Example for exact renormalization: The Kosterlitz renormalization group equations

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July 7, 2009

Outline

KT revisited

Hamiltonian for the XY–model with $\theta = \theta(x)$, $J := \frac{K}{\beta}$

$$H = \frac{J}{2} \int d^2x \; (\nabla\theta)^2$$

Rewrite Hamiltonian postulating vortex field

$$\oint \nabla\theta \, d^2 \, r = 2\,\pi\, q, \qquad q \in \mathbb{Z}$$

Vortex postulate

Rewrite Hamiltonian postulating vortex field
introduce "vortex charge density":

$$\rho(\vec{r}) = \sum_j q_j \, \delta(\vec{r} - \vec{r}_j)$$

and require that

$$\int \rho(\vec{r}) \, d^2 r = 0$$

Vortex postulate

Hamiltonian for the spin vortex interaction

$$H_v = 2\pi^2 J \sum_{i \neq j} q_i q_j G(\vec{r}_i - \vec{r}_j) + E_0$$

with winding number $q \in \mathbb{Z}$ and the Green's function and self energy defined by

$$G(\vec{r}_i - \vec{r}_j) = -\frac{1}{2\pi} \ln\left(\frac{r_i - r_j}{a}\right), \qquad E_0 = -\mu_0 \sum_i q_i^2$$

where a: "UV–cutoff", of the order of vortex core diameter
Define furthermore the "potential"

$$V(\vec{r}_i - \vec{r}_j) = 2\pi\, J\, Q_i\, Q_j\, \ln\left(\frac{r_i - r_j}{a}\right), \qquad Q_i = \sqrt{\pi}\, q_i$$

Vortices and their interactions

- vortex interaction has same form as 2D Coulomb potential
- winding number $\widehat{=}$ charge
- vortices created in pairs $\rightarrow$ dipoles

$\Rightarrow$ description of the classical 2D Coulomb gas with completely same formalism!

Partition sum

Partition sum for the vortices

$$Z_v = \sum_{N_+,N_-}^{\infty} \frac{1}{N_+!N_-!}\delta_{N_+,N_-}\, y^{N_++N_-} \int_{\Omega^{N_++N_-}} \prod_l \frac{d^2 r_l}{a^2}\, e^{-\beta V(|\vec{r}_i - \vec{r}_j|)}$$
$$\approx 1 + y^2 \frac{\Omega}{a^2} \int \frac{d^2 r}{\Omega}\, e^{-\beta V(r)}$$

with fugacity $y = e^{\beta\mu}$ of the system assumed to be small (low density)

Vortex dipoles

Treat $(+,-)$-pair of vortices as "dipole"
Calculate average number of dipoles in the distance r per volume:

$$
\begin{aligned}
n(r)\, d^2 r &= \frac{d^2 r}{\Omega} \left\langle \sum_{n>m} \delta(\vec{r} - (\vec{r}_n^{\,+} - \vec{r}_m^{\,-})) \right\rangle \\
&\approx \frac{y^2}{a^4}\, e^{-\beta V(r)}\, 2\, \pi\, r\, dr
\end{aligned}
$$

to order $O(y^2)$.

Dipole interactions

- take into account dipole–dipole interactions for low dipole density
- assume: screening is important
- introduce "dielectric function" $\epsilon(r) = 1 + 4\pi\chi(r)$ with susceptibility χ
- $V(r) = 2\pi J \, ln(\frac{\vec{r}}{a}) \quad \longrightarrow \quad \vec{F} = -\vec{\nabla} V = -\frac{2\pi J}{r}\hat{r}$
- $\frac{q_1 q_2}{r} \longrightarrow \frac{q_1 q_2}{\epsilon(r)\, r}$: $\quad V(r) \rightarrow \bar{v}(r)$

