

Protection of helical transport in Quantum Spin Hall samples: the rôle of symmetries on edges

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Time-reversal invariant two-dimensional topological insulators, often dubbed Quantum Spin Hall systems, possess helical edge modes whose ballistic transport is protected by physical symmetries. We argue that, though the time-reversal symmetry (TRS) of the bulk is needed for the existence of helicity, protection of the helical transport is actually provided by the spin conservation on the edges. This general statement is illustrated by some specific examples confirming the importance of the spin conservation. One of these examples demonstrates the ballistic conductance in the setup where the TRS on the edge is broken. It shows that attributing the symmetry protection exclusively to the TRS is not entirely correct. Analysis of the spin conservation may be important for understanding transport properties of the QSH samples which demonstrate a sub-ballistic conductance.

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The search of materials which may be used as a platform for realizing protected states has acquired nowadays the great importance because of potential applications in nanoelectronics, spintronics and quantum computers. A particular attention is drawn to systems whose transport properties are virtually not liable to effects of material imperfections, including backscattering and localization. This could provide a possibility to sustain the ballistic transport in relatively long samples.

One possibility for such a protection in one-dimensional (1D) conductors is provided by helicity. The latter is defined as a product of signs of the particle spin and chirality (denoting the direction of motion): $\eta = \text{sign}(\sigma_z)\text{sign}(\mu)$. Modes of the given helicity possess lock-in relation between their spin and momentum: electrons propagating in opposite directions have opposite spins. In the idealized case, this locking protects transport against potential disorder.

Helical transport originates on edges of time-reversal invariant two-dimensional topological insulators, the so-called Quantum Spin Hall (QSH) samples [1–3] which attract an enormous and continuous attention of experimentalists [4–24] and theoreticians [25–47] during past decade. The edge helicity requires the topologically non-trivial and time-reversal invariant state of the insulating bulk. That is why the protection is often referred to as the topological protection. This is a kind of historical term which should not be taken literally. There are example where the ballistic transport can be suppressed by a local perturbation on the edge, e.g. anisotropic spinfull impurities, which have no influence on the topologically nontrivial bulk. To understand the origin of protection, one should consider not only the state of the bulk but also all physical symmetries of the system, including the symmetries of the edge. In the simplest non-interacting case, helicity of the edge modes is related to the Kramers degeneracy which requires the time-reversal symmetry (TRS). In many papers, starting from classical reviews [1]

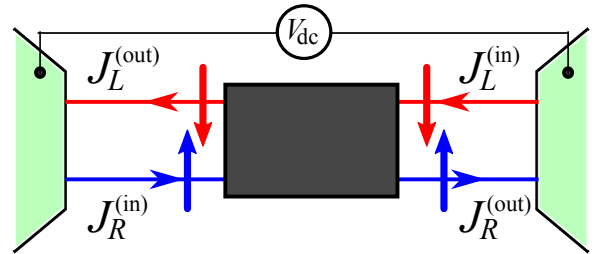


FIG. 1. (color on-line) Illustration of a generic two terminal setup: a wire with two modes of the same helicity (red and blue lines) is connected to 2D leads (colored trapezia). The central part of the wire includes "a black box" - an arbitrary conducting region which conserves charge and spin. The dc voltage, V_{dc} , is applied to outer ends of the helical wire.

ending by very recent papers [24, 48], the protection is attributed exclusively to the TRS and other physical symmetries are somehow forgotten or, at least, not present in the common terminology, though their importance is acknowledged by some members of the community. Note that, in the above mentioned example of the anisotropic spinfull impurities, the ballistic transport is suppressed though the TRS remains unbroken.

The goal of this Paper is to emphasize the rôle of another symmetry on the edges of the QSH sample. We will argue that, though the TRS symmetry of the insulating bulk is necessary for realization of the QSH samples, the necessary and sufficient condition for the symmetry protection of the helical transport is the spin conservation, i.e. the spin $U(1)$ symmetry, on the edge. Paradoxically, the TRS may be of the secondary importance. To illustrate this statement, we will explore an example of the helical fermions coupled to an array of localized spins. We will demonstrate that the helical transport is protected by the spin conservation even though the TRS on the edge can be broken by a spin ordering. Importance

of the spin conservation in this system was noticed in Refs.[32, 35] where a renormalized Drude peak in the dc conductivity was found for the simplest spin-conserving setup. Based on a straightforward though a kind of superficial analogy with the physics of usual (not helical) interacting 1D wires [49–51], a ballistic nature of the dc conductance was conjectured. Our analysis demonstrates that this conjecture is correct.

