steps, but the construction can be repeated ad infinitum. On this network we consider the Ising model with Hamiltonian

$$-\beta H(\{\sigma\}) = K \sum_{<ij>} \sigma_i \sigma_j + H_B \sum_{<ij>} (\sigma_i + \sigma_j) + H_S \sum_i \sigma_i \,,$$

where the brackets <> indicate that the sum is over nearest neighbours. Incidentally, the lattice may be associated the dimension 2 because its average shortest-path length scales as $l \sim N^{1/2}$, where N is the total number of sites. For comparison, this relation becomes $l \sim N^{1/d}$ for the hypercubic lattice of dimension d. Thus, as for the two dimensional Ising model on a square lattice we might expect a phase transition at non-zero temperature.

(a) Our goal is to calculate the free energy per lattice site. To do this, let us first tackle one of the quadrangles which the lattice is made up of, ie consider the sum

$$Z'(\sigma_1,\sigma_2) := \sum_{\sigma_+=\pm 1, \sigma_-=\pm 1} e^{-\beta H(\sigma_1,\sigma_+,\sigma_-,\sigma_2)} .$$



Prove that $\ln(Z') = K's_1s_2 + H'_B(s_1+s_2) + c' + H'_S(s_1+s_2)$ for some renormalized couplings K', H'_B, c' and $H'_S = H_S$. The reason why H_B and H_S are not subsumed in one parameter should become clear below. You are not required to find neat expressions for the new couplings, just show how in principle one can calculate them.

(b) Our renormalization-group transformation consists of decimating the two center sites in each quadrangle. This reduces a lattice after $n+1$ steps in the construction to a lattice after n steps in the construction. Using the result in (a) verify that the renormalized Hamiltonian is

$$-\beta H'(\{\sigma\}') = \sum_{<i'j'>} \{J'\sigma_{i'}\sigma_{j'} + H'_B(\sigma_{i'}+\sigma_{j'}) + c'\} + H'_S \sum_{i'} \sigma_{i'} .$$

(c) The renormalized interaction constants are

$$K' = \frac{1}{2}\ln(R_{++}R_{--}/R_{+-}^2), \quad H'_B = \frac{1}{2}\ln(R_{++}/R_{--}), \quad H'_S = H_S ,$$

$$c' = 4c + \frac{1}{2}\ln(R_{++}R_{--}R_{+-}^2),$$

with

$$R_{++} = xy^2z + x^{-1}z^{-1}, \quad R_{--} = x^{-1}z + xy^{-2}z^{-1},$$

$$R_{+-} = yz + y^{-1}z^{-1}, \quad x = e^{2K}, \quad y = e^{2H_B}, \quad z = e^{H_S} .$$

Here c is an additive constant per bond, equal to zero in the original Hamiltonian.

Consider the subspace $H_B = H_S = 0$. Explicitly check that the above relations are correct and show that there is exactly one unstable fixed point $0 < K_c < \infty$.[1] Describe the qualitative behaviour of the renormalization group flow for $K < K_c$ and $K > K_c$.

(d) Upon diagonalizing the renormalization group flow in the vicinity of the unstable fixed point, use standard renormalization group analysis to determine the nature of the singularity of the specific heat C and the magnetic susceptibility $\chi := \frac{\partial M}{\partial H_S}$, where M is the magnetization.
(Hint: Here it is crucial to assume that the free energy becomes additive for large systems such that the free energy per site becomes constant. Also notice that the ratio between the number of sites before and after the renormalization becomes $N_{n+1}/N_n \overset{n\to\infty}{\longrightarrow} 4 = b^d$, with the length rescaling factor $b = 2$ and dimension $d = 2$.)

[1] In fact, $K_c = \ln\left(\frac{1}{3}[1 + (19 - 3\sqrt{33})^{1/3} + (19 + 3\sqrt{33})^{1/3}]\right)$.