Scale dependent interaction

vortex dipole interaction with screening:

$$\bar{v}(r) = -\int_a^r \frac{2\,\pi\,J}{r'\,\epsilon(r')}\,dr',$$

define scale dependent interaction as

$$J \to J(r) := \frac{J}{\epsilon(r)},$$

average number of dipoles in distance r per volume

$$n(r) \to \bar{n}(r) := \frac{y^2}{a^4}\,2\,\pi\,r\,e^{-\beta \bar{V}(r)}$$

Susceptibility & polarizability

need to find dielectric function!
basic electrodynamics, linear response:

- susceptibility: $\chi(\vec{r}) = \int_a^r \alpha(r')\,\bar{n}(r')dr'$
- dipole moment: $\vec{p} := \vec{r}\,q$
- polarizability: $\alpha(r) = \frac{\partial\langle p cos(\theta)\rangle}{\partial E}|_{E=0} = \frac{\beta p^2}{2} = \frac{\beta\pi J}{2}r^2$

Differential equation for $\epsilon(r)$

Goal: find differential equation for the dielectric function

- comining equations for $\bar{v}$, $\bar{n}$, χ and α leads to complicated nonlinear integral equation
- thus try different approch: parameterize couplings by "flow parameter" $\lambda := ln\left(\frac{r}{a}\right)$

scaling form of the couplings:

$$K(\lambda) := \frac{\beta J}{\epsilon(\lambda)} =: \frac{K}{\epsilon(\lambda)}, \quad y(r) := y\,\frac{1}{\sqrt{2\pi}}\left(\frac{r}{a}\right)^2 e^{-\frac{\beta}{2}\bar{v}(r)}$$

where

$$\beta\,\bar{v}(r) = 2\pi \int_1^\lambda K(\lambda')d\lambda'$$

KT flow equations

Obtain flow equations by differentiation of the parameterized couplings with respect to λ:

$$\frac{dK^{-1}(\lambda)}{d\lambda} = \pi\, y^2(\lambda)$$

$$\frac{y(\lambda)}{d\lambda} = (2 - \pi\, K(\lambda))\; y(\lambda)$$

RG flow I

For convenience, define

$$x(\lambda) := \frac{2}{\pi\, K(\lambda)} - 1 = K_c\,\left(K^{-1}(\lambda) - K_c^{-1}\right), \qquad K_c^{-1} := \frac{\pi}{2}$$

rewrite flow equations

$$\frac{dx}{d\lambda} = 2\,y^2, \qquad \frac{dy}{d\lambda} = \frac{2}{1+x}\,x\,y$$

and thus

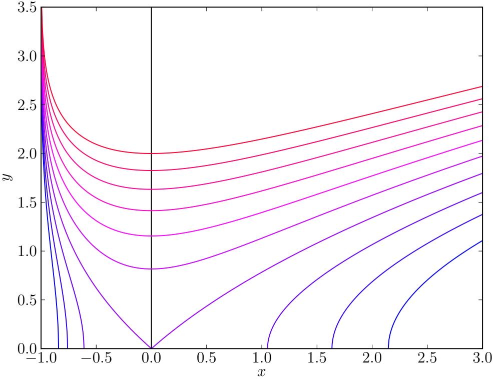
$$\frac{dy}{dx} = \frac{1}{1+x}\,\frac{x}{y}$$

RG flow II

solution of the flow equation

$$y_c(x) = \pm\sqrt{2}\,\sqrt{x + c - ln\,(1 + x)}, \qquad c \in \mathbb{R}, x \in (-1, \infty)$$

is a set of functions describing the RG flow



Further considerations

- theory unifying vortex and spin wave approach: no qualitative changes (T_c is different)
- treat velocity fluctuations in superfluid He^4-film in much the same way
  $\rightarrow$ universal jump discontinuity in the superfluid density

see literature:

- Nelson, “Two dimensional Superfludity and Melting” in Cohen, “Fundamental Problems in Statistical Mechanics V” (1980)
- P.M. Chaikin and T.C. Lubensky, “Principles of Condensed Matter Physics” (Cambridge University Press, Cambridge, 1995)

The End

Thanks for your attention!