Let us start from a simple elucidating proof of importance of the spin conservation. We focus only on the edge modes of the QSH sample and explore the dc conductance of the 1D helical wire attached to two leads. The Hamiltonian describing clean helical 1D Dirac fermions reads as:

$$\hat{H}_0 = -iv_F \int dx \hat{\psi}^\dagger(x) \hat{\sigma}_z \partial_x \hat{\psi}(x). \quad (1)$$

Here v_F is the Fermi velocity; $\hat{\psi} = \{\psi_+, \psi_-\}^T$ is the spinor constructed from the helical fermionic fields; $\hat{\sigma}_{x,y,z}$ denote the Pauli matrices operating in the spin space. A natural direction of the spin quantization is determined by the spin-orbit interaction (SOI) which governs helicity. We assume that it is directed along z-axis. We consider the wire whose left- and right parts are clean but the middle part contains “a black box” - a conducting region whose only known property is the absence of the spin and charge relaxation, see Fig.1.

The spin and charge conservation in the black box result in two equations:

$$\text{Charge conservation: } J_R^{(\text{in})} + J_L^{(\text{in})} = J_R^{(\text{out})} + J_L^{(\text{out})}; \quad (2)$$

$$\text{Spin conservation: } J_R^{(\text{in})} - J_L^{(\text{in})} = J_R^{(\text{out})} - J_L^{(\text{out})}. \quad (3)$$

Fermionic currents $J_\mu^{(\text{in}/\text{out})}$ are explained in Fig.1. It follows from Eqs.(2,3) that

$$J_\mu^{(\text{in})} = J_\mu^{(\text{out})}, \quad \mu = R (\equiv +), L (\equiv -); \quad (4)$$

i.e. the current through the entire system is ideal and the conductance of the helical wire with the inclusion of the charge/spin conserving black box is ballistic. Internal details of the black box, including the temperature, the presence or absence of the TRS and interactions, etc., do not matter. We emphasize that these properties are the direct consequence of helicity and, therefore, the physics is very distinct from that of usual (non-helical) 1D wires [49–51] where the number of currents is larger than that of conservation laws.

Now we will give two specific examples which illustrate the above generic statements.

1. Rotating SOI: ballistic conductance and hidden spin conservation. Consider at first the black box where there are no spinfull impurities but the direction of the spin quantization is locally rotated. This rotation could result from the inhomogeneously changed direction of the SOI inside the box. The Hamiltonian of this system can be written as

$$\hat{H}_{\text{rot}} = -\frac{i}{2} v_F \int dx \hat{\psi}^\dagger(x) \{(\mathbf{n} \cdot \boldsymbol{\sigma}), \partial_x\}_+ \hat{\psi}(x). \quad (5)$$

The coordinate-dependent unit vector $\mathbf{n} = \{\sin(\theta) \cos(\varphi), \sin(\theta) \sin(\varphi), \cos(\theta)\}$ corresponds to the direction of the SOI. Here θ and φ are the polar and azimuthal angles. The boundary condition is $\theta = 0 \Rightarrow \mathbf{n} = \{0, 0, 1\}$ in the helical wires which connect leads and the black box. Inside the box, the direction of \mathbf{n} is not fixed and angles θ and φ slowly (on the scale of the inverse Fermi momentum $1/k_F$) change in space. The anticommutator $\{\dots\}_+$ is needed for the Hermiticity of \hat{H}_{rot} .

\hat{H}_{rot} is time-reversal invariant but it does not commute with any spin operators and, therefore, the spin quantization axis cannot be globally defined in the basis of $\hat{\psi}$ -fermions. Nevertheless, the dc transport remains ballistic which can be erroneously understood as the TRS protection of transport in the absence of the spin conservation [48]. Such a conclusion, if correct, would contradict to our main statement concerning the role of the spin conservation. In reality, one can find the spin conservation which protects transport. This was reported in Ref.[52], where the interacting helical edge with a random SOI was explored by using the transfer matrix technique and the Luttinger liquid theory. The ballistic nature of transport in such a system was subsequently questioned and checked[39]. Let us show how one can trivially reduce the Hamiltonian (5) to the form which, after using Eqs.(2,3), straightforwardly explains robustness of the ballistic transport even in the presence of the spin-preserving interaction.

