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Flow equation renormalization of a spin-boson model with a structured bath

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Abstract

We discuss the dynamics of a spin coupled to a damped harmonic oscillator. This system can be mapped to a spin-boson model with a structured bath, i.e. the spectral function of the bath has a resonance peak. We diagonalize the model by means of infinitesimal unitary transformations (*flow equations*), thereby decoupling the small quantum system from its environment, and calculate spin–spin correlation functions.

Keywords: Flow equations; Quantum dissipative systems; Spin-boson; Structured bath

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1. Introduction—model

Recently, a new strategy for performing measurements on solid-state (Josephson) qubits was proposed which uses the entanglement of the qubit with states of a damped oscillator [1], with this oscillator representing the plasma resonance of the Josephson junction. This system of a spin coupled to a damped harmonic oscillator (see Fig. 1) can be mapped to a standard model for dissipative quantum systems, namely the spin-boson model [2]. Here the spectral function governing the dynamics of the spin has a resonance peak. Such structured baths were also discussed in connection with electron transfer processes [2]. We use the flow equation method introduced by Wegner [3] to analyze the system shown in Fig. 1, consisting of a two-level system coupled to a harmonic oscillator Ω, which is coupled to a bath of harmonic oscillators:

$$\tilde{\mathscr{H}} = -\frac{\Delta_0}{2}\,\sigma_x + \Omega B^\dagger B + g(B^\dagger + B)\sigma_z + \sum_k \tilde\omega_k \tilde b_k^\dagger \tilde b_k + (B^\dagger + B)\sum_k \kappa_k(\tilde b_k^\dagger + \tilde b_k) + (B^\dagger + B)^2 \sum_k \frac{\kappa_k^2}{\tilde\omega_k},$$

with the spectral function $J(\omega) \equiv \sum_k \kappa_k^2 \delta(\omega - \tilde\omega_k) = \Gamma\omega$. This system can be mapped to a spin-boson model [2]

$$\mathscr{H} = -\frac{\Delta_0}{2}\,\sigma_x + \frac{1}{2}\,\sigma_z \sum_k \lambda_k(b_k^\dagger + b_k) + \sum_k \omega_k b_k^\dagger b_k, \qquad (1)$$

where the dynamics of the spin depends only on the spectral function $J(\omega) \equiv \sum_k \lambda_k^2 \delta(\omega - \omega_k)$ given by

$$J(\omega) = \frac{2\alpha\omega\Omega^4}{(\Omega^2 - \omega^2)^2 + (2\pi\Gamma\omega\Omega)^2} \quad \text{with} \quad \alpha = \frac{8\Gamma g^2}{\Omega^2}. \qquad (2)$$

2. Method—results

Using the flow equation technique we approximately diagonalize the Hamiltonian $\mathscr{H}$ [Eq. (1)] by means of infinitesimal unitary transformations. The continuous sequence of unitary transformations $U(l)$ is labelled by a flow parameter l. Applying such a transformation to a given Hamiltonian, this Hamiltonian becomes a function of l : $\mathscr{H}(l)=$

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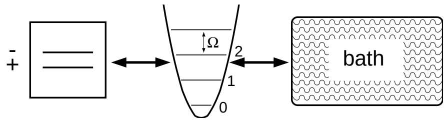

Fig. 1. A two-level system is coupled to a damped harmonic oscillator with frequency Ω.

$U(l)\mathscr{H}U^{\dagger}(l)$. Here $\mathscr{H}(l=0)=\mathscr{H}$ is the initial Hamiltonian and $\mathscr{H}(l=\infty)$ is the final diagonal Hamiltonian. Usually, it is more convenient to work with a differential formulation

$$\frac{\mathrm{d}\mathscr{H}(l)}{\mathrm{d}l}=[\eta(l),\mathscr{H}(l)] \quad \text{with}$$

$$\eta(l)=\frac{\mathrm{d}U(l)}{\mathrm{d}l}\,U^{-1}(l). \tag{3}$$

Using the flow equation approach one can decouple system and bath by diagonalizing $\mathscr{H}(l=0)$ [4]:

$$\mathscr{H}(l=\infty)=-\frac{\varDelta_\infty}{2}\,\sigma_x+\sum_k \omega_k b_k^{\dagger}b_k. \tag{4}$$

