Exact solution to the Ising model

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Exact Solution to the Ising-Model

Outline:

Starting point: Hamiltonian derived in previous lecture

Transformation to free fermion-style Hamiltonian

Diagonalization

Partition sum, free energy and specific heat

Starting point

$$H = \sum_n \sigma_3(n) - \lambda \sum_n \sigma_1(n)\sigma_1(n+1)$$

describing one-dimensional quantum chain in a transverse magnetic field with periodic boundary conditions

Remarks:

Unitary transformation $\sigma_1 \rightarrow -\sigma_3; \sigma_3 \rightarrow \sigma_1$

$\sigma_1(n) = 1 \times \ldots \times \sigma_1 \times \ldots \times 1$

Transformation I

$\sigma_n^{\pm} \equiv \frac{1}{2}\left(\sigma_1(n) \pm i\,\sigma_2(n)\right)$ satisfying $\left\{\sigma_n^{+}, \sigma_n^{-}\right\} = 1 \qquad \left[\sigma_n^{+}, \sigma_m^{-}\right] = 0;\;\; n \neq m$ and $(\sigma^{\pm})^2 = 0$

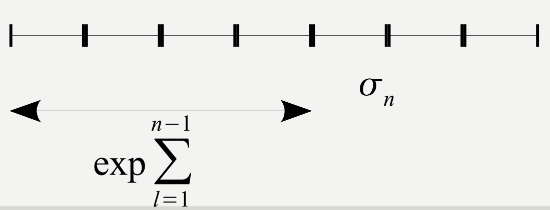
$$\Rightarrow \; H = \sum_n \left(1 - 2\sigma_n^{-}\sigma_n^{+}\right) - \lambda \sum_n \left(\sigma_n^{+} + \sigma_n^{-}\right)\left(\sigma_{n+1}^{+} + \sigma_{n+1}^{-}\right)$$

and then

$c_n \equiv \exp\left(i\pi \sum_{l=1}^{n-1} \sigma_l^{+}\sigma_l^{-}\right)\sigma_n^{-}$ and $c_n^{\dagger} \equiv \sigma_n^{+} \exp\left(-i\pi \sum_{l=1}^{n-1} \sigma_l^{+}\sigma_l^{-}\right)$

to arrive at Fermion operators with $\left\{c_n, c_m^{\dagger}\right\} = \delta_{nm}$

Motivation:



$$\left\{\sigma_n^{+}, \exp\left(i\pi\,\sigma_n^{+}\sigma_n^{-}\right)\right\} = 0$$

Transformation II

$$\Rightarrow \quad H=\sum_n 2c_n^\dagger c_n-\lambda\sum_n (c_n^\dagger-c_n)(c_{n+1}^\dagger+c_{n+1})$$

(Free fermion Hamiltonian)

in momentum space: $a_k=\frac{1}{\sqrt{2N+1}}\sum_{n=-N}^{N} e^{ikn}c_n;\quad k=0,\pm\frac{2\pi}{2N+1},\ldots,\pm\frac{2\pi N}{2N+1}$

$$\Rightarrow \quad H=\sum_{k>0} 2(1-\lambda\cos k)(a_k^\dagger a_k+a_{-k}^\dagger a_{-k})+2i\lambda\sum_{k>0}\sin k\,(a_k^\dagger a_{-k}^\dagger+a_k a_{-k})$$

after symmetrization.

Diagonalization

construct explicitly the matrix in Fock space

calculate the eigenvalues:

$$a = -2\mathrm{i}\,\lambda \sin k\,; \quad b = 1 - \lambda \cos k$$

$$\epsilon_{0/1} = 2b$$ (twice)

$$\epsilon_{2/3}(k) = 2b \pm 2\sqrt{1 + \lambda^2 - 2\lambda \cos k}$$

Partition sum and free energy

The partition sum yields $Z = tr(e^{-\tau H}) = \sum_{\nu} \alpha_{\nu}^{N} \underset{\text{TD limit}}{\rightarrow} \exp(-\tau N \int_0^{\pi} dq\, \epsilon_3(q))$

$\epsilon_3(k) = (1 - \lambda \cos k - \sqrt{1 + \lambda^2 - 2\lambda \cos k})$

Thus we get for the free energy density

$$\frac{F}{N} = -\frac{1}{\tau N} \ln Z = f = -\frac{1}{\pi} \int_0^{\pi} dk \sqrt{1 - 2\lambda \cos k + \lambda^2} + const.$$

Specific heat

We analyze the free energy integral for small values of k.
To do so, we cut the integral in two parts.

$$\int_0^{\pi} \rightarrow \int_0^{\epsilon} + \int_{\epsilon}^{\pi} \qquad \cos k \approx 1 - \frac{k^2}{2}$$