To uncover the spin conservation which protects transport, we simply rotate the fermionic basis. Firstly we note that

$$\hat{g}^\dagger (\mathbf{n} \cdot \boldsymbol{\sigma}) \hat{g} = \hat{\sigma}_3, \quad \hat{g} = e^{-i\frac{\varphi}{2} \hat{\sigma}_3} e^{-i\frac{\theta}{2} \hat{\sigma}_2}. \quad (6)$$

Thus, the unitary rotation of the fermionic basis by the matrix \hat{g} diagonalizes \hat{H}_{rot} . Note that $\hat{g} = 1$ outside the black box because of the boundary conditions for \mathbf{n} . Changing to the basis $\hat{\Psi} = \hat{g}^\dagger \hat{\psi}$, we find

$$\hat{H}_{\text{rot}} = -v_F \int dx \hat{\Psi}^\dagger \left\{ i\hat{\sigma}_3 \partial_x + \frac{1}{2} \cos(\theta) [\partial_x \varphi] \right\} \hat{\Psi}; \quad (7)$$

see algebraic details in Suppl.Mat.A. The spin of the fermions is manifestly conserved in the $\hat{\Psi}$ -basis. There is no backscattering in Eq.(7). Thus, conservation laws (2,3) hold true and the dc transport remains ballistic, as expected. Clearly, the spin preserving density-density electron interactions have no influence on our approach and cannot change the ballistic nature of the helical transport.

2. Helical modes interacting with a Kondo-Heisenberg array: ballistic conductance and broken TRS. Consider the helical edge modes which interact with the array of local magnetic moments – Kondo Spins (KSs), see Fig.2. The total Hamiltonian, $\hat{H} = \hat{H}_0 + \hat{H}_b + \hat{H}_H + \hat{H}_V$, includes the fermionic and spin parts, \hat{H}_0 and \hat{H}_H , respectively, and the voltage source part \hat{H}_V which depends only on the fermionic operators. \hat{H}_b is the Hamiltonian of the

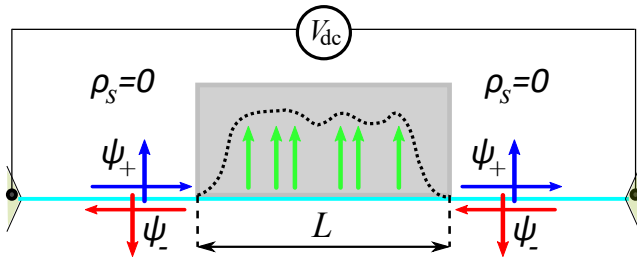


FIG. 2. (color on-line) 1D wire with helical electrons, ψ_{\pm} , coupled to the Kondo-Heisenberg array (green arrows) located inside the black box (shadowed rectangle). Density of the local spins (dotted line) vanishes outside the black box and fluctuates in space inside it. The interaction constants, $J_{K,H}$ can also depend on the coordinate.

fermion-spin interaction which originates from backscattering induced by KSs. In the spin-conserved setup, it reads as follows:

$$\hat{H}_b = \sum_j \hat{S}_j^{\pm} [J_K \psi_{\pm}^{\dagger} \psi_{\pm}] (x_j) + h.c. \quad (8)$$

Here the sum runs over KS positions x_j , J_K is the position dependent coupling constant between the KSs and the fermions; $\hat{S}_j^{\pm} = e^{\pm 2ik_F x_j} (\hat{S}_x \pm i\hat{S}_y)_j$ are rotated raising/lowering operators of the KSs. Eq.(8) corresponds to the XXZ-coupling with $J_K = J_K^{(x)} = J_K^{(y)}$. To simplify discussion, we take into account neither electron-electron interactions nor forward-scattering generated by the component $J_K^{(z)}$ though including them is straightforward and does not change our conclusions.

The spin Hamiltonian \hat{H}_H describes the direct Heisenberg exchange interaction between the z-components of the KSs:

$$\hat{H}_H = \sum_j J_H(x_j) \hat{S}_z(j) \hat{S}_z(j+1); \quad (9)$$

where J_H is the position dependent coupling constant. For concreteness, we have chosen the exchange interaction between nearest neighbors. The interspin distance $\xi_s(j) = x_{j+1} - x_j$ can fluctuate in space if the spin array is geometrically disordered.

The spin density is $\rho_s^{(d)}(x) = \sum_j \delta(x - x_j)$ for the discrete array. Its smeared counterpart, $\rho_s(x)$, vanishes outside the black box and is finite and coordinate-dependent inside it.