Here $\varDelta_\infty$ is the renormalized tunnelling frequency. For the generator of the flow we choose the Ansatz [4]

$$\begin{aligned}\eta=&\sum_k(\mathrm{i}\sigma_y\varDelta(b_k+b_k^{\dagger})+\sigma_z\omega_k(b_k-b_k^{\dagger}))\frac{\lambda_k}{2}\\&\times\left(\frac{\varDelta-\omega_k}{\varDelta+\omega_k}\right)\\&+\frac{\varDelta}{2}\sum_{q,k}\lambda_k\lambda_q I(\omega_k,\omega_q,l)(b_k+b_k^{\dagger})(b_q-b_q^{\dagger}),\end{aligned} \tag{5}$$

with

$$I(\omega_k,\omega_q,l)=\frac{\omega_q}{\omega_k^2-\omega_q^2}\left(\frac{\omega_k-\varDelta}{\omega_k+\varDelta}+\frac{\omega_q-\varDelta}{\omega_q+\varDelta}\right).$$

The flow equations for the effective Hamiltonian [Eq. (4)] then take the following form:

$$\begin{aligned}\frac{\partial J(\omega,l)}{\partial l}=&-2(\omega-\varDelta)^2J(\omega,l)\\&+2\Delta J(\omega,l)\int\mathrm{d}\omega'\,J(\omega',l)I(\omega,\omega',l),\end{aligned} \tag{6}$$

$$\frac{\mathrm{d}\varDelta}{\mathrm{d}l}=-\varDelta\int\mathrm{d}\omega\,J(\omega,l)\,\frac{\omega-\varDelta}{\omega+\varDelta}. \tag{7}$$

The unitary flow diagonalizing the Hamiltonian generates a flow for $\sigma_z(l)$ which takes the structure

$$\sigma_z(l)=h(l)\sigma_z+\sigma_x\sum_k\chi_k(l)(b_k+b_k^{\dagger}), \tag{8}$$

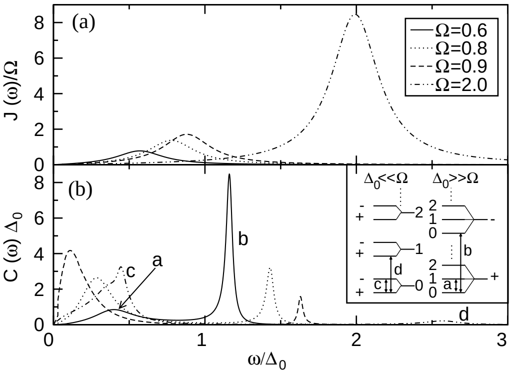

Fig. 2. (a) Different effective spectral functions $J(\omega,l=0)$ and (b) the corresponding $C(\omega)$ for $\Omega\Gamma=0.06$ and $\alpha=0.15$. The inset shows the term scheme of a two-level system coupled to a harmonic oscillator for the two limits $\varDelta_0\ll\Omega$ and $\varDelta_0\gg\Omega$.

where $h(l)$ and $\chi_k(l)$ obey the differential equations

$$\frac{\mathrm{d}h}{\mathrm{d}l}=-\varDelta\sum_k\lambda_k\chi_k\,\frac{\omega_k-\varDelta}{\omega_k+\varDelta}, \tag{9}$$

$$\begin{aligned}\frac{\mathrm{d}\chi_k}{\mathrm{d}l}=&\,\Delta h\lambda_k\,\frac{\omega_k-\varDelta}{\omega_k+\varDelta}\\&+\sum_q\chi_q\lambda_k\lambda_q\Delta I(\omega_k,\omega_q,l).\end{aligned} \tag{10}$$

One can show that the function $h(l)$ decays to zero as $l\to\infty$. Therefore, the observable σ_z decays completely into bath operators [4].

We integrated the flow equations numerically in order to calculate the Fourier transform, $C(\omega)$, of the spin–spin correlation function

$$C(t)\equiv\tfrac{1}{2}\langle\sigma_z(t)\sigma_z(0)+\sigma_z(0)\sigma_z(t)\rangle. \tag{11}$$

$C(t)$ can be used to calculate dephasing and relaxation times for measurements on qubits [1]. Fig. 2(a) shows $J(\omega,l=0)$ and Fig. 2(b) $C(\omega)$ for different values of Ω. $C(\omega)$ displays a double-peak structure, which can be understood from the term scheme shown in the inset. The arrows indicate the transitions responsible for the peaks in $C(\omega)$. Additional structure of $C(\omega)$ due to higher-order transitions in the term scheme is not seen in Fig. 2. This is due to our Ansatz for $\sigma_z(l)$ [see Eq. (8)], which does not include the corresponding higher-order terms. However, we do not expect the additional peaks to have much weight, as the sum rule [4] for the total spectral weight is fulfilled with an error of less than 5% for all the plots in Fig. 2(b). We leave the extension of the Ansatz for $\sigma_z(l)$ for future work.

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