6.4.1 Tolman solution

April 14, 2009

Let us consider a spherically symmetric inhomogeneity. In this case one can always find a coordinate system where

$$x^i = a\left(R, t\right) q^i$$

and $R \equiv |\mathbf{q}|$ is the radial Lagrangian coordinate. The strain tensor is then:

$$J^i_k = a\delta^i_k + a' R n^i n^k, \tag{1}$$

where $a' \equiv \partial a/\partial R$ and $n^i \equiv q^i/R$. For a point at a given distance from the center one can always rotate coordinate system to get $n^1 = 1$, $n^2 = n^3 = 0$, so that the strain tensor becomes diagonal:

$$\mathbf{J} = \left(\begin{array}{ccc} (aR)' & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{array} \right), \tag{2}$$

and hence

$$J = a^2 \left(aR\right)', \qquad \mathrm{tr}\left(\left(\dot{\mathbf{J}} \cdot \mathbf{J}^{-1}\right)^2\right) = \left(\frac{(\dot{a}R)'}{(aR)'}\right)^2 + 2\left(\frac{\dot{a}}{a}\right)^2. \tag{3}$$

Substituting these expressions into equation (6.89) on the page 281, we obtain

$$\frac{(\ddot{a}R)'}{(aR)'} + 2\frac{\ddot{a}}{a} = -\frac{4\pi G \varrho_0\left(R\right)}{a^2 \left(aR\right)'}, \tag{4}$$

which can be rewritten as

$$\left(aR\right)^2 (\ddot{a}R)' + \left(\left(aR\right)^2\right)' (\ddot{a}R) = -4\pi G \varrho_0 R^2. \tag{5}$$

Integrating this equation over R resutls to

$$\ddot{a} = -\frac{4\pi G \bar{\varrho}\left(R\right)}{3a^2}, \tag{6}$$

where

$$\bar{\varrho}\left(R\right) = \frac{3\int_0^R \varrho_0\left(\tilde{R}\right)\tilde{R}^2 d\tilde{R}}{R^3},$$

is the comoving density averaged over the sphere of radius R. Multiplying equation above by $\dot{a}$, we easily derive its first integral

$$\dot{a}^2(R,t) - \frac{8\pi G\bar{\varrho}(R)}{3a(R,t)} = F(R), \tag{7}$$

where $F(R)$ is a constant of integration. Note that for a homogeneous matter distribution $\bar{\varrho}$, a and F do not depend on R and equation (7) coincides with the Friedmann equation for a matter-dominated universe.

Problem 6.8. Verify that the solution of equation (7) can be written in the following parametric form:

$$a(R,\eta) = \frac{4\pi G\bar{\varrho}}{3|F|}(1-\cos\eta), \quad t(R,\eta) = \frac{4\pi G\bar{\varrho}}{3|F|^{3/2}}(\eta - \sin\eta) + t_0(R) \text{ for } F<0, \tag{8}$$

$$a(R,\eta) = \frac{4\pi G\bar{\varrho}}{3F}(\cosh\eta - 1), \quad t(R,\eta) = \frac{4\pi G\bar{\varrho}}{3F^{3/2}}(\sinh\eta - \eta) + t_0(R) \text{ for } F>0, \tag{9}$$

where $t_0(R)$ is a further integration constant. Note that the same "conformal time" η generally corresponds to different values of physical time t for different R. Assuming that the initial singularity ($a \to 0$) occurs at the same moment of physical time $t=0$ everywhere in space, we can set $t_0(R) = 0$.

Let us consider the evolution of a spherically symmetric overdense region in a flat, matter-dominated universe. Far away from the center of this region the matter remains undisturbed and hence $\bar{\varrho} = \varrho_0(R \to \infty) \to \varrho_\infty = const$. The condition of flatness requires $F \to 0$ as $R \to \infty$. Taking the limit $|F| \to 0$ so that the ratio $\eta/\sqrt{|F|}$ remains fixed, we immediately obtain from (8)

$$a(R \to \infty, t) = (6\pi G\varrho_\infty)^{1/3} t^{2/3}. \tag{10}$$

The energy density is consequently

$$\varepsilon(R \to \infty, t) = \frac{\varrho_0}{a^2(aR)'} = \frac{\varrho_\infty}{a^3} = \frac{1}{6\pi G t^2}, \tag{11}$$

in complete agreement with what one would expect for a flat dust-dominated universe. Inside the overdense region, F is negative and the energy density does not continually decrease. At the center of the cloud $\bar{\varrho} = \varrho_0$ and $a' = 0$. Because in this case $\varepsilon \propto a^{-3}$, the density takes its minimal value ε_m when $a(R=0,t)$ reaches its maximal value $a_m = 8\pi G\varrho_0/3|F|$ at $\eta = \pi$ (see (8)). This happens at the moment of physical time

$$t_m = \frac{4\pi^2 G\varrho_0}{3|F|^{3/2}}, \tag{12}$$

when the energy density is equal to

$$\varepsilon_m\left(R=0\right)=\frac{\varrho_0}{a_m^3}=\frac{27\left|F\right|^3}{\left(8\pi G\right)^3\varrho_0^2}=\frac{3\pi}{32Gt_m^2}. \tag{13}$$

Comparing this result with the averaged density at $t=t_m$, given by (11), we find that when the energy density in the center of the overdense region exceeds the averaged density by a factor of

$$\frac{\varepsilon_m}{\varepsilon\left(R\to\infty\right)}=\frac{9\pi^2}{16}\simeq 5.55, \tag{14}$$

the matter there detaches from the Hubble flow and begins to collapse.

Formally the energy density becomes infinite at $t=2t_m$; in reality, however, this does not happen because there always exist deviations from exact spherical symmetry. As a result a spherical cloud of particles virializes and forms a stationary spherical object.

Problem 6.9. Consider a homogeneous spherical cloud of particles at rest and, using the virial theorem, verify that after virialization its size is halved. Assuming that virialization is completed at $t=2t_m$, compare the density inside the cloud with the average density in the universe at this time. (*Hint*: The virial theorem states that at equilibrium, $U=-2K$, where U and K are the total potential and kinetic energies respectively.)

Problem 6.10. Assuming that $\eta\ll 1$ and expanding the expressions in (8) in powers of η, derive the following expansion for the energy density in the center of the spherical region in powers of $\left(t/t_m\right)^{2/3}\ll 1$:

$$\varepsilon=\frac{1}{6\pi t^2}\left(1+\frac{3}{20}\left(\frac{6\pi t}{t_m}\right)^{2/3}+O\left(\left(\frac{t}{t_m}\right)^{4/3}\right)\right), \tag{15}$$

where t_m is defined in (12). The second term inside the brackets is obviously the amplitude of the linear perturbation δ. Thus, when the actual density exceeds the averaged density by a factor of 5.5, according to the linearized theory $\delta\left(t_m\right)=3\left(6\pi\right)^{2/3}/20\simeq 1.06$. Later on, at $t=2t_m$, the Tolman solution formally gives $\varepsilon\to\infty$, while the linear perturbation theory predicts $\delta\left(2t_m\right)\simeq 1.69$.