Our approach is similar to that suggested in Refs.[49, 53]: We express the electric current via a convolution of the non-local conductivity, $\sigma(x, x'; \omega)$, and an inhomogeneous electric field $E(\omega, x')$:

$$j(x, t) = \int dx' \int \frac{d\omega}{2\pi} e^{-i\omega t} \sigma(x, x'; \omega) E(\omega, x'). \quad (10)$$

$E(\omega, x)$ is governed by the applied voltage. Next, we bosonize the theory, use the technique of the functional integrals on the imaginary time, and describe the

fermions by the standard Lagrangian of the helical Luttinger Liquid with the source term [26, 54]:

$$\mathcal{L}_{\text{HLL}} = [(\partial_{\tau}\phi)^2 + (v_F\partial_x\phi)^2] / 2\pi v_F + i\chi\phi. \quad (11)$$

The Fourier transform of the bosonic Green's function (GF), $G(x, x'; \tau) = -\langle \phi(x', \tau) \phi(x, 0) \rangle$, yields the Matsubara conductivity: $\sigma(x, x'; \bar{\omega}) = (e^2 \bar{\omega} / \pi) G(x, x'; \bar{\omega})$. Following Refs.[32, 35], we parameterize each Kondo spin by its azimuthal angle, α , and projection on z-axis, $|n_z| \leq 1$: $S^{\pm} = s e^{i\alpha} \sqrt{1 - n_z^2}$, $S_z = s n_z$; with s being the spin value. This approach requires the Wess-Zumino term in the Lagrangian [55, 56]:

$$\mathcal{L}_{\text{WZ}}[n_z, \alpha] = i s \rho_s (1 - n_z) \partial_{\tau} \alpha. \quad (12)$$

The Lagrangian describing the spin-conserving backscattering reads:

$$\mathcal{L}_b[n_z, \alpha, \phi] = 2s \rho_s(x) J_K(x) \sqrt{1 - n_z^2} \cos(\alpha - 2\phi). \quad (13)$$

Let us now shift the spin phase $\tilde{\alpha} = \alpha - 2\phi$. The full Lagrangian in the new variables is

$$\mathcal{L} = \mathcal{L}_{\text{HLL}}[\phi, \chi] + \mathcal{L}_{\text{WZ}}[n_z, 2\phi] + \mathcal{L}_{\text{KS}}[n_z, \tilde{\alpha}]. \quad (14)$$

Here $\mathcal{L}_{\text{KS}} \equiv \mathcal{L}_b[n_z, \tilde{\alpha}, 0] + \mathcal{L}_H[n_z] + \mathcal{L}_{\text{WZ}}[n_z, \tilde{\alpha}]$ is the spin Lagrangian; \mathcal{L}_H describes the Heisenberg interaction of the KSs.

In these new variables, the coupling between the fermionic and KS sectors is reflected only by $\mathcal{L}_{\text{WZ}}[n_z, 2\phi]$. Its contribution to the low-energy theory effectively vanishes and the KS variables drop out from the equation for the dc conductivity which reduces to that of the clean helical wire, see the proof in Suppl.Mat.B. Therefore, coupling between the helical modes and the spin-preserving Kondo-Heisenberg array is unable to change the helical dc conductance which remains ballistic.

A physical explanation for the robustness of the helical transport at $J_H \ll J_K$ can be found by exploiting an analogy with a single spin-1/2 Kondo impurity immersed in the helical wire [57]: Each spin-preserving backscattering is accompanied by the spin-flip. Therefore, the Kondo spin-1/2 is able to support only successive backscatterings of the helical fermions with alternating chirality: $\psi_+ \rightarrow \psi_-$, $\psi_- \rightarrow \psi_+$, ... This successive backscattering does not change the dc conductance. The KS array can be qualitatively considered as a single collective impurity. Its spin \mathcal{S} is large but finite. The helical fermions can transfer their spins to the collective impurity only during the time $\mathcal{T} \sim \mathcal{S} \tau_s$, where $1/\tau_s$ is the spin-flip rate for the individual KS. If the total spin is conserved and the observation time is large, $\gg \mathcal{T}$, the successive nature of backscatterings is restored and the KS array cannot change the dc conductance of the helical wire.

Direct and indirect (RKKY) exchange interactions between the KSs compete with each other [58]. If the Kondo-Heisenberg array is dense and large enough, such a competition leads to an exotic spin order on the edge of the QSH sample [59]. If $J_H \ll J_K$, one comes across

a nematic (or vector chiral) spin order. If J_H exceeds some critical value, J_H^* , the quantum phase transition of the Ising type occurs at zero temperature and the nematic spin order becomes the scalar chiral spin order. The component S_z acquires a non-zero semiclassical average value, $s\langle n_z \rangle$. This means that the TRS can be spontaneously broken on the QSH edge. Nevertheless, the helical dc transport remains ballistic and protected as soon as the total spin is conserved. These arguments show the secondary importance of the TRS on the edge for the protection of the helical transport. Note that the fermion backscattering by the magnetically ordered Kondo-Heisenberg array is suppressed because it requires an energy of the order of the energy of the domain wall, $\mathcal{E}_{\text{DW}} \sim [(J_H - J_H^*)/J_K]^2 \xi_s/v_F$ close to the critical region, see details in Ref.[59], and $\mathcal{E}_{\text{DW}} \sim J_H$ at $J_H \gg J_K$. Thus, the ballistic nature of transport is extended to finite frequencies which are much smaller than \mathcal{E}_{DW} .