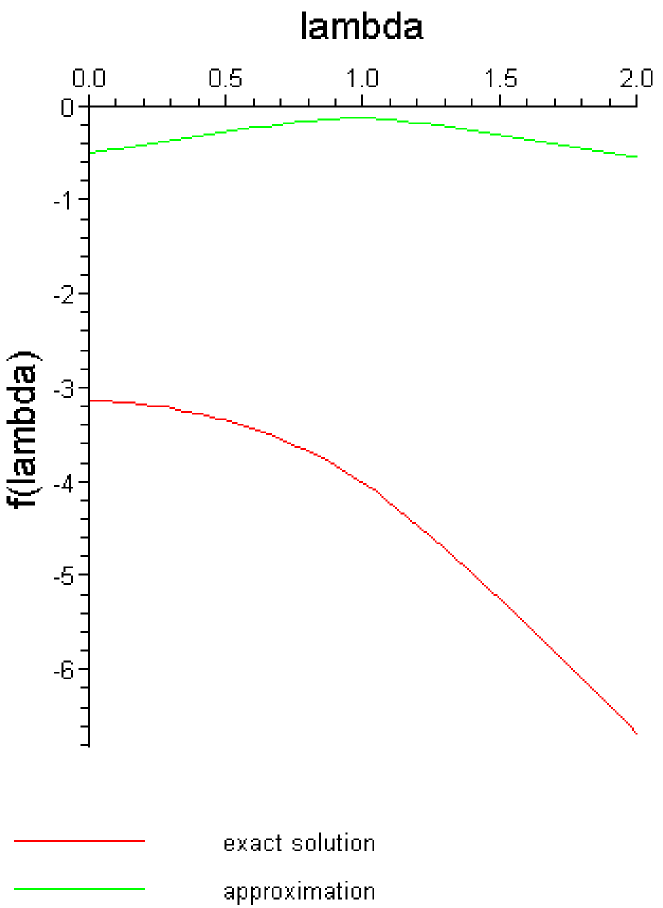
When we integrate this and identify

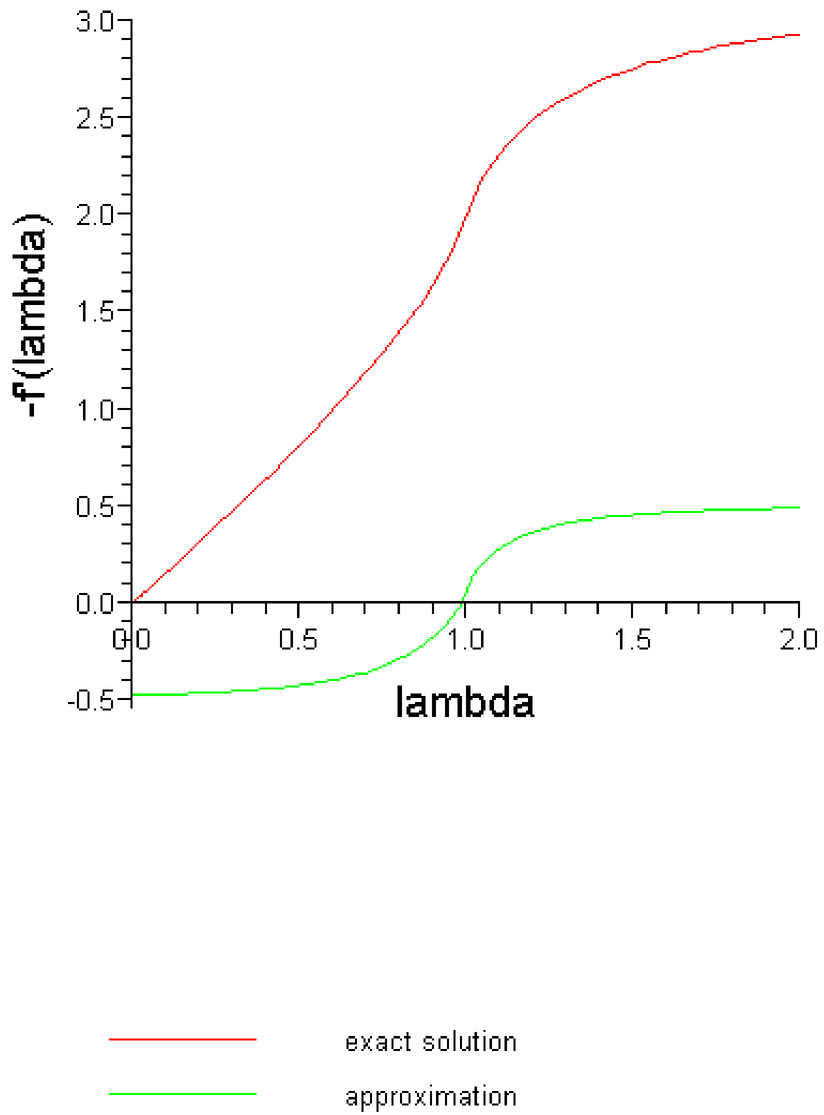
$$t \equiv |1-\lambda| \sim \frac{T-T_c}{T_c}$$

we get

$$f = -\int_0^{\epsilon} dk \sqrt{t^2 + 2(1-t)k^2} \sim -t^2 \ln|t| \;\Rightarrow\; c = \frac{-\partial^2}{\partial t^2} f \sim -\ln|t|$$

Free energy, entropy





Specific heat