Real-Space Renormalisation
Decimation & Blocking

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Abstract

Real-Space Renormalisation produces a growing number of new long-ranged interaction terms. Therefore only approximations to the exact Renormalisation Group Equations can be calculated. A systematic way to do that is to use the method of blocking on finite lattices.

Definitions

All the following will refer to *Ising Systems*, consisting of discrete spins σ on a lattice that can be oriented in two directions $\sigma = \pm 1$.

Decimation is the process of reducing the number of degrees of freedom.
This can happen in many ways, an example is

$$Z = \operatorname*{Tr}_{\sigma}^{N} \left\{ \mathrm{e}^{-\beta\mathcal{H}\{\sigma\}} \right\} = \operatorname*{Tr}_{\sigma'}^{N'} \operatorname*{Tr}_{\sigma''}^{N''} \left\{ \mathrm{e}^{-\beta\mathcal{H}\{\sigma',\sigma''\}} \right\} = \operatorname*{Tr}_{\sigma'}^{N'} \left\{ \mathrm{e}^{-\beta\mathcal{H}'\{\sigma'\}} \right\}$$

where Z is the partition sum, Tr is the trace operator for N sites σ, $\beta = (k_B T)^{-1}$ and $\mathcal{H}$ is the Hamiltonian. $\mathcal{H}'\{\sigma'\}$ shall have the same structure as the original Hamiltonian $\mathcal{H}\{\sigma\}$.

Blocking is the process of grouping spins to blocks.
This means that a certain number of spins σ of the original lattice are grouped together to a *block-spin* σ':

$$\mathrm{e}^{-\beta\mathcal{H}'\{\sigma'\}} = \operatorname*{Tr}_{\sigma}^{N} \left\{ P(\sigma',\sigma) \mathrm{e}^{-\beta\mathcal{H}\{\sigma\}} \right\}$$

where $\mathcal{H}'\{\sigma'\}$ shall again have the same structure as $\mathcal{H}\{\sigma\}$, and where $P(\sigma',\sigma)$ is called *blocking function* and shall have the following properties:

$$P(\sigma',\sigma) \geq 0 \qquad \forall \sigma', \sigma \qquad \text{and} \qquad \operatorname*{Tr}_{\sigma'}^{N} \left\{ P(\sigma',\sigma) \right\} = 1$$

From the second condition follows

$$Z' = \operatorname*{Tr}_{\sigma'}^{N'} \left\{ \mathrm{e}^{-\beta\mathcal{H}'\{\sigma'\}} \right\} = \operatorname*{Tr}_{\sigma'}^{N'} \operatorname*{Tr}_{\sigma}^{N} \left\{ P(\sigma',\sigma) \mathrm{e}^{-\beta\mathcal{H}\{\sigma\}} \right\} = \operatorname*{Tr}_{\sigma}^{N} \left\{ \mathrm{e}^{-\beta\mathcal{H}\{\sigma\}} \right\} = Z$$

which means that the partition sum does not change. This implies that the thermodynamic behaviour is the same.

The blocking function often has the following additional property, which makes actual computations easier

$$P(\sigma',\sigma) = \prod_{i}^{N'} P(\sigma'_i, \sigma_{i_j})$$

meaning that the block spins σ'_i only depend on the spins σ_{i_j} in the correspondent block.

Easy examples for the one dimensional Ising chain, where three spins σ are blocked together, are the *Majority Rule*

$$P(\sigma'_i, \sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}) = \begin{cases} 1 & \text{if } \sigma'_i \cdot (\sigma_{i_1} + \sigma_{i_2} + \sigma_{i_3}) > 0 \\ 0 & \text{otherwise} \end{cases}$$

or the simple chosing of the middle spin

$$P(\sigma'_i, \sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}) = \delta_{\sigma'_i, \sigma_{i_2}} \tag{1}$$

The latter opens the possibility to do some calculations analytically.

One dimensional Ising Chain

A block spin transformation with the blocking function of equation (1) is explicitly done now.

