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Compensation of decoherence from telegraph noise by means of an open-loop quantum-control technique

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With the growing efforts in isolating solid-state qubits from external decoherence sources, the origins of noise inherent to the material start to play a relevant role. One representative example is charged impurities in the device material or substrate, which typically produce telegraph noise and can hence be modeled as bistable fluctuators. In order to demonstrate the possibility of the active suppression of the disturbance from a *single* fluctuator, we theoretically implement an elementary bang-bang control protocol, a protocol based on sudden pulses. We numerically simulate the random walk of the qubit state on the Bloch sphere with and without bang-bang compensation by means of a stochastic Schrödinger equation and compare it with an analytical saddle-point solution of the corresponding Langevin equation in the long-time limit. We find that the deviation with respect to the noiseless case is significantly reduced when bang-bang pulses are applied, being scaled down approximately by the ratio of the bang-bang period to the typical flipping time of the bistable fluctuation. Our analysis gives not only the effect of bang-bang control on the variance of these deviations, but also their entire distribution. As a result, we expect that bang-bang control works as a high-pass filter on the spectrum of noise sources. This indicates how the influence of $1/f$ noise ubiquitous to the solid-state world can be reduced.

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In order to implement solid-state quantum information processing devices, the decoherence acting on the quantum states has to be carefully understood, controlled, and eliminated. So far, research has concentrated on decoupling from external noise sources (like thermal heat baths and electromagnetic noise). With the success of this effort, noise sources intrinsic to the material, such as defect states, increase in importance and have to be controlled in order to improve coherence even further.

Most external noise sources are composed of extended modes in the thermodynamic limit close to equilibrium such that their fluctuations are purely Gaussian. Thus, their influence can be modeled by an oscillator bath, see, e.g., [1]. However, there are physical situations when this assumption fails [2–4]. In particular, this is true for localized noise sources with bounded spectra as they occur in disordered systems for hopping defect states [5]. Physical examples for this situation are background charges in charge qubits [4,6,7] or traps in the oxide layers of Josephson tunnel junctions [8]. Such localized noise sources are more realistically represented by a collection of bistable fluctuators [4,9] (henceforth abbreviated bfls), as their noise spectrum is considerably non-Gaussian. If many of these noise sources with different flipping times are appropriately superimposed, they lead to $1/f$ noise [5,10,11]. With the progress of fabrication technology and miniaturization of qubits, we expect, however, that there might only be a few fluctuators in a qubit [8].

We analyze the impact of a single fluctuator in the semiclassical limit, where it acts as a source of telegraph noise. We apply an open loop quantum control technique, namely quantum bang-bang [12–14], which is designed suitably for slowly fluctuating noise sources. We simulate the noise-influenced qubit dynamics with and without bang-bang correction by integrating the time-dependent Schrödinger equation for each specific realization of the noise. We present the resulting random walks around the unperturbed signal on the Bloch sphere and analyze the quality of this suppression by a comparison of the ensemble-averaged deviations of these random walks with and without bang-bang correction.

We describe our system by the effective Hamiltonian

$$H_q^{\mathrm{eff}}(t) = H_q + H_{q,\mathrm{bfl}}^{\mathrm{noise}}(t), \tag{1}$$

$$H_q = \hbar\epsilon_q\hat{\sigma}_z^q + \hbar\Delta_q\hat{\sigma}_x^q \quad H_{q,\mathrm{bfl}}^{\mathrm{noise}}(t) = \hbar\alpha\xi_{\mathrm{bfl}}(t)\hat{\sigma}_z^q, \tag{2}$$

where α denotes the coupling strength between the fluctuator and the qubit and $\xi_{\mathrm{bfl}}(t)$ represents a symmetric telegraph process that is flipping between ±1, whose switching events are Poisson distributed with a mean separation τ_{bfl} between two flips.

On a microscopic level, such noise is typically generated by coupling the qubit to a two-state impurity, which is in turn coupled to a heat bath causing the two-state system to flip randomly and incoherently. Our model corresponds to the semiclassical limit and should be accurate whenever the coupling of the impurity to the bath is much stronger than its coupling to the qubit [2,4] such that the qubit does not act back on the noise source. The assumption of a *symmetrical* telegraph process corresponds to a high bath temperature compared to the impurity level spacing. This restriction is not essential for the following investigations, for an asymmetric noise signal would only produce an additional constant drift.