General arguments underlying Eqs.(2,3) suggest that the spin-conserving in-plane Heisenberg interaction of the KSSs, $\mathcal{L}_H^{(xy)} \propto \rho_s e^{-2ik_F \xi_s} S^+(x+\xi_s)S^-(x) + h.c.$, is also unable to suppress the ballistic helical dc transport. However, it is not simple to trace microscopic details of realization of that generic statement because the phase ϕ cannot be gauged out simultaneously from the Lagrangians \mathcal{L}_b and $\mathcal{L}_H^{(xy)}$. Detailed study of this case is beyond the scope of the present paper. Qualitative arguments in favor of the helical ballistic transport are given in Ref.[59] where the properties of the helical edge coupled to a large SU(2)-symmetric Kondo-Heisenberg array are considered in the continuous limit.

We emphasize that the dc helical transport is sensitive neither to the profile of the spin density, $\rho_s(x)$, nor to the spatial inhomogeneity of the coupling constants, $J_{K,H}(x)$. We conclude that, unlike the case of the usual (non-helical) wires [49–51], properties of the transition region between the clean helical wire and the black box do not matter. This is the essential difference between these two cases. We note that such a difference does not appear in the ac conductance which might be sensitive to details of the setup in the helical and non-helical systems.

Broken spin conservation: If the total spin in the black box is not conserved, transport is not protected any longer and the ballistic dc conductance becomes suppressed. For instance: (i) Coupling between the helical fermions and the infinitely long XY-anisotropic Kondo array, $J_H = 0, L \rightarrow \infty$, has been explored in Refs.[32, 35]. A constant XY-anisotropy opens a gap in the spectrum of

the helical modes while a random XY-anisotropy causes their Anderson localization. Moreover, even a single XY-anisotropic Kondo impurity has a pronounced destructive effect on the ballistic dc transport [60]. In both cases, the anisotropy impedes the helical dc transport and leads to the vanishing dc conductance at low temperatures. (ii) Electron interactions could generate two-particle umklapp backscattering which is consistent with the TRS but does not preserve the spin and, as a result, can suppress the ballistic helical transport [26, 61].

Conclusions: We have demonstrated that, though the time-reversal symmetry of the bulk is the necessary condition for formation of helical edge modes in 2D topological insulators, the necessary and sufficient condition for protection of the 1D helical transport is the total spin (and, obviously, charge) conservation on the edges. This conservation law makes the dc helical transport insensitive to disorder, properties of contacts and even to breaking the TRS on the edges. The generic result [Eqs.(2,3) and their discussion] has been exemplified by the microscopic analysis of two nontrivial examples: 1) the Quantum Spin Hall sample with inhomogeneous spin-orbit interaction, and 2) the helical edge coupled to the Kondo-Heisenberg array of localized spins. An Ising-like spin order, which breaks the TRS on the edge, can appear in the latter example. If the total spin is conserved, this broken TRS has no influence on the dc ballistic helical transport. Moreover, a single-electron backscattering in Ising-ordered spin arrays is suppressed and, therefore, transport can remain ballistic even at finite frequencies.

Analysis of the spin conservation is a powerful tool which allows one to identify mechanisms leading to suppression of the ballistic helical transport and to rule out irrelevant ones. Thus, our approach may be important for understanding transport properties of the QSH samples which demonstrate a sub-ballistic conductance.

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Supplemental Materials for the paper

“Symmetry protection of helical transport on edges of Quantum Spin Hall samples”

by O. M. Yevtushenko and V.I. Yudson

Suppl.Mat. A: Rotation of spinors

Consider the Hamiltonian Eq.(5):

$$\hat{H}_{\text{rot}} = -\frac{i}{2}v_F \int dx \hat{\psi}^\dagger(x) \{(\mathbf{n} \cdot \boldsymbol{\sigma}), \partial_x\}_+ \hat{\psi}(x) = -iv_F \int dx \hat{\psi}^\dagger(x) \left[(\mathbf{n} \cdot \boldsymbol{\sigma})\partial_x + \frac{1}{2}[\partial_x(\mathbf{n} \cdot \boldsymbol{\sigma})] \right] \hat{\psi}(x). \quad (\text{A1})$$