If it is done with a nearest neighbour interaction Hamiltonian $-\beta\mathcal{H}\{\sigma\} = K\sum_j \sigma_j\sigma_{j+1}$ one will have to introduce a constant offset to the free energy. As the Hamiltonians shall have identical structure, the starting Hamiltonian is chosen as

$$-\beta\mathcal{H}\{\sigma\} = A + K\sum_j^N \sigma_j\sigma_{j+1}$$

and therefore:

$$\begin{aligned}
Z &= \operatorname*{Tr}_{\sigma}^{N} \left\{ \mathrm{e}^{A+K\sum_j \sigma_j\sigma_{j+1}} \right\} \\
&= \operatorname*{Tr}_{\sigma}^{N} \operatorname*{Tr}_{\sigma'}^{N'} \left\{ \mathrm{e}^{A} \prod_i^{N'} \delta_{\sigma'_i,\sigma_{i_2}} \mathrm{e}^{\mathrm{K}\sum_{\mathrm{j}} \sigma_{\mathrm{j}}\sigma_{\mathrm{j}+1}} \right\} \\
&= \operatorname*{Tr}_{\sigma'}^{N'} \left\{ \mathrm{e}^{A} \prod_i^{N'} \sum_{\sigma_{i_3}} \sum_{\sigma_{i+1_1}} \mathrm{e}^{K(\sigma'_i\sigma_{i_3} + \sigma_{i_3}\sigma_{i+1_1} + \sigma_{i+1_1}\sigma'_{i+1})} \right\} \\
&\quad \text{because each } \sigma'_i \text{ contributes in the same way} \\
&= \operatorname*{Tr}_{\sigma'}^{N'} \left\{ \prod_i^{N'} \exp\Big[A + \ln\Big(\mathrm{e}^{K(\sigma'_i+1+\sigma'_{i+1})} + \mathrm{e}^{K(\sigma'_i-1-\sigma'_{i+1})} + \mathrm{e}^{K(-\sigma'_i-1+\sigma'_{i+1})} + \mathrm{e}^{K(-\sigma'_i+1-\sigma'_{i+1})} \Big) \Big] \right\} \\
&\quad \text{by carrying out the sums and introducing the identity } \exp\Big[\ln\Big(\ldots\Big)\Big] \\
&\overset{!}{=} \operatorname*{Tr}_{\sigma'}^{N'} \left\{ \prod_i^{N'} \exp\Big[A' + K'\sigma'_i\sigma'_{i+1} \Big] \right\} \\
&= \operatorname*{Tr}_{\sigma'}^{N'} \left\{ \mathrm{e}^{A'+K'\sum_i \sigma'_i\sigma'_{i+1}} \right\}
\end{aligned}$$

The values of K' and A' can be obtained by comparing all possible values of the exponentials:

σ_i	σ_{i+1}	$A + \ln\Big(\ldots\Big)$	$A' + K'\sigma'_i\sigma'_{i+1}$
$+$	$+$	$A + \ln\Big(\mathrm{e}^{3K} + 3\mathrm{e}^{-K}\Big)$	$A' + K'$
$-$	$-$		
$+$	$-$	$A + \ln\Big(3\mathrm{e}^{K} + \mathrm{e}^{-3K}\Big)$	$A' - K'$
$-$	$+$		

Therefore, the **Renormalisation Group Equation** $\{A, K\} \to \{A', K'\}$ for this transformation is

$$\begin{aligned} A' &= \frac{1}{2}\ln(10 + 6\cosh 4K) + A \\ K' &= \frac{1}{2}\ln\Big(\frac{\mathrm{e}^{3K} + 3\mathrm{e}^{-K}}{3\mathrm{e}^{K} + \mathrm{e}^{-3K}}\Big) \end{aligned}$$

The constant A only affects itself, and not the real coupling constant K. It only produces a constant offset for the free energy and leaves its derivatives untouched. It is irrelevant for the following procedures, namely the finding of fixpoints and the calculation of potential critical exponents, but this is not done here. Instead, a deeper look into the finding of Renormalisation Group Equations is taken.