We describe the resulting evolution of the noise-influenced qubit by a stochastic Schrödinger equation [15,16] with the time-dependent Hamiltonian (2). For any initial state and realization of the noise, we numerically integrate the Schrödinger equation. The result is a random walk on the Bloch sphere, which is centered around the free precession corresponding to ϵ_q and Δ_q.

We implement the following idealized open loop quantum control scheme: apply an infinite train of π pulses on the qubit with negligibly short pulse durations and a constant separation time τ_{bb} between successive pulses. (The assumption of negligibly short, perfectly applied π pulses is for technical convenience only). In doing so, we intend to average out the $\hat{\sigma}_z$ parts of the effective Hamiltonian (and thereby in particular the noise term) on time scales large compared to τ_{bb}. This is accomplished by iteratively spin flipping the qubit and thus effectively switching the sign of the noisy part of the Hamiltonian. This mechanism thus works analogously to the well-known spin-echo procedure, specifically the Carr-Purcell procedure of NMR [17]. The suppression of the telegraph noise effects should qualitatively scale as follows: The size of the random walk induced by the noise is determined by the typical time separation of the fluctuator's influence between two flips τ_{bfl} and its coupling strength α and scales roughly with $\alpha\tau_{\mathrm{bfl}}$ [4]. Using bang-bang, the bfls influence remains uncompensated for at most a single bang-bang period. Thus, we reduce the influence of the bfl by an average factor of $\tau_{\mathrm{bfl}}/\tau_{\mathrm{bb}}$.

As generic conditions of the system dynamics we consider for the numerical simulations $\epsilon_q = \Delta_q \equiv \Omega_0$. Without loss of generality, we assume $\langle \hat{\sigma}_z^q \rangle = +1$ as an initial state. If there were no noise, the spin would precess on the Bloch sphere around the rotation axis $\hat{\sigma}_x^q + \hat{\sigma}_z^q$. So we expect for not too large an interaction strength ($\alpha \ll 1$) a slight deviation of the individual quantum trajectory from the free evolution case. We take $\alpha = 0.1$ for our coupling strength. All the following time and energy measures are given in units of the unperturbed system Hamiltonian: our time unit is $\tau_{\mathrm{Sys}} = 1/\Omega_0$ and our energy unit is $\Delta E = \sqrt{\epsilon_q^2 + \Delta_q^2} = \sqrt{2}\Omega_0$. Note that in these units, a period lasts $\pi\tau_{\mathrm{sys}}/\sqrt{2}$. We have integrated the time-dependent Schrödinger equation and averaged over $N = 1000$ realizations. The time-scale ratio $\tau_{\mathrm{bfl}}/\tau_{\mathrm{bb}} = 10$ if not denoted otherwise. We characterize our results by the root-mean-square (rms) deviation from the unperturbed signal

$$\Delta\vec{\sigma}_{\mathrm{rms}}(t) = \sqrt{\frac{1}{N}\sum_j \left(\vec{\sigma}_j^q(t) - \vec{\sigma}^q_{\mathrm{noisy},j}(t)\right)^2}. \tag{3}$$

In order to characterize the degree of noise suppression by means of bang-bang control, we define the suppression factors for a given time t_0

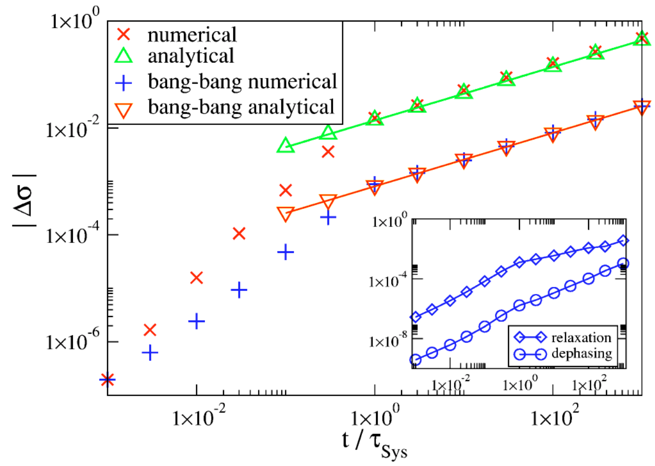

FIG. 1. (Color online) Time evolution of the mean deviations for bfl-induced random walks with and without bang-bang control. The straight lines are square-root fits of the analytical derived random-walk model variances (plotted as triangles). Inset: Transverse and perpendicular components of bang-bang suppressed noise.

