Übung zur Vorlesung
Mathematische Statistische Physik
Sommersemester 2008
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Problem 22

Consider the classical infinite spin Ising-Model with Hamiltonian $H = -J\sum_{i=1}^{N} x_i x_{i+1}$, where the indices are understood as mod N and the x_i take values in the range $-1 \leq x_i \leq 1$ (and not just $x \in \{-1, 1\}$).

(a) Show that the partition function can be written as $Z_N = \int_{-1}^{1} L^{(N)}(x,x)\mathrm{d}x$, where $L^{(N)}$ is the Nth iterate of the integral operator $L(x,y) = \exp(Kxy)$ ($K = \beta J$), which linearly maps $\psi(x)$ on $\int_{-1}^{1} L(x,y)\psi(y)\mathrm{d}y$.

(b) Suppose that $L(x,y)$ can be expressed in terms of a complete orthonormal set of (real) eigenfunctions ψ_j, ie $L(x,y) = \sum_{j=0}^{\infty} \lambda_j \psi_j(x)\psi_j(y)$ ($\lambda_0 < \lambda_1 < \lambda_2 < \ldots$). Show that in the thermodynamic limit the free energy per lattice site becomes $f = -kT\log\lambda_0$ and the pair correlation function

$$\rho(r) = \langle x_i x_{i+r}\rangle = \sum_{m=0}^{\infty} \left(\frac{\lambda_m}{\lambda_0}\right)^r \left(\int_{-1}^{1} \psi_0(x) x \psi_m(x)\mathrm{d}x\right)^2 .$$

(c) The eigenfunctions $\psi_j(x)$ of the integral operator $L(x,y)$ are the oblate spheroidal wave functions, which are even (odd) functions of x for even (odd) integral j. For low temperature (and large K) one can show that

$$\int_{-1}^{1} x^2\psi_0^2(x)\mathrm{d}x = 1 + O(K^{-1}),$$

and

$$\left(\int_{-1}^{1} \psi_0(x) x \psi_1(x)\mathrm{d}x\right)^2 = 1 + O(K^{-1}).$$

Take advantage of this fact and of the symmetry of the wave functions ψ_j to show that in the low temperature limit $\rho(r)$ decays exponentially with correlation length $1/\ln(\lambda_0/\lambda_1)$.

Problem 23

The partition function for the one-dimensional n-vector $O(n)$ model in the absence of an external field is

$$Z(T) = \int \mathrm{d}\vec{S}_1 \delta(\vec{S}_1^2 - 1)\ldots\int \mathrm{d}\vec{S}_N \delta(\vec{S}_N^2 - 1)\exp(K\vec{S}_1\cdot\vec{S}_2)\cdots\exp(K\vec{S}_{N-1}\cdot\vec{S}_N),$$

with $\vec{S}_i \in \mathbb{R}^n$.

(a) Show that

$$\zeta(T) = \int d\vec{S}_N \delta(\vec{S}_N^2 - 1) \exp(K \vec{S}_{N-1} \cdot \vec{S}_N) \,,$$

is independent of the orientation of $\vec{S}_{N-1}$, and since $\vec{S}_{N-1}^2 = 1$, independent of $\vec{S}_{N-1}$. Hence show that

$$Z(T) = \frac{1}{2} \mathcal{S}_n \left[\zeta(t)\right]^{N-1} \,,$$

where $\mathcal{S}_n$ is the surface area of a unit sphere in n dimensions.

(b) Show that for the Heisenberg model ($n = 3$), the free energy per lattice site in the thermodynamic limit equals

$$f(T) = -k_B T \ln\left(\frac{2\pi \sinh K}{K}\right) \,.$$

(c) In the case of the Heisenberg model, f can also be calculated using the transfer operator method of Problem 22. Show that the corresponding integral operator in spherical coordinates reads $L(r_1, \theta_1, \phi_1; r_2, \theta_2, \phi_2) := \delta(r_2^2 - 1) L(\theta_1, \phi_1; \theta_2, \phi_2) := \delta(r_2^2 - 1) \exp\left[K \cos(\Theta)\right]$, with $\cos(\Theta) = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\cos(\phi_2 - \phi_1)$.

Using the identity

$$\exp(K \cos(\Theta)) = (\pi/2K)^{1/2} \sum_{l=0}^{\infty} (2l+1) I_{l+\frac{1}{2}}(K) P_l(\cos(\Theta)) \,,$$

(where $I_{l+\frac{1}{2}}(x)$ are modified Bessel functions and $P_l(x)$ Legendre polynomials) and the addition theorem for spherical harmonics

$$P_l(\cos(\Theta)) = 4\pi(2l+1)^{-1} \sum_{m=-l}^{l} Y_{lm}^*(\theta_2, \phi_2) Y_{lm}(\theta_1, \phi_1) \,,$$

(the star denoting the complex conjugate) show that the eigenfunctions of the operator are the spherical harmonics Y_{lm}. Hence prove that for periodic boundary conditions, the thermodynamic limit yields the same result for f as for open boundary conditions (as considered in (b)).

Please notice that from now on the problem session takes place every Tuesday at 12:15 in room 249.