Two dimensional quadratic Ising lattice

When working in Real-Space, which simply means that it is not the Fourier-Space, a big problem arises in systems with more than one dimension. The transformation creates longer ranged interaction terms. This will be shown now:

Consider a quadratic lattice and a blocking function similar to equation (1), only with two spins: $P(\sigma_i', \sigma_{i_1}, \sigma_{i_2}) = \delta_{\sigma_i', \sigma_{i_1}}$

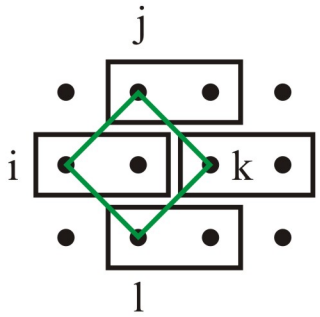

Figure 1: Part of the infinite lattice. The blocks contain two sites. The spins labeled with i, j, k and l span the new lattice, indicated in green.

Starting with a clean nearest neighbour interaction Hamiltonian, ignoring the constant free eneregy shift A,

$$-\beta\mathcal{H}\{\sigma\} = K \sum_{<ij>} \sigma_i \sigma_j$$

one gets:

$$
\begin{aligned}
Z &= \operatorname*{Tr}_{\sigma}^{N} \Big\{ \exp\Big[K \sum_{<ij>} \sigma_i \sigma_j \Big] \Big\} \\
&= \operatorname*{Tr}_{\sigma}^{N} \operatorname*{Tr}_{\sigma'}^{N'} \Big\{ \prod_{i'}^{N'} \delta_{\sigma_{i'}', \sigma_{i_1'}} \exp\Big[\mathrm{K} \sum_{<\mathrm{ij}>} \sigma_{\mathrm{i}} \sigma_{\mathrm{j}} \Big] \Big\} \\
&= \operatorname*{Tr}_{\sigma'}^{N'} \Big\{ \prod_{i}^{N'} \sum_{\sigma_{i_2}} \exp\Big[K \sigma_{i_2} (\sigma_i' + \sigma_j' + \sigma_k' + \sigma_l') \Big] \Big\}
\end{aligned}
$$

because again each σ_i' contributes in the same way. The four σ' are ordered quadratically.

$$
\begin{aligned}
&= \operatorname*{Tr}_{\sigma'}^{N'} \Big\{ \prod_{i}^{N'} \exp\Big[\ln\Big(2\cosh\big(K(\sigma_i' + \sigma_j' + \sigma_k' + \sigma_l') \big) \Big) \Big] \Big\} \\
&\stackrel{!}{=} \operatorname*{Tr}_{\sigma'}^{N'} \Big\{ \prod_{i}^{N'} \exp\Big[A' + \frac{K'}{2} \sum_{<mn>_i}^{4} \sigma_m' \sigma_n' \Big] \cdot \exp\Big[L' \sum_{\ll mn \gg_i}^{2} \sigma_m' \sigma_n' + M' \sigma_i' \sigma_j' \sigma_k' \sigma_l' \Big] \Big\}
\end{aligned}
$$

By comparing all possible values of the exponentials it is easily shown that new interaction terms L' and M' need to be introduced to find a solution. The index i indicates that the sums run over the (next-) nearest neighbours correspondent to the spin σ_i'.

$$
\begin{aligned}
&= \operatorname*{Tr}_{\sigma'}^{N'} \Big\{ \exp\Big[A' + K' \sum_{<ij>} \sigma_i' \sigma_j' \Big] \cdot \exp\Big[L' \sum_{\ll ij \gg} \sigma_i' \sigma_j' + M' \sum_{[[ij]]} \sigma_i' \sigma_j' \sigma_k' \sigma_l' \Big] \Big\}
\end{aligned}
$$

where the term with M' is called plaquette interaction.