$$\mathcal{S}_{t_0}(\tau_{\mathrm{bfl}}/\tau_{\mathrm{bb}}) = \frac{\Delta\vec{\sigma}^{\mathrm{bfl}}_{\mathrm{rms}}(t_0)}{\Delta\vec{\sigma}^{\mathrm{bb}}_{\mathrm{rms}}(t_0)}. \tag{4}$$

The deviation as a function of time is plotted in Fig. 1.

The total deviations at intermediate times are suppressed by a ratio of $\simeq 10$. Detailed analysis shows that the tangential (dephasing) and the orthogonal (relaxation) deviation are of the same size for the uncompensated case. In contrast, the bang-bang modulation mostly compensates the dephasing-type deviation, as shown in the inset of Fig. 1.

We now develop analytical random-walk models for our system. The two-dimensional random walk on the Bloch sphere is in general reduced to an effectively one-dimensional model by bang-bang control, representing the relevant perpendicular part. We restrict ourselves to the long-time limit.

For simplicity, we replace the fluctuating number of random-walk steps for a given time Δt of noisy evolution by its expectation value $\Delta t/\tau_{\mathrm{bfl}}$ [18]. This allows one to use the number of random-walk steps as a time parameter. We encounter different one-step distributions, depending on whether the number of steps is odd or even, corresponding to an "up" or "down" state of the bfl.[1] The step-size distribution of the bfl model in our small deviation regime is given by Poisson statistics

$$\Phi^{\mathrm{bfl}}_{\mathrm{odd/even}}(x) = \frac{e^{\mp x/\beta}\theta(\pm x)}{\beta}, \tag{5}$$

with $\beta = (\sqrt{5}/2)\alpha\tau_{\mathrm{bfl}}$ as a typical random-walk one-step deviation. The prefactor accounts for the geometrical situation. τ_{unit} is a time unit, corresponding to a discrete step length $x_{\mathrm{unit}} = \alpha\tau_{\mathrm{unit}}$ of the random walk. $\theta(x)$ denotes the Heaviside step function. We neglect the correlations between transverse

[1] We assume the bfl being initially in its "upper" state. This restriction is of no relevance for the long-time limit.

and perpendicular deviations as they average out in the long-time limit.

For the bang-bang suppressed random walk, the flipping positions of the bfl-noise sign in the bang-bang time slots are essentially randomly distributed as long as $\tau_{bb} \ll \tau_{bfl}$. We find a homogenous step-size distribution between zero deviation and a maximum $\gamma=\alpha 2\tau_{bb}/\sqrt{2}$,

$$\Phi^{bb}_{odd/even}(x) = \frac{\theta(\pm x)\theta[\pm(\gamma-x)]}{\gamma}. \quad (6)$$

The factor of $1/\sqrt{2}$ occurs because the bang-bang sequence also averages over the static ϵ_q term and hence slows down the free evolution.

By means of these one-step probability distributions, we are able to calculate via convolution the distributions for $2N$-step random walks. Specifically, they are the inverse Fourier transforms of the N-fold products of the Fourier transforms of the two-step distribution [18]. For the uncompensated case, we find

$$\Phi^{bfl}_{2N}(x) = \int_{-\pi}^{\pi} \frac{dk}{2\pi\beta^{2N}} e^{-ikx}\left(\frac{1}{1-2\cos(k)e^{-1/\beta}+e^{-2/\beta}}\right)^N, \quad (7)$$

whereas for the compensated case

$$\Phi^{bb}_{2N}(x) = \int_{-\pi}^{\pi} \frac{dk}{2\pi\gamma^{2N}} e^{-ikx}\left(\frac{[1-\cos((\gamma+1)k)]}{[1-\cos(k)]}\right)^N. \quad (8)$$

Already for random-walk step numbers on the order of ten, the resulting distributions are almost Gaussian. Their standard deviations give the rms deviations of the random-walk models plotted in Fig. 1. As expected, they grow as a square root of the number of steps.