Here $\mathbf{n} = \{\sin(\theta) \cos(\varphi), \sin(\theta) \sin(\varphi), \cos(\theta)\}$ is the unit vector. Let us introduce a unitary matrix

$$\hat{g} = \exp\left[-i\frac{\varphi}{2}\hat{\sigma}_3\right] \exp\left[-i\frac{\theta}{2}\hat{\sigma}_2\right], \quad (\text{A2})$$

which has the property

$$\hat{g}^\dagger(\mathbf{n} \cdot \boldsymbol{\sigma})\hat{g} = \hat{\sigma}_3 \Leftrightarrow \hat{g}\hat{\sigma}_3\hat{g}^\dagger = (\mathbf{n} \cdot \boldsymbol{\sigma}). \quad (\text{A3})$$

Changing to a new spinor $\hat{\Psi} = \hat{g}^\dagger\hat{\psi}$, we obtain the new Hamiltonian in the form

$$\hat{H}_{\text{rot}} = -iv_F \int dx \hat{\Psi}^\dagger(x)\hat{g}^\dagger \left[(\mathbf{n} \cdot \boldsymbol{\sigma})\partial_x + \frac{1}{2}[\partial_x(\mathbf{n} \cdot \boldsymbol{\sigma})] \right] \hat{g}\hat{\Psi}(x) = -iv_F \int dx \hat{\Psi}^\dagger(x) \left[\hat{\sigma}_3\partial_x + \hat{h} \right] \hat{\Psi}(x), \quad (\text{A4})$$

where

$$\hat{h} = \hat{g}^\dagger(\mathbf{n} \cdot \boldsymbol{\sigma})\hat{g}\hat{g}^\dagger\partial_x\hat{g} + \frac{1}{2}\hat{\sigma}_3(\partial_x\hat{g}^\dagger)\hat{g} + \frac{1}{2}\hat{g}^\dagger\partial_x\hat{g}\hat{\sigma}_3 = \frac{1}{2} [\hat{\sigma}_3\hat{g}^\dagger\partial_x\hat{g} + \hat{g}^\dagger\partial_x\hat{g}\hat{\sigma}_3]. \quad (\text{A5})$$

We have used Eq.(A3) to obtain Eq.(A5). The spatial derivative of the matrix $\hat{g}(x)$ reads as

$$\partial_x\hat{g} = -i\frac{1}{2} [\hat{\sigma}_3\hat{g}\partial_x\varphi + \hat{g}\hat{\sigma}_2\partial_x\theta]. \quad (\text{A6})$$

Inserting this expression in Eq.(A5), we find

$$\hat{h} = -\frac{i}{4} [\hat{\sigma}_3\hat{g}^\dagger\hat{\sigma}_3\hat{g} + \hat{g}^\dagger\hat{g}\hat{\sigma}_3] \partial_x\varphi = -\frac{i}{4}\hat{g}^\dagger [\hat{g}\hat{\sigma}_3\hat{g}^\dagger\hat{\sigma}_3 + \hat{\sigma}_3\hat{g}\hat{\sigma}_3\hat{g}^\dagger] \hat{g}\partial_x\varphi. \quad (\text{A7})$$

Using Eq.(A3), we simplify Eq.(A7):

$$\hat{h} = -\frac{i}{4}[(\mathbf{n} \cdot \boldsymbol{\sigma})\hat{\sigma}_3 + \hat{\sigma}_3(\mathbf{n} \cdot \boldsymbol{\sigma})]\partial_x\varphi = -\frac{i}{2} \cos(\theta)\partial_x\varphi. \quad (\text{A8})$$

Thus, the Hamiltonian in the rotated frame takes the form

$$\hat{H}_{\text{rot}} = -v_F \int dx \hat{\Psi}^\dagger(x) \left[i\hat{\sigma}_3\partial_x + \frac{1}{2} \cos(\theta)\partial_x\varphi \right] \hat{\Psi}(x), \quad (\text{A9})$$

see Eq.(7) in the main text.

Suppl.Mat. B: Helical wire coupled to a Kondo-Heisenberg array

Non-local Matsubara conductivity of a helical wire can be expressed in terms of the Green's function (GF) of bosonized excitations [54]:

$$\sigma(x, x'; \bar{\omega}) = (e^2\bar{\omega}/\pi) G(x, x'; \bar{\omega}). \quad (\text{B1})$$

If the wire is coupled to localized spins (a Kondo-Heisenberg array), the generating functional, $Z[\chi]$, for this GF reads as follows:

$$Z[\chi] = \frac{1}{Z[0]} \int \mathcal{D}\{n_z \tilde{\alpha}\} \exp\left(-S_{\text{spin}}[n_z, \tilde{\alpha}]\right) \int \mathcal{D}\{\phi\} \exp\left(-S_{\text{HLL}}[\phi, \chi] - S_{\text{WZ}}[n_z, 2\phi]\right); \quad (\text{B2})$$

$$G(x_1, x_2; \tau_1 - \tau_2) = \frac{\delta^2 Z[\chi]}{\delta\chi(\zeta_1)\delta\chi(\zeta_2)} \Big|_{\chi \rightarrow 0}, \quad \zeta_{1,2} \equiv \{x_{1,2}, \tau_{1,2}\}. \quad (\text{B3})$$