This is not self-consistent, as $\{K\} \to \{K', L', M'\}$. So, the starting Hamiltonian should already include these three couplings, but then even more interaction terms would be generated. Starting only with nearest and next nearest neighbour couplings, therefore ignoring the plaquette interaction, one already reaches infinite range interaction in a single step. This is shown now. Starting with

$$
-\beta \mathcal{H}\{\sigma\} = K \sum_{<ij>} \sigma_i \sigma_j + L \sum_{\ll ij \gg} \sigma_i \sigma_j
$$

the partition sum transforms

$$
\begin{aligned}
Z &= \operatorname*{Tr}_{\sigma}^{N} \operatorname*{Tr}_{\sigma'}^{N'} \Big\{ \prod_{i'}^{N'} \delta_{\sigma_{i'}', \sigma_{i_1'}} \exp\Big[\mathrm{K} \sum_{<\mathrm{ij}>} \sigma_{\mathrm{i}} \sigma_{\mathrm{j}} + \mathrm{L} \sum_{\ll \mathrm{ij} \gg} \sigma_{\mathrm{i}} \sigma_{\mathrm{j}} \Big] \Big\} \\
&\neq \operatorname*{Tr}_{\sigma'}^{N'} \Big\{ \prod_{i}^{N'} \sum_{\sigma_{i_2}} \exp\Big[\mathrm{K} \sigma_{\mathrm{i}_2} (\sigma_{\mathrm{i}}' + \sigma_{\mathrm{j}}' + \sigma_{\mathrm{k}}' + \sigma_{\mathrm{l}}') + \frac{\mathrm{L}}{2} (\sigma_{\mathrm{i}}' \sigma_{\mathrm{j}}' + \sigma_{\mathrm{j}}' \sigma_{\mathrm{k}}' + \sigma_{\mathrm{k}}' \sigma_{\mathrm{l}}' + \sigma_{\mathrm{l}}' \sigma_{\mathrm{i}}') \Big] \cdot \\
&\quad \cdot \exp\Big[L \sigma_{i_2} (\sigma_{j_2} + \sigma_{l_2}) \Big] \Big\}
\end{aligned}
$$

where the last exponential indicates the next nearest neighbour interaction with spins that have to be summed over. The factorization does not work here as this last term would require an infinite number of new interaction terms.

Ignoring every interaction besides K' and L' one gets an approximation to the exact Renormalisation Group Equation $\{K, L\} \to \{K', L'\}$:

$$
\begin{aligned}
K' &= \frac{1}{4} \ln(\cosh 4K) + L \\
L' &= \frac{1}{8} \ln(\cosh 4K)
\end{aligned}
$$

A systematic approach

The approximation above was obtained in a rather random way. A more systematic approach to find approximative Renormalisation Group Equations is to consider finite lattices. (The above examples contained numbers of spin sites N and N', but the calculations were done independent of them. All above holds for the thermodynamic limit $N \to \infty$.)

Now, consider a lattice with exactly 16 spins, that are to be grouped together into 4 block spins, with periodic boundary conditions.

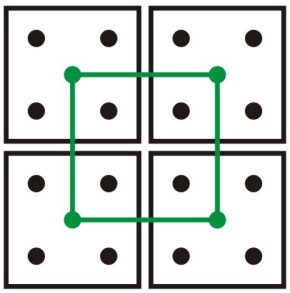

Figure 2: Finite lattice with 16 sites. The new lattice consists of 4 spins, indicated in green.

The blocking function can be defined in many ways, but it is important that the spin symmetries are conserved. This means that if all original spins are flipped, the block-spins need to flip as well:

$$\{\sigma\} \to -\{\sigma\} \qquad \Longrightarrow \qquad \{\sigma'\} \to -\{\sigma'\}$$

This can be achieved by defining

$$\sigma'_i = \begin{cases} 1 & \text{for } (++++), (+++-), (++-+), (+--++), (-++++), (+-+-), (++--), (+--+) \\ -1 & \text{for the flipped combinations} \end{cases}$$

This system can be solved exactly by carrying out the trace explicitly

$$\mathrm{e}^{-\beta\mathcal{H}'\{\sigma'\}} = \operatorname*{Tr}_{\sigma}^{N} \left\{ P(\sigma', \sigma) \mathrm{e}^{-\beta\mathcal{H}\{\sigma\}} \right\}$$

where the $2^{16} = 65536$ summands can be handled by modern computers. The number of block-spins defines the number of possible interactions. If odd couplings are abscent, the possible couplings are the same as above, namely $\{A', K', L', M'\}$.

This procedure of numerically calculating the Renormalisation Group Equation in a finite system can be done for arbitrary lattice forms and dimensions, and therefore provides a systematic approach.