Wetting & Logistic Equation
Hauptseminar: RG and Phase Transitions

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1 Wetting

We try to model a wetting - dewetting transition with a 1+1 dimensional model shown in (1). The liquid-gas interface is described by the following Hamiltonian:

$$\mathcal{H}[l_i] = \sum_{i=1}^{N} \frac{J}{2} \left(l_i - l_{i+1}\right)^2 + V(l_i)(1)$$

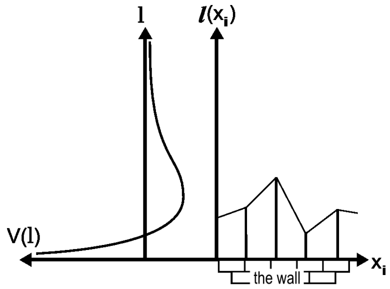

Figure 1: A discretized 1+1 dimensional model.

The notation of the partition sum $\mathcal{Z} = \int \Pi_{n=1}^{N} dl_n exp(-\mathcal{H}/\mathcal{T})$ can be simplified by the transfermatrix T

$$T(z,z') = exp\left[-\frac{1}{2}(z-z')^2 - \frac{1}{2}\left(U(z)+U(z')\right)\right] \tag{2}$$

where we have rescaled the variables l, l': $z = \sqrt{\beta J}$.
Now we integrate over every second lattice site. Additionally, the system has to be rescaled by a factor of $\sqrt{2}$ in the l or z-direction. In this way we get a new transfermatrix T':

$$T'(z,z'') = \int_{-\infty}^{\infty} dz' T(\sqrt{2}z, z') T(z', \sqrt{2}z'') \tag{3}$$

Since we are able to extract important thermodynamic quantities like critical exponents from the fixpoint of the RG flow we have to look for fixpoints of equation (3). A trivial fixpoint is the Gaussian fixpoint that belongs to the free interface without external wall potential $U(z)$:

$$T^*(z,z') = exp\left(-\frac{1}{2}(z-z')^2\right) \tag{4}$$

Since the wall potential $U(z)$ is the central object that gouverns the description of the phase we only consider the renormalization of the symmetrized potential $U(z,z') = \frac{1}{2}\left(U(z)+U(z')\right)$

$$exp[-U'(z,z'')] = \int_{0}^{\infty} \frac{dz'}{\sqrt{\pi}} exp\left[-\left(z' + \frac{z+z''}{\sqrt{2}}\right)^2 - U(z', \sqrt{2}z) - U(z', \sqrt{2}z'')\right] \tag{5}$$

Note that equations (3) and (5) are completely general. They lead to a line of fixpoint solutions. These fixpoint potentials are looking quite different from the bare potential $U^{(0)}(z,z') = \frac{1}{2}\left(U(z)+U(z')\right)$. Like in the one dimensional Ising model we can extract all thermodynamic properties from the eigenvalues of the

transfermatrix.
The first step torwards a spectral resolution of the transfermatrix T(z,z') will be to formulate everything in a more functional analysis like operator language. T(z,z') is like a matrix but with continuous indices. It maps a vector from a function space like L^2-the space of all square integrable functions to another vector of this space: $T\psi = \phi$

$$T\psi = \int_{-\infty}^{\infty} dz' T(z, z')\psi(z') = \phi(z) \tag{6}$$

We have already seen that integrating out every second degree of freedom involves a proper rescaling of the z-direction. Let's define a unitary dilation operator: $S_{\sqrt{2}}\psi(z) = \frac{1}{\sqrt{\sqrt{2}}}\psi\left(\frac{z}{\sqrt{2}}\right)$. Thus the fixpoint equation (3) can be written in the following form:

$$T = S_{\sqrt{2}}^{-1} T^2 S_{\sqrt{2}} \tag{7}$$

This is a nonlinear equation in order to reduce it to a linear one, we define the operator H:

$$T = exp[-H] \tag{8}$$

Equation (7) becomes

$$\frac{1}{2}H = S_{\sqrt{2}}^{-1} H S_{\sqrt{2}} \tag{9}$$

The following operator H solves this equation

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\rho}{x^2} \tag{10}$$

In order to apply the spectral theorem from functional analysis H has to be symmetric and bounded from below. These conditions restrict the possible choices for H. For example a term $\frac{1}{x}\partial_x$ would also satisfy equation (9) but it violates the symmetry condition.
We check the symmetry condition for our H:

$$(\psi, H\phi) = \int dx \psi(x) H\phi(x) = \int dx H\psi(x)\phi(x) = (H\psi, \phi) \tag{11}$$

In the second equality we used an integration by parts to get the derivative operator from the ϕ to the ψ. The spectral resolution of the potential part of the transfermatrix is:

$$exp[-\bar{U}(z^2)] = \sqrt{2\pi}z \cdot exp[-z^2]I_\nu(z^2) \tag{12}$$

The parameter ν can be related to the parameter ρ:

$$2\rho = \nu^2 - \frac{1}{4} \tag{13}$$

2 Logistic Equation

The logistic equation is a discrete nonlinear iterative equation:

$$x_{n+1} = \lambda \cdot x_n(1 - x_n) \tag{14}$$

This type of equation can appear under many circumstances. To be concrete, let's concider a the development of a population. Reproduction gives exponential growth with a reproduction rate of r. Then the population at one later time step is $q_r = 1 + r$ times bigger:

$$x_{n+1} = q_r \cdot x_n \tag{15}$$

In a realistic model there is only a limited amount of food. This limit in the food supply also limits the population size to a biggest amount P. As the population increases further and further it will feel more and more the limited food supply. Hence it may even decline by a factor which depends on the population size itself:

$$q_f = q_v \cdot (P - x_n) \tag{16}$$

Combining the reproduction rate (15) and the rate due to the limited amount of food (16) we get the logistic equation (14) with a single paramter $\lambda = q_r \cdot q_v \cdot P$ after rescaling of all variables. This equation has an interesting behaviour for different values from λ. As shown in figure (2) the logistic equation shows period doubling. Many nonlimear systems exhibit period doubling, for example the driven pendelum.