The above integrals can be evaluated analytically using the saddle-point approximation. We find variances of

$$\sigma_{bfl}(N) = \sqrt{2N}\beta = \frac{\sqrt{5N}}{2}\alpha\tau_{bfl} \quad (9)$$

for the pure bfl random walk and

$$\sigma_{bb}(N) = \sqrt{\frac{N}{2}}\gamma = \sqrt{\frac{N}{2}}\alpha\tau_{bb} \quad (10)$$

for the compensated one. In the large-N limit, these results show excellent agreement with numerics.

Beyond predicting the variance, our analysis also allows evaluation of the full distribution. We compared evolution with and without bang-bang compensation via simulations with 10^4 realizations and calculated the full distribution function for an evolution time $t_0=\tau_{Sys}$. The numerical histograms of the deviation with their respective one- and two-dimensional Gaussian fits are shown in Fig. 2.

We observe that not only the bang-bang compensated distribution is much narrower than the uncompensated distribution, but also that its shape is qualitatively different: its maximum is at zero error, whereas the uncompensated distribution has its maximum at a finite error $|\Delta\sigma|\approx 0.01$ and zero probability of zero error.

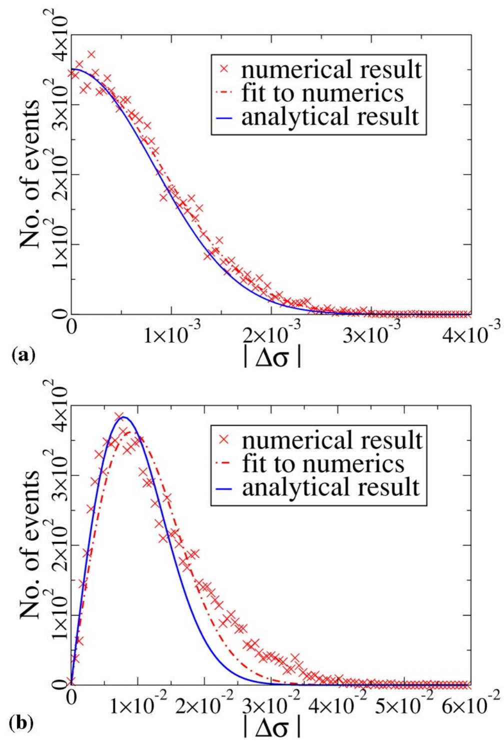

FIG. 2. (Color online) Histograms of the deviation from free evolution with and without bang-bang control and respective fits to the expected two-dimensional (pure bfl case) and one-dimensional (bang-bang corrected) random-walk statistics. Numerical data consists of 10^4 realizations at a fixed time $t_0=\tau_{Sys}$. With $\tau_{bfl}=0.01\tau_{Sys}$ the random-walk distributions are calculated for $N=\tau_{Sys}/\tau_{bfl}=100$ steps. (NB: The x-axis scale of the right graph is 15 times smaller than that of the left graph.)

We have systematically studied suppression factors for different ratios between times, τ_{bfl}/τ_{bb}, at a constant fluctuator flipping rate $\tau_{bfl}=10^{-2}\tau_{sys}$ and evolution time $t_0=\tau_{sys}$. The numerical data in Fig. 3 show that the suppression efficiency is linear in the bang-bang repetition rate, $S=\mu\tau_{bfl}/\tau_{bb}$. The numerically derived value of the coefficient, $\mu_{numerical} \approx 1.679$, is in good agreement with our analytical result $\mu_{analytical}=\sigma_{bfl}(N)/\sigma_{bb}(N)=\sqrt{5/2}\simeq 1.581$ from the saddle-point approximation, Eqs. (9) and (10). This small discrep-