Here $Z[0]$ is the partition function; actions $S_{\text{HLL}, \text{WZ}, \text{KS}}$ correspond to the Lagrangians $\mathcal{L}_{\text{HLL}, \text{WZ}, \text{KS}}$, see Sect. "Helical modes interacting with the Kondo-Heisenberg array" in the main text:

$$S_{\text{HLL}} = \int d\zeta \left\{ \frac{1}{2\pi v_F} [(\partial_\tau \phi)^2 + (v_F \partial_x \phi)^2] + i\chi \phi \right\}; \quad (\text{B4})$$

$$S_{\text{WZ}}[n_z, 2\phi] = 2is \int d\zeta \rho_s (1 - n_z) \partial_\tau \phi = 2is \int d\zeta \rho_s \phi \partial_\tau n_z; \quad (\text{B5})$$

$$S_{\text{KS}} = \int d\zeta \rho_s(x) \left(2s J_K(x) \sqrt{1 - n_z^2} \cos(\tilde{\alpha}) + s^2 J_H(x) n_z(x, \tau) n_z(x + \xi_s, \tau) \right) + S_{\text{WZ}}[n_z, \tilde{\alpha}]. \quad (\text{B6})$$

The Gaussian integral over ϕ in Eq.(B2) can be calculated straightforwardly:

$$\frac{\int \mathcal{D}\{\phi\} \exp\left(-S_0[\phi, \chi] + S_{\text{WZ}}[n_z, 2\phi]\right)}{\int \mathcal{D}\{\phi\} \exp\left(-S_0[\phi, \chi = 0]\right)} = \exp\left(-S_{\text{zz}} + \frac{1}{2} \int d\zeta d\zeta' \left[\chi(\zeta) G_0(\zeta - \zeta', \omega) \chi^*(\zeta') + 2s \chi(\zeta) G_0(\zeta - \zeta') \rho_s(x') \partial_{\tau'} n_z(\zeta') \right]\right); \quad (\text{B7})$$

with

$$S_{\text{zz}} \equiv - \int d\zeta d\zeta' \left[2s^2 \rho_s(x) \partial_\tau n_z(\zeta) G_0(\zeta - \zeta') \rho_s(x') \partial_{\tau'} n_z(\zeta') \right]. \quad (\text{B8})$$

The variational derivative over the source field, Eq.(B3), yields:

$$G(x_1, x_2; \tau_1 - \tau_2) = G_0(\zeta_1 - \zeta_2) + s^2 \int d\zeta d\zeta' G_0(\zeta_1 - \zeta) \rho_s(x) [\partial_{\tau, \tau'}^2 \mathcal{N}(\zeta, \zeta')] \rho_s(x') G_0(\zeta' - \zeta_2); \quad (\text{B9})$$

where

$$\mathcal{N}(\zeta, \zeta') \equiv \langle \langle n_z(\zeta) n_z(\zeta') \rangle \rangle_{\text{spin}}; \quad (\text{B10})$$

G_0 is the bare bosonic GF of the clean wire, and $\langle \langle \mathcal{AB} \rangle \rangle \equiv \langle \mathcal{AB} \rangle - \langle \mathcal{A} \rangle \langle \mathcal{B} \rangle$. Decoupled part of \mathcal{N} does not contribute to Eq.(B9) because $\partial_\tau \langle n_z \rangle = 0$. The averaging in Eq.(B10) is performed over the full spin action $S_{\text{KS}} + S_{\text{zz}}$. Integrating by parts and using translational invariance of G_0 , we find:

$$G(x_1, x_2; \tau_1 - \tau_2) = G_0(\zeta_1 - \zeta_2) + s^2 \int d\zeta d\zeta' [\partial_{\tau_1} G_0(\zeta_1 - \zeta)] \rho_s(x) \mathcal{N}(\zeta, \zeta') \rho_s(x') [\partial_{\tau_2} G_0(\zeta' - \zeta_2)]. \quad (\text{B11})$$

If there is no exchange interaction between the itinerant electrons and the localized spins, $J_K = 0$, Eq.(B9) manifestly reproduces G_0 because $S_{\text{zz}} = \partial_{\tau, \tau'}^2 \mathcal{N}(\zeta, \zeta') = 0$. In the momentum-frequency representation, the expression for G_0 reads:

$$G_0(q, \bar{\omega}) = -\langle \phi^*(q, \bar{\omega}) \phi(q, \bar{\omega}) \rangle = -\frac{\pi v_F}{\bar{\omega}^2 + (v_F q)^2}. \quad (\text{B12})$$

