Übung zur Vorlesung
Mathematische Statistische Physik
Sommersemester 2008
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Problem 17
Consider the Ising model as discussed in the lecture. In the mean-field approximation, the Hamiltonian for the spin-configuration σ is given by $H(\sigma) = -J\sum_{\langle xy\rangle}\sigma_x\sigma_y - h\sum_x \sigma_x$, with the coupling constant J and the magnetic field h. We assume that the number of sites N in the system is finite.

(a) One way to perform a mean-field approximation is by replacing the spins σ_x, σ_y with their average values at a certain instant in time, ie

$$H_{MF}(\sigma) = -\frac{N}{2}Jm(\sigma)^2 - Nhm(\sigma),$$

with the magnetization $m(\sigma) = \frac{1}{N}\sum_x \sigma_x$ which is dependent on the configuration σ. Show that, in the limit $N \to \infty$, the free energy per site is given by the global minimum of

$$\beta\cdot f(m) = -\beta\left(\frac{J}{2}m^2 + hm\right) + \frac{1+m}{2}\ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\ln\left(\frac{1-m}{2}\right), m \in]-1,1[. \quad (1)$$

Check that the location of the extrema of f satisfies

$$m = \tanh\left[\beta(Jm+h)\right] . \quad (2)$$

Hint: You might want to use (without proof) Stirling's formula $\log N! = N(\log(N)-1) + o(N)$, $\lim_{N\to\infty} o(N)/N = 0$.

(b) For the moment, suppose that $h = 0$. Find the critical temperature T_c below which (2) has nonzero solutions and check that these make up the minima of f. For $h > 0$, prove that the location of the global minimum m_0 of f is given by some $m_0 > 0$.

(c) Using the free energy $f(m_0)$, calculate the average magnetization $m \equiv m(T,h)$. Qualitatively, plot m against the temperature T for $h = 0$ and $h > 0$. Show that $m(T,0) \sim (T_c - T)^\beta$ in the vicinity of the critical point and find the critical exponent β. Also calculate the entropy S and the specific heat c_h in terms of m_0 and plot these quantities against T for $h = 0$ and $h > 0$. Convince yourself that, at least for small h, the maximum of c_h is not at the critical temperature $T = T_c$.

Problem 18
(a) By the transfer matrix method, find an expression, (in terms of the eigenvalues and eigenvectors of the transfer matrix, T) for the correlation function

$$C_{ij} \equiv \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle ,$$

of the one-dimensional Ising model with Hamiltonian

$$H = -\sum_i (h\sigma_i + J\sigma_i\sigma_{i+1}) \ .$$

Show that the correlation function decays exponentially as $e^{-|i-j|/\xi}$. Find a simple expression for ξ in terms of T for $h = 0$.

(b) Express the susceptibility, $\chi(h, T)$, in terms of the correlation function and thus, find the low temperature behaviour of $\chi(h = 0, T)$.

(c) Calculate χ directly from $F(h, T)$ and compare with part (b).

Problem 19

The specific heat of the Ising model (and other quantities such as the susceptibility) can be expanded in high temperature series expansions in powers of $K \equiv J/T$.

(a) Show that the Ising model partition function in zero magnetic field can be written as

$$Z = [g(K)]^{N_B} \sum_{\{\sigma_i = \pm 1\}} \prod_{\langle ij \rangle} [1 + t(K)\sigma_i\sigma_j]$$

with N_B the number of bonds in the lattice (the product is over nearest neighbours i, j). Find $g(K)$ and $t(K)$.

(b) Use the expression in (a) to expand the free energy and hence the specific heat per site in powers of t up to order t^4 for (i) a square lattice and (ii) a cubic lattice. You should find a very small number of terms; these can be expressed graphically as diagrams on these lattices.

(c) Using (a) and (b), obtain the specific heat per site up to order K^4.

(d) Onsager's solution of the two dimensional square lattice Ising model yields the exact result for the specific heat.

$$f = \frac{-T}{2} \int_{-\pi}^{\pi} \frac{d\phi_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_2}{2\pi} \ln \left[\cosh^2 2K - \sinh 2K \left(\cos\phi_1 + \cos\phi_2\right)\right] \ . \tag{3}$$

Expanding this in powers of K, check your result from part (c).

(e) From (3), show that the specific heat diverges logarithmically at a critical value of $K = K_c$. Find K_c and the amplitude of the logarithmic part.