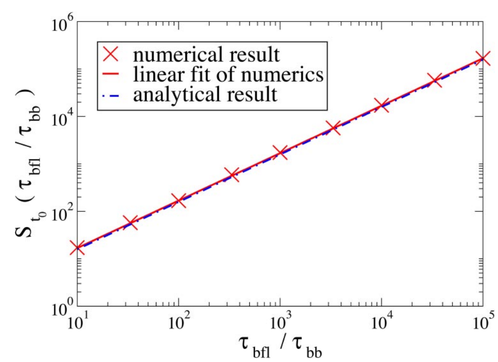

FIG. 3. (Color online) The suppression factor $\mathcal{S}_{t_0}(\tau_{bfl}/\tau_{bb}) = \Delta\vec{\sigma}^{bfl}_{rms}(t_0)/\Delta\vec{\sigma}^{bb}_{rms}(t_0)$ evaluated for $t_0=\tau_{Sys}$ as a function of the ratio of the flipping time τ_{bfl} and the bang-bang pulse separation τ_{bb}.

ancy reflects the correlations between the transverse and longitudinal random walk in the uncompensated case (see Fig. 2).

We have demonstrated the ability of a bang-bang protocol to compensate environmental fluctuations with frequencies $\omega \ll 1/\tau_{\mathrm{bb}}$. Thus, bang-bang control acts as a high pass filter for noise with a roll-off frequency of $1/\tau_{\mathrm{bb}}$. Evidently, the bang-bang correction is suitable for suppressing the impact of telegraph noise on qubits and can enhance the coherence by orders of magnitude. The application of the scheme which we outlined requires a relatively strict separation of time scales: One has to be able to flip the spin very rapidly, typically two orders of magnitude faster than τ_{bfl}. The stability of the bang-bang suppression efficiency regarding finite pulse lengths (instead of infinitesimal, as assumed here for technical convenience) as well as nonperfect (i.e., erroneous) pulses will be examined in another more explicit paper. Moreover, we have assumed that the environment produces symmetric telegraph noise regardless of the qubit dynamics. Clearly, the issue of when one may neglect feedback effects between the qubit and bfl must be critically revisited in the low-temperature limit. We conjecture that the set up is promising for $1/f$ noise, as in particular the most harmful and predominantly low-frequency fraction of a corresponding ensemble of fluctuators would be compensated most strongly. Finally, one has to be aware that also the static term of the Hamiltonian is averaged out. This does reduce the number of control options. However, by a combination of (i) slow pulses which commute with the bang-bang sequence and (ii) fast pulses on the time scale of the bang-bang sequence. One can still achieve full control and universal computation as shown in Ref. [14].

Another approach for decoupling from slow noise is to choose an appropriate working point with a dominant term $\Omega\sigma_x$ in the static Hamiltonian. The action of this term can be understood as a rapid flipping of the spin, similar to our bang-bang protocol. Using a Gaussian approximation of the noise from the bfl with standard rate expressions (e.g., [19]), it can be shown that the dephasing rate reads $\Gamma_\phi = (\alpha/\tau_{\mathrm{bfl}})\Omega^2$ with compensation, instead of $\Gamma_\phi = \alpha\tau_{\mathrm{bfl}}$ without. This corresponds to the same amount of reduction as in our case. This scheme has been implemented in superconducting qubits [20]. In that case, it turned out that because the σ_x term was limited by fabrication, this consideration led to a major redesign. Our compensation scheme purely relies on external control and thus keeps the hardware design flexible.

A related problem has been addressed in Ref. [21], which deals with bang-bang suppression of Gaussian $1/f$ noise, i.e., a bosonic bath with an appropriate sub-Ohmic spectrum. That system is treated in the weak-coupling approximation, i.e., it assumes $S(\omega)/\omega \ll 1$ at low frequencies where $S(\omega)$ is the noise spectral function. Both assumptions are serious constraints in the $1/f$ case [4,5]. Our work is not constrained to weak coupling, takes the full non-Gaussian statistics of telegraph noise into account, and gives the full resulting distribution of errors.

In summary, we examined the decoherence of a single qubit from a single symmetric telegraph noise source and proposed an adequate open quantum control compensation protocol for suppressing its impact. We simulated the qubits dynamics using a stochastic Schrödinger equation and analyzed its deviation from free evolution. We formulated analytically solvable one- and two-dimensional random-walk models, which are in excellent agreement with the simulations in the long-time limit. Specifically, we showed quantitatively, how the degree of noise compensation is controlled by the ratio between bfl flipping time scale and bang-bang pulse length. We gave the full statistics of deviations in both cases.

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