We are interested in the dc response of the wire at zero temperature. Changing from the momentum to the coordinate, we obtain in the low-frequency limit:

$$|\bar{\omega}(x_1 - x_2)|/v_F \ll 1: \quad \bar{\omega} G_0(x_1 - x_2; \bar{\omega}) = -\frac{\pi}{2} \text{sign}(\bar{\omega}). \quad (\text{B13})$$

If $J_K \neq 0$, n_z acquires dynamics and $S_{zz} \neq 0$. The dc response requires only the low-energy sector of the entire theory where Eq.(B8) for S_{zz} can be simplified by using Eq.(B13):

$$S_{zz} = -2s^2 \int \frac{d\bar{\omega}}{2\pi} \int dx dx' \left[\bar{\omega}^2 \rho_s(x) n_z(x, \bar{\omega}) G_0(x - x', \bar{\omega}) \rho_s(x') n_z(x', -\bar{\omega}) \right] \simeq \int \frac{d\bar{\omega}}{2} |\bar{\omega}| \times \left| \int dx s \rho_s(x) n_z(x, \bar{\omega}) \right|^2. \quad (\text{B14})$$

Let us now Fourier-transform Eq.(B11) for the GF, analytically continue it to the upper half-plane to obtain the physical retarded correlation function, G^R , and simplify the product ωG^R in the dc limit:

$$\omega G^R(x_1, x_2; \omega) = \omega \left[G_0^R(x_1 - x_2; \omega) + \delta G^R(\omega) \right]. \quad (\text{B15})$$

We have to analyze the low frequency limit of the product

$$\omega \delta G^R(\omega) = \frac{\pi}{2} s^2 \omega \int dx_1 dx_2 \rho_s(x_1) \mathcal{N}^R(x_1, x_2; \omega) \rho_s(x_2), \quad (\text{B16})$$

which yields a correction to the nonlocal conductivity of the clean wire, see Eq.(B1). Low-energy properties of the Matsubara correlation function $\mathcal{N}(x_1, x_2; \bar{\omega})$ are governed mainly by the action S_{KS} , Eq.(B6). This is because S_{zz} yields vanishing contributions at small energies, see Eq.(B14). S_{KS} corresponds to the quantum 1D Ising model in a transverse (effective) magnetic field, $B_x \sim s \rho_s J_K$. The competition between the magnetic field and the Ising interaction results in formation of phases with different spins order: the spins become aligned with the magnetic field at $J_K \gg J_H$ while the Ising order dominates in the opposite case with $J_K \ll J_H$. n_z is gapped variable in both phases – see, e.g., the analysis of the continuous case in Ref.[59] and the exact solution for the spin-1/2 quantum Ising model in the textbook [56]. These two phases are separated by the point of the quantum phase transition. Far from criticality, one can estimate $\mathcal{N}^R(x_1, x_2; \omega \rightarrow 0) \sim 1/\Delta$, with Δ being the gap of spin excitation. Δ shrinks to zero at the transition. Nevertheless, $\mathcal{N}^R(x_1, x_2; \omega \rightarrow 0)$ remains finite due to the finite size L of the spin array. Close to criticality, the estimate becomes $\mathcal{N}^R(x_1, x_2; \omega \rightarrow 0) \sim 1/E_s^{(\min)}$, where $E_s^{(\min)}$ is the minimal energy of the spin excitations with the momentum of order $\sim 1/L$. Thus, we arrive at inequality

$$\omega \delta G^R(\omega) \lesssim (\mathcal{S}_{\text{tot}})^2 \frac{\omega}{\max[\Delta, E_s^{(\min)}]}. \quad (\text{B17})$$

Here $\mathcal{S}_{\text{tot}} \equiv \int dx s \rho_s(x)$ is the total spin of the array. Eq.(B17) shows that

$$\lim_{\omega \rightarrow 0} (\omega \delta G^R(\omega)) = 0 \quad (\text{B18})$$

for any finite spin conserving Kondo-Heisenberg array and, hence, the dc conductance of the helical wire coupled to such an array coincides with the ballistic conductance of the clean helical wire regardless of (i) properties of contacts between the wire and the region of localized spins, (ii) a spatial inhomogeneity or a spin disorder of the Kondo-Heisenberg array, etc.

If the spin array is infinite, the right-hand side of Eq.(B17) may diverge, our approach is not applicable any longer and the theory of Ref.[32] for the infinite Kondo array should be used instead. This theory predicts the dc conductivity (not conductance!) with the renormalized Drude weight.