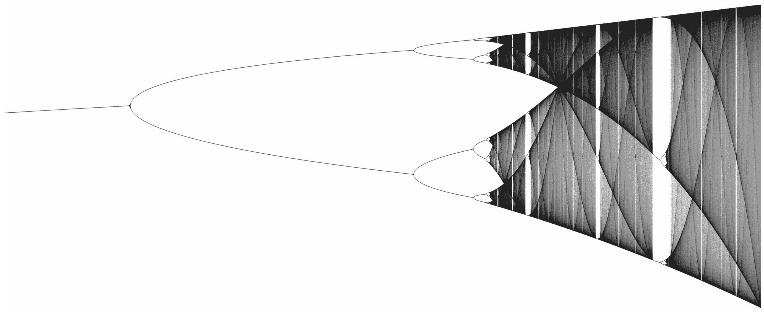

Figure 2: x-axis: the values of the parameter λ from ca. 3 to 4 is plotted. The y-axis shows the accumulation points of the sequence generated by the logistic mapping. One can easily notice that for $\lambda < 3$ the sequence converges to a single point. That means the orbit is of length one. At around $\lambda_2 = 3.449\ldots$ the period doubles and the sequence is going to alternate between two values. Now the orbit has length two. This period doubling goes on until the generated sequence is completely chaotic. At $\lambda > 4$ the sequence diverges, hence the diagram is not drawn beyond this value.

Interestingly, this behaviour vanishes when one considers the continuous limit. In that case we get the following differential equation which can be solved analytically:

$$f'(t) = h \cdot f(t) \cdot (P - f(t)) \tag{17}$$

The solution of this differential equation is a smooth curve which starts at 0 and converges to P. There is no oscillatory behaviour, etc..
The logistic equation has universal constants. The Feigenbaum constant δ is the ratio of the length of the interval where an orbit of length 2^{n-1} is observed through the length of the interval where an orbit of length 2^n is observed.

$$\frac{\Delta\lambda_{n-1}}{\Delta\lambda_n} = \delta = 4.6692... \tag{18}$$

We get another universal constant by looking on the difference between the starting point $x_0 = 1/2$ and the point after the half of the whole orbit of length 2^n is passed $x_{2^{n-1}} = f_{\lambda_n}^{2^{n-1}}(x_0)$. The constant α is:

$$\frac{\frac{1}{2} - f_{\lambda_n}^{2^{n-1}}(x_0 = \frac{1}{2})}{\frac{1}{2} - f_{\lambda_{n+1}}^{2^n}(x_0 = \frac{1}{2})} \to -\alpha = 2.502907875... \tag{19}$$

The convergence is geometrically with $n \to \infty$.
Feigenbaum discovered that the two exponents α and δ are universal for a large

class of systems and that this fact is related to the existence of a universal function $g(z)$ which satisfies

$$g(z) = -\alpha g(g(z/\alpha)) \tag{20}$$

The universality of the exponents follows because whenever g has a quadratic maximum, this equation determines α and δ uniquely.
In the following we want to empirically derive that equation for the logistic mapping. Let us go to the chaotic λ-parameter regime and generate the sequence of points in figure (3)

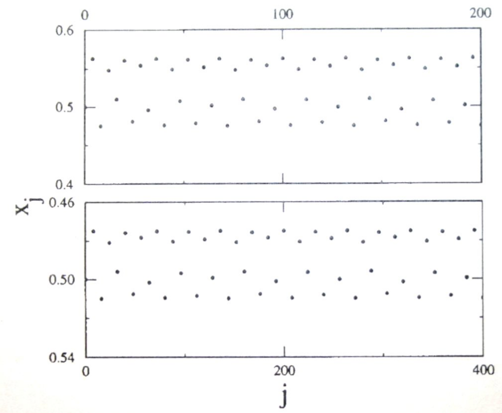

Fig. 1. Two plots of the time series of x values for the logistic map ($x_{j+1} = \lambda x_j(1-x_j)$) with the parameter value $\lambda = \lambda_\infty = 3.569\,945\,669...$, starting with $x_{j=0} = \frac{1}{2}$. Note that the axes have different scales in the lower panel than in the upper panel: the horizontal axis is compressed by a factor of 2, and the vertical axis is inverted and expanded by a factor of 2.5.

Figure 3: a sequence of points generated from $x_0 = 0.5$ after applying the logistic mapping in the parameter regime with an infinite period length. In the lower picture the y-axis is inverted at 0.5 and multiplied by α. The x-axis is multiplied by 2 and every second point is not shown.

At first we notice that the two sequences shown in the upper and lower picture of (3) have exactly the same pattern. We formulate this empirical observation into an equation and obtain:

$$-\alpha(x_{2k} - \frac{1}{2}) = x_k - \frac{1}{2} \tag{21}$$

We define new variables $z_k = x_k - \frac{1}{2}$. The equation becomes: $-\alpha z_{2k} = z_k$. Furthermore we define a function $g(z_k) = z_{k+1}$ which generates the sequence of points in figure(3).

$$-\alpha z_{2(k+1)} = z_{k+1} \tag{22}$$

$$-\alpha g(g(z_{2k})) = g(z_k)$$

$$\Rightarrow -\alpha g(g(-z_k/\alpha)) = g(z_k) \tag{23}$$

This is Feigenbaum's renormalization group equation. The method presented here is analogous to the decimation renormalization group for the one-dimensional Ising model.

References

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